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Solvability of a Boundary Value Problem for Transonic Flow in a Nozzle

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Abstract. A nonlinear perturbation problem for steady two-dimensional inviscid transonic flow in a nozzle is studied. The existence of a smooth solution to the problem is proved under the condition of positive acceleration of the given flow. The proof involves the method of singular perturbations for solving a linear problem associated with the nonlinear one. The technique for obtaining a priori estimates is simpler than that used in previous papers.

Keywords: *Transonic flows, perturbations, boundary value problems, equations of mixed type, Sobolev spaces, a priori estimates*

AMS subject classification: Primary 35 M 10, secondary 76 H 05

1. Introduction

We consider gas flow in a Laval nozzle, i.e. a channel of variable cross section area which has a minimum at some value of the longitudinal coordinate. In such a nozzle, there exists a flow regime with subsonic–supersonic transition, which is typically shock-free. This fact is well documented by experimental and numerical studies.

Mathematical conditions of the existence of a smooth solution to the perturbation problem for flow in a nozzle were established by Kuz'min [14 - 16] and Larkin [19]. The most simple condition is the positiveness of the acceleration of the given transonic flow. The solvability was proved using a theory of the equation of mixed type

$$Lu := k(x,t)u_{tt} + \sum_{i,j=1}^{n} [a_{ij}(x,t)u_{x_i}]_{x_j} - \alpha(x,t)u_t + c(x,t)u = f(x,t) \qquad (1.1)$$

where the subscripts t, x_i and x_j denote partial derivatives, the coefficient $k(x,t)$ is of variable sign in a given cylindrical domain, while the differential operator $\sum a_{ij}u_{x_ix_j}$ with respect to the variables $x = (x_1, ..., x_n)$ is a uniformly elliptic one.

In the theory of equation (1.1), of great importance is a *basic boundary value problem*, in which the type of that equation is supposed to be elliptic, i.e. $k > 0$, on both foundations of the cylindrical domain. Solvability of the basic problem was studied in [7, 15] with a method of singular perturbations, so that a solution u^0 was obtained as

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limit of solutions u^ε of the third-order equation $\varepsilon u_{ttt} + Lu = f$ as $\varepsilon \to 0$. The study of the basic problem has made it possible to analyse other problems for equation (1.1) and a perturbation problem for transonic flow in the nozzle [15 - 16]. Further developments of the theory provided a framework for analysis of transonic flow with a local supersonic region under a boundary condition that prevents formation of shock waves [17].

We note that a number of attempts to study transonic flow admitting shock waves [8 - 13, 21] faced huge mathematical difficulties because of the entropy condition to be held across the shocks. In a recent paper [11], Gamba and Morawetz established solvability of a viscous model for a system of equations governing the velocity potential $\varphi(x, y)$ and the gas density $\rho(x, y)$. Also, they proved the existence of a convergent subsequence $(\varphi^\varepsilon, \rho^\varepsilon)$ with a limit (φ^0, ρ^0) as $\varepsilon \to 0$. However, it has not been proved that the limit is a weak solution to the inviscid system.

In this paper, we pursue the study of transonic flow without shock waves. The technique for obtaining a priori estimates is modified, and the analysis of the basic problem is presented in a form which is most convenient for transonic flow research. In Section 2, we formulate a problem for flow with subsonic-supersonic transition in a nozzle. Sections 3 - 5 are concerned with solvability of the basic problem for equation (1.1) in the case of two independent variables. In contrast with [15, 17], we avoid using the variables (x, t), which are relevant in the general theory of mixed type equations, and analyse the basic problem directly in the physical (x, y)-plane where the flow is governed by the von Karman equation. In Section 6, solvability of the nonlinear perturbation problem for flow with positive acceleration is established.

2. Formulation of the problem for transonic flow in a nozzle

For simplicity, we consider the nearsonic approximation, in which a steady potential flow of inviscid gas is governed by the von Karman equation [4]

$$(\gamma + 1)(1 - \varphi_x)\varphi_{xx} + \varphi_{yy} = 0 \tag{2.1}$$

where $\varphi = \varphi(x, y)$ is the non-dimensional velocity potential and $\gamma > 1$ is the ratio of specific heats. We prescribe the standard slip condition $\varphi_n = (\nabla\varphi, \vec{n}) = 0$ on the walls of a nozzle, where $\vec{n}$ is the normal vector. Using asymptotic expansions with the deviation of the wall contour from a straight segment [5], one can replace the condition $\varphi_n = 0$ on the walls by $\varphi_y(x, \pm 1) = \beta^\pm(x)$, where $\beta^\pm$ are the angles made by the tangent to the upper and lower walls with the x-axis. Hence, the problem for flow in a nozzle can be formulated in a rectangle of the (x, y)-plane as follows (see Figure 1). Find a solution φ of equation (2.1) in the domain

$$G_{noz} = \Big\{(x, y) \in \mathbb{R}^2 : 0 < x < l_{noz} \text{ and } -1 < y < 1\Big\}$$

subject to the boundary conditions

$$\left.\begin{array}{ll} \varphi(0, y) = 0,\ \varphi_x(0, y) < 1 & \text{for } -1 < y < 1 \\ \varphi_x(l_{noz}, y) > 1 & \text{for } -1 < y < 1 \\ \varphi_y(x, \pm 1) = \beta^\pm(x) & \text{for } 0 < x < l_{noz}. \end{array}\right\} \tag{2.2}$$

By virtue of (2.2), the flow velocity is subsonic at the inlet and supersonic at the outlet of the nozzle. However, its values are not prescribed.

Figure 1

Let φ be a solution to problem (2.1), (2.2). If the contour of the walls is slightly changed, then the angles $\beta^{\pm}$ and the flowfield undergo small perturbations. Both φ and the potential φ^f of the perturbed flow satisfy equation (2.1). Therefore, the difference $u = \varphi - \varphi^f$ is governed by the equation

$$(1 - \varphi_x + u_x)u_{xx} - \varphi_{xx}u_x + \tfrac{1}{1+\gamma}u_{yy} = 0. \tag{2.3}$$

The boundary conditions with respect to u follow from (2.2):

$$\left.\begin{aligned} u(0,y) &= 0 \\ u_y(x,\pm 1) &= f^{\pm}(x) \end{aligned}\right\} \tag{2.4}$$

where the functions $f^{\pm}$ are perturbations of $\beta^{\pm}$. Along with (2.3), let us consider the linear equation

$$L_g u := (1 - \varphi_x + g)u_{xx} - \varphi_{xx}u_x + \tfrac{1}{1+\gamma}u_{yy} = 0 \tag{2.5}$$

in which g is a given function. At $g \equiv 0$, equation (2.5) reduces to

$$L_0 u := [(1 - \varphi_x)u_x]_x + \tfrac{1}{1+\gamma}u_{yy} = 0. \tag{2.6}$$

Theorem 1. *Assume that* $\varphi \in W^{2,2}(G_{noz})$, $\varphi_x(0,y) < 1$ *and* $\varphi_x(l_{noz},y) > 1$. *In addition, let* $\varphi_{xx}(x,y) > 0$ *a.e. in the domain* G_{noz}*, that is the acceleration of the given flow be positive in the nozzle. Then the linearized problem* (2.4), (2.6) *can have at most one solution* $u \in W^{2,2}(G_{noz})$.

Hereafter $W^{p,2}(G_{noz})$ denotes the usual Sobolev space of functions with derivatives up to the order p, which are square-integrable over G_{noz}. The assertion of the theorem follows from [22: Lemma 2]. Below we present a simple proof of Theorem 1 (see [15]) for methodology purposes.

Proof. If there are two solutions u_1 and u_2 of problem (2.4),(2.6), then $u = u_1 - u_2$ satisfies equation (2.6) and boundary conditions (2.4) with $f^{\pm} = 0$. Integrating $L_0 u \cdot u_x$ over G_{noz} and using Green's formula, one obtains the identity

$$
\begin{aligned}
&-2\int_{G_{noz}} L_0 u \cdot u_x \, dxdy \\
&\quad = \int_{G_{noz}} \varphi_{xx} u_x^2 dxdy + \int_{\Gamma_{noz}} \tfrac{1}{1+\gamma} 2u_x u_y \, dx + \big[(\varphi_x - 1)u_x^2 + \tfrac{1}{1+\gamma} u_y^2\big] dy
\end{aligned}
\tag{2.7}
$$

where the integration over the boundary Γ_{noz} of G_{noz} is carried out in the counterclockwise direction. From (2.7), due to the boundary conditions, we find

$$
\begin{aligned}
-2\int_{G_{noz}} L_0 u \cdot u_x \, dxdy = \int_{G_{noz}} \varphi_{xx} u_x^2 dxdy &+ \int_{-1}^{1} [(1-\varphi_x)u_x^2]\big|_{x=0} dy \\
&+ \int_{-1}^{1} \big[(\varphi_x - 1)u_x^2 + \tfrac{1}{1+\gamma} u_y^2\big]\big|_{x=l_{noz}} dy
\end{aligned}
\tag{2.8}
$$

where the left-hand side vanishes, while all the integrals in the right-hand side are non-negative. Therefore, each of the integrals vanishes, and $\varphi_{xx} u_x^2 = 0$ a.e. in G_{noz}. Since $\varphi_{xx} > 0$, we obtain $u_x = 0$. Due to $u(0,y) = 0$, one arrives at $u(x,y) = 0$ a.e. in G_{noz} ∎

3. Solvability of a basic problem

In Sections 3 and 4, we treat a more general equation as compared to (2.5):

$$
Lu := k(x,y)u_{xx} + [a(x,y)u_y]_y - \alpha(x,y)u_x + c(x,y)u = f(x,y) \tag{3.1}
$$

in the rectangle

$$
G = \Big\{(x,y) \in \mathbb{R}^2 : 0 < x < l \text{ and } -1 < y < 1\Big\}.
$$

Suppose that

$$
\left.
\begin{array}{ll}
a(x,y) \ge \delta > 0 & \text{in } G \\
\left.\begin{array}{r} k(0,y) > 0 \\ k(l,y) > 0 \end{array}\right\} & \text{at } -1 \le y \le 1
\end{array}
\right\}
\tag{3.2}
$$

i.e. the type of equation (3.1) is elliptic on both the right- and the left-hand sides of the rectangle G, while the type may be variable in G and at $y = \pm 1$, $0 < x < l$ (see Figure 2). We notice that the x-axis plays the same role in the case of equation (3.1) as the t-axis in (1.1). Actually, one can reduce equation (3.1) to (1.1) by substituting t for x and $-x$ for y.

The boundary conditions are:

$$
\left.
\begin{array}{rl}
u(0,y) = 0 & \text{for } -1 < y < 1 \\
u_x(l,y) = 0 & \text{for } -1 < y < 1 \\
u_y(x,\pm 1) = 0 & \text{for } 0 < x < l.
\end{array}
\right\}
\tag{3.3}
$$

Figure 2

We suppose that the coefficients of equation (3.1) are smooth enough: $k, a \in C^3(\bar{G})$ and $\alpha, c \in C^2(\bar{G})$.

Theorem 2. *Let inequalities* (3.2) *and*

$$\left.\begin{aligned} 2\alpha \pm k_x \geq \delta > 0 \\ a_x \leq 0 \\ c \leq 0 \\ c_x \geq 0 \end{aligned}\right\} \tag{3.4}$$

hold in the domain G. Then there exists a unique solution $u \in W^{2,2}(G)$ of problem $(3.1), (3.3)$ *for all $f \in W^{1,2}(G)$. If, in addition,*

$$\left.\begin{aligned} 2\alpha - 3k_x \geq \delta > 0 \\ 2\alpha - 5k_x \geq \delta > 0 \end{aligned}\right\} \tag{3.5}$$

in G and $f \in W^{3,2}(G)$, then the solution belongs to $W^{4,2}(G_{in})$ where $G_{in} = G \cap \{x : \sigma < x < l - \sigma\}$ for any small $\sigma > 0$.

Theorem 2 was proved with different methods in [6, 7, 15]. Below we use the method of singular perturbations described in [15]. However, we modify the proof and employ simpler cut-off functions.

Proof of Theorem 2. First, let us establish the estimate

$$\|u_x\|_0 \leq m\|Lu\|_0 \tag{3.6}$$

valid for any function $u \in W^{2,2}(G)$ satisfying the boundary conditions (3.3), where $\|\cdot\|_0$ is the norm of the space $L^2(G)$ and $m > 0$. Integrating $Lu \cdot u_x$ over G and using Green's formula, one obtains

$$
\begin{aligned}
-2\int_G Lu \cdot u_x \, dG = \int_G \big[(2\alpha + k_x)u_x^2 - a_x u_y^2 + c_x u^2\big] dG \\
+ \int_\Gamma \big[2a u_x u_y \, dx - \big(k u_x^2 - a u_y^2 + c u^2\big) dy\big]
\end{aligned}
\tag{3.7}
$$

where $dG = dxdy$ and Γ is the boundary of G. Due to (3.3), identity (3.7) reduces to

$$
\begin{aligned}
-2\int_G Lu \cdot u_x \, dG = \int_G \big[(2\alpha + k_x)u_x^2 - a_x u_y^2 + c_x u^2\big] \, dG \\
+ \int_{-1}^{1} (a u_y^2 - c u^2)\big|_{x=l} \, dy + \int_{-1}^{1} (k u_x^2)\big|_{x=0} \, dy.
\end{aligned}
\tag{3.8}
$$

Using conditions (3.2), (3.4) and omitting the non-negative terms on the right-hand side, we find

$$
-2\int_G Lu \cdot u_x \, dG \ge \int_G (2\alpha + k_x)u_x^2 dG \ge \delta \int_G u_x^2 dG.
$$

The left-hand side can be estimated with Young's inequality as

$$
\frac{1}{\gamma}\|Lu\|_0^2 + \gamma\|u_x\|_0^2 \ge -2\int_G Lu \cdot u_x \, dG \ge \delta \int_G u_x^2 dG \tag{3.9}
$$

where γ is a positive parameter. At $\gamma < \delta$, inequality (3.9) yields estimate (3.6), which proves the uniqueness of the solution to problem (3.1), (3.3).

In order to prove the existence of the solution, we insert the term εu_{xxx} into equation (3.1) and pose an extra condition at $x = 0$:

$$
L_\varepsilon u \equiv \varepsilon u_{xxx} + Lu = f \quad (\varepsilon > 0) \tag{3.10}
$$

$$
\left.\begin{aligned}
u(0,y) = u_x(0,y) = 0 \\
u_x(l,y) = 0
\end{aligned}\right\} \tag{3.11}
$$

$$
u_y(x,\pm 1) = 0 \tag{3.12}
$$

An approximate solution $u^{N,\varepsilon}$ of problem (3.10) - (3.12) is sought in Galerkin's form [18: Section 4.3]

$$
u^{N,\varepsilon}(x,y) = \sum_{i=1}^{N} X_i^{N,\varepsilon}(x)\, Y_i(y) \tag{3.13}
$$

where $\{Y_i\}_{i=1}^N$ is a complete system in $W^{2,2}(-1,1)$ which is orthonormal in $L^2(-1,1)$. The functions $X_i^{N,\varepsilon}$ are to be found from the relations $\int_{-1}^{1}(L_\varepsilon u^{N,\varepsilon} - f)\, Y_j dy = 0$ which reduce to

$$
\int_{-1}^{1} \Big[\big(\varepsilon u_{xxx}^{N,\varepsilon} + k u_{xx}^{N,\varepsilon} - \alpha u_x^{N,\varepsilon} + c u^{N,\varepsilon} - f\big) Y_j - a u_y^{N,\varepsilon} \cdot (Y_j)_y\Big] dy = 0 \tag{3.14}
$$

if one integrates by parts the term $(au_y^{N,\varepsilon})_y Y_j$ and omits $(au_y^{N,\varepsilon}Y_j)\big|_{y=\pm 1}$ in view of (3.12).

Relations (3.14) are a system of third-order ordinary differential equations with respect to $X_i^{N,\varepsilon}$. Boundary conditions at $x=0$ and $x=l$ are prescribed according to (3.11) by

$$X_i^{N,\varepsilon}(0) = \frac{dX_i^{N,\varepsilon}}{dx}\bigg|_{x=0} = \frac{dX_i^{N,\varepsilon}}{dx}\bigg|_{x=l} = 0. \tag{3.15}$$

The a priori estimate

$$\|u^{N,\varepsilon}\|_1^2 + \varepsilon\|u_{xx}^{N,\varepsilon}\|_0^2 \le m\|f\|_0^2 \tag{3.16}$$

is true where $\|\cdot\|_p = \|\cdot\|_{W^{p,2}(G)}$ is the norm of the space $W^{p,2}(G)$. Hereafter we denote by $m>0$ constants which are independent of $u^{N,\varepsilon}$ and may be different in different formulae. In order to prove (3.16), we multiply each of equations (3.14) by $e^{-\mu x}(X_j^{N,\varepsilon})_x$ $(\mu>0)$, then sum up the results from 1 to N, and integrate with respect to x from 0 to l. This yields the relation

$$\int_G \Big[\big(\varepsilon u_{xxx}^{N,\varepsilon} + ku_{xx}^{N,\varepsilon} - \alpha u_x^{N,\varepsilon} + cu^{N,\varepsilon}\big)u_x^{N,\varepsilon} - au_y^{N,\varepsilon}u_{xy}^{N,\varepsilon}\Big]e^{-\mu x}dG$$

$$= \int_G fu_x^{N,\varepsilon}e^{-\mu x}dG.$$

From the latter, (3.16) can be derived in the same way as (3.6) from (3.7). Estimate (3.16) proves the uniqueness of the solution to problem (3.14), (3.15). The uniqueness ensures the existence of the solution to the system of N third-order equations endowed with $3N$ boundary conditions, as known from the theory of linear ordinary differential equations. Moreover, the solution $X_i^{N,\varepsilon}$ belongs to $C^4[0,l]$ due to the smoothness conditions imposed for the coefficients of equation (3.1). Thus, the existence of the approximate solution (3.13) is established.

Consider the sequence $u^{N,\varepsilon}$ as $N\to\infty$. Owing to (3.16), the norm $\|u^{N,\varepsilon}\|_1$ is bounded uniformly in N. Therefore, due to the weak compactness of a bounded set in the Hilbert space, there exists a subsequence with a weak limit $u^\varepsilon \in W^{1,2}(G)$. In order to prove that u^ε is a solution of equation (3.10), we multiply each of equations (3.14) by a function $\chi_j \in C^\infty(0,l)$ vanishing in the vicinities of $x=0$ and $x=l$, then sum up from $j=1$ to $j=N$, and integrate with respect to x from 0 to l. This yields the identity

$$\int_G \Big[\big(\varepsilon u_{xxx}^{N,\varepsilon} + ku_{xx}^{N,\varepsilon} - \alpha u_x^{N,\varepsilon} + cu^{N,\varepsilon} - f\big)\chi^N - au_y^{N,\varepsilon}\chi_y^N\Big]dG = 0 \tag{3.17}$$

where $\chi^N(x,y) = \sum_{j=1}^N \chi_j(x)Y_j(y)$. Now we pass to the above mentioned subsequence of $u^{N,\varepsilon}$, integrate by parts the first and second terms in (3.17), and set $N\to\infty$ to get

$$\int_G \Big[\varepsilon u_x^\varepsilon\chi_{xx} - u_x^\varepsilon(k\chi)_x - au_y^\varepsilon\chi_y + (-\alpha u_x^\varepsilon + cu^\varepsilon - f)\chi\Big]dG = 0. \tag{3.18}$$

Relation (3.18) is valid for any function $\chi \in W^{1,2}(G)$ vanishing at $x = 0$ and $x = l$. Consequently, u^ε is a weak solution to equation (3.10) [18: Section 4.3]. That solution satisfies the boundary conditions (3.11), because $u^\varepsilon_{xx} \in L^2(G)$ due to (3.16), therefore, traces of u^ε and u^ε_x at $x =$ const belong to $L^2(-1, 1)$ and satisfy the same conditions at $x = 0$ and $x = l$ as $u^{N,\varepsilon}$ and $u^{N,\varepsilon}_x$ do owing to (3.15). The boundary condition (3.12) is involved in (3.18) as verified below (see (3.23)). Hence, u^ε is a weak solution to problem (3.10) - (3.12).

Now, let us consider a sequence $u^\varepsilon = u^\varepsilon(x, y)$ as $\varepsilon \to 0$. Due to (3.16), the norm $\|u^\varepsilon\|_1$ is bounded uniformly in ε. Therefore, there exists a subsequence with a weak limit $u^0 \in W^{1,2}(G)$. This limit is a weak solution of the mixed type equation (3.1), as seen from (3.18) with $\varepsilon = 0$, and it vanishes along with u^ε on the left-hand side of $G : u^0(0, y) = 0$. In order to prove that the Neumann condition $u^0_x(l, y) = 0$ holds on the right-hand side of G, we need to use the estimate

$$\|u^\varepsilon_x\|_{W^{1,2}(G_{l-\sigma})} \le m\|f\|_0 \tag{3.19}$$

where $G_{l-\sigma} = G \cap \{x : l - \sigma < x < l\}$. The validity of (3.19) is established in Section 4 with a technique of cut-off functions in the same way as in the theory of elliptic equations. Estimate (3.19) shows that $u^0_x = 0$ along with $u^\varepsilon_x = 0$ at $x = l$.

In order to prove the regularity of the obtained solution, i.e. $u^0 \in W^{2,2}(G)$, we need one more estimate

$$\|u^\varepsilon_{xx}\|_{L^2(G_{in})} \le m\|f\|_1 \tag{3.20}$$

where $G_{in} = G \cap \{x : \sigma < x < l - \sigma\}$. The validity of (3.20) under the additional condition $2\alpha - k_x \ge \delta > 0$ will be established in Section 5. In the strip $G_\sigma = G \cap \{x : 0 < x < \sigma\}$ adjacent to the left-hand side of G, the type of equation (3.1) is elliptic, therefore $u^0 \in W^{1,2}(G_\sigma)$ yields [18]

$$u^0 \in W^{2,2}(G_\sigma). \tag{3.21}$$

By combining (3.19) - (3.21), we find $u^0_{xx} \in L^2(G)$. This enables one to represent (3.18) at $\varepsilon = 0$ in the form

$$\int_G \Big[u^0_{xx}\chi - au^0_y\chi_y + (\underline{ku^0_{xx} - u^0_{xx} - \alpha u^0_x + cu^0 - f})\chi\Big] dG = 0. \tag{3.22}$$

Having denoted by $-\tilde f$ the underlined terms, one can interpret u^0 as a weak solution of the equation $u_{xx} + (au_y)_y = \tilde f \in L^2(G)$. On the other hand, the latter equation under the boundary conditions (3.3) has a solution $\tilde u \in W^{2,2}(G)$ as follows from the theory of equations of the elliptic type [18]. Because of the uniqueness in $W^{1,2}(G)$, we obtain $u^0 \equiv \tilde u \in W^{2,2}(G)$. As a consequence, the second term in (3.22) can be integrated by parts:

$$\begin{aligned} &\int_G \big[ku^0_{xx} + (au^0_y)_y - \alpha u^0_x + cu^0 - f\big]\chi dG \\ &\quad + \int_0^l (au^0_y\chi)\big|_{x=-1} dx - \int_0^l (au^0_y\chi)\big|_{x=1} dx = 0. \end{aligned} \tag{3.23}$$

Due to the arbitrariness in the choice of $\chi \in W^{1,2}(G)$ we conclude that equation (3.1) and the boundary condition $u^0_y(x, \pm 1) = 0$ are satisfied a.e.

We notice that estimate (3.20) is not valid over the whole domain G uniformly in ε. Hence, the boundary condition $u^\varepsilon_x(0, y) = 0$ is lost with respect to u^0 so that only the condition $u^0(0, y) = 0$ is true (see Figure 3). Consequently, the "viscous solutions" u^ε do not converge to u^0 in the norm $\| \cdot \|_{W^{2,2}(G)}$.

One can prove that $u^0 \in W^{p,2}(G_{in})$ $(p = 3, 4)$ using a priori estimates

$$\left.\begin{aligned} \|u^\varepsilon_{xxx}\|_{L^2(G_{in})} &\le m\|f\|_2 \\ \|u^\varepsilon_{xxxx}\|_{L^2(G_{in})} &\le m\|f\|_3 \end{aligned}\right\} \tag{3.24}$$

which are similar to (3.20) and valid under conditions (3.5). However, the third and fourth order derivatives of u^0 are not, in general, square integrable over the entire G because of singularities at the corner points (vertices) of the rectangle G ∎

Figure 3

4. A proof of estimate (3.19)

Since $k(x, y) > 0$ at $x = 0$ and $x = l$, one can choose a small $\sigma > 0$ so that $k(x, y) > 0$ in the strips $0 \le x \le 2\sigma$ and $l - 2\sigma \le x \le l$, $|y| \le 1$. Consider a non-negative cut-off function $\eta \in C^\infty[0, l]$ vanishing at $x \le l - 2\sigma$ and equal to $e^{\mu x}$ at $l - \sigma \le x \le l$ where $\mu > 0$ is large enough so that $a_x\eta + a\eta_x \ge 0$ at $x = l$. Let us multiply (3.14) by $\eta(X^{N,\varepsilon}_j)_{xx}$, then sum up over j from 1 to N, and integrate with respect to x from 0 to l. In this way we arrive at

$$\int_G \Big[\big(\varepsilon w_{xxx} + k w_{xx} - \alpha w_x + c w - f \big) w_{xx} - a w_y w_{xxy} \Big] \eta \, dG = 0 \tag{4.1}$$

where w denotes the approximate solution $u^{N,\varepsilon}$ for brevity. Relation (4.1) can be represented in the form

$$\int_G f w_{xx}\eta\,dG = \int_G \Big[\tfrac{\varepsilon}{2}(w_{xx}^2)_x + k w_{xx}^2 - \alpha w_x w_{xx} + c w w_{xx} - a w_y w_{xxy}\Big]\eta\,dG. \tag{4.2}$$

Integrating by parts the first and the last terms in the right-hand side and taking into consideration that $w_{xy}(l,y)=0$ due to (3.15), one obtains

$$\begin{aligned}\int_G f w_{xx}\eta\,dG &\\ = \int_{-1}^{1} &\tfrac{\varepsilon}{2} w_{xx}^2(l,y)\eta(l)\,dy\\ &+ \int_G \Big[-\tfrac{\varepsilon}{2}\eta_x w_{xx}^2 + k\eta w_{xx}^2 - \alpha\eta w_x w_{xx} + c\eta w w_{xx} + (a\eta w_y)_x w_{xy}\Big]dG \\ \geq \int_G &\Big[-\tfrac{\varepsilon}{2}\eta_x w_{xx}^2 + k\eta w_{xx}^2 - \alpha\eta w_x w_{xx} + c\eta w w_{xx} + (a\eta)_x w_y w_{xy} + a\eta w_{xy}^2\Big]dG.\end{aligned} \tag{4.3}$$

The integral $\int\big[-\tfrac{\varepsilon}{2}\eta_x w_{xx}^2\big]\,dG$ can be estimated by $\|f\|_0^2$ in view of (3.16):

$$\begin{aligned}\int_G f w_{xx}\eta\,dG + m\|f\|_0^2 &\\ \geq \int_G \Big[k\eta w_{xx}^2 + a\eta w_{xy}^2 &- \alpha\eta w_x w_{xx} + c\eta w w_{xx} + (a\eta)_x w_y w_{xy}\Big]dG.\end{aligned} \tag{4.4}$$

Integrating by parts the last three terms in (4.4), we find

$$\begin{aligned}\int_G f w_{xx}\eta\,dG + m\|f\|_0^2 &\\ \geq \tfrac{1}{2}\int_{-1}^{1} &(a\eta)_x w_y^2\big|_{x=l}dy\\ &+ \int_G \Big[k\eta w_{xx}^2 + a\eta w_{xy}^2 + \tfrac{1}{2}(\alpha\eta)_x w_x^2 - (c\eta w)_x w_x - \tfrac{1}{2}(a\eta)_{xx} w_y^2\Big]dG.\end{aligned} \tag{4.5}$$

The first integral in the right-hand side is non-negative due to the choice of η and can be omitted. The terms involving the first-order derivatives of w can be estimated by $\|f\|_0^2$ owing to (3.16). That is why (4.5) reduces to

$$\int_G f w_{xx}\eta\,dG + m\|f\|_0^2 \geq \int_G \big[k\eta w_{xx}^2 + a\eta w_{xy}^2\big]\,dG \tag{4.6}$$

(we recall that $m>0$ is independent of w and may be different in different formulae). Using Young's inequality in the left-hand side, we obtain

$$\frac{1}{2\gamma}\int_G f^2 dG + \frac{\gamma}{2}\int_G w_{xx}^2\eta^2 dG + m\|f\|_0^2 \geq \int_G \big[k\eta w_{xx}^2 + a\eta w_{xy}^2\big]\,dG. \tag{4.7}$$

Since $\eta \equiv 0$ at $x \leq l - 2\sigma$, the integration in the right-hand side is virtually carried out over the strip $G_{l-2\sigma} = G \cap \{x : x > l - 2\sigma\}$. In this strip, $k \geq k_{min} > 0$, therefore we arrive at the estimate

$$\tfrac{1}{2\gamma}\|f\|_0^2 + m\|f\|_0^2 \geq \int_{G_{l-2\sigma}} \left[(k_{min}\eta - \tfrac{1}{2}\gamma\eta^2)w_{xx}^2 + a\eta w_{xy}^2\right] dG. \tag{4.8}$$

The parameter $\gamma > 0$ can be chosen small enough to provide $k_{min}\eta - \frac{1}{2}\gamma\eta^2 > 0$, hence

$$m\|f\|_0^2 \geq \int_{G_{l-\sigma}} [w_{xx}^2 + w_{xy}^2]\, dG. \tag{4.9}$$

The latter means

$$\|w_x\|_{W^{1,2}(G_{l-\sigma})} \leq m\|f\|_0 \tag{4.10}$$

where $G_{l-\sigma} = G \cap \{x : x > l - \sigma\}$. Recalling that $w = u^{N,\varepsilon}$ and setting $N \to \infty$, from (4.10) we get (3.19).

5. A proof of estimate (3.20)

Let σ and w be the same as in the previous section. We choose now a non-negative cut-off function $\eta \in C^\infty[0, l]$ that equals unity at $\sigma \leq x \leq l - \sigma$ and vanishes at $0 \leq x \leq \frac{1}{2}\sigma$ and $l - \frac{1}{2}\sigma \leq x \leq l$. Let us multiply (3.14) by $\eta(X_j^{N,\varepsilon})_{xxx}$, then sum up from 1 to N, and integrate with respect to x from 0 to l. In this way, we obtain

$$\int_G \Big[\big(\varepsilon w_{xxx} + k w_{xx} - \alpha w_x + c w - f\big) w_{xxx} - a w_y w_{xxxy}\Big] \eta\, dG = 0, \tag{5.1}$$

i.e.

$$\begin{aligned} &\int_G f w_{xxx}\eta\, dG \\ &\quad = \int_G \Big[\varepsilon w_{xxx}^2 + k w_{xx} w_{xxx} - \alpha w_x w_{xxx} + c w w_{xxx} - a w_y w_{xxxy}\Big] \eta\, dG. \end{aligned} \tag{5.2}$$

By omitting the term εw_{xxx}^2 and integrating by parts the others, due to $\eta(0) = \eta(l) = 0$ we arrive at the estimate

$$\begin{aligned} &- \int_G (f\eta)_x w_{xx} dG \\ &\quad \geq \int_G \Big[- \tfrac{1}{2}(k\eta)_x w_{xx}^2 + (\alpha w_x \eta)_x w_{xx} - (c w \eta)_x w_{xx} - (a w_y \eta)_{xx} w_{xy}\Big]\, dG. \end{aligned} \tag{5.3}$$

Owing to the choice of η, the derivatives η_x and η_{xx} vanish over G except for two strips $G \cap \{x : \frac{1}{2}\sigma < x < \sigma\}$ and $G \cap \{x : l - \sigma < x < l - \frac{1}{2}\sigma\}$ in which $k \geq k_{min} > 0$. The integrals over the second strip can be estimated by invoking inequality (4.10), while those over the first strip can be estimated using a similar inequality obtained in the same way as (4.10) with a cut-off function that vanishes at $x \le \frac{1}{4}\sigma$ and $x \ge \frac{3}{2}\sigma$. Then (5.3) yields

$$\begin{aligned} &-\int_G f_x w_{xx} \eta \, dG + m\|f\|_0^2 \\ &\ge \int_G \Big[-\tfrac{1}{2} k_x w_{xx}^2 + (\alpha w_x)_x w_{xx} - (cw)_x w_{xx} - (a w_y)_{xx} w_{xy} \Big] \eta \, dG. \end{aligned} \tag{5.4}$$

Now we use the condition $2\alpha - k_x \ge \delta > 0$ valid due to (3.4):

$$\begin{aligned} -\int_G f_x w_{xx} \eta \, dG + m\|f\|_0^2 \ge \int_G \Big[&\tfrac{1}{2}\delta w_{xx}^2 + \underline{\alpha_x w_x w_{xx}} - \underline{(c_x w + c w_x) w_{xx}} \\ &- (\underline{a_{xx} w_y} + 2 a_x w_{xy} + \underline{a w_{xxy}}) w_{xy} \Big] \eta \, dG. \end{aligned} \tag{5.5}$$

Integrating by parts the underlined terms in the right-hand side and estimating the terms with η_x again by $\|f\|_0^2$, we get

$$\begin{aligned} &-\int_G f_x w_{xx} \eta \, dG + m\|f\|_0^2 \\ &\ge \tfrac{1}{2} \int_G \Big[\delta w_{xx}^2 - \alpha_{xx} w_x^2 + 2(c_x w)_x w_x + c_x w_x^2 + a_{xxx} w_y^2 - 3 a_x w_{xy}^2 \Big] \eta \, dG. \end{aligned} \tag{5.6}$$

The integral of the terms involving first order derivatives of w can be estimated by invoking (3.16). Also, we notice that $a_x \le 0$ due to conditions (3.4). That is why (5.6) reduces to

$$-\int_G f_x w_{xx} \eta \, dG + m\|f\|_0^2 \ge \tfrac{\delta}{2} \int_G w_{xx}^2 \eta \, dG. \tag{5.7}$$

Using Young's inequality in the left-hand side, we find

$$\tfrac{1}{2\gamma} \|f_x\|_0^2 + \tfrac{\gamma}{2} \|w_{xx}\eta\|_0^2 + m\|f\|_0^2 \ge \tfrac{\delta}{2} \int_G w_{xx}^2 \eta \, dG. \tag{5.8}$$

At $\gamma \le \frac{1}{2}\delta$, the second term is less than half the right-hand side:

$$m\|f\|_1^2 \ge \tfrac{\delta}{4} \int_G w_{xx}^2 \eta \, dG. \tag{5.9}$$

Since $\eta = 1$ in the subdomain $G_{in} = G \cap \{x : \sigma < x < l - \sigma\}$, we obtain

$$\|w_{xx}\|_{L^2(G_{in})} \le m\|f\|_1 \tag{5.10}$$

where $w = u^{N,\varepsilon}$. At $N \to \infty$, (5.10) yields (3.20).

6. Solvability of the transonic flow problem (2.3), (2.4)

Let us pass to the nonlinear perturbation problem (2.3) - (2.4). By using the solvability of the linear problem established in Theorem 2, we shall construct a sequence of approximate solutions and prove its convergence in an appropriate Sobolev space with the principle of contractive mappings.

Theorem 3. *Assume that $\varphi \in W^{6,2}(G_{noz})$ is a given function such that $\varphi_x(0,y) < 1, \varphi_x(l_{noz},y) > 1$ and in addition $\varphi_{xx} \geq \delta > 0$ in G_{noz}. Then for any perturbation $f^{\pm}$ vanishing in the vicinity of $x = 0$ and having sufficiently small norm $\|f^{\pm}\|_{W^{3,2}(0,l_{noz})}$ there exists a solution $u \in W^{4,2}(G_{noz})$ of the nonlinear problem (2.3), (2.4). That solution is unique in the class of functions $\|u\|_{W^{3,2}(G_{noz})} < r$ where r depends on φ and G_{noz}.*

Proof. We employ the approach outlined in [16]. The solution will be constructed through the following iteration scheme:

$$(1 - \varphi_x + g_i)u_{xx}^{(i+1)} - \varphi_{xx}u_x^{(i+1)} + \tfrac{1}{1+\gamma}u_{yy}^{(i+1)} = 0 \tag{6.1}$$

$$g_i = u_x^{(i)} \ (i \in \mathbb{N}), \quad g_0 \equiv 0$$

under the boundary conditions (2.4). The proof is split into four steps.

Step 1. The linear problem (2.4), (2.5) can be easily reduced to the problem for the non-homogeneous equation

$$L_g u := (1 - \varphi_x + g)u_{xx} - \varphi_{xx}u_x + \tfrac{1}{1+\gamma}u_{yy} = f \in W^{3,2}(G_{noz}) \tag{6.2}$$

given in G_{noz} and endowed with the homogeneous boundary conditions

$$\left.\begin{aligned} u(0,y) &= 0 \\ u_y(x,\pm 1) &= 0 \end{aligned}\right\} \tag{6.3}$$

Indeed, one can obtain this by substituting $u + \hat{u}$ for u in (2.4), (2.5) where $\hat{u} = \frac{1}{4}f^+(x)(1+y)^2 - \frac{1}{4}f^-(x)(1-y)^2$ is a function satisfying the boundary conditions (2.4).

Equation (6.2) coincides with (3.1) in the special case $k = 1 - \varphi_x + g, \ \alpha = \varphi_{xx}, \ a = \frac{1}{1+\gamma}, \ c \equiv 0$. Obviously, the expressions

$$\left.\begin{aligned} 2\alpha + k_x &= \varphi_{xx} + g_x & \qquad 2\alpha - k_x &= 3\varphi_{xx} - g_x \\ 2\alpha - 3k_x &= 5\varphi_{xx} - 3g_x & \qquad 2\alpha - 5k_x &= 7\varphi_{xx} - 5g_x \end{aligned}\right\} \tag{6.4}$$

are positive at sufficiently small g due to $\varphi_{xx} \geq \delta > 0$ in G_{noz}. In order to apply Theorem 2, we need to modify problem (6.2), (6.3) so as to gain the elliptic type of the equation on the right-hand side of a rectangle as well as on the left-hand side. Let us choose $l > l_{noz}$ and prolong the functions φ, f, g into the domain $\{(x,y) : \ l_{noz} < x < l \text{ and } -1 < y < 1\}$ in such a way that $\varphi_x(l,y) < 1, \ \varphi \in W^{6,2}(G), f \in W^{3,2}(G)$ and (6.4) remain true in the extended domain $G = \{(x,y) : \ 0 < x < l \text{ and } -1 < y < 1\}$. Since $\varphi_x(l,y) < 1$, the type of equation (6.2) is elliptic on the right-hand side of G at sufficiently small g. That is why we prescribe the boundary condition

$$u_x(l, y) = 0 \tag{6.5}$$

in addition to (6.3). As seen, in the extended domain G, the coefficients of equation (6.2) satisfy all the inequalities required in Theorem 2. The condition $\varphi \in W^{6,2}(G)$ yields $\varphi_x \in C^3(\bar{G})$ due to the imbedding theorems for the two-dimensional domain [1]. Therefore, at $g \equiv 0$ the coefficients of (6.2) are smooth enough, and Theorem 2 establishes the existence of a solution $u \in W^{2,2}(G) \cap W^{4,2}(G_{in})$ to problem (6.2), (6.3), (6.5). The restriction of u to the subdomain G_{noz} gives a solution to problem (6.2), (6.3). Theorem 1 ensures the uniqueness of the solution $u \in W^{2,2}(G_{noz}) \cap W^{4,2}(G_{in})$ obtained in the case $g \equiv 0$.

Step 2. The condition of vanishing $f^{\pm}(x)$ in the vicinity of $x = 0$ yields $f(x, y) \equiv 0$ near the left-hand side of G and provides $u \in W^{4,2}(G_{noz})$. In order to prove the latter, we use the even prolongation of $u \in W^{2,2}(G_{noz})$ and smooth prolongation of the coefficients of (6.2) across the segments $y = \pm 1,\ 0 \le x \le \sigma$, where σ is small enough, into domains $1 < |y| < 1 + \sigma,\ 0 < x < \sigma$ (see Figure 2, in which the upper domain is indicated by the dashed segments). By formal differentiation of (6.2) with respect to y, we find that u_y is a weak solution of Dirichlet's problem for the equation of elliptic type $L_0 u_y = \bar{f}$ in the strip $0 < x < \sigma,\ |y| < 1 + \sigma$, where $\bar{f}$ is square-integrable. A weak solution of such a problem is necessarily regular [18], hence $u_y \in W^{2,2}(G_\sigma)$, $G_\sigma = \{(x, y) :\ 0 < x < \sigma \text{ and } |y| < 1\}$.

Similarly, the second differentiation of (6.2) shows that $u_{yy} \in W^{2,2}(G_\sigma)$. Then equation (6.2) provides $u_{xx} \in W^{2,2}(G_\sigma)$, consequently $u \in W^{4,2}(G_\sigma)$. Taking into consideration that $u \in W^{4,2}(G_{in})$, where G_{in} and G_σ can overlap, one obtains $u \in W^{4,2}(G_{noz})$.

Step 3. In the case $0 \not\equiv g \in W^{3,2}(G_{noz})$, which is crucial to the validity of scheme (6.1), the coefficient in front of u_{xx} in (6.2) does not belong to $C^3(\bar{G}_{noz})$. However, $g \in C^1(\bar{G}_{noz})$ due to imbedding theorems [1]. Therefore, the assertion of Theorem 2 in the part of the existence of the solution $u \in W^{2,2}(G)$ remains valid, because estimate (3.20) is true at $k \in C^1(\bar{G})$, as seen from inequality (5.4) which does not involve higher order derivatives of k. Consequently, there exists the solution $u \in W^{2,2}(G_{noz})$ of problem (6.2), (6.3) owing to the arguments of Step 1.

In order to prove that the solution belongs to $W^{4,2}(G_{noz})$ in the case $g \in W^{3,2}(G_{noz})$ we first validate a priori estimates

$$\|u\|_{W^{p,2}(G_{noz})} \le m \|L_g u\|_{W^{p-1,2}(G_{noz})} \qquad (p = 1, 2, 3, 4;\ m > 0) \tag{6.6}$$

for sufficiently small $\|g\|_{W^{3,2}(G_{noz})}$ and any function $u \in W^{p,2}(G_{noz})$ satisfying boundary conditions (6.3) and such that $L_g u \in W^{p-1,2}(G_{noz})$ and $L_g u \equiv 0$ in the vicinity of $x = 0$. The validity of (6.6) at $p = 1$ follows from the analysis of the integral $\int_{G_{noz}} L_g u \cdot u_x e^{\mu x} dG$ similar to (2.8). Let us prove that (6.6) is true at $p = 4$ if it holds at $p = 1, 2, 3$. The condition $\varphi_x(0, y) < 1$ provides $\varphi_x < 1$ in the strip $0 \le x \le 4\sigma, |y| \le 1$ at sufficiently small $\sigma > 0$. Consider a non-negative cut-off function $\eta \in C^\infty[0, l]$ vanishing at $\{x \le \sigma\}$

and equal to unity at $\{x \geq 2\sigma\}$. It can be easily checked that the function $\tilde{u} = \eta u$ satisfies the equation

$$L_g \tilde{u} = \tilde{f} := \eta L_g u + (1 - \varphi_x + g)(\eta_{xx} u + 2\eta_x u_x) - \varphi_{xx}\eta_x u \in W^{3,2}(G_{noz}).$$

By differentiating $L_g \tilde{u} = \tilde{f}$ twice with respect to x, we obtain

$$\begin{aligned} &(1 - \varphi_x + g)(\tilde{u}_{xx})_{xx} - 3\varphi_{xx}(\tilde{u}_{xx})_x + \tfrac{1}{1+\gamma}(\tilde{u}_{xx})_{yy} \\ &\quad = \tilde{f}_{xx} - g_{xx}\tilde{u}_{xx} - 2g_x \tilde{u}_{xxx} + 3\varphi_{xxx}\tilde{u}_{xx} + \varphi_{xxxx}\tilde{u}_x. \end{aligned} \tag{6.7}$$

Hence, $\tilde{u}_{xx}$ can be considered as a solution of the second order equation that only differs from (6.2) by the multiplier 3 in front of φ_{xx}, which does not influence the validity of (6.6). In addition, $\tilde{u}_{xx}$ satisfies the boundary conditions (6.3), because $\tilde{u}_{xx} = 0$ at $x \leq \sigma$. That is why one can use (6.6) with $p = 2$ in order to estimate $\|\tilde{u}_{xx}\|_{W^{2,2}(G_{noz})}$ by the right-hand side of (6.7):

$$\|\tilde{u}_{xx}\|_2 \leq m \left\| \tilde{f}_{xx} - g_{xx}\tilde{u}_{xx} - 2g_x\tilde{u}_{xxx} + 3\varphi_{xxx}\tilde{u}_{xx} + \varphi_{xxxx}\tilde{u}_x \right\|_1 \tag{6.8}$$

where $\|\cdot\|_p$ denotes $\|\cdot\|_{W^{p,2}(G_{noz})}$. Due to Remark presented below, the second and third terms in the right-hand side of (6.8) can be estimated as

$$\left.\begin{aligned} \|g_{xx}\tilde{u}_{xx}\|_1 \leq m\|g_{xx}\|_1\|\tilde{u}_{xx}\|_2 \leq m\|g\|_3\|\tilde{u}_{xx}\|_2 \\ \|g_x\tilde{u}_{xxx}\|_1 \leq m\|g_x\|_2\|\tilde{u}_{xxx}\|_1 \leq m\|g\|_3\|\tilde{u}_{xx}\|_2 \end{aligned}\right\}$$

where $m > 0$ may as usual be different in different formulae. Therefore, under sufficiently small $\|g\|_3$, inequality (6.8) reduces to

$$\|\tilde{u}_{xx}\|_2 \leq m\|\tilde{f}_{xx} + 3\varphi_{xxx}\tilde{u}_{xx} + \varphi_{xxxx}\tilde{u}_x\|_1. \tag{6.9}$$

Since $\varphi_{xxx} \in C^1(\bar{G}_{noz})$, the second and third terms on the right-hand side of (6.9) can be estimated by (6.6) with $p = 3$ and $p = 2$. Then the equation $L_g\tilde{u} = \tilde{f}$ makes it possible to estimate $\|\tilde{u}_{yy}\|_2$ by $\|\tilde{u}_{xx}\|_2$, and we arrive at $\|\tilde{u}\|_4 \leq m\|\tilde{f}\|_3$. Recalling that $\eta = 1$ and $\tilde{u} = u$ at $x \geq 2\sigma$, we obtain

$$\|u\|_{W^{4,2}(G_{noz} \cap \{x : x > 2\sigma\})} \leq m\|\tilde{f}\|_3. \tag{6.10}$$

A similar inequality $\|u\|_{W^{4,2}(G_{noz}\cap\{x:x<3\sigma\})} \leq m\|L_g u\|_3$ holds due to the elliptic type of the operator $L_g u$ at $0 \leq x \leq 4\sigma$ and the considerations of Step 2. By combining the two inequalities, we get (6.6) with $p = 4$.

Remark. For $u \in W^{2,2}(G)$ and $v \in W^{1,2}(G)$, the estimate $\|uv\|_1 \leq m\|u\|_2\|v\|_1$ is true. Indeed,

$$\begin{aligned} \|uv\|_1^2 &= \int_G (u_x v + u v_x)^2 + (u_y v + u v_y)^2 + u^2 v^2 \; dG \\ &\leq 2\int_G u_x^2 v^2 + u^2 v_x^2 + u_y^2 v^2 + u^2 v_y^2 \; dG + \int_G u^2 v^2 dG \\ &\leq 2\int_G (u_x^2 + u_y^2) v^2 dG + 2 \; \max_G u \;^2 \|v\|_1^2 \\ &\leq 2\|u_x^2 + u_y^2\|_0 \|v^2\|_0 + m\|u\|_2^2\|v\|_1^2 \\ &\leq m\|u\|_2^2\|v\|_1^2 \end{aligned}$$

owing to multiplicative inequalities [1, 18], which yield for a rectangle G

$$\|v^2\|_0 = \|v\|^2_{L^4(G)} \le m\|v\|_1\|v\|_0 \le m\|v\|_1^2,$$

consequently $\|u_x^2 + u_y^2\|_0 \le m\|u\|_2^2$ where the constants $m > 0$ are independent of u and v.

Now, we use (6.6) in order to prove that the solution $u \in W^{2,2}(G_{noz})$ of problem (6.2), (6.3) belongs to $W^{4,2}(G_{noz})$. Let us approximate g in the norm $\|\cdot\|_{W^{3,2}(G_{noz})}$ by a sequence $g_j \in C^3(\bar{G}_{noz})$ ($j \in \mathbb{N}$). The equations

$$(1 - \varphi_x + g_j)u_{xx} - \varphi_{xx}u_x + \tfrac{1}{1+\gamma}u_{yy} = f \tag{6.11}$$

endowed with boundary conditions (6.3) have solutions $u_j \in W^{4,2}(G_{noz})$ due to the result of Step 2. These solutions are bounded in the norm $\|\cdot\|_4$ owing to (6.6). Hence, there exists a subsequence with a weak limit $\hat{u} \in W^{4,2}(G_{noz})$. On the other hand, the sequence u_j converges to the solution $u \in W^{2,2}(G_{noz})$, because by subtracting (6.11) from (6.2), one obtains

$$L_g(u - u_j) = (g_j - g)u_{jxx},$$

therefore

$$\|u - u_j\|_2 \le m\|L_g(u - u_j)\|_1 = m\|(g_j - g)u_{jxx}\|_1 \to 0 \qquad \text{as } j \to \infty.$$

That is why $u = \hat{u} \in W^{4,2}(G_{noz})$.

Step 4. Scheme (6.1) implies solving a sequence of problems (6.1), (2.4) for $g_i \in B_r = \{g \in W^{3,2}(G_{noz}) : \ \|g\|_3 < r\}$. On account of Step 3, the solutions $u^{(i+1)}$ exist if the radius r of the ball B_r is small enough. Then scheme (6.1) can be represented in the form $g_{i+1} = Tg_i$, in which the operator Tg is defined by $Tg = u_x$ for all $g \in B_r$ where u is the solution of the equation $L_g u = 0$ under boundary conditions (2.4). Consequently, the nonlinear problem (2.4), (2.5) can be rewritten as $g = Tg$.

Under sufficiently small $\|f^{\pm}\|$, the operator Tg maps the ball B_r into B_r because

$$\begin{aligned}\|Tg\|_3 &= \|u_x\|_3 \\ &\le \|u_x - \hat{u}_x\|_3 + \|\hat{u}_x\|_3 \\ &\le m\|f\|_3 + \|\hat{u}_x\|_3 \\ &\le m\|\,|f^+| + |f^-|\,\|_{W^{3,2}(0,l_{noz})} \\ &< r.\end{aligned} \tag{6.12}$$

Let us prove that the operator Tg is a contractive one in the norm $\|\cdot\|_1$ if the radius r is small enough. Due to (6.6) with $p = 2$, one obtains for two elements g_1 and g_2

$$\|Tg_1 - Tg_2\|_1 = \|u_{1x} - u_{2x}\|_1 \le \|u_1 - u_2\|_2 \le m\|L_{g_1}(u_1 - u_2)\|_1. \tag{6.13}$$

From equation (2.5) we find

$$L_{g_1}u_1 - L_{g_2}u_2 = L_{g_1}(u_1 - u_2) + (g_1 - g_2)u_{2xx} = 0.$$

Therefore, the right-hand side of (6.13) can be estimated by

$$\begin{aligned} m\|(g_1 - g_2)u_{2xx}\|_1 &\le m\|g_1 - g_2\|_1 \, \|u_{2xx}\|_2 \\ &\le m\|g_1 - g_2\|_1 \, \|u_{2x}\|_3 \\ &\le mr\|g_1 - g_2\|_1 \end{aligned}$$

owing to (6.12). Hence, at $r < \frac{1}{m}$ the operator Tg is a contractive one. Then the principle of contractive mappings shows that sequential approximations $g_{i+1} = Tg_i$ ($i \in \mathbb{N}_0$) converge to the unique solution $g \in W^{1,2}(G_{noz})$ of the equation $g = Tg$. On the other hand, because of the weak compactness of B_r, there exists a subsequence which weakly converges to $\hat{g} \in W^{3,2}(G_{noz})$. Due to the uniqueness, one gets $g = \hat{g} \in W^{3,2}(G_{noz})$. Then $u = \int_0^x g\,dx$ is the solution of the nonlinear problem (2.3) - (2.4) ∎

Conclusion. A shock-free flow with subsonic-supersonic transition in a nozzle was considered. A nonlinear perturbation problem for the von Karman equation was formulated and studied. The obtained results can contribute to the analysis of finite element approximations of transonic flow pursued in [2, 3, 20].

References

[1] Adams, R. A.: *Sobolev spaces.* New York: Academic Press 1975.

[2] Berger, H.: *A convergent finite element formulation for transonic flow.* Num. Math. 56 (1989), 425 – 447.

[3] Berger, H., Warnecke, G. and W. L. Wendland: *Analysis of a FEM/BEM coupling method for transonic flow computations.* Math. of Comput. 66 (1997), 1407 – 1440.

[4] Bers, L.: *Mathematical aspects of subsonic and transonic gas dynamics.* Surv. Appl. Math. 3 (1958), 1 – 208.

[5] Cole, J. D. and L. P. Cook: *Transonic Aerodynamics.* Amsterdam: North-Holland Publ. Comp. 1986.

[6] Dachev, G. D.: *On the existence of smooth solutions of boundary value problems for a class of equations of mixed type.* Diff. Equ. 27 (1991), 435 – 441.

[7] Egorov, I. E.: *Solvability of a boundary value problem for a high–order equation of mixed type* (in Russian). Diff. Uravn. 23 (1987), 1560 – 1567.

[8] Feistauer, M. and J. Nečas: *On the solvability of transonic potential flow problems.* Z. Anal. Anw. 4 (1985), 305 – 329.

[9] Feistauer, M. and J. Nečas: *Viscosity method in a transonic flow.* Comm. Part. Diff. Equ. 13 (1988), 775 – 812.

[10] Feistauer, M. and J. Nečas: *Remarks on the solvability of transonic flow problems.* Man. Math. 61 (1988), 417 – 428.

[11] Gamba, I.M. and C. S. Morawetz: *A viscous approximation for a 2-D steady semiconductor or transonic gas dynamic flow: Existence theorem for potential flow.* Comm. Pure Appl. Math. 49 (1996), 999 – 1049.

[12] Gittel, H.-P.: *A variational approach to boundary value problems for a class of nonlinear systems of mixed type.* Z. Anal. Anw. 14 (1995), 575 – 591.

[13] Klouček, P.: *The existence of the entropic solution for the transonic flow problem.* Nonlin. Anal.: Meth & Appl. 22 (1994), 467 – 480.

[14] Kuz'min, A. G.: *The existence of a solution of the direct Lavalle nozzle problem in variations.* USSR Comp. Math. & Math. Phys. 24 (1984), 171 – 174.

[15] Kuz'min, A. G.: *Non-Classical Equations of Mixed Type and Their Applications in Gas Dynamics* (Internat. Series of Numerical Math.: Vol. 109). Basel - Boston - Berlin: Birkhäuser Verlag 1992.

[16] Kuz'min, A. G.: *Solvability of boundary value problems for the Chaplygin equation of mixed type.* In: Proc. Intern. Conf. "Functional Spaces. Differential Operators. Problems of Math. Education", Moscow: Russian People Friendship Univ. 2b (1998), pp. 226 – 230.

[17] Kuz'min, A. G.: *Solvability of a problem for transonic flow with a local supersonic region.* J. Nonlin. Diff. Equ. Appl. (to appear).

[18] Ladyzhenskaia, O. A.: *The Boundary Value Problems of Mathematical Physics* (Applied Mathematical Sciences: Vol. 49). New York: Springer-Verlag 1985.

[19] Larkin, N. A.: *Smooth Solutions to the Equations of Transonic Gas Dynamics* (in Russian). Novosibirsk: Nauka 1991.

[20] Mandel, J. and J. Nečas: *Convergence of finite elements for transonic potential flow.* SIAM J. Numer. Anal. 24 (1987), 985 – 996.

[21] Morawetz, C. S.: *On steady transonic flow by compensated compactness.* Methods Appl. Anal. 2 (1995), 257 – 268.

[22] Vragov, V. N.: *On boundary value problems for a class of mixed type equations in three-dimensional space.* Diff. Uravn. 11 (1975), 27 – 32.

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