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Some Oscillation and Non-Oscillation Theorems for Fourth Order Difference Equations

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Abstract. Sufficient conditions are established for oscillation of all solutions of the fourth order difference equation

$$\Delta \;\; a_n\Delta(b_n\Delta(c_n\Delta y_n)) \;\; + q_n f(y_{n+1}) = h_n \qquad (n \in \mathbb{N}_0)$$

where Δ is the forward difference operator $\Delta y_n = y_{n+1} - y_n$, $\{a_n\}, \{b_n\}, \{c_n\}, \{q_n\}, \{h_n\}$ are real sequences, and f is a real-valued continuous function. Also, sufficient conditions are provided which ensure that all non-oscillatory solutions of the equation approach zero as $n \to \infty$. Examples are inserted to illustrate the results.

Keywords: *Fourth order difference equations, oscillation, non-oscillation*

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1. Introduction

In the past two decades there has been an increasing interest in studying the oscillatory and non-oscillatory behavior of solutions of difference equations. However, most of the work on the subject has been restricted to first and second order equations (see [1] and the references cited therein). It should be noted that almost all the results concerning the oscillatory behavior of difference equations are obtained as discrete analogues of those for differential equations. The ideas behind the analogues are similar but different due to the discrete nature. Motivation of the present study also stems from the works of Lovelady [8] and Kusano and Onose [6] who considered the differential equations

$$\big(p_3(p_2(p_1u')')'\big)' + qu = 0 \qquad (E_1)$$

$$\big(p_3(p_2(p_1u')')'\big)' + qf(u) = b(t) \qquad (E_2)$$

and obtained conditions for oscillation of all solutions of equation (E_1) and for non-oscillatory solutions of equation (E_2) to tend to zero as $t \to \infty$, respectively.

In this paper we consider the fourth order difference equation

$$\Delta\big(a_n\Delta(b_n\Delta(c_n\Delta y_n))\big) + q_n f(y_{n+1}) = h_n \qquad (n \in \mathbb{N}_0) \qquad (1)$$

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where Δ is the forward difference operator defined as $\Delta y_n = y_{n+1} - y_n$, $\{a_n\}, \{b_n\}, \{c_n\}$, $\{q_n\}, \{h_n\}$ are sequences of real numbers and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $uf(u) > 0$ for $u \neq 0$. By a solution of equation (1) we mean a real sequence $\{y_n\}$ satisfying equation (1) so that $\sup_{n \geq m} |y_n| > 0$ for any $m \in \mathbb{N}_0$. We always assume that such solutions exist. A solution of equation (1) is called *oscillatory* if there is no end of n_1 and n_2 $(n_1 < n_2)$ in $\mathbb{N}$ such that $y_{n_1} y_{n_2} \leq 0$; otherwise it is called *non-oscillatory.* Clearly, a non-oscillatory solution of equation (1) must be eventually of one sign.

Our purpose in this paper is to obtain conditions for oscillation of all solutions of equation (1), and for non-oscillatory solutions of equation (1) to tend to zero as $n \to \infty$. In Section 2 we obtain conditions for oscillation of all solutions of equation (1) when $h_n \equiv 0$ and Section 3 contains sufficient conditions which ensure that all non-oscillatory solutions of equation (1) tend to zero as $n \to \infty$. For more results regarding oscillation and asymptotic behavior of fourth order difference equations we refer, in particular, to [3 - 5, 9 - 14]. Further, our equation is quite general and therefore the results of this paper even in some special cases complement and generalize some of the results in the literature [3, 4, 9, 14, 16].

2. Oscillation results

In this section we study the oscillatory behavior of equation (1) under the following additional conditions:

(c_1) $h_n \equiv 0$.

(c_2) $\{a_n\}, \{b_n\}, \{c_n\}, \{q_n\}$ are real positive sequences such that $\sum_{n=0}^{\infty} \frac{1}{a_n} = \sum_{n=0}^{\infty} \frac{1}{b_n} = \sum_{n=0}^{\infty} \frac{1}{c_n} = \infty$.

(c_3) f is non-decreasing and $\frac{f(u)}{u} \geq M > 0$ for $u \neq 0$.

Theorem 1. *Let conditions* (c_1) - (c_3) *hold and suppose that each of the following hypotheses* (H_1) - (H_3) *is true:*

(H_1) $\sum_{n=0}^{\infty} \big(\sum_{s=0}^{n-1} \big(\frac{1}{a_s} \sum_{t=0}^{s-1} \big(\frac{1}{b_t} \sum_{r=0}^{t-1} \frac{1}{c_r} \big)\big)\big) q_n = \infty$.

(H_2) *If* $\sum_{n=0}^{\infty} q_n < \infty$ *and* $\sum_{n=0}^{\infty} \big(\frac{1}{a_n} \sum_{s=n}^{\infty} q_s \big) < \infty$, *then* $\sum_{n=0}^{\infty} \big(\frac{1}{b_n} \sum_{s=n}^{\infty} \big(\frac{1}{a_s} \sum_{t=s}^{\infty} q_t \big)\big) = \infty$.

(H_3) $\sum_{n=0}^{\infty} \big(\sum_{s=0}^{n-1} \big(\frac{1}{c_s} \sum_{t=0}^{s-1} \frac{1}{b_t} \big)\big) q_n = \infty$.

Then every solution of equation (1) *is oscillatory.*

Proof. Let $\{y_n\}$ be a non-oscillatory solution of equation (1). Without loss of generality we may assume that $\{y_n\}$ is eventually positive since the proof is similar when $\{y_n\}$ is eventually negative. Therefore there is an integer $n_0 \in \mathbb{N}_0$ such that $y_n > 0$ for all $n \geq n_0$. For $n \geq n_0$, let

$$\left.\begin{aligned} u_n &= y_n \\ v_n &= c_n \Delta u_n \\ w_n &= b_n \Delta v_n \\ z_n &= a_n \Delta w_n \end{aligned}\right\}.$$

Now the system

$$\left.\begin{aligned}\Delta u_n &= \frac{v_n}{c_n}\\ \Delta v_n &= \frac{w_n}{b_n}\\ \Delta w_n &= \frac{z_n}{a_n}\\ \Delta z_n &= -q_n f(u_{n+1})\end{aligned}\right\} \tag{2}$$

is satisfied. Clearly, $\{z_n\}$ is non-increasing. If there is an integer $n_1 \geq n_0$ such that $z_{n_1} < 0$, then

$$\left.\begin{aligned}w_n &= w_{n_0} + \sum_{s=n_0}^{n-1} \frac{z_s}{a_s}\\ v_n &= v_{n_0} + \sum_{s=n_0}^{n-1} \frac{w_s}{b_s}\\ u_n &= u_{n_0} + \sum_{s=n_0}^{n-1} \frac{v_s}{c_s}\end{aligned}\right\} \tag{3}$$

and from condition (c_2) we have that

$$\left.\begin{matrix} w_n \\ v_n \\ u_n \end{matrix}\right\} \to -\infty \qquad (n \to -\infty)$$

which is a contradiction. Thus $z_n \geq 0$ for all $n \geq n_0$, so $\lim_{n\to\infty} z_n = z_\infty$ exists and $z_\infty \geq 0$. Also, $z_{n_1} > 0$ if $n_1 > n_0$. Then $z_n = 0$ whenever $n \geq n_1$. Thus, from (2), $\Delta z_n = 0$ and $q_n = 0$ whenever $n \geq n_1$. But this contradicts hypothesis (H_1), so $z_n > 0$ for $n \geq n_0$. Thus $\{z_n\}$ is increasing for $n \geq n_0$.

Now we take different cases.

Suppose $w_n < 0$ for $n \geq n_0$. Now $w_\infty \leq 0$, and if $w_\infty < 0$, then (3) again gives a contradiction, so $w_\infty = 0$. Now v_n is decreasing for $n \geq n_0$, and $v_\infty < 0$ is impossible, so $v_\infty \geq 0$. If $j \geq n \geq n_0$, then $z_j - z_n = -\sum_{s=n}^{j-1} q_s f(u_{s+1})$, so

$$z_\infty - z_n = -\sum_{s=n}^{\infty} q_s f(u_{s+1}) \qquad \text{or} \qquad z_n \geq \sum_{s=n}^{\infty} q_s f(u_{s+1}) \geq \sum_{s=n}^{\infty} q_s f(u_s).$$

Since $v_n > 0$, u_n is increasing, so $z_n \geq f(u_{n_0}) \sum_{s=n}^{\infty} q_s$ for $n \geq n_0$. If $\sum_{n=0}^{\infty} q_n < \infty$ fails in Hypothesis (H_2), this is a contradiction, hence assume $\sum_{n=0}^{\infty} q_n < \infty$ holds. Since $w_\infty = 0$, we have $w_n = -\sum_{s=n}^{\infty} \frac{z_s}{a_s}$ for $n \geq n_0$. But the last inequality says that if $\sum_{n=0}^{\infty} \left(\frac{1}{a_n} \sum_{s=n}^{\infty} q_s\right) < \infty$ in hypothesis (H_2) fails, this is a contradiction, so assume $\sum_{n=0}^{\infty} \left(\frac{1}{a_n} \sum_{s=n}^{\infty} q_s\right) < \infty$ holds. If $n \geq n_0$, then

$$v_n - v_{n_0} = \sum_{s=n_0}^{n-1} \frac{w_s}{b_s} = -\sum_{s=n_0}^{n-1} \frac{1}{b_s} \left(\sum_{t=s}^{\infty} \frac{z_t}{a_t} \right)$$

and so

$$-v_{n_0} \leq - \sum_{s=n_0}^{n-1} \frac{1}{b_s} \left(\sum_{t=s}^{\infty} \frac{z_t}{a_t} \right)$$

$$v_{n_0} \geq \sum_{s=n_0}^{n-1} \frac{1}{b_s} \left(\sum_{t=s}^{\infty} \frac{z_t}{a_t} \right) \geq f(u_0) \sum_{s=n_0}^{n-1} \frac{1}{b_s} \left(\sum_{t=s}^{\infty} \frac{1}{a_t} \left(\sum_{i=t}^{\infty} q_i \right) \right).$$

However, this contradicts hypothesis (H_2), and we are through the case $w_n < 0$ for $n \geq n_0$.

Since $\{w_n\}$ is increasing and $w_n < 0$ is false ensure that there is an integer $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$ and $w_n > 0$ for all $n \geq n_1$. Now $\{v_n\}$ is increasing for all $n \geq n_1$. If $v_n \leq 0$ for all $n \geq n_1$, then $\{u_n\}$ is bounded. But hypothesis (H_1) and a result in [15] say that every bounded solution of equation (1) is oscillatory, so there is an integer $n_2 \geq n_1$ such that $v_n > 0$ for all $n \geq n_2$. Now if $n \geq n_2$, then

$$\begin{aligned} u_n &= u_{n_2} + \sum_{s=n_2}^{n-1} \frac{v_s}{c_s} \\ &\geq \sum_{s=n_2}^{n-1} \frac{v_s}{c_s} \\ &= \sum_{s=n_2}^{n-1} \frac{1}{c_s} \left(v_{n_2} + \sum_{t=n_2}^{s-1} \frac{w_t}{b_t} \right) \\ &\geq \sum_{s=n_2}^{n-1} \frac{1}{c_s} \left(\sum_{t=n_2}^{s-1} \frac{w_t}{b_t} \right) \\ &\geq w_{n_2} \sum_{s=n_2}^{n-1} \frac{1}{c_s} \left(\sum_{t=n_2}^{s-1} \frac{1}{b_t} \right). \end{aligned}$$

If $n \geq n_2$, then

$$0 < z_n = z_{n_2} + \sum_{s=n_2}^{n-1} \Delta z_s = z_{n_2} - \sum_{s=n_2}^{n-1} q_s f(u_{s+1}).$$

So

$$z_{n_2} \geq \sum_{s=n_2}^{n-1} q_s f(u_s) \geq M w_{n_2} \sum_{s=n_2}^{n-1} q_s \left(\sum_{t=n_2}^{s-1} \frac{1}{c_s} \sum_{j=n_2}^{t-1} \frac{1}{b_j} \right). \tag{4}$$

But, according to Stolz's Theorem [2], we have

$$\lim_{s \to \infty} \frac{\sum_{t=n_2}^{s-1} \frac{1}{c_s} \sum_{j=n_2}^{t-1} \frac{1}{b_j}}{\sum_{t=0}^{s-1} \frac{1}{c_s} \sum_{j=0}^{t-1} \frac{1}{b_j}} = 1$$

and so hypothesis (H_3) implies the divergence of the summations in (4) as $n \to \infty$. This contradiction completes the proof of the theorem ∎

Corollary 2. *Assume hypothesis* (H_3) *holds and*

$$\sum_{n=0}^{\infty}\left(\sum_{s=0}^{n-1}\frac{1}{a_s}\left(\sum_{t=0}^{s-1}\frac{1}{b_s}\right)\right)q_s=\infty. \tag{5}$$

Then every solution of equation (1) *is oscillatory.*

Proof. Let $\{y_n\}$ be a non-oscillatory solution of equation (1). Without loss of generality we may assume that $\{y_n\}$ is eventually positive. If hypothesis (H_2) holds and $n\in\mathbb{N}_0$, then two successive applications of summation by parts give

$$\begin{aligned}
&\sum_{s=0}^{\infty}\frac{1}{b_s}\left(\sum_{t=s}^{\infty}\frac{1}{a_t}\left(\sum_{j=t}^{\infty}q_j\right)\right)\\
&=\left(\sum_{t=0}^{n-1}\frac{1}{b_t}\right)\left(\sum_{t=n}^{\infty}\frac{1}{a_t}\sum_{j=t}^{\infty}q_t\right)+\sum_{s=0}^{n-1}\frac{1}{a_s}\left(\sum_{t=0}^{s}\frac{1}{b_t}\right)\sum_{j=s}^{\infty}q_j\\
&\geq\sum_{s=0}^{n-1}\left(\frac{1}{a_s}\sum_{t=0}^{s-1}\frac{1}{b_t}\right)\sum_{j=s}^{\infty}q_j\\
&=\sum_{t=0}^{n-1}\left(\frac{1}{a_t}\sum_{j=0}^{t-1}\frac{1}{b_j}\right)\left(\sum_{j=n}^{\infty}q_j\right)+\sum_{s=0}^{n-1}q_s\left(\sum_{t=0}^{s}\frac{1}{a_t}\sum_{j=0}^{t-1}\frac{1}{b_j}\right)\\
&\geq\sum_{s=0}^{n-1}q_s\left(\sum_{t=0}^{s-1}\frac{1}{a_t}\sum_{j=0}^{t-1}\frac{1}{b_j}\right).
\end{aligned}$$

Thus (5) implies hypothesis (H_2). Now condition (c_2) and two applications of Stolz's Theorem imply that

$$\lim_{i\to\infty}\frac{\sum_{s=0}^{i-1}\frac{1}{a_s}\sum_{t=0}^{s-1}\frac{1}{b_t}}{\sum_{s=0}^{i-1}\frac{1}{a_s}\sum_{t=0}^{s-1}\frac{1}{b_t}\sum_{j=0}^{t-1}\frac{1}{c_j}}=0,$$

so there is an integer $N\in\mathbb{N}_0$ such that

$$\sum_{s=0}^{i-1}\frac{1}{a_s}\sum_{t=0}^{s-1}\frac{1}{b_t}\sum_{j=0}^{t-1}\frac{1}{c_j}\geq\sum_{s=0}^{i-1}\frac{1}{a_s}\sum_{t=0}^{s-1}\frac{1}{b_t}$$

whenever $i\geq N$, and we see that (5) implies hypothesis (H_1), and the result now follows from Theorem 1 ∎

Remark 1. If $a_n\equiv c_n$, then (5) is the same as hypothesis (H_3), so in this case (5) implies that every solution of equation (1) is oscillatory. If $a_n=c_n=1$ and $b_n=r_n$, then (5) is equivalent to $\sum_{n=0}^{\infty}\sum_{s=0}^{n-1}\frac{n-s-1}{r_s}q_n=\infty$ and hence Corollary 2 implies that every solution of equation (1) is oscillatory. This is [3: Theorem 6.11].

Example 1. Consider the difference equation

$$\Delta\left((n+1)\Delta\left(\frac{1}{n}\Delta(n\Delta y_n)\right)\right)+\left(8n+14+\frac{(2n+1)}{n(n+1)}\right)y_{n+1}(1+|y_{n+1}|)=0 \qquad (6)$$

for $n \geq 1$ where $a_n = n+1, b_n = \frac{1}{n}, c_n = n, q_n = 8n+14+\frac{2n+1}{n(n+1)}$ and $f(u) = u(1+|u|)$. It is easy to see that all conditions of Corollary 2 are satisfied and hence every solution of equation (6) is oscillatory. In fact, $\{y_n\} = \{(-1)^n\}$ is such a solution.

3. Asymptotic behavior of non-oscillatory solutions

Here we discuss the asymptotic behavior of non-oscillatory solutions of equation (1) under the following conditions:

(c_4) $\{a_n\}, \{b_n\}, \{c_n\}, \{q_n\}$ are real and positive sequences such that $\sum_{n=0}^{\infty}\frac{1}{a_n} < \infty, \sum_{n=0}^{\infty}\frac{1}{b_n} < \infty, \sum_{n=0}^{\infty}\frac{1}{c_n} < \infty$.

(c_5) $\lim_{n\to\infty}\rho_i(n) = 0$ where $\rho_i(n) = \sum_{s=n+1}^{\infty}\frac{\rho_{i-1}(s)}{r_i(s)}$ $(i = 1,2,3)$ with $\rho_0(n) \equiv 1$ and $r_1(n) = c_n, r_2(n) = b_n, r_3(n) = a_n$.

We begin with two lemmas that will be needed in the proof of our main result of this section.

Lemma 3. *Consider the difference equation*

$$\Delta u_n - \frac{\Delta\rho(n)}{\rho(n)}u_n + \frac{\Delta\rho(n)}{\rho(n)}\phi_n = 0 \qquad (7)$$

where $\{\phi_n\}, \{\rho(n)\}$ are real sequences defined for $n \geq N \in \mathbb{N}_0$ and $\rho(n) > 0, \Delta\rho(n) < 0, \lim_{n\to\infty}\rho(n) = 0$. Let $\{u_n\}$ be the solution of equation (7) defined for $n \geq N$ and satisfying $u_N = 0$. Then

$$\begin{aligned}\lim_{n\to\infty}\phi_n = \infty &\implies \lim_{n\to\infty}u_n = \infty\\ \lim_{n\to\infty}\phi_n = -\infty &\implies \lim_{n\to\infty}u_n = -\infty\end{aligned}.$$

Proof. The solution $\{u_n\}$ of equation (7) is given by the formula

$$u_n = -\rho(n)\sum_{s=N}^{n-1}\frac{\Delta\rho(s)}{\rho(s)\rho(s+1)}\phi_s \qquad (n \geq N).$$

If $\lim_{n\to\infty}\phi_n = \pm\infty$, then it is obvious that

$$\lim_{n\to\infty}\left(-\sum_{s=N}^{n-1}\frac{\Delta\rho(s)}{\rho(s)\rho(s+1)}\phi_s\right) = \begin{cases}+\infty\\ -\infty\end{cases}$$

Hence, by Stolz's theorem,

$$\lim_{n\to\infty}u_n = \lim_{n\to\infty}\left|\frac{\Delta\left(-\sum_{s=N}^{n-1}\frac{\Delta\rho(s)}{\rho(s)\rho(s+1)}\phi_s\right)}{\Delta\left(\frac{1}{\rho(n)}\right)}\right| = \lim_{n\to\infty}\phi_n = \begin{cases}+\infty\\ -\infty\end{cases}$$

and the lemma is proved ∎

Lemma 4. *Let* $\{\rho_n\}$ *and* $\{v_n\}$ *be real sequences defined for* $n \geq N \in \mathbb{N}_0$. *If the limit* $\lim_{n\to\infty}(\rho_n \Delta v_n + v_n)$ *exists in the extended real line* $\mathbb{R}^*$, *then the limit* $\lim_{n\to\infty} v_n$ *exists in* $\mathbb{R}^*$.

Proof. Let $z_n = \rho_n \Delta v_n + v_n$. Then

$$\begin{aligned}\lim_{n\to\infty} z_n &= \lim_{n\to\infty} (\rho_n v_{n+1} - \rho_n v_n + v_n) \\ &= \lim_{n\to\infty} \rho_n \lim_{n\to\infty} v_{n+1} - \lim_{n\to\infty} \rho_n \lim_{n\to\infty} v_n + \lim_{n\to\infty} v_n \\ &= \lim_{n\to\infty} v_n.\end{aligned}$$

So $\lim_{n\to\infty} z_n \in \mathbb{R}^*$ implies $\lim_{n\to\infty} v_n \in \mathbb{R}^*$ ∎

Theorem 5. *Let conditions* $(c_4), (c_5)$ *hold, and assume that* $\liminf_{y\to\infty} f(y) > 0$ *and* $\limsup_{y\to-\infty} f(y) < 0$. *If*

$$\sum_{n=N}^{\infty} \rho_3(n) q_n = \infty \qquad \textit{and} \qquad \sum_{n=N}^{\infty} \rho_3(n)|h_n| < \infty, \tag{8}$$

then all non-oscillatory solutions of equation (1) *are bounded and tend to zero as* $n \to \infty$.

Proof. Let $\{y_n\}$ be a non-oscillatory solution of equation (1). We may suppose that $y_n > 0$ for $n \geq N_1 \in \mathbb{N}$. Define

$$\left.\begin{aligned} G_0(n) &= y_n \\ G_i(n) &= r_i(n)\Delta G_{i-1}(n) \quad (i = 1, 2, 3) \end{aligned}\right\}$$

and

$$u_k(n) = \sum_{s=N_1+1}^{n} \rho_{3-k}(s) \Delta G_{3-k}(s) \quad (k = 0, 1, 2, 3). \tag{9}$$

We shall first show that $\{y_n\}$ is bounded above. From equation (1) we obtain

$$G_3(n) - G_3(N_1) + \sum_{s=N_1}^{n-1} q_s f(y_{s+1}) = \sum_{s=N_1}^{n-1} h_s.$$

Since herein the first sum is positive and by (8_2) the second sum is bounded, there exists a constant k_3 such that

$$G_3(n) = r_3(n)\Delta G_2(n) \leq k_3 \qquad (n \geq N_1).$$

Dividing the last inequality by $r_3(n)$ and summing from N_1 to $n-1$, we obtain

$$G_2(n) - G_2(N_1) \leq k_3 \sum_{s=N_1}^{n-1} \frac{1}{r_3(n)} \qquad (n \geq N_1)$$

which shows in view of condition (c_4) that there exists a constant k_2 such that

$$G_2(n) = r_2(n)\Delta G_1(n) \le k_2 \qquad (n \ge N).$$

Applying the above arguments repeatedly, we obtain

$$\left.\begin{aligned} G_1(n) &\le k_1 \\ G_0(n) &\le k_0 \end{aligned}\right\} \qquad (n \ge N_1)$$

where k_1 and k_0 are constants. It follows that $\{y_n\}$ is bounded above for $n \ge N_1$. Summation by parts yields

$$\begin{aligned} & u_{k-1}(n) \\ &= \sum_{s=N_1+1}^{n} \rho_{4-k}(s)\Delta G_{4-k}(s) \\ &= \rho_{4-k}(n+1)G_{m-k}(n+1) - \rho_{4-k}(N_1+1)G_{4-k}(N_1+1) + \sum_{s=N_1+1}^{n} \frac{\rho_{3-k}(s)}{r_{4-k}(s)} G_{4-k}(s) \\ &= -\frac{\rho_{3-k}(n+1)}{\Delta\rho_{4-k}(n)}\Delta u_k(n) + \Delta u_k(n) + u_k(n) - 2\rho_{4-k}(N_1+1)G_{4-k}(N_1+1) \\ &= -\frac{\rho_{4-k}(n)}{\Delta\rho_{4-k}(n)}\Delta u_k(n) + u_k(n) - 2\rho_{4-k}(N_1+1)G_{4-k}(N_1+1). \end{aligned}$$

This shows that $\{u_k(n)\}$ satisfies the difference equation

$$\frac{\rho_{4-k}(n)}{\Delta\rho_{4-k}(n)}\Delta u_k(n) - u_k(n) + \phi_k(n) = 0 \tag{10}$$

or, equivalently,

$$\Delta u_k(n) - \frac{\Delta\rho_{4-k}(n)}{\rho_{4-k}(n)} u_k(n) + \frac{\Delta\rho_{4-k}(n)}{\rho_{4-k}(n)}\phi_k(n) = 0 \tag{11}$$

where

$$\phi_k(n) = u_{k-1}(n) + 2\rho_{4-k}(N_1+1)G_{4-k}(N_1+1).$$

Since $u_k(N_1) = 0$ by (9) and since

$$\left.\begin{aligned} \rho_{4-k}(n) &> 0 \\ \Delta\rho_{4-k}(n) &< 0 \\ \lim_{n\to\infty} \rho_{4-k}(n) &= 0 \end{aligned}\right\}$$

by condition (c_5) we apply Lemma 3 to (11) to conclude that $\lim_{n\to\infty} u_{k-1}(n) = \pm\infty$ implies $\lim_{n\to\infty} u_k(n) = \pm\infty$. Further, applying Lemma 4 to (10) we conclude that $\lim_{n\to\infty} u_k(n)$ exists in R^* whenever $\lim_{n\to\infty} u_{k-1}(n)$ exists in R^*.

We now multiply both sides of equation (1) by $\rho_3(n)$, and summing from N_1+1 to n we get

$$\sum_{s=N_1+1}^{n} \rho_3(s)\Delta G_3(s) + \sum_{s=N_1+1}^{n} \rho_3(s)q_s f(y_{s+1}) = \sum_{s=N_1+1}^{n} \rho_3(s)h_s. \tag{12}$$

We consider the two cases

$$\sum_{s=N_1+1}^{\infty} \rho_3(s)q_s f(y_{s+1}) = \begin{cases} +\infty \\ -\infty \end{cases}. \tag{13}$$

Suppose (13_1) holds. In view of (8_2) the right-hand side of (12) tends to a finite limit as $n \to \infty$, so from (12) we see that $\lim_{n\to\infty} u_0(n) = -\infty$. Hence by Lemma 3 applied to (11) with $k = 1$ we have $\lim_{n\to\infty} u_1(n) = -\infty$. Applying Lemma 3 again to (11) with $k = 2$ we find $\lim_{n\to\infty} u_2(n) = -\infty$. Repeating the same argument we conclude that $\lim_{n\to\infty} u_3(n) = -\infty$ which implies that $\lim_{n\to\infty} y_n = -\infty$. However, this contradicts the positivity of y_n. Hence (13_1) is impossible. Now letting $n \to \infty$ in (12) and using (13_2) we see that $\lim_{n\to\infty} u_0(n)$ is finite. From Lemma 4 applied to (10) with $k = 1$ it follows that $\lim_{n\to\infty} u_1(n)$ exists in R^*. This limit must be finite since $\lim_{n\to\infty} u_1(n) = -\infty$ would imply $\lim_{n\to\infty} y_n = -\infty$ which is a contradiction, and $\lim_{n\to\infty} u_1(n) = \infty$ would imply $\lim_{n\to\infty} y_n = \infty$ which is a contradiction to the boundedness of y_n. Repeating the same argument we conclude that $\lim_{n\to\infty} u_3(n)$ is finite. Hence $\lim_{n\to\infty} y_n$ exists as a finite number. On the other hand, from (8_1) and (13_2) we see that $\lim_{n\to\infty} y_n = 0$. Therefore we conclude that $y_n \to 0$ as $n \to \infty$ ∎

We conclude this section with the following example.

Example 2. Consider the difference equation

$$\Delta\big(2^n\Delta(2^n\Delta(2^n\Delta y_n))\big) + 8^n y_{n+1}^3 = \frac{1}{8} \qquad (n \geq 0). \tag{14}$$

In this case $\rho_1(n) = \frac{1}{2^n}, \rho_2(n) = \frac{1}{3}\left(\frac{1}{4^n}\right), \rho_3(n) = \frac{1}{21}\left(\frac{1}{8^n}\right)$. Since all conditions of Theorem 5 are satisfied, every non-oscillatory solution of equation (14) tends to zero as $n \to \infty$. Especially, this equation has the non-oscillatory solution $\{y_n\} = \left\{\frac{1}{2^n}\right\}$ which tends to zero as $n \to \infty$.

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