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A Multiplicity Fixed Point Theorem in Fréchet Spaces

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Abstract. A new multiplicity result is presented for maps between Fréchet spaces. Our argument relies on fixed point results in Banach spaces together with a result on hemicompact maps. An application is also given to illustrate how the theory can be applied in practice.

Keywords: *Fixed point theory, multiplicity results*

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1. Introduction

This paper is concerned with the existence of multiple fixed points to multi-valued maps between Fréchet spaces. The paper is divided into two main sections. In Section 2 the existence of multiple fixed points is established by means of a "diagonal type" process together with a result on hemicompact [1, 8] maps. It is worth remarking here that the theory presented in this paper to guarantee the existence of one or more fixed points is more general, and in particular more applicable, than the theory presented in the literature [3, 6, 9]. To illustrate the generality of the fixed point theorem from Section 2, we establish in Section 3 a new result which guarantees the existence of twin non-negative solutions to a very general integral equation on the semi-infinite interval.

For the remainder of this section we gather together some definitions and known results. Let (X,d) be a metric space and Ω_X the system of all bounded subsets of X. The Kuratowski measure of non-compactness is the map $\alpha : \Omega_X \to [0,\infty]$ defined by

$$\alpha(B) = \inf \Big\{ r > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and } \operatorname{diam}(B_i) \le r \Big\} \qquad (B \in \Omega_X).$$

Let S be a non-empty subset of X. For each $x \in X$, define $d(x,S) = \inf_{y \in S} d(x,y)$. Now suppose $G : S \to 2^X$; here 2^X denotes the family of non-empty subsets of X. Then:

(i) $G : S \to 2^X$ is k-set contractive (here $k \ge 0$) if $\alpha(G(W)) \le k\,\alpha(W)$ for all non-empty, bounded sets W of S.

(ii) $G : S \to 2^X$ is condensing if G is 1-set contractive and $\alpha(G(W)) < \alpha(W)$ for all bounded sets W of S with $\alpha(W) \ne 0$.

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(iii) $G : S \to 2^X$ is hemicompact if each sequence $(x_n)_{n=1}^{\infty}$ in S has a convergent subsequence whenever $d(x_n, Gx_n) \to 0$ as $n \to \infty$.

We now state three results from the literature which will be used in Section 2.

Theorem 1.1 (see [8]). *Let (X, d) be a metric space, D a non-empty, complete subset of X, and $G : D \to 2^X$ a condensing map with $G(D)$ bounded. Then G is hemicompact.*

Theorem 1.2 (see [2, 5, 7]). *Let C be a closed, convex subset of a Banach space with U an open subset of C and $0 \in U$. Suppose $F : \overline{U} \to CK(C)$ is an upper semi-continuous, k-set contractive (here $0 \le k < 1$) map with $F(\overline{U})$ bounded; here $CK(C)$ denotes the family of non-empty, compact, convex subsets of C. Then either*

(A1) *there exists $x \in \overline{U}$ with $x \in F(x)$*

or

(A2) *there exists $u \in \partial_C U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Let $E = (E, \|\cdot\|)$ and, for $\rho > 0$,

$$\Omega_\rho = \big\{x \in E : \|x\| < \rho\big\}.$$

Theorem 1.3 (see [2, 7]). *Let $E = (E, \|\cdot\|)$ be a Banach space, $C \subseteq E$ a cone and let $\|\cdot\|$ be increasing with respect to C. Also, r and R are constants with $0 < r < R$. Suppose $F : \overline{\Omega_R} \cap C \to CK(C)$ is an upper semicontinuous, k-set contractive (here $0 \le k < 1$) map and assume the conditions*

$$\|y\| \ge \|x\| \qquad \text{for all } y \in F(x) \text{ and } x \in \partial_E \Omega_R \cap C \tag{1.1}$$

$$\|y\| \le \|x\| \qquad \text{for all } y \in F(x) \text{ and } x \in \partial_E \Omega_r \cap C \tag{1.2}$$

hold. Then F has a fixed point in $C \cap \{x \in E : r \le \|x\| \le R\}$.

2. Fixed point theory

Let $\mathbb{N} = \{1, 2, ...\}$. In this section we assume E is a Fréchet space endowed with a family of seminorms $\{|\cdot|_n\}_{n\in\mathbb{N}}$ with

$$|x|_1 \le |x|_2 \le \dots \qquad \text{for all } x \in E.$$

Also, assume for each $n \in \mathbb{N}$ that $(E_n, |\cdot|_n)$ is a Banach space and suppose

$$E_1 \supseteq E_2 \supseteq \dots$$

with $E = \cap_{n=1}^{\infty} E_n$ and $|x|_n \le |x|_{n+1}$ for all $x \in E_{n+1}$. For each $n \in \mathbb{N}$ let C_n be a cone in E_n and assume $|\cdot|_n$ is increasing with respect to C_n. In addition, assume

$$C_1 \supseteq C_2 \supseteq \dots\ .$$

For $\rho > 0$ and $n \in \mathbb{N}$ let

$$U_{n,\rho} = \{x \in E_n : |x|_n < \rho\} \qquad \text{and} \qquad \Omega_{n,\rho} = U_{n,\rho} \cap C_n.$$

Notice

$$\partial_{C_n}\Omega_{n,\rho} = \partial_{E_n}U_{n,\rho} \cap C_n \qquad \text{and} \qquad \overline{\Omega_{n,\rho}} = \overline{U_{n,\rho}} \cap C_n$$

(the first closure is with respect to C_n whereas the second is with respect to E_n). In addition, notice since $|x|_n \le |x|_{n+1}$ for all $x \in E_{n+1}$ that

$$\Omega_{1,\rho} \supseteq \Omega_{2,\rho} \supseteq \dots \qquad \text{and} \qquad \overline{\Omega_{1,\rho}} \supseteq \overline{\Omega_{2,\rho}} \supseteq \dots .$$

We now establish a general result which guarantees that the inclusion

$$y \in Fy \tag{2.1}$$

has two solutions in E.

The main points needed to establish the existence of solutions to (2.1) are the following:

(1) The existence of upper semicontinuous maps $F_n : \overline{U_{n,R}} \cap C_n \to CK(C_n)$.

(2) The sequence of maps $\{F_n\}$ has the property that a convergent sequence of fixed points $\{y_n\}$ of $\{F_n\}$ converges to a fixed point of F.

(3) The assumptions on F_n are such that the Krasnoselskii-Petryshyn theorem [2, 7] in a Banach space can be applied.

We note here that F_n need not be the restriction of F to E_n (see Section 3).

Definition 2.1. Fix $k \in \mathbb{N}$. If $x, y \in E_k$, then we say $x = y$ in E_k if $|x - y|_k = 0$ (i.e. if $x - y = 0$; here 0 is the zero in E_k).

Definition 2.2. If $x, y \in E$, then we say $x = y$ in E if $x = y$ in E_k for each $k \in \mathbb{N}$.

Definition 2.3. Fix $k \in \mathbb{N}$. We say $x \in Fy$ in E_k if there exists $w \in Fy$ with $x = w$ in E_k.

Theorem 2.1. *Let L, γ, r, R be constants with $0 < L < \gamma < r < R$. Assume the following conditions are satisfied for each $n \in \mathbb{N}$:*

$$F_n : \overline{U_{n,R}} \cap C_n \to CK(C_n) \quad \text{is an upper semicontinuous map} \tag{2.2}$$

$$|y|_n \le |x|_n \quad \text{for all } y \in F_n(x) \text{ and } x \in \partial_{E_n}U_{n,L} \cap C_n \tag{2.3}$$

$$|y|_n \le |x|_n \quad \text{for all } y \in F_n(x) \text{ and } x \in \partial_{E_n}U_{n,r} \cap C_n \tag{2.4}$$

$$|y|_n \ge |x|_n \quad \text{for all } y \in F_n(x) \text{ and } x \in \partial_{E_n}U_{n,R} \cap C_n. \tag{2.5}$$

Further, assume the following:

$$\left\{ \begin{array}{l} \text{For each } n \in \mathbb{N}, \text{ the map } \mathcal{K}_n : \overline{U_{n,R}} \cap C_n \to 2^{C_n} \text{ given by} \\ \mathcal{K}_n y = \cup_{m=n}^{\infty} F_m y \text{ is } k\text{-set contractive (here } 0 \le k < 1) \end{array} \right\}. \tag{2.6}$$

$$\left\{\begin{array}{l} \textit{For every } k\in\mathbb{N} \textit{ and any subsequence } A\subseteq\{k,k+1,\ldots\}, \\ \textit{if } x\in C_n \ (n\in A) \textit{ is such that } R\geq|x|_n\geq r, \textit{ then } |x|_k\geq\gamma \end{array}\right\}. \tag{2.7}$$

$$\left\{\begin{array}{l} \textit{If there exists a } v\in E \textit{ and a sequence } \{u_n\}_{n\in\mathbb{N}} \textit{ with} \\ u_n\in\overline{U_{n,L}}\cap C_n \textit{ and } u_n\in F_nu_n \textit{ in } E_n \textit{ such that for every} \\ k\in\mathbb{N} \textit{ there exists a subsequence } S\subseteq\{k+1,k+2,\ldots\} \textit{ of } \mathbb{N} \\ \textit{with } u_n\to v \textit{ in } E_k \textit{ as } n\to\infty \textit{ in } S, \textit{ then } v\in Fv \textit{ in } E \end{array}\right\}. \tag{2.8}$$

$$\left\{\begin{array}{l} \textit{If there exists a } z\in E \textit{ and a sequence } \{w_n\}_{n\in\mathbb{N}} \textit{ with} \\ w_n\in(\overline{U_{n,L}}\setminus U_{n,r})\cap C_n \textit{ and } w_n\in F_nw_n \textit{ in } E_n \textit{ such that for} \\ \textit{every } k\in\mathbb{N} \textit{ there exists a subsequence } P\subseteq\{k+1,k+2,\ldots\} \\ \textit{of } \mathbb{N} \textit{ with } w_n\to z \textit{ in } E_k \textit{ as } n\to\infty \textit{ in } P, \textit{ then } z\in Fz \textit{ in } E \end{array}\right\}. \tag{2.9}$$

Then (2.1) *has at least two solutions* x_0 *and* x_1 *with*

$$x_0\in\cap_{n=1}^{\infty}(\overline{U_{n,L}}\cap C_n) \qquad \textit{and} \qquad x_1\in\cap_{n=1}^{\infty}\big((\overline{U_{n,R}}\backslash U_{n,\gamma})\cap C_n\big).$$

Remark. The definition of $\mathcal{K}_n$ in (2.6) is as follows. If $y\in\overline{U_{n,R}}\cap C_n$ and $y\notin\overline{U_{n+1,R}}\cap C_{n+1}$, then $\mathcal{K}_ny=F_ny$, whereas if $y\in\overline{U_{n+1,R}}\cap C_{n+1}$ and $y\notin\overline{U_{n+2,R}}\cap C_{n+2}$, then $\mathcal{K}_ny=F_ny\cup F_{n+1}y$, and so on.

Remark. If F is defined on E_1 and $F_n=F|_{E_n}$ for each $n\in\mathbb{N}$, then (2.8) and (2.9) are automatically satisfied.

Proof of Theorem 2.1. Fix $n\in\mathbb{N}$. Note (2.3) implies $x\notin\lambda F_n(x)$ for all $\lambda\in(0,1)$ and $x\in\partial_{E_n}U_{n,L}\cap C_n$. To see this suppose there exist $x\in\partial_{E_n}U_{n,L}\cap C_n$ and $\lambda\in(0,1)$ with $x\in\lambda F_n(x)$. Then there exists $y\in F_n(x)$ with $x=\lambda y$ and so $L=|x|_n=|\lambda|\,|y|_n<|y|_n\leq|x|_n=L$ which is a contradiction. Theorem 1.2 guarantees that $y\in F_ny$ has a fixed point $u_n\in\overline{U_{n,L}}\cap C_n$. In particular, $|u_n|_n\leq L$. Theorem 1.3 (note (2.4) and (2.5)) guarantees that $y\in F_ny$ has a fixed point $w_n\in(\overline{U_{n,R}}\backslash U_{n,r})\cap C_n$. In particular, $r\leq|w_n|_n\leq R$.

Let us look at $\{u_n\}_{n\in\mathbb{N}}$. Note $u_n\in\overline{U_{1,L}}$ for each $n\in\mathbb{N}$. To see this notice $|u_n|_n\leq L$ and $|x|_1\leq|x|_n$ for all $x\in E_n$ implies $|u_n|_1\leq L$. Now Theorem 1.1 (with $X=E_1$, $G=\mathcal{K}_1$, $D=\overline{U_{1,L}}\cap C_1$ and note $d_1(u_n,\mathcal{K}_1u_n)=0$ for each $n\in\mathbb{N}$ since $|x|_1\leq|x|_n$ for all $x\in E_n$ and $u_n\in F_nu_n$ in E_n; here $d_1(x,S)=\inf_{y\in S}|x-y|_1$ if S is a non-empty subset of X) guarantees that there exists a subsequence $N_1^\star$ of $\mathbb{N}$ and $v_1\in\overline{U_{1,L}}\cap C_1$ with $u_n\to v_1$ in E_1 as $n\to\infty$ in $N_1^\star$. Notice in particular that $|v_1|_1\leq L$. Let us now look at $\{w_n\}_{n\in\mathbb{N}}$. Note $w_n\in\overline{U_{1,R}}\backslash U_{1,\gamma}$ for each $n\in\mathbb{N}$. To see this notice $|w_n|_n\leq R$ and $|x|_1\leq|x|_n$ for all $x\in E_n$ implies $|w_n|_1\leq R$. Thus $w_n\in\overline{U_{1,R}}$ for each $n_0\in\mathbb{N}$. On the other hand, $|w_n|_n\geq r$ and $w_n\in C_n$ together with (2.7) imply $|w_n|_1\geq\gamma$. Now Theorem 1.1 (with $X=E_1$, $G=\mathcal{K}_1$, $D=(\overline{U_{1,R}}\backslash U_{1,\gamma})\cap C_1$ and note $d_1(u_n,\mathcal{K}_1u_n)=0$ for each $n\in\mathbb{N}$) guarantees that there exists a subsequence $P_1^\star$ of $\mathbb{N}$ and $z_1\in(\overline{U_{1,R}}\backslash U_{1,\gamma})\cap C_1$ with $w_n\to z_1$ in E_1 as $n\to\infty$ in $P_1^\star$. Notice in particular that $\gamma\leq|z_1|_1\leq R$.

Let $N_1=N_1^\star\backslash\{1\}$ and $P_1=P_1^\star\backslash\{1\}$. Look at $\{u_n\}_{n\in N_1}$. Notice $u_n\in\overline{U_{2,L}}$ for each $n\in N_1$. Now Theorem 1.1 with $X=E_2$, $G=\mathcal{K}_2$, $D=\overline{U_{2,L}}\cap C_2$ and note

$d_2(u_n, \mathcal{K}_2 u_n) = 0$ for each $n \in N_1$; here $d_2(x, S) = \inf_{y \in S} |x - y|_2$ if S is a non-empty subset of X) guarantees that there exist a subsequence $N_2^\star$ of N_1 and $v_2 \in \overline{U_{2,L}} \cap C_2$ with $u_n \to v_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Notice in particular that $|v_2|_2 \le L$. Note $|v_2 - v_1|_0 = 0$ since $N_2^\star \subseteq N_1$ and $E_1 \supseteq E_2$. Thus $v_2 = v_1$ in E_1. Let us now look at $\{w_n\}_{n \in P_1}$. Notice it is easy to see using $r \le |w_n|_n \le R$ and (2.7) that $w_n \in \overline{U_{2,R}} \backslash U_{2,\gamma}$ for each $n \in P_1$. Now Theorem 1.1 (with $X = E_2$, $G = \mathcal{K}_2$, $D = (\overline{U_{2,R}} \backslash U_{2,\gamma}) \cap C_2$ and note $d_2(u_n, \mathcal{K}_2 u_n) = 0$ for each $n \in P_1$) guarantees that there exist a subsequence $P_2^\star$ of P_1 and $z_2 \in (\overline{U_{2,R}} \backslash U_{2,\gamma}) \cap C_2$ with $w_n \to z_2$ in E_2 as $n \to \infty$ in $P_2^\star$. Notice in particular that $\gamma \le |z_2|_2 \le R$ and $z_2 = z_1$ in E_1.

Let $N_2 = N_2^\star \backslash \{2\}$ and $P_2 = P_2^\star \backslash \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \dots, \qquad N_k^\star \subseteq \{k, k+1, \dots\}$$
$$P_1^\star \supseteq P_2^\star \supseteq \dots, \qquad P_k^\star \subseteq \{k, k+1, \dots\}$$

and $v_k \in \overline{U_{k,L}} \cap C_k$, $z_k \in (\overline{U_{k,R}} \backslash U_{k,\gamma}) \cap C_k$ with $u_n \to v_k$ in E_k as $n \to \infty$ in $N_k^\star$ and $w_n \to z_k$ in E_k as $n \to \infty$ in $P_k^\star$. Note $v_{k+1} = v_k$ in E_k and $z_{k+1} = z_k$ in E_k for $k \in \mathbb{N}$. Also, let $N_k = N_k^\star \backslash \{k\}$ and $P_k = P_k^\star \backslash \{k\}$.

Let $y_1 = v_k$ in E_k and $y_2 = z_k$ in E_k. Notice y_1 and y_2 are well defined and $y_1, y_2 \in E_k$ for each $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$. Now $u_n \in F_n u_n$ in E_n for $n \in N_k$ and $u_n \to y_1$ in E_k as $n \to \infty$ in N_k (since $y_1 = v_k$ in E_k). This together with (2.8) implies $y_1 \in F y_1$. Similarly, $w_n \in F_n w_n$ in E_n for $n \in N_k$ and $w_n \to y_2$ in E_k as $n \to \infty$ in P_k together with (2.9) imply $y_2 \in F y_2$ ∎

Remark. It is also possible to use Theorem 1.3 and [2: Theorem 2.4] or [7: Theorem 3] together with the ideas in Theorem 2.1 to obtain other multiplicity results.

3. Integral equations on the semi-infinite interval

In this section the results of Section 2 are used to establish the existence of twin solutions to the integral equation

$$y(t) = \int_0^\infty K(t,s) f(s, y(s))\, ds \qquad \text{for } t \in [0,\infty). \tag{3.1}$$

Theorem 3.1. *Suppose the following conditions are satisfied:*

$$\text{For each } t \in [0,\infty), \text{ the map } s \mapsto K(t,s) \text{ is measurable.} \tag{3.2}$$

$$\sup_{t \in [0,\infty)} \int_0^\infty |K(t,s)|\, ds < \infty. \tag{3.3}$$

$$\int_0^\infty |K(t',s) - K(t,s)|\, ds \to 0 \text{ as } t \to t', \text{ for each } t' \in [0,\infty). \tag{3.4}$$

$$\left\{ \begin{array}{l} f : [0,\infty) \times \mathbb{R} \to \mathbb{R} \text{ is a continuous function and} \\ \text{for each } b > 0 \text{ there exists } M_b > 0 \text{ such that} \\ |y| \le b \text{ implies } |f(s,y)| \le M_b \text{ for all } s \in [0,\infty) \end{array} \right\}. \tag{3.5}$$

$$\text{For each } t \in [0,T], K(t,s) \geq 0 \text{ for a.e. } s \in [0,t]. \tag{3.6}$$

$$f : [0,\infty) \times \mathbb{R} \to [0,\infty) \text{ with } f(s,u) > 0 \text{ for } (s,u) \in [0,\infty) \times (0,\infty). \tag{3.7}$$

$$\left\{ \begin{array}{l} \text{There exists } g : [0,\infty) \to (0,\infty) \text{ with } g \in L^1_{loc}[0,\infty) \\ \text{and with } K(t,s) \leq g(s) \text{ for } t \in [0,\infty) \end{array} \right\}. \tag{3.8}$$

$$\left\{ \begin{array}{l} \text{There exists } \delta, \varepsilon,\ 0 \leq \delta < \varepsilon \leq 1 \text{ and } M,\ 0 < M < 1, \\ \text{with } K(t,s) \geq M g(s) \text{ for } t \in [\delta, \varepsilon] \end{array} \right\}. \tag{3.9}$$

$$\left\{ \begin{array}{l} f(s,y) \leq w(y) \text{ for a.e. } t \in [0,\infty) \text{ and all } y \in [0,\infty); \\ \text{here } w \geq 0 \text{ is continuous and non-decreasing on } (0,\infty) \end{array} \right\}. \tag{3.10}$$

$$\text{There exists } r > 0 \text{ with } \frac{r}{w(r) \sup_{t\in[0,\infty)} \int_0^\infty K(t,s)ds} > 1. \tag{3.11}$$

$$\text{There exists } L,\ 0 < L < r,\ \text{with } \frac{L}{w(L) \sup_{t\in[0,\infty)} \int_0^\infty K(t,s)ds} > 1. \tag{3.12}$$

$$\left\{ \begin{array}{l} \text{There exists } \tau \in C[\delta,\varepsilon] \text{ with } \tau > 0 \text{ on } [\delta,\varepsilon] \text{ and} \\ \text{with } f(s,y) \geq \tau(s) w(y) \text{ on } [\delta,\varepsilon] \times (0,\infty) \end{array} \right\}. \tag{3.13}$$

$$\text{There exists } R > r \text{ with } \frac{x}{w(x)} \leq M \int_\delta^\varepsilon \tau(s) K(\sigma,s)\, ds \text{ for } x \in [MR, R]. \tag{3.14}$$

$$Mr > L. \tag{3.15}$$

$$\left\{ \begin{array}{l} \text{There exists } \sigma,\ 0 \leq \sigma < \infty,\ \text{with} \\ \displaystyle\int_\delta^\varepsilon \tau(s) K(\sigma,s)\, ds = \sup_{t\in[0,\infty)} \int_\delta^\varepsilon \tau(s) K(t,s)\, ds \end{array} \right\}. \tag{3.16}$$

Then equation (3.1) *has two solutions* $y_1, y_2 \in C[0,\infty)$ *with* $y_1, y_2 \geq 0$ *on* $[0,\infty)$ *and with* $|y_1|_n \leq L < |y_2|_n \leq R$ *for each* $n \in \mathbb{N}$ *large enough (here* $|u|_n = \sup_{t\in[0,n]} |u(t)|$ *for each* $n \in \mathbb{N}$*).*

Remark. In (3.7) one could replace $f : [0,\infty) \times \mathbb{R} \to [0,\infty)$ with $f : [0,\infty) \times [0,\infty) \to [0,\infty)$.

Proof of Theorem 3.1. Let $n \in \mathbb{N}$ and

$$F_n y(t) = \int_0^n K(t,s) f(s,y(s))\, ds \qquad \text{for } t \in [0,n]. \tag{3.17}$$

To show the existence of solutions y_1 and y_2 we will apply Theorem 2.1 with $E = C[0,\infty)$. Before we apply Theorem 2.1 let us look at $\sigma \in [0,\infty)$ defined in (3.16). We will assume without loss of generality that $\sigma \in [0,1]$ (otherwise $\sigma \in (m, m+1]$ for some $m \in \mathbb{N}$ and then we would apply Theorem 2.1 with F_{n+m} instead of F_n). Fix $n \in \mathbb{N}$ and let $E_n = C[0,n]$ and

$$C_n = \left\{ y \in C[0,n] : y(t) \geq 0 \text{ for } t \in [0,n] \text{ and } \min_{t\in[\delta,\varepsilon]} y(t) \geq M|y|_n \right\};$$

here δ, ε, M are as in (3.9). Also, from (3.15) let γ, $L < \gamma < r$, be chosen so that

$$Mr \geq \gamma. \tag{3.18}$$

For $\rho = L$, $\rho = r$ or $\rho = R$ let

$$U_{n,\rho} = \big\{ x \in C[0,n] : |u|_n < \rho \big\}.$$

We first show $F_n : C_n \to C_n$. To see this let $y \in C_n$. Then (3.6) - (3.7) imply $F_n y(t) \geq 0$ for $t \in [0,n]$. Also, (3.8) implies

$$|F_n y(t)| \leq \int_0^n g(s) f(s, y(s))\, ds \qquad \text{for } t \in [0,n]$$

and so

$$|F_n y|_n \leq \int_0^n g(s) f(s, y(s))\, ds. \tag{3.19}$$

On the other hand, (3.9) and (3.19) imply

$$\begin{aligned} \min_{t\in[\delta,\varepsilon]} F_n y(t) &= \min_{t\in[\delta,\varepsilon]} \int_0^n K(t,s) f(s,y(s))\, ds \\ &\geq M \int_0^n g(s) f(s,y(s))\, ds \\ &\geq M |F_n y|_n. \end{aligned}$$

As a result, $F_n : C_n \to C_n$. In addition, $F_n : \overline{U_{n,R}} \cap C_n \to C_n$ is a continuous map (see [4: p. 70]). Thus (2.2) holds. Next we show (2.3) is satisfied. To see this let $u \in \partial_{E_n} U_{n,L} \cap C_n$. Then $|u|_n = L$ and $u(t) \geq 0$ for $t \in [0,n]$, and so $u(t) \leq L$ for $t \in [0,n]$. Now (3.10) implies

$$|F_n u(t)| \leq \int_0^n K(t,s) w(u(s))\, ds \qquad \text{for } t \in [0,n]. \tag{3.20}$$

In addition, (3.12) implies

$$w(u(s)) \leq w(L) < \frac{L}{\sup_{t\in[0,\infty)} \int_0^\infty K(t,s)\, ds}$$

and this together with (3.20) yields

$$|F_n u(t)| \leq \frac{L}{\sup_{t\in[0,\infty)} \int_0^\infty K(t,s)\, ds} \int_0^n K(t,s)\, ds \leq L = |u|_n$$

for $t \in [0,n]$. Consequently, $|F_n u|_n \leq |u|_n$ for $u \in \partial_{E_n} U_{n,L} \cap C_n$ and so (2.3) holds. Essentially the same argument shows (2.4) is satisfied. We next show (2.5) holds. To see this let $u \in \partial_{E_n} U_{n,R} \cap C_n$. Then $|u|_n = R$ and $\min_{t\in[\delta,\varepsilon]} u(t) \geq M|u|_n \geq MR$ (in particular, $u(t) \in [MR, R]$ for $t \in [\delta, \varepsilon]$). Now (3.13) - (3.14) imply (here σ is as in (3.16) and also note we assumed at the beginning of the proof that $\sigma \in [0,1]$)

$$\begin{aligned} F_n u(\sigma) &= \int_0^n K(\sigma,s) f(s,u(s))\,ds \\ &\geq \int_\delta^\varepsilon K(\sigma,s) f(s,u(s))\,ds \\ &\geq \int_\delta^\varepsilon K(\sigma,s)\tau(s) w(u(s))\,ds \\ &\geq \frac{1}{M\int_\delta^\varepsilon K(\sigma,s)\tau(s)ds}\int_\delta^\varepsilon K(\sigma,s)\tau(s)u(s)\,ds \\ &\geq \frac{MR}{M} \\ &= R \\ &= |u|_n. \end{aligned}$$

Thus $|F_n u|_n \geq |u|_n$ for $u \in \partial_{E_n} U_{n,R} \cap C_n$ and so (2.5) holds. The argument in [4: p. 70] immediately guarantees that (2.6) is satisfied. Thus we have shown for each $n \in \mathbb{N}$ that (2.2) - (2.6) hold.

To show (2.7) fix $k \in \mathbb{N}$ and take any subsequence $A \subseteq \{k, k+1, \ldots\}$. Now if $x \in C_n$ ($n \in A$) is such that $R \geq |x|_n \geq r$, then $\min_{t\in[\delta,\varepsilon]} x(t) \geq M|x|_n \geq Mr$, and this together with (3.18) give $\min_{t\in[\delta,\varepsilon]} x(t) \geq \gamma$. Thus $|x|_k = \sup_{t\in[0,k]} |x(t)| \geq \gamma$ and so (2.7) holds.

Next we show (2.8) is satisfied. Suppose there exists a $v \in C[0,\infty)$ and a sequence $\{u_n\}_{n\in\mathbb{N}}$ with $u_n \in \overline{U_{n,L}} \cap C_n$ and $u_n(t) = F_n u_n(t)$ ($t \in [0,n]$) such that for every $k \in \mathbb{N}$ there exists a subsequence $S \subseteq \{k+1, k+2, \ldots\}$ of $\mathbb{N}$ with $u_n \to v$ in $C[0,k]$ as $n \to \infty$ in S. If we show

$$v(t) = \int_0^\infty K(t,s) f(s,v(s))\,ds \qquad \text{for } t \in [0,\infty), \tag{3.21}$$

then (2.8) holds. Fix $t \in [0,\infty)$. Consider $k \geq t$ and $n \in S$ (as described above). Then $u_n(t) = F_n u_n(t)$ for $n \in S$ and so

$$u_n(t) - \int_0^k K(t,s) f(s,u_n(s))\,ds = \int_k^n K(t,s) f(s,u_n(s))\,ds.$$

From (3.5) there exists M_L such that $|f(s,u_n(s))| \leq M_L$ for all $s \in [0,n]$ and so

$$\left| u_n(t) - \int_0^k K(t,s) f(s,u_n(s))\,ds \right| \leq \int_k^n M_L K(t,s)\,ds \leq M_L \int_k^\infty K(t,s)\,ds. \tag{3.22}$$

Let $n \to \infty$ through S in (3.22), and use the Lebesgue Dominated Convergence Theorem to obtain

$$\left| v(t) - \int_0^k K(t,s) f(s,v(s))\,ds \right| \leq M_L \int_k^\infty K(t,s)\,ds$$

since $u_n \to v$ in $C[0,k]$. Finally, let $k \to \infty$ (note (3.3)) to conclude

$$v(t) - \int_0^\infty K(t,s) f(s,v(s))\, ds = 0.$$

Thus (3.21) holds and so (2.8) is satisfied. A similar argument shows (2.9) holds. Theorem 2.1 now guarantees the result ∎

Remark. It is easy to see that (3.14) could be replaced by

$$\frac{R}{w(MR)} \leq \sup_{t\in[0,\infty)} \int_\delta^\varepsilon \tau(s) K(t,s)\, ds. \tag{3.14*}$$

The only change in the proof of Theorem 3.1 occurs when we show (2.5). In this case we have

$$F_n u(\sigma) \geq \int_\delta^\varepsilon K(\sigma,s)\tau(s) w(u(s))\, ds \geq w(MR) \int_\delta^\varepsilon K(\sigma,s)\tau(s)\, ds \geq R = |u|_n.$$

Example. Consider

$$y(t) = \int_0^\infty e^{-10(t+s)} \big(|y(s)|^\alpha + y^2(s) + 1\big) ds \qquad (t \in [0,\infty)) \tag{3.23}$$

with $0 < \alpha < 1$. Then (3.23) has two solutions $y_1, y_2 \in C[0,\infty)$ with $y_1, y_2 \geq 0$ on $[0,\infty)$ and with $0 < |y_1|_n \leq 1 < |y_2|_n$ for each $n \in \mathbb{N}$.

This follows from Theorem 3.1. Let $K(t,s) = e^{-10(t+s)}$, $f(s,y) = |y|^\alpha + y^2 + 1$, $w(y) = y^\alpha + y^2 + 1$, $g(s) = e^{-10s}$ and $\tau = 1$. Notice (3.2) - (3.8) and (3.10) are clearly satisfied. In addition (3.11) - (3.12) hold with $L = 1$ and $r = 2$ since

$$\frac{x}{w(x) \sup_{t\in[0,\infty)} \int_0^\infty K(t,s)\, ds} = \frac{10x}{x^\alpha + x^2 + 1}.$$

Let $\delta = 0$ and $\varepsilon = \frac{\ln 2}{20}$. Notice (3.9) is true with $M = e^{-10\varepsilon} = \frac{1}{\sqrt{2}}$ and (3.15) is also true since $Mr = \frac{2}{\sqrt{2}} > L = 1$. In addition, (3.13) is satisfied and (3.16) is true with $\sigma = 0$. Next notice

$$\frac{x}{w(x)} = \frac{x}{x^\alpha + x^2 + 1} \to 0 \qquad \text{as } x \to \infty,$$

so it is easy to see that there exists $R > r = 2$ such that (3.14) holds. Theorem 3.1 implies that there exist solutions $y_1, y_2 \in C[0,\infty)$ of (3.23) with $y_1, y_2 \geq 0$ on $[0,\infty)$ and with $0 < |y_1|_n \leq 1 < |y_2|_n$ for each $n \in \mathbb{N}$.

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