Fractional Derivatives
Non-Symmetric and Time-Dependent Dirichlet Forms and the Drift Form

N. Jacob and R. L. Schilling

Abstract. Using fractional derivatives we show that the drift form $\int_{-\infty}^{\infty} u(x) \frac{dv(x)}{dx} \, dx$ can be approximated by non-symmetric Dirichlet forms. A similar result holds for the drift form in $\mathbb{R}^n$ with variable coefficients if the coefficient functions satisfy certain regularity and commutator conditions. Since time-dependent Dirichlet forms (in the sense of Y. Oshima) can be interpreted as sums of a drift form (in τ-direction) and a mixture of τ-parametrized Dirichlet forms over $\mathbb{R}^n$, our results show that time-dependent Dirichlet forms arise as limits of ordinary non-symmetric Dirichlet forms in $\mathbb{R} \times \mathbb{R}^n$-space. An abstract result on fractional powers of Markov generators allows to extend this observation to generalized Dirichlet forms. Another consequence is that the bilinear form induced by an arbitrary Lévy process is the limit of non-symmetric Dirichlet forms.

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0. Introduction

It is well-known that a Lévy process $\{X_t\}_{t \geq 0}$ with values in $\mathbb{R}^n$ can be completely described by its characteristic exponent. This is a *negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$* that is determined by the relation

$$\mathbb{E}^x\big(e^{i(X_t - x)\xi}\big) = e^{-t\psi(\xi)}. \tag{0.1}$$

If ψ is real-valued, then $\{X_t\}_{t \geq 0}$ is associated to the symmetric Dirichlet form

$$\mathbf{B}^{\psi}(u, v) = \int_{\mathbb{R}^n} \psi(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)} \, d\xi \tag{0.2}$$

where $\widehat{u}$ denotes the Fourier transform of u (see [6: pp. 29 - 31]). The case of general ψ was considered by C. Berg and G. Forst [2]. They showed that the bilinear form

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$\mathbf{B}^\psi(\cdot, \cdot)$ for a complex-valued ψ extends to a *non-symmetric* Dirichlet form if and only if ψ satisfies the sector condition, that is, if

$$|\mathrm{Im}\psi(\xi)| \leq c\,\mathrm{Re}\psi(\xi) \tag{0.3}$$

is valid for some constant $c > 0$. Clearly, the corresponding stochastic processes are again Lévy processes.

Let us briefly recall the definition of a non-symmetric Dirichlet form:

Definition. A closed, densely defined bilinear form $\big(\mathbf{E}(\cdot, \cdot), \mathcal{D}(\mathbf{E})\big)$ on $L^2(\mathbb{R}^n, \mathbb{R})$ is called *non-symmetric Dirichlet form* if it satisfies the following properties:

(DF.1) $\mathbf{E}(u, u) \geq 0$.

(DF.2) $|\mathbf{E}(u, v)| \leq K\,\sqrt{\mathbf{E}(u, u) + \langle u, u\rangle_0}\,\sqrt{\mathbf{E}(v, v) + \langle v, v\rangle_0}$ (weak sector condition) for some constant $K > 0$.

(DF.3) $u^+ \wedge 1 \in \mathcal{D}(\mathbf{E})$ for all $u \in \mathcal{D}(\mathbf{E})$.

(DF.4) $\mathbf{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0$ and $\mathbf{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0$.

(DF.5) $\mathcal{D}(\mathbf{E})$ is complete under $\widetilde{\mathbf{E}}_1(u, v) = \mathbf{E}(u, v) + \mathbf{E}(v, u) + \langle u, v\rangle_0$.

That we can associate a Markov process to any non-symmetric Dirichlet form was shown by C. Carillo-Menendez [5]. Standard introductory texts to the theory of non-symmetric Dirichlet forms are the lecture notes of Y. Oshima [19, 22] and the textbook [17] by Z.-M. Ma and M. Röckner.

The continuous negative definite function $\mathbb{R} \ni \xi \mapsto -ib\xi$ $(b \in \mathbb{R})$ clearly corresponds to a Lévy process – the (deterministic) drift process of speed $-b$ – but it does not satisfy (0.3) and it is not possible to associate the drift process with a non-symmetric Dirichlet form. We can, however, associate with $-ib\xi$ a bilinear form that we will call *drift form*,

$$\int_\mathbb{R} (-ib\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,d\xi = b\int_\mathbb{R} u(x)\,\frac{dv(x)}{dx}\,dx. \tag{0.4}$$

The usual way to estimate this form is an application of the Cauchy-Schwarz inequality to the right-hand side,

$$\left|b\int_\mathbb{R} u(x)\,\frac{dv(x)}{dx}\,dx\right| \leq |b|\,\|u\|_0 \left\|\frac{dv}{dx}\right\|_0 \leq |b|\,\|u\|_0\,\|v\|_{H^1(\mathbb{R})}$$

where $\|\cdot\|_0$ denotes the L^2-norm and $H^s(\mathbb{R})$ $(s \in \mathbb{R})$ is the usual L^2-Sobolev space. Already in [9], however, it was pointed out that the estimate

$$\left|b\int_\mathbb{R} u(x)\,\frac{dv(x)}{dx}\,dx\right| \leq \int_\mathbb{R} |b\xi|\,|\widehat{u}(\xi)|\,|\widehat{v}(\xi)|\,d\xi \leq |b|\,\|u\|_{H^{\frac{1}{2}}(\mathbb{R})}\|v\|_{H^{\frac{1}{2}}(\mathbb{R})}$$

can be advantageous and that the drift form should be considered as a continuous bilinear form over $H^{\frac{1}{2}}(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ (which is clearly different from $L^2(\mathbb{R}) \times H^1(\mathbb{R})$).

Observe that for any $0 < \alpha < 1$ the function $\xi \mapsto (-ib\xi)^\alpha$ is again continuous negative definite. Moreover, it satisfies (0.3) since

$$\mathrm{Im}(-i\xi)^\alpha = |\xi|^\alpha \sin\frac{\alpha\pi}{2} = \tan\frac{\alpha\pi}{2}\,|\xi|^\alpha \cos\frac{\alpha\pi}{2} = c_\alpha\,\mathrm{Re}(-i\xi)^\alpha.$$

Thus,

$$\int_{\mathbb{R}} (-i\xi)^{\alpha} \widehat{u}(\xi) \overline{\widehat{v}(\xi)} \, d\xi$$

is a non-symmetric Dirichlet form with domain $H^{\frac{\alpha}{2}}(\mathbb{R})$. Using fractional derivatives we can rewrite this form as

$$\int_{\mathbb{R}} (-i\xi)^{\alpha} \widehat{u}(\xi) \overline{\widehat{v}(\xi)} \, d\xi = \int_{\mathbb{R}} D_{-}^{\frac{\alpha}{2}} u(x) D_{+}^{\frac{\alpha}{2}} v(x) \, dx \tag{0.5}$$

and, at least formally, the limit

$$\lim_{\alpha \uparrow 1} \int_{\mathbb{R}} D_{-}^{\frac{\alpha}{2}} u(x) D_{+}^{\frac{\alpha}{2}} v(x) \, dx = \int_{\mathbb{R}} u(x) \frac{dv(x)}{dx} \, dx \tag{0.6}$$

should exist for suitable functions u and v. This shows that the drift form, although it is itself not a Dirichlet form, can be obtained as a limit of non-symmetric Dirichlet forms. Note that the Lévy process associated with the characteristic exponent $\xi \mapsto (i\xi)^{\alpha}$ is a (non-symmetric) stable process of order α (see J. Bertoin [4: Chapter VIII.1]).

In this paper we will systematically investigate the observations made above and examine certain types of Dirichlet forms. In Section 1 we collect the material needed from the theory of fractional integrals and derivatives. The above considerations for one-dimensional drift forms will be made rigorous in Section 2. Exploiting the properties of fractional powers of (continuous) negative definite functions we show in Section 3 that *any* Lévy process with state space $\mathbb{R}^n$ is described by a bilinear form which is obtained as a limit of non-symmetric Dirichlet forms.

There are several generalizations of the notion of Dirichlet forms:

(1) *Semi-Dirichlet forms* (cf. Z.-M. Ma, L. Overbeck and M. Röckner [16].

(2) Stimulated by results of M. Pierre [23, 24], Y. Oshima [20 - 22] introduced *time-dependent* or *parabolic Dirichlet forms.*

(3) W. Stannat [29, 30] introduced *generalized Dirichlet forms.*

In Section 4 we consider the time-dependent forms of Y. Oshima which have the structure

$$\int_{\mathbb{R}} E^{(\tau)}(u(\tau, \cdot), v(\tau, \cdot)) \, d\tau + \int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{\partial u(\tau, x)}{\partial \tau} v(\tau, x) \, dx d\tau \tag{0.7}$$

where $\{E^{(\tau)}\}_{\tau \in \mathbb{R}}$ is a one-parameter family of Dirichlet forms on $\mathcal{D}(E^{(\tau)}) = \mathcal{D}(E^{(0)}) \subset L^2(\mathbb{R}^n)$. The second term in (0.7) is a kind of drift form. We prove that for $0 < \alpha < 1$ the form

$$\int_{\mathbb{R}} E^{(\tau)}(u(\tau, \cdot), v(\tau, \cdot)) \, d\tau + \int_{\mathbb{R}} \int_{\mathbb{R}^n} D_{-,\tau}^{\frac{\alpha}{2}} u(\tau, x) \, D_{+,\tau}^{\frac{\alpha}{2}} v(\tau, x) \, dx d\tau \tag{0.8}$$

is a non-symmetric Dirichlet form (in $\mathbb{R} \times \mathbb{R}^n$-space) and that (0.7) is obtained as a limit of (0.8) for $\alpha \to 1$. (It is an interesting open question whether the theory of fractional diffusions of W. R. Schneider and W. Wyss [28] can be treated within this framework.)

Section 5 is devoted to higher-dimensional drift forms with variable coefficients,

$$\mathbf{D}(u, v) = \sum_{j=1}^{n} \int_{\mathbb{R}^n} b_j(x)u(x) \frac{\partial v(x)}{dx_j} \, dx. \tag{0.9}$$

In order to get analoguous results to the (one-dimensional) constant coefficient case of Section 2, we need to establish commutator estimates for partial fractional derivatives $D^{\alpha}_{-,j}$ in direction x_j. For $b_j \in C^1_b(\mathbb{R}^n)$ we get as above

$$|\mathbf{D}(u, v)| \leq c \, \|u\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \, \|v\|_{H^{\frac{1}{2}}(\mathbb{R}^n)}.$$

The bilinear forms $\mathbf{D}^{(\alpha)}(\cdot, \cdot)$,

$$\mathbf{D}^{(\alpha)}(u, v) = \sum_{j=1}^{n} \int_{\mathbb{R}^n} b_j(x)u(x) \, D^{\alpha}_{+,j} v(x) \, dx \qquad (0 < \alpha < 1)$$

can be extended to Dirichlet forms if the coefficient functions satisfy the additional conditions

$$\sum_{j=1}^{n} D^{\alpha}_{-,j} b_j \geq 0 \qquad \text{and} \qquad \sum_{j=1}^{n} \left(D^{\alpha}_{j,\mathrm{Im}}(b_j u) - b_j D^{\alpha}_{j,\mathrm{Im}} u - u D^{\alpha}_{j,\mathrm{Im}} b_j \right) = 0$$

with $D^{\alpha}_{j,\mathrm{Im}} := \frac{1}{2}(D^{\alpha}_{-,j} - D^{\alpha}_{+,j})$. This is, for example, the case if $b_j(x_1, \ldots, x_n)$ is independent of x_j.

The final section is in a more abstract and general setting. In Theorem 6.2 we discuss in the context of this paper a result known for $\alpha = \frac{1}{2}$ (see T. Kato [13]): we show that for the generator of a Markov semigroup $(A, \mathcal{D}(A))$ on a complex Hilbert space the fractional powers $(-A)^{\alpha}$ $(0 < \alpha < 1)$ are sectorial operators. This enables us to deduce that the form $\langle -Au, v \rangle$ is the limit of sectorial forms (in fact, non-symmetric Dirichlet forms) $\langle (-A)^{\alpha} u, v \rangle$ as $\alpha \to 1$. This observation suggests that the results of Section 4 carry over to generalized Dirichlet forms in the sense of Stannat. It seems to be reasonable to conjecture that many notions form potential theory as well as the Markov processes – considered by Y. Oshima in the theory of time-dependent Dirichlet forms and by W. Stannat in the theory of generalized Dirichlet forms – can be obtained in the limit from the corresponding objects given by the approximating non-symmetric Dirichlet forms.

Finally, let us mention that recently several authors started to investigate relations of fractional derivatives with Markov processes, for example R. Gorenflo and F. Mainardi [7, 8] or A. Krägeloh [14].

1. Fractional integrals and derivatives

In the following sections we will need several results from the theory of fractional integrals and derivatives which we collect here. Most of the material is taken from the monograph [26] by S. G. Samko et al., another standard reference is the book [25] by B. Rubin.

Definition 1.1. For $0 < \alpha < 1$ the *fractional integrals* $I_+^\alpha \phi$ and $I_-^\alpha \phi$ are given by

$$I_+^\alpha \phi(x) = \tfrac{1}{\Gamma(\alpha)} \int_{-\infty}^x \frac{\phi(t)}{(x-t)^{1-\alpha}}\, dt \tag{1.1}$$

$$I_-^\alpha \phi(x) = \tfrac{1}{\Gamma(\alpha)} \int_x^\infty \frac{\phi(t)}{(t-x)^{1-\alpha}}\, dt. \tag{1.2}$$

If we denote

$$t_+ = \mathbf{1}_{(0,\infty)}(t)t$$
$$t_- = \mathbf{1}_{(-\infty,0)}(t)|t|$$

we find that

$$I_\pm^\alpha \phi(x) = \tfrac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} t_\pm^{\alpha-1}\, \phi(x-t)\, dt = \tfrac{1}{\Gamma(\alpha)} \int_{\mathbb{R}_+} t^{\alpha-1}\, \phi(x \mp t)\, dt, \tag{1.3}$$

i.e. fractional integrals are in fact convolution operators. Splitting the integration into two parts

$$I_\pm^\alpha \phi(x) = \tfrac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}\phi(x \mp t)\, dt + \tfrac{1}{\Gamma(\alpha)} \int_1^\infty t^{\alpha-1}\phi(x \mp t)\, dt$$

and applying the Minkowski integral inequality to the first term, the Hölder inequality to the second term, immediately shows that $I_\pm^\alpha \phi$ is (for almost every $x \in \mathbb{R}$) well-defined for any $\phi \in L^p(\mathbb{R})$ with $1 \le p < \frac{1}{\alpha}$ and $0 < \alpha < 1$. In particular, $I_\pm^\alpha \phi$ is defined for all $\phi \in \mathcal{S}(\mathbb{R})$, i.e. the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.

Definition 1.2. For $0 < \alpha < 1$ the *fractional derivatives* $\mathcal{D}_+^\alpha$ and $\mathcal{D}_-^\alpha$ are defined by

$$\mathcal{D}_+^\alpha \phi(x) = \tfrac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^x \frac{\phi(t)}{(x-t)^\alpha}\, dt \tag{1.4}$$

$$\mathcal{D}_-^\alpha \phi(x) = -\tfrac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^\infty \frac{\phi(t)}{(t-x)^\alpha}\, dt. \tag{1.5}$$

For $\alpha \ge 1$ we set $n = [\alpha] + 1$ and

$$\mathcal{D}_\pm^\alpha \phi(x) = \tfrac{(\pm 1)^n}{\Gamma(n-1)} \frac{d^n}{dx^n} \int_{\mathbb{R}_+} t^{n-\alpha-1}\, \phi(x \mp t)\, dt. \tag{1.6}$$

It is not hard to see that $\mathcal{D}_\pm^\alpha \phi$ is well-defined for all $\phi \in \mathcal{S}(\mathbb{R})$. Moreover, the following *Marchaud representation* holds.

Lemma 1.3. *Let $0 < \alpha < 1$. Then the operators*

$$D_\pm^\alpha \phi(x) = \tfrac{\alpha}{\Gamma(1-\alpha)} \int_{\mathbb{R}_+} \frac{\phi(x) - \phi(x \mp t)}{t^{1+\alpha}} \, dt \qquad (1.7)$$

are well-defined for $\phi \in \mathcal{S}(\mathbb{R})$ and satisfy $D_\pm^\alpha \phi = \mathcal{D}_\pm^\alpha \phi$ for all $\phi \in \mathcal{S}(\mathbb{R})$.

Note that the (maximal) domains of the operators $D_\pm^\alpha$ and $\mathcal{D}_\pm^\alpha$ do not coincide. The convergence of the integral in (1.7) depends on a Hölder condition for ϕ at 0, whereas (1.4) and (1.5) require certain growth conditions of ϕ at infinity. For our purposes (1.7) seems to be slightly more general, and we will therefore take $D_\pm^\alpha$ as *fractional derivative*. In particular, the operators $D_\pm^\alpha$ are well-defined for constant functions $f(x) \equiv c$, and (1.7) gives in this case $D_\pm^\alpha f(x) \equiv 0$.

For $\phi \in \mathcal{S}(\mathbb{R})$ we have always

$$D_\pm^\alpha I_\pm^\alpha \phi = I_\pm^\alpha D_\pm^\alpha \phi = \phi. \qquad (1.8)$$

Theorem 1.4. *For $\phi \in \mathcal{S}(\mathbb{R})$ and $0 < \alpha < 1$ the Fourier transforms of $I_\pm^\alpha \phi$ and $D_\pm^\alpha \phi$ are*

$$\widehat{I_\pm^\alpha \phi}(\xi) = \frac{1}{(\pm i\xi)^\alpha} \, \widehat{\phi}(\xi) \qquad (1.9)$$

$$\widehat{D_\pm^\alpha \phi}(\xi) = (\pm i\xi)^\alpha \, \widehat{\phi}(\xi). \qquad (1.10)$$

By (1.10) and Plancherel's theorem we get

$$\int_{\mathbb{R}} u(x) D_+^\alpha v(x) \, dx = \int_{\mathbb{R}} v(x) D_-^\alpha u(x) \, dx \qquad (u, v \in \mathcal{S}(\mathbb{R})). \qquad (1.11)$$

Since $|(\pm i\xi)^\alpha| \le |\xi|^\alpha$, Theorem 1.4 implies that $D_\pm^\alpha|_{\mathcal{S}(\mathbb{R})}$ can be extended onto the Sobolev space $H^\alpha(\mathbb{R})$ such that the mapping $D_\pm^\alpha : H^\alpha(\mathbb{R}) \to L^2(\mathbb{R})$ is continuous.

2. Fractional powers of the translation invariant drift form in one dimension

The bilinear form

$$b \int_{\mathbb{R}} u(x) \frac{dv(x)}{dx} \, dx \qquad (b \in \mathbb{R} \setminus \{0\}) \qquad (2.1)$$

defined on $\mathcal{S}(\mathbb{R})$ is called translation invariant *drift form*. Integration by parts yields

$$b \int_{\mathbb{R}} u(x) \frac{dv(x)}{dx} \, dx = -b \int_{\mathbb{R}} \frac{du(x)}{dx} v(x) \, dx \qquad (2.2)$$

which means that the drift form is completely antisymmetric. Therefore, setting $v = u$ in (2.2) shows

$$b \int_{\mathbb{R}} u(x) \frac{du(x)}{dx} \, dx = 0.$$

An application of Plancherel's theorem gives

$$b \int_{\mathbb{R}} u(x) \frac{dv(x)}{dx} \, dx = b \int_{\mathbb{R}} \widehat{u}(\xi) \, \overline{\left(\frac{d}{dx}v\right)\widehat{}(\xi)} \, d\xi,$$

and this implies

$$b \int_{\mathbb{R}} u(x) \frac{dv(x)}{dx} \, dx = b \int_{\mathbb{R}} \widehat{u}(\xi) \, \overline{i\xi \, \widehat{v}(\xi)} \, d\xi = \int_{\mathbb{R}} (-ib\xi) \widehat{u}(\xi) \, \overline{\widehat{v}(\xi)} \, d\xi. \qquad (2.3)$$

It is well-known that $\xi \mapsto -ib\xi$ $(b \in \mathbb{R})$ is a continuous negative definite function, but since $\mathrm{Re}(-ib\xi) \equiv 0$, the associated bilinear form is *no* translation invariant (non-symmetric) Dirichlet form. (By definition, a Dirichlet form $(\mathbf{B}(\cdot,\cdot), \mathcal{D}(\mathbf{B}))$ on $L^2(\mathbb{R}^n)$ is *translation invariant*, if for all translations τ_a $(a \in \mathbb{R}^n)$ and all $u, v \in \mathcal{D}(\mathbf{B})$ one has $\mathbf{B}(\tau_a u, \tau_a v) = \mathbf{B}(u,v)$.) This follows from a result due to C. Berg and G. Forst [2: Theorem 3.7] where it is shown that for a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ the associated bilinear form

$$\mathbf{B}^{\psi}(u,v) = \int_{\mathbb{R}}^{n} \psi(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)} \, d\xi \qquad (u,v \in \mathcal{S}(\mathbb{R}^n)) \qquad (2.4)$$

extends to a non-symmetric Dirichlet form if and only if

$$|\mathrm{Im}\psi(\xi)| \le c \, \mathrm{Re}\psi(\xi) \qquad (\xi \in \mathbb{R}^n) \qquad (2.5)$$

holds for some constant $c > 0$. Recall that a function $\psi : \mathbb{R}^n \to \mathbb{C}$ is said to be *negative definite*, if $\psi(0) \ge 0$ and if $\xi \mapsto e^{-t\psi(\xi)}$ is for all $t > 0$ positive definite in the usual sense. In this case, the domain of the Dirichlet form turns out to be

$$\mathcal{D}(\mathbf{B}^{\psi}) = \left\{ u \in L^2(\mathbb{R}^n) : \sqrt{\mathrm{Re}\psi} \, \widehat{u} \in L^2(\mathbb{R}^n) \right\}.$$

A standard way to construct new continuous negative definite functions from a given one is the composition with Bernstein functions: for any Bernstein function f and any (continuous) negative definite function ψ the function $f \circ \psi$ is again (continuous) negative definite (cf. C. Berg and G. Forst [3: p. 69]).

We are interested in the special Bernstein functions $f_{\alpha}(x) = x^{\alpha}$ $(0 < \alpha < 1)$. Because of the Lévy-Khinchin representation

$$x^{\alpha} = \tfrac{\alpha}{\Gamma(1-\alpha)} \int_{\mathbb{R}_+} (1 - e^{-sx}) \, s^{-\alpha-1} \, ds$$

we find for a continuous negative definite function $\psi : \mathbb{R} \to \mathbb{C}$ that

$$\psi(\xi)^{\alpha} = (f_{\alpha} \circ \psi)(\xi) = \tfrac{\alpha}{\Gamma(1-\alpha)} \int_{\mathbb{R}_+} \left(1 - e^{-s\psi(\xi)}\right) s^{-\alpha-1} \, ds.$$

We get, in particular, for the function $\psi(\xi) = -i\xi$

$$
\begin{aligned}
(-i\xi)^\alpha &= \tfrac{\alpha}{\Gamma(1-\alpha)} \int_{\mathbb{R}_+} \left(1 - \cos(\xi s)\right) \frac{ds}{s^{\alpha+1}} + \tfrac{i\alpha}{\Gamma(1-\alpha)} \int_{\mathbb{R}_+} \sin(\xi s) \frac{ds}{s^{\alpha+1}} \\
&= |\xi|^\alpha \, \cos\left(\frac{\alpha\pi}{2} \operatorname{sgn}\xi\right) - i|\xi|^\alpha \, \sin\left(\frac{\alpha\pi}{2} \operatorname{sgn}\xi\right).
\end{aligned}
\tag{2.6}
$$

Since $0 < \alpha < 1$, we see

$$
|\xi|^\alpha \, \cos\left(\frac{\alpha\pi}{2} \operatorname{sgn}\xi\right) = |\xi|^\alpha \, \cos\left(\frac{\alpha\pi}{2}\right)
$$

and

$$
|\xi|^\alpha \, \sin\left(\frac{\alpha\pi}{2} \operatorname{sgn}\xi\right) = \operatorname{sgn}\xi \, |\xi|^\alpha \, \sin\left(\frac{\alpha\pi}{2}\right)
$$

and this implies, in turn,

$$
|\mathrm{Im}(-i\xi)^\alpha| = |\xi|^\alpha \, \sin\left(\frac{\alpha\pi}{2}\right) = \tan\left(\frac{\alpha\pi}{2}\right) |\xi|^\alpha \, \cos\left(\frac{\alpha\pi}{2}\right) = c_\alpha \operatorname{Re}(-i\xi)^\alpha
$$

with the constant $c_\alpha = \tan\frac{\alpha\pi}{2} < \infty$ since $\alpha < 1$. Thus, the continuous negative definite function $(-i\xi)^\alpha$ fulfills (2.5), and – by the general theorem of Berg and Forst – the associated bilinear form

$$
\int_{\mathbb{R}} (-ib\xi)^\alpha \, \widehat{u}(\xi)\overline{\widehat{v}(\xi)} \, d\xi \qquad (b \in \mathbb{R} \setminus \{0\})
\tag{2.7}
$$

extends to a non-symmetric translation invariant Dirichlet form.

We want to give a different representation of form (2.7). Without loss of generality we will assume from now on that $b = 1$. By (2.6) we find for $u, v \in \mathcal{S}(\mathbb{R})$

$$
\begin{aligned}
\int_{\mathbb{R}} (-i\xi)^\alpha \, \widehat{u}(\xi)\overline{\widehat{v}(\xi)} \, d\xi = \tfrac{\alpha}{\Gamma(1-\alpha)} \Bigg(&\int_{\mathbb{R}}\int_{\mathbb{R}_+} (1 - \cos(s\xi))\widehat{u}(\xi)\overline{\widehat{v}(\xi)} \, \frac{ds}{s^{\alpha+1}} \, d\xi \\
&+ i\int_{\mathbb{R}}\int_{\mathbb{R}_+} \sin(s\xi) \, \widehat{u}(\xi)\overline{\widehat{v}(\xi)} \, \frac{ds}{s^{\alpha+1}} \, d\xi \Bigg).
\end{aligned}
$$

Observe that for $u \in \mathcal{S}(\mathbb{R})$

$$
e^{iy\xi} \, \widehat{u}(\xi) = \widehat{u(y + \cdot)}(\xi) = \widehat{\tau_{-y}u}(\xi) \qquad (y, \xi \in \mathbb{R})
$$

holds and that we have the identities

$$
1 - \cos(s\xi) = \operatorname{Re}\left(1 - e^{-is\xi}\right) = \tfrac{1}{2}\left(1 - e^{-is\xi}\right)\overline{\left(1 - e^{-is\xi}\right)}
$$

respectively

$$
\sin(s\xi) = \tfrac{1}{2i}\left(e^{is\xi} - e^{-is\xi}\right) = \tfrac{1}{2i}\left(e^{i\frac{s}{2}\xi} - e^{-i\frac{s}{2}\xi}\right)\left(e^{i\frac{s}{2}\xi} + e^{-i\frac{s}{2}\xi}\right).
$$

Using (several times) Plancherel's theorem and Fubini's theorem, we get for $u, v \in \mathcal{S}(\mathbb{R})$

$$\int_{\mathbb{R}}\int_{\mathbb{R}_+} (1 - \cos(s\xi))\, \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, \frac{ds}{s^{\alpha+1}}\, d\xi$$

$$= \tfrac{1}{2} \int_{\mathbb{R}_+}\int_{\mathbb{R}} (u - \tau_s u)\widehat{}(\xi)\overline{(v - \tau_s v)\widehat{}(\xi)}\, d\xi\, \frac{ds}{s^{\alpha+1}}$$

$$= \tfrac{1}{2} \int_{\mathbb{R}_+}\int_{\mathbb{R}} (u(x) - u(x - s))(v(x) - v(x - s))\, dx\, \frac{ds}{s^{\alpha+1}}$$

$$= \tfrac{1}{4} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x - y) - u(x))(v(x - y) - v(x))}{|y|^{\alpha+1}}\, dxdy$$

and, in a similar way,

$$i \int_{\mathbb{R}}\int_{\mathbb{R}_+} \sin(s\xi)\, \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, \frac{ds}{s^{\alpha+1}}\, d\xi$$

$$= \tfrac{1}{2} \int_{\mathbb{R}_+}\int_{\mathbb{R}} (\tau_{-\frac{s}{2}} u - \tau_{\frac{s}{2}} u)\widehat{}(\xi)\overline{(\tau_{-\frac{s}{2}} v + \tau_{\frac{s}{2}} v)\widehat{}(\xi)}\, d\xi\, \frac{ds}{s^{\alpha+1}}$$

$$= \tfrac{1}{2} \int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(u(x + \tfrac{s}{2}) - u(x - \tfrac{s}{2})\right)\left(v(x + \tfrac{s}{2}) + v(x - \tfrac{s}{2})\right) dx\, \frac{ds}{s^{\alpha+1}}$$

$$= \frac{1}{2^{\alpha+1}} \int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(u(x + t) - u(x - t)\right)\left(v(x + t) + v(x - t)\right) dx\, \frac{dt}{t^{\alpha+1}}.$$

This is almost the proof of our next theorem.

Theorem 2.1. *For any $0 < \alpha < 1$ the bilinear form*

$$\mathbf{E}^{(\alpha)}(u, v)$$

$$= \int_{\mathbb{R}} (-i\xi)^{\alpha}\, \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi$$

$$= \frac{\alpha}{4\Gamma(1 - \alpha)} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x - y) - u(x))(v(x - y) - v(x))}{|y|^{\alpha+1}}\, dxdy \qquad (2.8)$$

$$+ \frac{\alpha}{2^{\alpha+1}\Gamma(1 - \alpha)} \int_{\mathbb{R}}\int_{\mathbb{R}_+} \frac{(u(x + y) - u(x - y))(v(x + y) + v(x - y))}{y^{\alpha+1}}\, dydx$$

is a non-symmetric translation invariant Dirichlet form with domain $H^{\alpha/2}(\mathbb{R})$.

Proof. Identity (2.8) follows from the calculations preceding the theorem. That the bilinear form is indeed a non-symmetric Dirichlet form can be either concluded from the general theory by C. Berg and G. Forst [2] or directly verified using the right-hand side of (2.8): bilinearity, closedness, properties (DF.1) and (DF.2) are obvious, $H^{\frac{\alpha}{2}}(\mathbb{R})$ is a Hilbert space (with respect to the symmetric part of $\mathbf{E}^{(\alpha)}(\cdot, \cdot)$) which is invariant under Lipschitz maps, and the contraction property (DF.4) is easily (although tediously) checked by a direct computation ∎

Using the results on fractional derivatives summarized in Section 1 we can rewrite (2.8) in the following form.

Theorem 2.2. *For $0 < \alpha < 1$ and $u, v \in \mathcal{S}(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} v(x) D_-^\alpha u(x)\, dx = \int_{\mathbb{R}} D_-^{\frac{\alpha}{2}} u(x) D_+^{\frac{\alpha}{2}} v(x)\, dx = \int_{\mathbb{R}} (-i\xi)^\alpha\, \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi \qquad (2.9)$$

where the second equality holds even for $u, v \in H^{\alpha/2}(\mathbb{R})$.

Proof. Since we know that

$$\int_{\mathbb{R}} (-i\xi)^\alpha\, \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi = \int_{\mathbb{R}} (-i\xi)^{\frac{\alpha}{2}}\widehat{u}(\xi)\, \overline{(i\xi)^{\frac{\alpha}{2}}\widehat{v}(\xi)}\, d\xi$$

the assertion follows from (1.10) and Plancherel's theorem ∎

Note that Theorem 2.2 remains valid if $\alpha = 1$. This follows essentially form (2.3).

Corollary 2.3. *For all $u, v \in \mathcal{S}(\mathbb{R})$ or $u, v \in H^{\frac{1}{2}}(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} u(x) \frac{dv(x)}{dx}\, dx = \int_{\mathbb{R}} D_-^{\frac{1}{2}} u(x) D_+^{1/2} v(x)\, dx. \qquad (2.10)$$

Observe that (2.10) implies, in particular, for all $b \in \mathbb{R}$

$$\left| b \int_{\mathbb{R}} u(x) \frac{dv(x)}{dx}\, dx \right| \le |b|\, \|u\|_{H^{\frac{1}{2}}(\mathbb{R})} \|v\|_{H^{\frac{1}{2}}(\mathbb{R})}.$$

Moreover, the following convergence result holds true.

Theorem 2.4. *For all $u, v \in \mathcal{S}(\mathbb{R})$ or $u, v \in H^{\frac{1}{2}}(\mathbb{R})$ we have*

$$\lim_{\alpha \to 1} \int_{\mathbb{R}} D_-^{\frac{\alpha}{2}} u(x) D_+^{\frac{\alpha}{2}} v(x)\, dx = \int_{\mathbb{R}} D_-^{\frac{1}{2}} u(x) D_+^{\frac{1}{2}} v(x)\, dx. \qquad (2.11)$$

Proof. Since for all $\xi \in \mathbb{R}$

$$\left| (-i\xi)^\alpha \widehat{u}(\xi)\overline{\widehat{v}(\xi)} \right| \le |\xi|^\alpha\, |\widehat{u}(\xi)|\, |\widehat{v}(\xi)| \le (1 + |\xi|)\, |\widehat{u}(\xi)|\, |\widehat{v}(\xi)|$$

we can apply the dominated convergence theorem and conclude

$$\lim_{\alpha \to 1} \int_{\mathbb{R}} D_-^{\frac{\alpha}{2}} u(x) D_+^{\frac{\alpha}{2}} v(x)\, dx = \lim_{\alpha \to 1} \int_{\mathbb{R}} (-i\xi)^\alpha\, \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi$$

$$= \int_{\mathbb{R}} (-i\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi$$

$$= \int_{\mathbb{R}} D_-^{\frac{1}{2}} u(x) D_+^{\frac{1}{2}} v(x)\, dx$$

for all $u, v \in H^{\frac{1}{2}}(\mathbb{R})$ ∎

Corollary 2.5. *For all $u \in H^{\frac{1}{2}}(\mathbb{R})$ and $v \in H^1(\mathbb{R})$ we have*

$$\lim_{\alpha \to 1} \int_{\mathbb{R}} D_-^{\frac{\alpha}{2}} u(x) D_+^{\frac{\alpha}{2}} v(x)\, dx = \int_{\mathbb{R}} u(x) \frac{dv(x)}{dx}\, dx. \qquad (2.12)$$

We have thus found that the translation invariant drift form is the pointwise limit of non-symmetric translation invariant Dirichlet forms.

Obviously, the generator of the Dirichlet form $\langle D_-^{\alpha/2}\cdot, D_+^{\alpha/2}\cdot\rangle_0$ is the operator $-D_-^\alpha$ with domain $H^\alpha(\mathbb{R})$. A calculation similar to that in the proof of Theorem 2.4 shows again that for all $u \in H^1(\mathbb{R})$

$$\lim_{\alpha \uparrow 1} D_-^\alpha u = -\frac{d}{dx} u \qquad \text{strongly in } H^1(\mathbb{R}).$$

3. A remark on the closure of the set of translation invariant non-symmetric Dirichlet forms

Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function. It is well known that one can associate with every such ψ a uniquely determined Lévy process $\{X_t^\psi\}_{t \geq 0}$ with state space $\mathbb{R}^n$. If ψ is *real-valued*, the process $\{X_t^\psi\}_{t \geq 0}$ is also associated to the symmetric Dirichlet form

$$\int_{\mathbb{R}^n} \psi(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi$$

with domain $H^{\psi,1}(\mathbb{R}^n)$ which is given by

$$H^{\psi,1}(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \|u\|_{H^{\psi,1}(\mathbb{R}^n)} < \infty \right\} \tag{3.1}$$

where

$$\|u\|^2_{H^{\psi,1}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + \psi(\xi))|\widehat{u}(\xi)|^2\, d\xi \tag{3.2}$$

(see [3: p. 92]). For an *arbitrary* continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ we introduce the spaces $H^{\psi,s}(\mathbb{R}^n)$ $(s \in \mathbb{R})$ as in (3.1) but with norm

$$\|u\|_{H^{\psi,s}(\mathbb{R}^n)} = \left\| (1 + |\psi|)^{\frac{s}{2}}\, \widehat{u} \right\|_0 . \tag{3.3}$$

This scale of anisotropic Sobolev spaces is studied in [11] and [12: Chapter III.10].

Suppose now that $\psi : \mathbb{R}^n \to \mathbb{C}$ satisfies also (2.5). Since the real part $\mathrm{Re}\,\psi$ is itself a continuous negative definite function, the result of C. Berg and G. Forst [2] combined with the theory of non-symmetric Dirichlet forms (cf. [17]) imply that also in this case the Lévy process $\{X_t^\psi\}_{t \geq 0}$ is associated to the non-symmetric Dirichlet form

$$\mathbf{B}^\psi(u,v) = \int_{\mathbb{R}^n} \psi(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi$$

with domain $H^{\psi,1}(\mathbb{R}^n) = H^{\mathrm{Re}\,\psi,1}(\mathbb{R}^n)$.

Assume for the moment that $\psi : \mathbb{R}^n \to \mathbb{C}$ is an arbitrary continuous negative definite function and define the form $\mathbf{B}^\psi(\cdot,\cdot)$ for this ψ as above. From the estimate

$$\left| \int_{\mathbb{R}^n} \psi(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi \right| \leq \left(\int_{\mathbb{R}^n} |\psi(\xi)|\, |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\psi(\xi)|\, |\widehat{v}(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

where $u, v \in \mathcal{S}(\mathbb{R}^n)$ we easily deduce

$$|\mathbf{B}^\psi(u,v)| \leq \|u\|_{H^{\psi,1}(\mathbb{R}^n)} \|v\|_{H^{\psi,1}(\mathbb{R}^n)}$$

for all $u, v \in H^{\psi,1}(\mathbb{R}^n)$. Observe now that for any $0 < \alpha < 1$ and $z = x + iy \in \mathbb{C}$ we have

$$z^\alpha = |z|^\alpha e^{i\alpha \arg z} = |z|^\alpha \big(\cos(\alpha \arg z) + i \sin(\alpha \arg z) \big).$$

If $x = \mathrm{Re} z \geq 0$, then $\arg z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, hence

$$|\mathrm{Im} z^\alpha| = \tan(\alpha \arg z)\mathrm{Re} z^\alpha \leq \tan\left(\frac{\alpha\pi}{2}\right)\mathrm{Re} z^\alpha.$$

In particular, for *any* continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ the function $\psi^\alpha : \mathbb{R}^n \to \mathbb{C}$ $(0 < \alpha < 1)$ is again continuous negative definite (see, e.g., [3: p. 45]) and, since $\mathrm{Re}\psi \geq 0$, we have

$$|\mathrm{Im}\psi^\alpha(\xi)| \leq \tan\left(\frac{\alpha\pi}{2}\right)\mathrm{Re}\psi^\alpha(\xi) \qquad (0 < \alpha < 1). \tag{3.4}$$

Thus the fractional power ψ^α $(0 < \alpha < 1)$ of an arbitrary continuous negative definite function satisfies automatically condition (2.5). Moreover, there exists a Lévy process $\{X_t^{\psi^\alpha}\}_{t\geq 0}$ associated with the non-symmetric translation invariant Dirichlet form

$$\int_{\mathbb{R}^n} \psi^\alpha(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi$$

with domain $H^{\psi^\alpha,1}(\mathbb{R}^n) = H^{\mathrm{Re}\psi^\alpha,1}(\mathbb{R}^n)$. It is worth being noticed that the process $\{X_t^{\psi^\alpha}\}_{t\geq 0}$ has a realization as the process *subordinate* (in the sense of Bochner) to $\{X_t^\psi\}_{t\geq 0}$ with respect to the one-sided α-stable subordinator $\{\tau_t\}_{t\geq 0}$ which is by definition an independent increasing Lévy process with the Bernstein function $s \mapsto s^\alpha$ as characteristic exponent (see [4] or [27] for details).

The elementary estimate

$$1 + \mathrm{Re}\psi^\alpha(\xi) \leq 1 + |\psi(\xi)|^\alpha \leq 2(1 + |\psi(\xi)|)^\alpha \leq 2(1 + |\psi(\xi)|)$$

implies that $H^{\psi,1}(\mathbb{R}^n) \subset H^{\psi^\alpha,1}(\mathbb{R}^n)$ for all $0 < \alpha \leq 1$. Since for all $u, v \in H^{\psi,1}(\mathbb{R}^n)$

$$|\psi^\alpha(\xi)|\, |\widehat{u}(\xi)|\, |\widehat{v}(\xi)| \leq (1 + |\psi(\xi)|)|\widehat{u}(\xi)|\, |\widehat{v}(\xi)|,$$

we get, as in Theorem 2.4, by dominated convergence

$$\lim_{\alpha \to 1} \int_{\mathbb{R}^n} \psi^\alpha(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi = \mathbf{B}^\psi(u, v) \qquad (u, v \in H^{\psi,1}(\mathbb{R}^n)). \tag{3.5}$$

We have thus shown that for *any* Lévy process $\{X_t^\psi\}_{t\geq 0}$ there exists a bilinear form $(\mathbf{B}^\psi(\cdot,\cdot), H^{\psi,1}(\mathbb{R}^n))$ that can be obtained as a pointwise limit in $H^{\psi,1}(\mathbb{R}^n)$ of the non-symmetric translation invariant Dirichlet forms $(\mathbf{B}^{\psi^\alpha}(\cdot,\cdot), H^{\psi^\alpha,1}(\mathbb{R}^n))$ $(0 < \alpha < 1)$.

4. Time-dependent and generalized Dirichlet forms

Building on results by J. L. Lions and E. Magenes [15], M. Pierre introduced in [23, 24] *parabolic or time-dependent Dirichlet spaces*, and in [20, 21] Y. Oshima was able to construct a Markov process associated with a time-dependent Dirichlet space. Oshima's considerations were taken up by W. Stannat [29, 30] who followed the work by Lions and Magenes closer than Pierre and Oshima did and who introduced *generalized Dirichlet forms* and the associated stochastic processes.

We want to show how fractional derivatives enter naturally in the characterization of time-dependent Dirichlet spaces. Moreover, we will show that time-dependent Dirichlet forms arise as limits of certain non-symmetric Dirichlet forms. Similar statements hold also for generalized Dirichlet forms. Our presentation here follows, to some extent, the paper [10].

Let us briefly recall the notion of *time-dependent Dirichlet forms*. Let $V \subset L^2(\mathbb{R}^n)$ be a dense subspace such that $(V, \|\cdot\|_V)$ is a Hilbert space and assume that the test functions $C_c^\infty(\mathbb{R}^n) \subset V$ are a dense subset. Assume, moreover, that $\|u\|_0 \leq c\|u\|_V$ for all $u \in V$ and some constant $c > 0$. As usual, we identify $L^2(\mathbb{R}^n)$ with its topological dual and we get

$$V \hookrightarrow L^2(\mathbb{R}^n) \hookrightarrow V'$$

where "$\hookrightarrow$" stands for dense and continuous embedding. As usual, we have

$$\|u\|_{V'} = \sup_{0 \neq v \in V} \frac{|\langle u, v \rangle_0|}{\|v\|_V}.$$

Assume, furthermore, that V is stable under normal contractions, i.e. $u^+ \wedge 1 \in V$ for all $u \in V$. For $\tau \in \mathbb{R}$ let $E^{(\tau)} : V \times V \to \mathbb{R}$ be symmetric bilinear forms satisfying the following conditions:

(D.1) For all $u, v \in V$ the real-valued function $\tau \mapsto E^{(\tau)}(u, v)$ is measurable.

(D.2) The bilinear form $E_\lambda^{(\tau)}(u, v) = E^{(\tau)}(u, v) + \lambda \langle u, v \rangle_0$ is uniformly continuous on $V \times V$ with respect to τ, that is, there is a constant $M_\lambda > 0$ such that

$$|E_\lambda^{(\tau)}(u, v)| \leq M_\lambda \|u\|_V \|v\|_V$$

for all $u, v \in V$ and $\tau \in \mathbb{R}$.

(D.3) There exist two constants $\lambda_1 \geq 0$ and $\gamma_0 > 0$ such that

$$E^{(\tau)}(u, u) \geq \gamma_0 \|u\|_V^2 - \lambda_1 \|u\|_0^2$$

for all $u \in V$ and $\tau \in \mathbb{R}$.

(D.4) For all $u \in V$ and $\tau \in \mathbb{R}$ we have $E^{(\tau)}(u^+ \wedge 1, u^+ \wedge 1) \leq E^{(\tau)}(u, u)$.

Define on $\mathbb{R} \times \mathbb{R}^n$ the function spaces

$$\mathcal{H} = L^2(\mathbb{R}; L^2(\mathbb{R}^n)) \simeq L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^n)$$
$$\mathcal{V} = L^2(\mathbb{R}; V) \simeq L^2(\mathbb{R}) \otimes V$$
$$\mathcal{V}' = L^2(\mathbb{R}; V') \simeq L^2(\mathbb{R}) \otimes V'$$

which are equipped with their natural norms

$$\|u\|_{\mathcal{H}}^2 = \int_{\mathbb{R}} \|u(\tau, \cdot)\|_0^2 \, d\tau$$

$$\|u\|_{\mathcal{V}}^2 = \int_{\mathbb{R}} \|u(\tau, \cdot)\|_V^2 \, d\tau$$

$$\|u\|_{\mathcal{V}}'^2 = \int_{\mathbb{R}} \|u(\tau, \cdot)\|_{V'}^2 \, d\tau.$$

The next definition is due to Y. Oshima [20].

Definition 4.1. Let $E^{(\tau)}(\cdot, \cdot)$ be as above such that conditions (D.1) - (D.4) are satisfied. The *time-dependent Dirichlet space* $(\mathbf{E}(\cdot, \cdot), \mathcal{F})$ associated with the family $\{E^{(\tau)}(\cdot, \cdot)\}_{\tau \in \mathbb{R}}$ is given by

$$\mathcal{F} = \left\{ u \in \mathcal{V} : \frac{\partial u}{\partial \tau} \in \mathcal{V}' \right\}, \qquad \|u\|_{\mathcal{F}}^2 = \|u\|_{\mathcal{V}}^2 + \left\| \frac{\partial u}{\partial \tau} \right\|_{\mathcal{V}}'^2$$

and, for $u \in \mathcal{F}$ and $v \in \mathcal{V}$,

$$\mathbf{E}(u, v) := \int_{\mathbb{R}} E^{(\tau)}\big(u(\tau, \cdot), v(\tau, \cdot)\big) \, d\tau - \int_{\mathbb{R}} \left\langle \frac{\partial u(\tau, \cdot)}{\partial \tau}, v(\tau, \cdot) \right\rangle_0 d\tau$$

while for $u \in \mathcal{V}$ and $v \in \mathcal{F}$

$$\mathbf{E}(u, v) := \int_{\mathbb{R}} E^{(\tau)}\big(u(\tau, \cdot), v(\tau, \cdot)\big) \, d\tau + \int_{\mathbb{R}} \left\langle \frac{\partial v(\tau, \cdot)}{\partial \tau}, u(\tau, \cdot) \right\rangle_0 d\tau.$$

We are interested in the case where $V = H^{\psi,1}(\mathbb{R}^n)$ for a fixed continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$. Let us therefore assume that a family of symmetric Dirichlet forms $\{E^{(\tau)}(\cdot, \cdot), H^{\psi,1}(\mathbb{R}^n)\}_{\tau \in \mathbb{R}}$ is given such that:

(D.1′) $\tau \mapsto E^{(\tau)}(u, v)$ is measurable for all $u, v \in H^{\psi,1}(\mathbb{R}^n)$.

(D.2′) $|E^{(\tau)}(u, v)| \leq M \|u_{H^{\psi,1}(\mathbb{R}^n)}\|v\|_{H^{\psi,1}(\mathbb{R}^n)}$ with M independent of τ.

(D.3′) $E^{(\tau)}(u, u) \geq \gamma_0 \|u\|_{H^{\psi,1}(\mathbb{R}^n)}^2 - \lambda_1 \|u\|_0$ with γ_0, λ_1 independent of τ.

Accordingly, the spaces $\mathcal{H}, \mathcal{V}, \mathcal{V}'$ become

$$\mathcal{H} = L^2\big(\mathbb{R}; L^2(\mathbb{R}^n)\big)$$

$$\mathcal{V} = L^2\big(\mathbb{R}; H^{\psi,1}(\mathbb{R}^n)\big)$$

$$\mathcal{V}' = L^2\big(\mathbb{R}; H^{\psi,-1}(\mathbb{R}^n)\big)$$

and with

$$\mathcal{F} = \left\{ u \in L^2\big(\mathbb{R}; H^{\psi,1}(\mathbb{R}^n)\big) : \frac{\partial u}{\partial \tau} \in L^2(\mathbb{R}; H^{\psi,-1}(\mathbb{R}^n)) \right\}$$

we get the time-dependent Dirichlet form

$$\mathbf{E}(u, v) = \int_{\mathbb{R}} E^{(\tau)}\big(u(\tau, \cdot), v(\tau, \cdot)\big) \, d\tau - \int_{\mathbb{R}} \left\langle \frac{\partial u}{\partial \tau}(\tau, \cdot), v(\tau, \cdot) \right\rangle_0 d\tau \qquad (4.1)$$

whenever $u \in \mathcal{F}$ and $v \in \mathcal{V}$; if $u \in \mathcal{V}$ and $v \in \mathcal{F}$ we set

$$\mathbf{E}(u,v) = \int_{\mathbb{R}} E^{(\tau)}\big(u(\tau,\cdot), v(\tau,\cdot)\big)\, d\tau + \int_{\mathbb{R}} \Big\langle \frac{\partial v}{\partial \tau}(\tau,\cdot), u(\tau,\cdot) \Big\rangle_0 d\tau. \tag{4.2}$$

For (further) concrete examples we refer to the paper [10]. We want to get hold on the terms

$$\int_{\mathbb{R}} \Big\langle \frac{\partial u}{\partial \tau}(\tau,\cdot), v(\tau,\cdot) \Big\rangle_0 d\tau \qquad \text{and} \qquad \int_{\mathbb{R}} \Big\langle \frac{\partial v}{\partial \tau}(\tau,\cdot), u(\tau,\cdot) \Big\rangle_0 d\tau.$$

In order to do so we consider the anisotropic Sobolev spaces

$$H^{\rho_\alpha, s}(\mathbb{R} \times \mathbb{R}^n), \quad \rho_\alpha(\sigma, \xi) := |\sigma|^\alpha + \psi(\xi) \qquad (0 < \alpha \leq 1,\, s \in \mathbb{R}) \tag{4.3}$$

with norm given by

$$\|u\|^2_{H^{\rho_\alpha, s}(\mathbb{R} \times \mathbb{R}^n)} = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \big(1 + \rho_\alpha(\sigma, \xi)\big)^s |\widehat{u}(\sigma, \xi)|^2 d\xi d\sigma. \tag{4.4}$$

Note that ρ_α is again a continuous negative definite function on $\mathbb{R} \times \mathbb{R}^n$. Denote by $\widetilde{u}(\sigma, x)$ the partial Fourier transform in the first variable,

$$\widetilde{u}(\sigma, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\sigma\tau} u(\tau, x)\, d\tau \tag{4.5}$$

whereas $\widehat{u}(\sigma, \xi)$ denotes the (full) Fourier transform in the $\mathbb{R} \times \mathbb{R}^n$-space. It is not hard to see that

$$\|u\|^2_{\mathcal{F}} = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \big(1 + \psi(\xi)\big)|\widehat{u}(\sigma, \xi)|^2 d\xi d\sigma + \int_{\mathbb{R}} \int_{\mathbb{R}^n} \big(1 + \psi(\xi)\big)^{-1} |\sigma|^2 |\widehat{u}(\sigma, \xi)|^2 d\xi d\sigma.$$

Therefore, we have $H^{\rho_1, 2}(\mathbb{R} \times \mathbb{R}^n) \subset \mathcal{F}$, and for $u, v \in H^{\rho_1, 2}(\mathbb{R} \times \mathbb{R}^n)$ we find using the Plancherel theorem

$$\begin{aligned}
\mathbf{E}(u,v) &= \int_{\mathbb{R}} E^{(\tau)}\big(u(\tau,\cdot), v(\tau,\cdot)\big)\, d\tau - \int_{\mathbb{R}} \Big\langle \frac{\partial u}{\partial \tau}(\tau,\cdot), v(\tau,\cdot) \Big\rangle_0 d\tau \\
&= \int_{\mathbb{R}} E^{(\tau)}\big(u(\tau,\cdot), v(\tau,\cdot)\big)\, d\tau + \int_{\mathbb{R}} \int_{\mathbb{R}^n} (-i\sigma)\widetilde{u}(\sigma, x)\, \overline{\widetilde{v}(\sigma, x)}\, dx d\sigma \\
&= \int_{\mathbb{R}} E^{(\tau)}\big(u(\tau,\cdot), v(\tau,\cdot)\big)\, d\tau + \int_{\mathbb{R}} \int_{\mathbb{R}^n} D^{\frac{1}{2}}_{-,\tau} u(\tau, x)\, D^{\frac{1}{2}}_{+,\tau} v(\tau, x)\, dx d\tau.
\end{aligned}$$

Moreover, if $u, v \in H^{\rho_1, 1}(\mathbb{R} \times \mathbb{R}^n)$, we find by a density argument

$$|\mathbf{E}(u,v)| \leq M \int_{\mathbb{R}} \|u(\tau,\cdot)\|_{H^{\psi,1}(\mathbb{R}^n)} \|v(\tau,\cdot)\|_{H^{\psi,1}(\mathbb{R}^n)}\, d\tau$$

$$+ \|D^{\frac{1}{2}}_{-,\tau} u\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \|D^{\frac{1}{2}}_{+,\tau} v\|_{L^2(\mathbb{R} \times \mathbb{R}^n)}.$$

Since $L^2\big(\mathbb{R}, H^{\psi,1}(\mathbb{R}^n)\big) \simeq L^2(\mathbb{R}) \otimes H^{\psi,1}(\mathbb{R}^n)$, an application of the Cauchy-Schwarz inequality yields

$$\begin{aligned}
|\mathbf{E}(u,v)| &\leq M\|u\|_{H^{\rho_1, 1}(\mathbb{R} \times \mathbb{R}^n)} \|v\|_{H^{\rho_1, 1}(\mathbb{R} \times \mathbb{R}^n)} \\
&\quad + \|D^{\frac{1}{2}}_{-,\tau} u\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \|D^{\frac{1}{2}}_{+,\tau} v\|_{L^2(\mathbb{R} \times \mathbb{R}^n)}.
\end{aligned} \tag{4.6}$$

On the other hand,

$$\|D^{\frac{1}{2}}_{\mp} u\|^2_{L^2(\mathbb{R} \times \mathbb{R}^n)} \leq \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\sigma|\, |\widehat{u}(\sigma, \xi)|^2 d\xi d\sigma \leq \|u\|^2_{H^{\rho_1, 1}(\mathbb{R} \times \mathbb{R}^n)}. \tag{4.7}$$

Together, (4.6) and (4.7) prove our next theorem.

Theorem 4.2. *Let* $\left\{E^{(\tau)}(\cdot,\cdot), H^{\psi,1}(\mathbb{R}^n)\right\}_{\tau \geq 0}$, $(\mathbf{E}(\cdot,\cdot), \mathcal{F})$, *and* $\psi : \mathbb{R}^n \to \mathbb{R}$ *be as above. Then the estimate*

$$|\mathbf{E}(u,v)| \leq c\|u\|_{H^{\rho_1,1}(\mathbb{R}\times\mathbb{R}^n)}\|v\|_{H^{\rho_1,1}(\mathbb{R}\times\mathbb{R}^n)} \tag{4.8}$$

is valid for all $u,v \in H^{\rho_1,1}(\mathbb{R}\times\mathbb{R}^n)$ *with* $\rho_1(\sigma,\xi) = |\sigma| + \psi(\xi)$.

As in the case of the translation invariant drift form considered in Section 2 above, the form $\left(\mathbf{E}, H^{\rho_1,1}(\mathbb{R}\times\mathbb{R}^n)\right)$ does not satisfy the sector condition on $H^{\rho_1,1}(\mathbb{R}\times\mathbb{R}^n)$ since the term $\left\langle D_{-,\tau}^{\frac{1}{2}}u, D_{+,\tau}^{\frac{1}{2}}v\right\rangle_{L^2(\mathbb{R}\times\mathbb{R}^n)}$ does neither satisfy the sector condition nor can it be controlled by the expression $\int_{\mathbb{R}} E^{(\tau)}\big(u(\tau,\cdot), v(\tau,\cdot)\big)\,d\tau$. However, as in Section 2, we get the following assertion.

Theorem 4.3. *For every* $0 < \alpha < 1$ *the bilinear form*

$$\mathbf{E}^{(\alpha)}(u,v) = \int_{\mathbb{R}} E^{(\tau)}\big(u(\tau,\cdot), v(\tau,\cdot)\big)d\tau + \int_{\mathbb{R}}\int_{\mathbb{R}^n} D_{-,\tau}^{\frac{\alpha}{2}}u(\tau,x)\, D_{+,\tau}^{\frac{\alpha}{2}}v(\tau,x)\,dxd\tau \tag{4.9}$$

satisfies on $H^{\rho_\alpha,1}(\mathbb{R}\times\mathbb{R}^n)$ *the sector condition, i.e.*

$$|\mathbf{E}^{(\alpha)}(u,v)| \leq c_\alpha \|u\|_{H^{\rho_\alpha,1}(\mathbb{R}\times\mathbb{R}^n)}\|v\|_{H^{\rho_\alpha,1}(\mathbb{R}\times\mathbb{R}^n)} \tag{4.10}$$

for all $u,v \in H^{\rho_\alpha,1}(\mathbb{R}\times\mathbb{R}^n)$. *In particular,* $\left(\mathbf{E}^{(\alpha)}(\cdot,\cdot), H^{\rho_\alpha,1}(\mathbb{R}\times\mathbb{R}^n)\right)$ *is a non-symmetric Dirichlet form.*

Proof. It is enough to control the antisymmetric part of (4.9) by its symmetric part. Since by our assumptions $\int_{\mathbb{R}} E^{(\tau)}\big(u(\tau,\cdot), v(\tau,\cdot)\big)\,d\tau$ is symmetric, we only have to estimate the antisymmetric part of $\left\langle D_{-,\tau}^{\frac{\alpha}{2}}u, D_{+,\tau}^{\frac{\alpha}{2}}v\right\rangle_{L^2(\mathbb{R}\times\mathbb{R}^n)}$ in terms of its symmetric part. But this is done by exactly the same calculations as for the subordinated drift form (cf. Section 2).

The proof of (4.10) is an obvious adaptation of the proof of Theorem 4.2. That $\left(\mathbf{E}^{(\alpha)}(\cdot,\cdot), H^{\rho_\alpha,1}(\mathbb{R}\times\mathbb{R}^n)\right)$ is indeed an ordinary non-symmetric Dirichlet form follows easily: bilinearity and closedness of the form are clear, the sector condition has just been established, and (DF.1), (DF.3) - (DF.5) can be seen just as in Theorem 2.1 ∎

With the same methods as in Section 2 get again an approximation result:

Corollary 4.4. *In the situation of Theorem* 4.3 *we have*

$$\lim_{\alpha\uparrow 1}\mathbf{E}^{(\alpha)}(u,v) = \mathbf{E}(u,v) \tag{4.11}$$

for all $u,v \in H^{\rho_1,1}(\mathbb{R}\times\mathbb{R}^n)$.

In the case of generalized Dirichlet forms (see W. Stannat [29, 30]) the operator $\frac{\partial}{\partial\tau}$ in the definition of $\mathbf{E}(\cdot,\cdot)$ is replaced by the generator Λ_τ of any sub-Markovian semigroup on $L^2(\mathbb{R})$. If the operator Λ_τ is of the type

$$\Lambda_\tau u(\tau,x) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\sigma\tau}\theta(\sigma)\widetilde{u}(\sigma,x)\,d\sigma$$

with some continuous negative definite function $\theta : \mathbb{R} \to \mathbb{C}$, we may almost literally apply the considerations made before for time-dependent Dirichlet forms. More general situations will be treated in Section 6 below.

5. The drift form in $\mathbb{R}^n$

We will now consider the bilinear form

$$\mathbf{D}(u,v) = \sum_{j=1}^{n} \int_{\mathbb{R}^n} b_j(x)\, u(x)\, \frac{\partial v(x)}{\partial x_j}\, dx. \tag{5.1}$$

By $D_{\pm,j}^{\alpha}$ we denote the partial fractional derivative of order α $(0 < \alpha \le 1)$ in direction x_j $(1 \le j \le n)$ given by the analogue of (1.7). If the coefficient functions b_j are smooth enough, we find

$$\sum_{j=1}^{n} \int_{\mathbb{R}^n} b_j(x) u(x) (D_{-,j}^{\frac{1}{2}} D_{-,j}^{\frac{1}{2}} v)(x)\, dx$$

$$= \sum_{j=1}^{n} \int_{\mathbb{R}^n} b_j(x) D_{+,j}^{\frac{1}{2}} u(x) \cdot D_{-,j}^{\frac{1}{2}} v(x)\, dx + \sum_{j=1}^{n} \int_{\mathbb{R}^n} u(x) [b_j; D_{-,j}^{\frac{1}{2}}] D_{-,j}^{\frac{1}{2}} v(x)\, dx.$$

By $[b_j; D_{-,j}^{\frac{1}{2}}]$ we denote the *commutator*:

$$[b_j; D_{-,j}^{\frac{1}{2}}]u = b_j(x) D_{-,j}^{\frac{1}{2}} u(x) - D_{-,j}^{\frac{1}{2}}(b_j u)(x).$$

We will see below for which b_j these formal manipulations can be justified. If $b_j \in L^\infty(\mathbb{R}^n)$, we have the obvious estimate

$$\left| \sum_{j=1}^{n} \int_{\mathbb{R}^n} b_j(x) D_{+,j}^{\frac{1}{2}} u(x) \cdot D_{-,j}^{\frac{1}{2}} v(x)\, dx \right|$$

$$\le \sum_{j=1}^{n} \|b_j\|_\infty \|D_{+,j}^{\frac{1}{2}} u\|_0 \, \|D_{-,j}^{\frac{1}{2}} v\|_0$$

$$\le \sum_{j=1}^{n} \|b_j\|_\infty \|u\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \|v\|_{H^{\frac{1}{2}}(\mathbb{R}^n)}.$$

In order to estimate the second term in the first calculation above we need the following commutator estimate.

Proposition 5.1. *Let $b \in C_b^1(\mathbb{R}^n)$. Then we have for all $0 < \alpha \le 1$ and $1 < p < \infty$ the estimate*

$$\left\| [D_{-,j}^{\alpha}; b]u \right\|_{L^p(\mathbb{R}^n)} \le c_{\alpha,p} \|b\|_{C_b^1(\mathbb{R}^n)} \|u\|_{L^p(\mathbb{R}^n)} \tag{5.2}$$

where $[D_{-,j}^{\alpha}; b]u = D_{-,j}^{\alpha}(bu) - b D_{-,j}^{\alpha} u$ stands for the commutator of $D_{-,j}^{\alpha}$ and b.

Proof. For notational convenience, we write $(\hat{x}, x_j)$ instead of $(x_1, \ldots, x_j, \ldots, x_n)$; this abuse of notation should not cause any problems. Since

$$D_{-,j}^{\alpha} v(x) = \tfrac{\alpha}{\Gamma(1-\alpha)} \int_{\mathbb{R}_+} \big(v(\hat{x}, x_j) - v(\hat{x}, x_j + t) \big) t^{-\alpha-1} dt$$

we find

$$[D^\alpha_{-,j}; b]u(\hat{x}, x_j) = \int_{\mathbb{R}_+} \frac{b(\hat{x}, x_j) - b(\hat{x}, x_j + t)}{t^{\alpha+1}} u(\hat{x}, x_j + t)\, dt$$

$$= \int_0^1 \frac{(b(\hat{x}, x_j) - b(\hat{x}, x_j + t))}{t^{\alpha+1-\lambda}} \frac{u(\hat{x}, x_j + t)}{t^\lambda}\, dt$$

$$+ \int_1^\infty \frac{(b(\hat{x}, x_j) - b(\hat{x}, x_j + t))}{t^{\alpha+1-\mu}} \frac{u(\hat{x}, x_j + t)}{t^\mu}\, dt$$

with parameters λ, μ satisfying $\alpha - 1 + \frac{1}{p} < \lambda < \frac{1}{p} < \mu < \alpha + \frac{1}{p}$. If q is such that $\frac{1}{p} + \frac{1}{q} = 1$, we get by the Hölder inequality

$$\left| [D^\alpha_{-,j}; b]u(\hat{x}, x_j) \right|$$

$$\leq \left(\int_0^1 \frac{|b(\hat{x}, x_j) - b(\hat{x}, x_j + t)|^q}{t^{(\alpha+1-\lambda)q}}\, dt \right)^{\frac{1}{q}} \left(\int_0^1 \frac{|u(\hat{x}, x_j + t)|^p}{t^{\lambda p}}\, dt \right)^{\frac{1}{p}}$$

$$+ \left(\int_1^\infty \frac{|b(\hat{x}, x_j) - b(\hat{x}, x_j + t)|^q}{t^{(\alpha+1-\mu)q}}\, dt \right)^{\frac{1}{q}} \left(\int_1^\infty \frac{|u(\hat{x}, x_j + t)|^p}{t^{\mu p}}\, dt \right)^{\frac{1}{p}}$$

$$\equiv I_1(x) + I_2(x).$$

Taking the $L^p(\mathbb{R}^n)$-norm we find

$$\|I_1\|_{L^p(\mathbb{R}^n)}$$

$$\leq \sup_{x \in \mathbb{R}^n} \left(\int_0^1 \frac{|b(\hat{x}, x_j) - b(\hat{x}, x_j + t)|^q}{t^{(\alpha+1-\lambda)q}}\, dt \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} \int_0^1 \frac{|u(\hat{x}, x_j + t)|^p}{t^{\lambda p}}\, dt\, dx \right)^{\frac{1}{p}}$$

$$= \sup_{x \in \mathbb{R}^n} \left(\int_0^1 \frac{|b(\hat{x}, x_j) - b(\hat{x}, x_j + t)|^q}{t^{(\alpha+1-\lambda)q}}\, dt \right)^{\frac{1}{q}} \left(\int_0^1 \frac{dt}{t^{\lambda p}} \right)^{1/p} \|u\|_{L^p(\mathbb{R}^n)}.$$

Observing that

$$\sup_{x \in \mathbb{R}^n} \left(\int_0^1 \frac{|b(\hat{x}, x_j) - b(\hat{x}, x_j + t)|^q}{t^{(\alpha+1-\lambda)q}}\, dt \right)^{\frac{1}{q}} \leq \left\| \frac{\partial b}{\partial x_j} \right\|_\infty \left(\int_0^1 \frac{dt}{t^{(\alpha-\lambda)q}} \right)^{\frac{1}{q}}$$

and that, by our choice of λ, $q(\alpha - \lambda) < 1$, we get

$$\|I_1\|_{L^p(\mathbb{R}^n)} \leq c_1 \left\| \frac{\partial b}{\partial x_j} \right\|_\infty \|u\|_{L^p(\mathbb{R}^n)}.$$

By a similar calculation,

$$\|I_2\|_{L^p(\mathbb{R}^n)}$$

$$\leq \sup_{x \in \mathbb{R}^n} \left(\int_1^\infty \frac{|b(\hat{x}, x_j) - b(\hat{x}, x_j + t)|^q}{t^{(\alpha+1-\mu)q}}\, dt \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n}\!\int_1^\infty \frac{|u(\hat{x}, x_j + t)|^p}{t^{\mu p}}\, dt dx \right)^{\frac{1}{p}}$$

$$\leq \left(\int_1^\infty \frac{\|b(\hat{x}, x_j) - b(\hat{x}, x_j + t)\|_\infty^q}{t^{(\alpha+1-\mu)q}}\, dt \right)^{\frac{1}{q}} \left(\int_1^\infty \frac{dt}{t^{\mu p}} \right)^{\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^n)}$$

$$\leq 2^{1-\frac{1}{q}}\|b\|_\infty \left(\int_1^\infty \frac{dt}{t^{(\alpha+1-\mu)q}} \right)^{\frac{1}{q}} \left(\int_1^\infty \frac{dt}{t^{\mu p}} \right)^{\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^n)}$$

implying – by our choice of λ and μ – that

$$\|I_2\|_{L^p(\mathbb{R}^n)} \leq c_2 \|b\|_\infty \|u\|_{L^p(\mathbb{R}^n)}$$

and the assertion follows ∎

The next theorem shows, in particular, for which b_j the formal calculations of the first paragraph in this section can be justified.

Theorem 5.2. *Let $\mathbf{D}(u,v)$ be given by (5.1). For $b_j \in C_b^1(\mathbb{R}^n)$ the estimate*

$$|\mathbf{D}(u,v)| \leq c\, \|u\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \|v\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \tag{5.3}$$

holds for all $u, v \in H^{\frac{1}{2}}(\mathbb{R}^n)$.

Proof. Let $u, v \in H^{\frac{1}{2}}(\mathbb{R}^n)$. Then

$$|\mathbf{D}(u,v)| \leq \sum_{j=1}^n \|b_j\|_\infty \|u\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \|v\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} + \sum_{j=1}^n \left| \int_{\mathbb{R}^n} u(x)[b_j; D^{\frac{1}{2}}_{-,j}] D^{\frac{1}{2}}_{-,j} v(x)\, dx \right|$$

$$\leq \sum_{j=1}^n \|b_j\|_\infty \|u\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \|v\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} + \sum_{j=1}^n c_j \|b_j\|_{C_b^1(\mathbb{R}^n)} \|u\|_0 \|D^{\frac{1}{2}}_{-,j} v\|_0$$

$$\leq c \sum_{j=1}^n \|b_j\|_{C_b^1(\mathbb{R}^n)} \|u\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \|v\|_{H^{\frac{1}{2}}(\mathbb{R}^n)}$$

and we are done ∎

Remark 5.3. For $u \in H^1(\mathbb{R}^n)$ we get

$$\mathbf{D}(u,u) = \tfrac{1}{2} \sum_{j=1}^n \int_{\mathbb{R}^n} b_j(x) \frac{\partial u^2(x)}{\partial x_j}\, dx = -\tfrac{1}{2} \int_{\mathbb{R}^n} \left(\sum_{j=1}^n \frac{\partial b_j(x)}{\partial x_j} \right) u^2(x)\, dx.$$

Thus, $\sum_{j=1}^n \frac{\partial b_j(x)}{\partial x_j} \leq 0$ implies $\mathbf{D}(u,u) \geq 0$. By Theorem 5.2 we get even $\mathbf{D}(u,u) \geq 0$ for $u \in H^{\frac{1}{2}}(\mathbb{R}^n)$.

Let $0 < \alpha < 1$ and $b_j \in C_b^1(\mathbb{R}^n)$. We consider the form

$$
\begin{aligned}
\mathbf{D}^{(\alpha)}(u, v) &:= \sum_{j=1}^n \int_{\mathbb{R}^n} b_j(x) v(x) D_{-,j}^\alpha u(x) \, dx \\
&= \sum_{j=1}^n \int_{\mathbb{R}^n} b_j(x) D_{+,j}^{\frac{\alpha}{2}} v(x) \cdot D_{-,j}^{\frac{\alpha}{2}} u(x) \, dx \\
&\quad + \sum_{j=1}^n \int_{\mathbb{R}^n} v(x) [b_j; D_{-,j}^{\frac{\alpha}{2}}] D_{-,j}^{\frac{\alpha}{2}} u(x) \, dx.
\end{aligned}
\tag{5.4}
$$

From Proposition 5.1 we know that this form is well-defined and we immediately get the following analogue to Theorem 5.2.

Theorem 5.4. *Let $0 < \alpha < 1$ and $\mathbf{D}^{(\alpha)}(u, v)$ be given by (5.4). For $b_j \in C_b^1(\mathbb{R}^n)$ the estimate*

$$
|\mathbf{D}^{(\alpha)}(u, v)| \le c \, \|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)} \|v\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)}
\tag{5.5}
$$

holds for all $u, v \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$. Moreover, we have

$$
\lim_{\alpha \uparrow 1} \mathbf{D}^{(\alpha)}(u, v) = \mathbf{D}(u, v)
\tag{5.6}
$$

for all $u, v \in H^{\frac{1}{2}}(\mathbb{R}^n)$.

Proof. Since (5.5) follows just as (5.3), we will only check (5.6). Using (5.4) we get

$$
\begin{aligned}
\mathbf{D}^{(\alpha)}(u, v) &= \sum_{j=1}^n \int_{\mathbb{R}^n} b_j(x) D_{+,j}^{\frac{\alpha}{2}} v(x) \cdot D_{-,j}^{\frac{\alpha}{2}} u(x) \, dx \\
&\quad + \sum_{j=1}^n \int_{\mathbb{R}^n} [D_{+,j}^{\frac{\alpha}{2}}; b_j] v(x) \cdot D_{-,j}^{\frac{\alpha}{2}} u(x) \, dx.
\end{aligned}
$$

For $u, v \in H^{\frac{1}{2}}(\mathbb{R}^n)$ and $b_j \in C_b^1(\mathbb{R}^n)$ the limits

$$
\lim_{\alpha \uparrow 1} D_{-,j}^{\frac{\alpha}{2}} u = D_{-,j}^{\frac{1}{2}} u \qquad \text{and} \qquad \lim_{\alpha \uparrow 1} [D_{+,j}^{\frac{\alpha}{2}}; b_j] v = [D_{+,j}^{\frac{1}{2}}; b_j] v
$$

exist strongly in $L^2(\mathbb{R}^n)$ and the assertion follows ∎

In order to examine the form $\mathbf{D}^{(\alpha)}(\cdot, \cdot)$ in greater detail, we note that we can rewrite (5.4) as

$$
\mathbf{D}^{(\alpha)}(u, v) = \sum_{j=1}^n \mathbf{E}^{(\alpha)}(u, b_j v)
$$

where $\mathbf{E}^{(\alpha)}(\cdot, \cdot)$ is the higher-dimensional counterpart of the form introduced in (2.8). Set $\gamma_\alpha = \frac{\alpha}{\Gamma(1-\alpha)}$. Applying Theorem 2.1 to each of the terms $\mathbf{E}^{(\alpha)}(u, b_j v)$ $(j = 1, \ldots, n)$

we find

$$\mathbf{E}^{(\alpha)}(u, b_j v)$$

$$= \int_{\mathbb{R}^n} b_j(x) v(x) D^\alpha_{-,j} u(x)\, dx$$

$$= \tfrac{\gamma_\alpha}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}} \frac{\left(u(\hat{x}, x_j - t) - u(\hat{x}, x_j)\right)\left((b_j v)(\hat{x}, x_j - t) - (b_j v)(\hat{x}, x_j)\right)}{|t|^{\alpha+1}}\, dt dx$$

$$+ \tfrac{\gamma_\alpha}{2^{\alpha+1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} \frac{\left(u(\hat{x}, x_j + t) - u(\hat{x}, x_j - t)\right)\left((b_j v)(\hat{x}, x_j + t) + (b_j v)(\hat{x}, x_j - t)\right)}{t^{\alpha+1}}\, dt dx$$

$$\equiv \tfrac{1}{4} I_1^j + \tfrac{1}{2^{\alpha+1}} I_2^j.$$

Elementary but rather lengthy calculations yield

$$I_1^j = \gamma_\alpha \int_{\mathbb{R}^n} \int_{\mathbb{R}} b_j(\hat{x}, x_j) \frac{\left(u(\hat{x}, x_j + t) - u(\hat{x}, x_j)\right)\left(v(\hat{x}, x_j + t) - v(\hat{x}, x_j)\right)}{|t|^{\alpha+1}}\, dt dx$$

$$- \int_{\mathbb{R}^n} v(x) [D^\alpha_{+,j} + D^\alpha_{-,j}; b_j] u(x)\, dx + \int_{\mathbb{R}^n} v(x) u(x) (D^\alpha_{+,j} + D^\alpha_{-,j}) b_j(x)\, dx$$

and

$$I_2^j =$$

$$\gamma_\alpha \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} b_j(\hat{x}, x_j + t) \frac{\left(u(\hat{x}, x_j + t) - u(\hat{x}, x_j - t)\right)\left(v(\hat{x}, x_j + t) + v(\hat{x}, x_j - t)\right)}{t^{\alpha+1}}\, dt dx$$

$$+ 2^\alpha \int_{\mathbb{R}^n} v(x) [D^\alpha_{+,j}; b_j] u(x)\, dx - 2^\alpha \int_{\mathbb{R}^n} v(x) u(x) D^\alpha_{+,j} b_j(x)\, dx.$$

This proves our next theorem.

Theorem 5.5. *Let* $\mathbf{D}^{(\alpha)}(u, v)$ *denote the bilinear form given by (5.4). If* $b_j \in C^1_b(\mathbb{R}^n)$, *we have the alternative representation*

$$\mathbf{D}^{(\alpha)}(u, v) = \sum_{j=1}^n \left(\tfrac{\gamma_\alpha}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}} b_j(\hat{x}, x_j) \right.$$

$$\times \frac{\left(u(\hat{x}, x_j + t) - u(\hat{x}, x_j)\right)\left(v(\hat{x}, x_j + t) - v(\hat{x}, x_j)\right)}{|t|^{\alpha+1}}\, dt dx$$

$$+ \tfrac{\gamma_\alpha}{2^{\alpha+1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} b_j(\hat{x}, x_j + t) \qquad\qquad (5.7)$$

$$\left. \times \frac{\left(u(\hat{x}, x_j + t) - u(\hat{x}, x_j - t)\right)\left(v(\hat{x}, x_j + t) + v(\hat{x}, x_j - t)\right)}{t^{\alpha+1}}\, dt dx \right)$$

$$+ \frac{1}{2} R^{(\alpha)}(u, v)$$

where $\gamma_\alpha = \frac{\alpha}{\Gamma(1-\alpha)}$ *and*

$$R^{(\alpha)}(u, v) =$$

$$\tfrac{1}{2} \sum_{j=1}^n \int_{\mathbb{R}^n} v(x) \left([D^\alpha_{+,j} - D^\alpha_{-,j}; b_j] u(x) - u(x)(D^\alpha_{+,j} - D^\alpha_{-,j}) b_j(x) \right) dx \qquad (5.8)$$

for all $u, v \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$.

Corollary 5.6. *Define $D^\alpha_{j,\mathrm{Im}} = \frac{1}{2}(D^\alpha_{-,j} - D^\alpha_{+,j})$. In the situation of Theorem 5.5 we find for $u \in H^\alpha(\mathbb{R}^n)$ that*

$$R^{(\alpha)}(u, v) = -\sum_{j=1}^{n} \int_{\mathbb{R}^n} v(x) \Big(D^\alpha_{j,\mathrm{Im}}(b_j u)(x) - b_j(x)D^\alpha_{j,\mathrm{Im}}u(x) - u(x)D^\alpha_{j,\mathrm{Im}}b_j(x) \Big) dx.$$

If, in particular, for all $u \in H^\alpha(\mathbb{R}^n)$

$$\sum_{j=1}^{n} \Big(D^\alpha_{j,\mathrm{Im}}(b_j u)(x) - b_j(x)D^\alpha_{j,\mathrm{Im}}u(x) - u(x)D^\alpha_{j,\mathrm{Im}}b_j(x) \Big) = 0 \tag{5.9}$$

is satisfied, we have

$$\mathbf{D}^{(\alpha)}(u, v) = \sum_{j=1}^{n} \Bigg(\frac{\gamma_\alpha}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}} b_j(\hat{x}, x_j)$$

$$\times \frac{\big(u(\hat{x}, x_j + t) - u(\hat{x}, x_j)\big)\big(v(\hat{x}, x_j + t) - v(\hat{x}, x_j)\big)}{|t|^{\alpha+1}} \, dtdx$$

$$+ \frac{\gamma_\alpha}{2^{\alpha+1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} b_j(\hat{x}, x_j + t)$$

$$\times \frac{\big(u(\hat{x}, x_j + t) - u(\hat{x}, x_j - t)\big)\big(v(\hat{x}, x_j + t) + v(\hat{x}, x_j - t)\big)}{t^{\alpha+1}} \, dtdx \Bigg)$$

with $\gamma_\alpha = \frac{\alpha}{\Gamma(1-\alpha)}$. Under the additional assumptions

$$b_j \geq 0 \;\; (1 \leq j \leq n) \qquad and \qquad \sum_{j=1}^{n} D^\alpha_{-,j} b_j \geq 0 \tag{5.10}$$

the bilinear form $\mathbf{D}^{(\alpha)}(\cdot, \cdot)$ is positive definite in the sense that $\mathbf{D}^{(\alpha)}(u, u) \geq 0$ for all $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$.

Proof. It remains to show the positive definiteness of the form. Let $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$. Then

$$\mathbf{D}^{(\alpha)}(u, u) = \frac{\alpha}{4\Gamma(1-\alpha)} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \int_{\mathbb{R}} b_j(\hat{x}, x_j) \frac{\big(u(\hat{x}, x_j + t) - u(\hat{x}, x_j)\big)^2}{|t|^{\alpha+1}} \, dtdx$$

$$+ \frac{\alpha}{2^{\alpha+1}\Gamma(1-\alpha)} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} b_j(\hat{x}, x_j) \frac{u^2(\hat{x}, x_j) - u^2(\hat{x}, x_j - 2t)}{t^{\alpha+1}} \, dtdx$$

and

$$\frac{\alpha}{2^{\alpha+1}\Gamma(1-\alpha)} \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} b_j(\hat{x}, x_j) \frac{u^2(\hat{x}, x_j) - u^2(\hat{x}, x_j - 2t)}{t^{\alpha+1}} \, dtdx$$

$$= \frac{1}{2} \int_{\mathbb{R}^n} b_j(x) D^\alpha_{+,j}(u^2)(x) \, dx$$

$$= \frac{1}{2} \int_{\mathbb{R}^n} D^\alpha_{-,j} b_j(x) \cdot u^2(x) \, dx.$$

Summation over $j = 1, \ldots, n$ proves the assertion ∎

Note that – by the remark following Lemma 1.3 at the end of Section 1 – (5.9) is automatically fulfilled if each b_j is independent of x_j, i.e. if $b_j(\hat{x}, x_j) \equiv b_j(\hat{x})$ $(1 \leq j \leq n)$ where we used our shorthand $(\hat{x}, x_j)$ for x.

Corollary 5.7. *Suppose, in addition to the assumptions made in Theorem 5.5, that $b_j(x) \geq \kappa > 0$ and that b_j is independent of x_j, i.e. $b_j(\hat{x}, x_j) = b_j(\hat{x})$. Then $\left(\mathbf{D}^{(\alpha)}(\cdot, \cdot), H^{\frac{\alpha}{2}}(\mathbb{R}^n)\right)$ is a (non-symmetric) Dirichlet form.*

Proof. Observe that, under our assumptions, (5.9) and (5.10) hold. From Theorem 5.5 we deduce that $\mathbf{D}^{(\alpha)}(\cdot, \cdot)$ takes the form

$$\mathbf{D}^{(\alpha)}(u, v) =$$

$$\sum_{j=1}^{n} \left(\frac{\gamma_\alpha}{4} \int_{\mathbb{R}^{n-1}} b_j(\hat{x}) \right.$$

$$\times \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\left(u(\hat{x}, x_j + t) - u(\hat{x}, x_j)\right)\left(v(\hat{x}, x_j + t) - v(\hat{x}, x_j)\right)}{|t|^{\alpha+1}} \, dt dx_j d\hat{x}$$

$$+ \frac{\gamma_\alpha}{2^{\alpha+1}} \int_{\mathbb{R}^{n-1}} b_j(\hat{x})$$

$$\left. \times \int_{\mathbb{R}} \int_{\mathbb{R}_+} \frac{\left(u(\hat{x}, x_j + t) - u(\hat{x}, x_j - t)\right)\left(v(\hat{x}, x_j + t) + v(\hat{x}, x_j - t)\right)}{t^{\alpha+1}} \, dt dx_j d\hat{x} \right)$$

with $\gamma_\alpha = \frac{\alpha}{\Gamma(1-\alpha)}$. This implies the contraction property (DF.4) for $\mathbf{D}^{(\alpha)}(\cdot, \cdot)$, just as in the setting disscussed in Section 2. It remains to prove that $\left(\mathbf{D}^{(\alpha)}(\cdot, \cdot), H^{\frac{\alpha}{2}}(\mathbb{R}^n)\right)$ is closed. By Theorem 5.4 we know already that

$$\mathbf{D}_1^{(\alpha)}(u, u) := \mathbf{D}^{(\alpha)}(u, u) + \langle u, u \rangle_0 \leq c \, \|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)}^2.$$

It is therefore enough to show

$$\|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)}^2 \leq \tilde{c} \mathbf{D}_1^{(\alpha)}(u, u)$$

for some constant $\tilde{c} > 0$. Using the lower bound of the coefficient functions we find

$$\mathbf{D}^{(\alpha)}(u, u) = \sum_{j=1}^{n} \frac{\gamma_\alpha}{4} \int_{\mathbb{R}^{n-1}} b(\hat{x}) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\left(u(\hat{x}, x_j + t) - u(\hat{x}, x_j)\right)^2}{|t|^{\alpha+1}} \, dt dx_j d\hat{x}$$

$$\geq \kappa \sum_{j=1}^{n} \frac{\gamma_\alpha}{4} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\left(u(\hat{x}, x_j + t) - u(\hat{x}, x_j)\right)^2}{|t|^{\alpha+1}} \, dt dx_j d\hat{x}$$

$$\geq c(\alpha, \kappa) \sum_{j=1}^{n} \int_{\mathbb{R}^n} |\xi_j|^\alpha |\widehat{u}(\xi)|^2 \, d\xi$$

$$\geq \tilde{c}(\alpha, \kappa, n) \int_{\mathbb{R}^n} \left(\sum_{j=1}^{n} |\xi_j|^2 \right)^{\frac{\alpha}{2}} |\widehat{u}(\xi)|^2 d\xi$$

where we used very much the same calculation that led to Theorem 2.1 ∎

Remark 5.8. The *most general* conditions such that $\mathbf{D}^{(\alpha)}(\cdot, \cdot)$ given by (5.4) extends to a non-symmetric Dirichlet form are yet unknown. In particular, the closedness of $\mathbf{D}^{(\alpha)}(\cdot, \cdot)$ and the contraction property are difficult to prove.

On the other hand, the condition that the coefficients $b_j(\hat{x}, x_j) = b_j(\hat{x})$ are independent of x_j seems not to be too restrictive. Note that $[D^\alpha_{\pm,j}; b_j] \equiv 0$ already implies $D^\alpha_{\mp,j} b_j(x) = 0$ for Lebesgue-almost every x. This, in turn, gives $b_j(\hat{x}, x_j) = b_j(\hat{x})$ whenever b_j is smooth enough.

6. Fractional powers of dissipative operators

We will now consider arbitrary closed dissipative operators and, in particular, infinitesimal generators of arbitrary strongly continuous contraction semigroups. Assume throughout this section that $(A, \mathcal{D}(A))$ is a closed operator on the complex Hilbert space $\mathfrak{H}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The operator is *dissipative*, that is

$$\|\lambda u - Au\| \geq \mathrm{Re}\lambda \|u\| \qquad (u \in \mathcal{D}(A)).$$

Clearly, this implies that such operators are non-positive in the sense that $\mathrm{Re}\langle Au, u \rangle \leq 0$, but A is not necessarily sectorial. Fractional powers of a closed dissipative operator can be defined by

$$(-A)^\alpha u = \tfrac{\sin(\alpha\pi)}{\pi} \int_{\mathbb{R}_+} (-A)(s - A)^{-1} u \, \frac{ds}{s^{1-\alpha}} \qquad (u \in \mathcal{D}(A), \, 0 < \alpha < 1) \qquad (6.1)$$

where $\mathcal{D}(A)$ is a core for $\big((-A)^\alpha, \mathcal{D}((-A)^\alpha)\big)$ (see, e.g., Yosida [31: Section IX.11] or [1, 18, 27]).

We want to show that $(-A)^\alpha$ is always *sectorial*, that is to say that

$$\langle (-A)^\alpha u, u \rangle \in S_{\frac{\alpha\pi}{2}} := \left\{ z \in \mathbb{C} : \, |\arg z| \leq \tfrac{\alpha\pi}{2} \right\} \qquad (6.2)$$

($\arg z$ takes values in $(-\pi, \pi]$). Let us start with some preparations.

Lemma 6.1. *Let $(A, D(A))$ be a closed dissipative operator on a complex Hilbert space. Then its fractional power has the representation*

$$e^{-i\alpha\vartheta}(-A)^\alpha u = \frac{\sin(\alpha\pi)}{\pi} \int_{\mathbb{R}_+} (-A)(se^{i\vartheta} - A)^{-1} u \, \frac{ds}{s^{1-\alpha}} \quad (u \in \mathcal{D}(A)) \qquad (6.3)$$

with $0 < \alpha < 1$ and any $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$.

Proof. It is sufficient to consider the case where $0 < \vartheta < \frac{\pi}{2}$, since we may always change from ϑ to $-\vartheta$, and $\vartheta = 0$ is just (6.1). The resolvent $z \mapsto (z - A)^{-1}$ is an analytic

function in the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, so we may integrate along the paths

and obtain

$$\int_{\Gamma'} (-A)(z - A)^{-1} u \, \frac{dz}{z^{1-\alpha}} = \int_{\Gamma_\epsilon \cup \Gamma \cup \Gamma_n} (-A)(z - A)^{-1} u \, \frac{dz}{z^{1-\alpha}}.$$

Parametrizing the curves and changing variables gives

$$\int_{\Gamma'} (-A)(z - A)^{-1} u \, \frac{dz}{z^{1-\alpha}} = e^{i\alpha\vartheta} \int_\epsilon^n (-A)(se^{i\vartheta} - A)^{-1} \frac{ds}{s^{1-\alpha}}$$

$$\int_\Gamma (-A)(z - A)^{-1} u \, \frac{dz}{z^{1-\alpha}} = \int_\epsilon^n (-A)(s - A)^{-1} \frac{ds}{s^{1-\alpha}}.$$

Note that for $u \in \mathcal{D}(A)$ we have

$$\|(-A)(z - A)^{-1} u\| \le 2\|u\| \qquad \text{and} \qquad \|(-A)(z - A)^{-1} u\| \le (\operatorname{Re} z)^{-1} \|Au\|.$$

Therefore,

$$\left\| \int_n^\infty (-A)(s - A)^{-1} u \, \frac{ds}{s^{1-\alpha}} \right\| \le \int_n^\infty \frac{ds}{s^{2-\alpha}} \|Au\| = \frac{n^{\alpha-1}}{1 - \alpha} \|Au\|$$

and

$$\left\| \int_0^\epsilon (-A)(s - A)^{-1} u \, \frac{ds}{s^{1-\alpha}} \right\| \le 2 \int_0^\epsilon \frac{ds}{s^{1-\alpha}} \|u\| = \frac{\epsilon^\alpha}{\alpha} \|u\|,$$

and this implies that the strong limit $\lim_{n,\epsilon} \int_\epsilon^n (-A)(s - A)^{-1} u s^{\alpha-1} ds$ exists (at least for $u \in \mathcal{D}(A)$) and equals $(-A)^\alpha u$. The assertion thus follows once we have shown that on $\mathcal{D}(A)$

$$\lim_{\epsilon \to 0} \left\| \int_{\Gamma_\epsilon} (-A)(z - A)^{-1} u \, \frac{dz}{z^{1-\alpha}} \right\| = 0 \tag{6.4}$$

$$\lim_{n \to \infty} \left\| \int_{\Gamma_n} (-A)(z - A)^{-1} u \, \frac{dz}{z^{1-\alpha}} \right\| = 0. \tag{6.5}$$

Again parametrizing the arcs and changing variables we arrive at

$$\left\| \int_{\Gamma_\epsilon} (-A)(z - A)^{-1} u \, \frac{dz}{z^{1-\alpha}} \right\| \leq \epsilon^\alpha \int_0^\vartheta \left\| (-A)(\epsilon e^{i(\vartheta - \phi)} - A)^{-1} u \right\| d\phi$$

$$\leq \epsilon^\alpha \int_0^\vartheta \left(1 + \frac{1}{\cos(\vartheta - \phi)} \right) \|u\| \, d\phi$$

where we used the fact that for $u \in \mathfrak{H}$

$$\left\| (-A)(\epsilon e^{i(\vartheta - \phi)} - A)^{-1} u \right\| \leq \|u\| + \epsilon \left\| (\epsilon e^{i(\vartheta - \phi)} - A)^{-1} u \right\| \leq \left(1 + \frac{\epsilon}{\operatorname{Re}\epsilon e^{i(\vartheta - \phi)}} \right) \|u\|.$$

Since $\vartheta < \frac{\pi}{2}$, the above integral is finite, and (6.4) follows. The limit (6.5) can be seen in a similar way using the estimate $\|(-A)(ne^{i\phi} - A)^{-1} u\| \leq (\operatorname{Re} ne^{i\phi})^{-1} \|Au\|$ ∎

We are now ready for the main theorem of this section. For $\alpha = \frac{1}{2}$ and a different representation of the square-root of a dissipative operator this result can be found in Kato's book [13: p. 281]. For the readers' convenience, we give here a somewhat different proof that is adapted to our situation and holds for an arbitrary $0 < \alpha < 1$.

Theorem 6.2. *Let $(A, \mathcal{D}(A))$ be a closed dissipative operator on the complex Hilbert space $\mathfrak{H}$. Then its fractional power $(-A)^\alpha$ is sectorial, i.e. $\langle (-A)^\alpha u, u \rangle \in S_{\frac{\alpha\pi}{2}}$ holds for all $u \in \mathcal{D}((-A)^\alpha)$ and $0 < \alpha < 1$.*

Proof. Fix some $\delta > 0$, $0 < \vartheta < \frac{\pi}{2}$ and apply Lemma 6.1 to the operator $A_\delta := A - \delta \operatorname{id}$ to find

$$e^{-i\alpha\vartheta}(-A_\delta)^\alpha u = \frac{\sin(\alpha\pi)}{\pi} \int_{\mathbb{R}_+} (-A_\delta)(se^{i\vartheta} - A_\delta)^{-1} u \, \frac{ds}{s^{1-\alpha}} \quad \left(u \in \mathcal{D}(A_\delta) = \mathcal{D}(A) \right).$$

Let $u \in \mathcal{D}(A)$ and set $v = -A_\delta u$. Then

$$(-A_\delta)^{\alpha-1} v = (-A_\delta)^{\alpha-1}(-A_\delta) u = (-A_\delta)^\alpha u \qquad \left(v \in \mathcal{D}((-A_\delta)^{\alpha-1}) \right)$$

and we find

$$e^{-i\alpha\vartheta} \langle (-A_\delta)^{\alpha-1} v, v \rangle = \frac{\sin(\alpha\pi)}{\pi} \int_{\mathbb{R}_+} \langle (se^{i\vartheta} - A_\delta)^{-1} v, v \rangle \, \frac{ds}{s^{1-\alpha}}.$$

Substituting $w = (se^{i\vartheta} - A_\delta)^{-1} v$, we see that for the above integrand

$$\operatorname{Re}\langle (se^{i\vartheta} - A_\delta)^{-1} v, v \rangle = \operatorname{Re}\langle w, (se^{i\vartheta} - A_\delta) w \rangle$$
$$= s \cos\vartheta \, \|w\|^2 + \delta \|w\|^2 + \operatorname{Re}\langle w, -Aw \rangle$$
$$\geq 0$$

by our assumptions on ϑ and A. Thus,

$$\operatorname{Re}\left(e^{-i\alpha\vartheta} \langle (-A_\delta)^{\alpha-1} v, v \rangle \right) \geq 0 \tag{6.6}$$

for all $v = -A_\delta u$ and $u \in \mathcal{D}(A)$. Since multiplication by $e^{-i\alpha\vartheta}$ is a rotation of angle $\alpha\vartheta$, (6.6) implies

$$\langle(\delta - A)^{\alpha-1}v, v\rangle \in S_{\frac{\pi}{2} - \alpha\vartheta}. \tag{6.7}$$

Writing $w := (-A_\delta)^{\alpha-1}v = (-A_\delta)^\alpha u$ for $u \in \mathcal{D}(A)$, we observe because of the identities

$$(-A_\delta)^\alpha(-A_\delta)^{1-\alpha} = (-A_\delta)^{1-\alpha}(-A_\delta)^\alpha = -A_\delta$$

(in the sense of closed operators; see, e.g., [18, 27]) that $\mathcal{D}((-A_\delta)^{1-\alpha}) = (-A_\delta)^\alpha[\mathcal{D}(A_\delta)]$. Since $\vartheta \in (0, \frac{\pi}{2})$ was arbitrary, we may let $\vartheta \to \frac{\pi}{2}$ and (6.7) becomes

$$\langle w, (\delta - A)^{1-\alpha}w\rangle \in S_{(1-\alpha)\frac{\pi}{2}} \qquad \left(w \in \mathcal{D}((-A_\delta)^{1-\alpha})\right).$$

Since $\mathcal{D}((-A_\delta)^\alpha) = \mathcal{D}((-A)^\alpha)$ is independent of δ for all $0 \le \alpha \le 1$ and since $\lim_{\delta\to 0} A_\delta u = Au$ strongly, and the Theorem is established as $\delta \to 0$ ∎

Corollary 6.3. *In the situation of Theorem 6.2 we have*

$$\left|\mathrm{Im}\langle(-A)^\alpha u, u\rangle\right| \le \tan\left(\tfrac{\alpha\pi}{2}\right)\mathrm{Re}\langle(-A)^\alpha u, u\rangle$$

for all $u \in \mathcal{D}((-A)^\alpha)$.

Corollary 6.4. *Let $(A, D(A))$ be the generator of any strongly continuous contraction semigroup on the real Hilbert space $\mathfrak{H}$. Then every fractional power $(-A)^\alpha$ $(0 < \alpha < 1)$ satisfies the sector condition* (DF.2). *In fact, we even have*

$$\left|\langle(-A)^\alpha u, v\rangle\right| \le c_\alpha \sqrt{\langle(-A)^\alpha u, u\rangle}\sqrt{\langle(-A)^\alpha v, v\rangle}$$

for all $u, v \in \mathcal{D}((-A)^\alpha)$.

Proof. Denote by $A^\mathbb{C}$ the complexification of A with natural domain $\mathcal{D}(A) + i\mathcal{D}(A) \subset \mathfrak{H}^\mathbb{C}$. Since

$$[(-A)(s - A)^{-1}]^\mathbb{C} = (-A^\mathbb{C})(s - A^\mathbb{C})^{-1} \qquad (s > 0)$$

we get from formula (6.1) that $[(-A)^\alpha]^\mathbb{C} = (-A^\mathbb{C})^\alpha$ and that $\mathcal{D}(A) + i\mathcal{D}(A)$ is a core for $[(-A)^\alpha]^\mathbb{C}$. The assertion now follows from Theorem 6.2 by standard results from abstract functional analysis (see, e.g., Ma and Röckner [17: Proposition 2.17]) ∎

For our next result we need a generalization of non-symmetric Dirichlet forms that can also be found in [17]. A closed bilinear form satisfying the conditions (DF.1) - (DF.3) and (DF.5), but only one alternative in (DF.4) is called *semi-Dirichlet form.*

Observe that $-(-A)^\alpha$ generates a strongly continuous contraction semigroup whenever $-A$ does (see [31: Section IX.11]).

Corollary 6.5. *Let $(A, \mathcal{D}(A))$ be the generator of a strongly continuous sub-Markovian contraction semigroup on the real Hilbert spcae $\mathfrak{H}$. Then $\mathbf{E}^{(\alpha)}(u, v) := \langle(-A)^\alpha u, v\rangle$ $(u, v \in \mathcal{D}((-A)^\alpha))$ extends to a semi-Dirichlet form. The domain $\mathcal{D}(\mathbf{E}^{(\alpha)})$ is the completion of $\mathcal{D}(A)$ with respect to the scalar product*

$$\langle\!\langle u, v\rangle\!\rangle_\alpha = \langle(-A)^\alpha u, v\rangle + \langle(-A)^\alpha v, u\rangle + \langle u, v\rangle.$$

If the adjoint semigroup is also sub-Markovian, $\mathbf{E}^{(\alpha)}(\cdot,\cdot)$ extends to a non-symmetric Dirichlet form. Moreover, we have

$$\lim_{\alpha\uparrow 1}\langle(-A)^\alpha u, v\rangle = \langle(-A)u, v\rangle \tag{6.8}$$

for all $u, v \in \mathcal{D}(A)$.

Proof. As generator of a semigroup $-(-A)^\alpha$ is a maximal dissipative operator. Since it also satisfies the sector condition (DF.2), the form $\mathbf{E}^{(\alpha)}(u, v) := \langle(-A)^\alpha u, v\rangle$ extends to a semi-Dirichlet form (see [17: p. 39]).

On a Hilbert space, the adjoint operator of a maximal dissipative operator is again maximal dissipative. Clearly, $(-A^*)^\alpha = (-A)^{\alpha\,*}$, and applying Corollary 6.4 to the operator $(-A^*)$ shows that $(-A)^{\alpha\,*}$ is sectorial. If $\{T_t^*\}_{t\geq 0}$ is sub-Markovian, we conclude as above that the form

$$\mathbf{F}^{(\alpha)}(u, v) := \langle(-A)^{\alpha\,*}u, v\rangle = \langle u, (-A)^\alpha v\rangle = \mathbf{E}^{(\alpha)}(v, u)$$

extends to a semi-Dirichlet form; this shows that $\mathbf{E}^{(\alpha)}(\cdot,\cdot)$ satisfies both conditions in (DF.4), i.e. it has an extension to a non-symmetric Dirichlet form.

The general theory shows that $\mathcal{D}(\mathbf{E}^{(\alpha)})$ is obtained as completion of $\mathcal{D}((-A)^\alpha)$ with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\alpha$. Since $\mathcal{D}(A)$ is an operator core for $((-A)^\alpha, \mathcal{D}((-A)^\alpha))$ – that is to say that every $u \in \mathcal{D}((-A)^\alpha)$ can be approximated in graph-norm by a sequence contained in $\mathcal{D}(A)$ – it is obvious that $\mathcal{D}(\mathbf{E}^{(\alpha)})$ arises also as completion of $\mathcal{D}(A)$ with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\alpha$.

Statement (6.8) can be rephrased as

$$\lim_{\alpha\uparrow 1}(-A)^\alpha u = (-A)u \qquad \text{weakly in the space } \mathfrak{H}.$$

This, however, follows from the known left-continuity (in the strong topology) of the map $(0, 1] \ni \alpha \mapsto (-A)^\alpha u$ $(u \in \mathcal{D}(A))$ (see Nollau [18: Folgerung 2] or Balakrishnan [1: Lemma 2.3]; the latter proves only left-continuity at $\alpha = 1$ which is but sufficient) ∎

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