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On the Existence of C^1 Functions with Perfect Level Sets

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Abstract. Given a closed set $M \subset [0,1]$ of Lebesgue measure zero, we construct a C^1 function f with the property that $f^{-1}(\{y\})$ is a perfect set for every y in M.

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1. Introduction

Bruckner and Garg in [1] give a full description of the level set structure of a typical continuous function. In [2] the description of the level set structure of a typical C^1 function is given. It follows from [2: Theorem 2] that a typical C^1 function is either strictly monotone or has the property that all of its level sets are countable.

In this article we investigate the behaviour of C^1 functions in the other direction. It is easy to show that given a C^1 function $f : [0,1] \to [0,1]$, the set of points where the level sets of f are uncountable has Lebesgue measure zero [3: p. 226/Lemma 6.3]. We show that given a closed set $M \subset [0,1]$ of Lebesgue measure zero, there exists a C^1 function $f : [0,1] \to [0,1]$ with the property that $f^{-1}(\{y\})$ is a perfect set for every y in M as well as finite for every y in $[0,1]\backslash M$.

2. Preliminaries

In this section a few definitions, notations and lemmas are stated that are used throughout the article.

Definition 2.1. A subset $B \subseteq \mathbb{R}^2$ is a *box* if $B = I \times J$ for some compact intervals I and J of the real line $\mathbb{R}$. A *signed box* is an ordered pair $(B, *)$ where B is a box and $* \in \{+, -\}$. We will generally use B^* to denote the signed box $(B, *)$.

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Throughout π_1 and π_2 will be the coordinate projections. We shall say that two intervals of the real line are *disjoint* if their intersection is empty or coincides with a single point.

Definition 2.2. Let $B = [a, b] \times [c, d]$ be a box. A C^1 function $f : [a, b] \to [c, d]$ is *diagonal to* B^+, denoted by $f \square B^+$, if

1. f is increasing on $[a, b]$ with $f(a) = c$ and $f(b) = d$

2. $f'(a) = f'(b) = 0$

where by the derivatives at the end points we mean the right and the left derivatives, respectively. Further, we say that f is *diagonal to* B^-, denoted by $f \square B^-$, if condition (2) above is satisfied and the condition

3. f is decreasing on $[a, b]$ with $f(a) = d$ and $f(b) = c$

is satisfied. Finally, we say that f is *diagonal to* B, denoted by $f \square B$, if $f \square B^+$ or $f \square B^-$. If $f \square B^*$, we say that $*$ is the *sign induced on* B *by* f.

Definition 2.3. Let $B = I \times J$ be a box. Then the *slope* of B, denoted by $\mathrm{sl}(B)$, is $\frac{\lambda(J)}{\lambda(I)}$.

The proof of the following lemma is a very easy exercise.

Lemma 2.4. *Let B^* be a signed box and $\varepsilon > 0$. Then there exists a C^1 function f such that $f \square B^*$ and $|f'(x)| < \mathrm{sl}(B) + \varepsilon$ for all $x \in \pi_1(B)$.*

Definition 2.5. Let $B = I \times J$ be a box. We say:

(i) $f : I \to J$ is *jagged inside* B^+ if $f|_{B_L} \square B_L^+$, $f|_{B_M} \square B_M^-$ and $f|_{B_R} \square B_R^+$, where $B_i = I_i \times J$ and $\{I_L, I_M, I_R\}$ is the partition of I into three equal pieces ordered from the left to the right.

(ii) f is *jagged inside* B^- if $f|_{B_L} \square B_L^-$, $f|_{B_M} \square B_M^+$ and $f|_{B_R} \square B_R^-$, where the B_i's are the same as above.

(iii) f is *jagged inside* B if f is jagged inside B^+ or B^-.

We call $\{B_L, B_M, B_R\}$ *vertical splitting* of B into three equal pieces.

Lemma 2.6. *Let B^* be a signed box and $\varepsilon > 0$. Then there is a C^1 function g jagged inside B^* such that $|g'(x)| < 3\,\mathrm{sl}(B) + \varepsilon$ for all $x \in \pi_1(B)$.*

Proof. If $* = +$, we apply Lemma 2.4 to B_L^+, B_M^- and B_R^+. If $* = -$, we apply Lemma 2.4 to B_L^-, B_M^+ and B_R^- ∎

3. Proof of the main result

In this section we first prove a lemma essential to our main result.

Lemma 3.1. *Suppose B is a box, $*$ is a sign (i.e. $* \in \{+,-\}$), $M \subseteq \pi_2(B)$ is a closed set with $\lambda(M) = 0$, $N > 0$ is an integer and $\delta > 0$. Then there exist a C^1 function h and a sequence of pairwise disjoint boxes $B_1, \dots, B_n$ contained in B such that:*

1. $h \,\square\, B^*$.
2. $M \subseteq \pi_2(\cup_{i=1}^n B_i)$, $\pi_2(B_i) \cap M \neq \emptyset$, *and* $\lambda(\cup_{i=1}^n \pi_2(B_i)) < \delta$.
3. $\{\pi_2(B_i)\}$ *is a pairwise disjoint finite sequence, and so is* $\{\pi_1(B_i)\}$.
4. $\mathrm{sl}(B_i) = \frac{1}{N} \quad (1 \le i \le n)$.
5. $h|_{B_i} \,\square\, B_i^* \quad (1 \le i \le n)$.
6. $|h'(x)| < \mathrm{sl}(B) + \delta$ *for all* $x \notin \cup_{i=1}^n \pi_1(B_i)$.
7. $|h'(x)| < \frac{1}{N} + \frac{\delta}{2}$ *for all* $x \in \cup_{i=1}^n \pi_1(B_i)$.

Proof. Let $B = I \times J$, where $I = [a,b]$ and $J = [c,d]$. Without loss of generality let us assume that $* = +$. Since M is a closed subset of a compact set and $\lambda(M) = 0$, there exist a finite number of disjoint closed intervals $J_1, \dots, J_n$ such that $M \subseteq \cup_{i=1}^n J_i$, $J_i \cap M \neq \emptyset$, $\lambda(\cup_{i=1}^n J_i) < \delta$ and

$$0 < \frac{\lambda(J) - \lambda(\cup_{i=1}^n J_i)}{\lambda(I) - N\lambda(\cup_{i=1}^n J_i)} < \frac{\lambda(J)}{\lambda(I)} + \frac{\delta}{2}.$$

We shall find a finite sequence of pairwise disjoint intervals $I_1, \dots, I_n$ contained in I and $h \in C^1(I)$ such that

(i) $\lambda(I_i) = N\lambda(J_i) \quad (1 \le i \le n)$

(ii) h is diagonal to B^+

(iii) $h|_{B_i} \,\square\, B_i^+ \quad (1 \le i \le n)$ where $B_i = I_i \times J_i$

(iv) $0 \le h'(x) < \frac{\lambda(J) - \lambda(\cup_{i=1}^n J_i)}{\lambda(I) - N\lambda(\cup_{i=1}^n J_i)} + \frac{\delta}{2} \quad (x \in I \setminus \cup_{i=1}^n I_i)$.

Without loss of generality we may assume that $z \ge y$ for all $z \in J_{i+1}$ and all $y \in J_i$ $(1 \le i \le n)$. Let

$$m = \frac{\lambda(J) - \lambda(\cup_{i=1}^n J_i)}{\lambda(I) - N\lambda(\cup_{i=1}^n J_i)}$$

and set

$$d_1 = d(J_1, c),\ I_1 = \left[a + \frac{d_1}{m}, a + \frac{d_1}{m} + N\lambda(J_1)\right]$$

$$d_2 = d(J_1, J_2),\ I_2 = \left[a + \frac{d_1}{m} + \frac{d_2}{m} + N\lambda(J_1), a + \frac{d_1}{m} + \frac{d_2}{m} + N(\lambda(J_1) + \lambda(J_2))\right]$$

$$\vdots$$

$$d_n = d(J_{n-1}, J_n),\ I_n = \left[a + \sum_{i=1}^n \frac{d_i}{m} + N\sum_{i=1}^{n-1} \lambda(J_i), a + \sum_{i=1}^n \frac{d_i}{m} + N\sum_{i=1}^n \lambda(J_i)\right].$$

Apply Lemma 2.4 to B_i^+ and δ, and also to $[a, a + \frac{d_1}{m}] \times [c, c + d_1]^+$ and $\frac{\delta}{2}$, to

$$\left[a + \sum_{i=1}^{n} \frac{d_i}{m} + N \sum_{i=1}^{n} \lambda(J_i), b\right] \times [\max J_n, d]^+$$

and $\frac{\delta}{2}$, and to

$$\tilde{B}_t^+ = \left[a + \sum_{i=1}^{t} \frac{d_i}{m} + N \sum_{i=1}^{t} \lambda(J_i), a + \sum_{i=1}^{t+1} \frac{d_i}{m} + N \sum_{i=1}^{t} \lambda(J_i)\right] \times [\max J_t, \min J_{t+1}]^+$$

and $\frac{\delta}{2}$, for $t \in \{1, \dots, n-1\}$. We paste together the resulting functions and call this function h. Then h is a C^1 function which satisfies the desired properties (1) - (7) ∎

Theorem 3.2. *Let $M \subseteq [0,1]$ be a closed set with $\lambda(M) = 0$. Then there exists a C^1 function $f : [0,1] \to [0,1]$, onto, such that $f^{-1}(\{y\})$ is a nowhere dense perfect set for all $y \in M$ as well as finite for all $y \in [0,1] \backslash M$.*

Proof. We shall construct the desired function f as C^1 limit of a sequence of piecewise monotone C^1 functions. At stage n we shall have a piecewise monotone function f_n with $f_n(0) = 0$, $f_n(1) = 1$ and a finite collection of boxes. Then f_{n+1} will be an appropriate modification of f_n inside these boxes. This sequence $\{f_n\}$ will be constructed inductively.

Let us first construct f_1. We apply Lemma 3.1 with $B = [0,1] \times [0,1]$, $* = +$, $M = M$, $N = 3 \cdot 2$ and $\delta = \frac{1}{2}$ and obtain a function h and a finite collection of boxes $B_1, \dots, B_n$ which satisfy the conclusions of Lemma 3.1. Using Lemma 2.6 with $\varepsilon = \frac{1}{2}$, for each i, $1 \le i \le n$, we obtain a C^1 function g_i which is jagged inside B_i^+. Let

$$f_1(x) = \begin{cases} h(x) & \text{if } x \in [0,1] \backslash \cup_{i=1}^{n} \pi_1(B_i) \\ g_i(x) & \text{if } x \in \pi_1(B_i) \text{ for some } i. \end{cases}$$

Now, for each i, let $B_{i,L}$, $B_{i,M}$, $B_{i,R}$ be the vertical splitting of B_i into three equal pieces and let $\mathcal{G}_1 = \{B_{i,L}, B_{i,M}, B_{i,R} : 1 \le i \le n\}$. At the end of stage 1, the following properties hold:

1. f_1 is a piecewise monotone C^1 function with $f_1(0) = 0$ and $f_1(1) = 1$.
2. $f_1|_B \,\square\, B$ for all $B \in \mathcal{G}_1$.
3. $|f_1'(x)| < \mathrm{sl}(B) + \delta = 1 + \frac{1}{2}$ for all $x \in [0,1] \backslash \cup_{i=1}^{n} \pi_1(B_i)$.
4. $|f_1'(x)| < 3 \cdot \frac{1}{N} + \delta = 3 \cdot \frac{1}{3 \cdot 2} + \frac{1}{2} = 1$ for all $x \in \cup_{i=1}^{n} \pi_1(B_i)$.

Now let us assume that we are at stage $k > 1$, f_k and $\mathcal{G}_k$ have been constructed so that the following properties are satisfied:

(i) $M \subseteq \pi_2(\cup \mathcal{G}_k)$, $\pi_2(B) \cap M \ne \emptyset$ for all $B \in \mathcal{G}_k$, and $\lambda(\pi_2(\cup \mathcal{G}_k)) < \frac{1}{(k+1)!}$.

(ii) $\mathrm{sl}(B) = \frac{1}{2^k}$ for all $B \in \mathcal{G}_k$.

(iii) Suppose that $y \in M$ and $B \in \mathcal{G}_{k-1}$ are such that $y \in \pi_2(B)$. Then there exist disjoint boxes B_1 and B_2 in $\mathcal{G}_k$ contained in B such that $y \in \pi_2(B_1) \cap \pi_2(B_2)$.

(iv) f_k is a piecewise monotone C^1 function with $f_k(0) = 0$ and $f_k(1) = 1$.

(v) $f_k(x) = f_{k-1}(x)$ if $x \in [0,1]\backslash\pi_1(\cup\mathcal{G}_{k-1})$.

(vi) $f_k|_B \,\square\, B$ for all $B \in \mathcal{G}_k$.

(vii) $|f_k'(x)| < \frac{1}{2^k} + \frac{1}{2^k}$ for all $x \in \cup_{B\in\mathcal{G}_k}\pi_1(B)$.

(viii) $|f_k'(x)| < \frac{1}{2^{k-1}} + \frac{1}{2^k}$ for all $x \in \pi_1(\cup\mathcal{G}_{k-1})\backslash\pi_1(\cup\mathcal{G}_k)$.

(ix) If $(x, f_k(x))$ is such that $f_k(x) \in M$, then $(x, f_k(x)) \in \cup\mathcal{G}_k$.

Now we proceed to construct f_{k+1}. Fix a box $B \in \mathcal{G}_k$ and let $*$ be the sign induced on B by $f_k|_B$. Now we apply Lemma 3.1 to B^*, $M \cap \pi_2(B)$, $N = 3 \cdot 2^{k+1}$ and $\delta = \frac{1}{(k+2)!} \cdot \lambda(\pi_2(B))$ and obtain a function $h : \pi_1(B) \to \pi_2(B)$, onto, and a finite collection of pairwise disjoint boxes $B_1, \ldots, B_n$ which satisfy the conclusions of Lemma 3.1.

Using Lemma 2.6 with $\varepsilon = \frac{1}{2^{k+1}}$, for each i, $1 \le i \le n$, we obtain a C^1 function g_i which is jagged inside B_i^*. Now let $h_B : \pi_1(B) \to \pi_2(B)$ be defined as

$$h_B(x) = \begin{cases} h(x) & \text{if } x \in \pi_1(B)\backslash \cup_{i=1}^n \pi_1(B_i) \\ g_i(x) & \text{if } x \in \pi_1(B_i) \text{ for some } i. \end{cases}$$

We observe the following:

1. h_B is a piecewise monotone C^1 function.
2. $|h_B'(x)| < \mathrm{sl}(B) + \delta < \frac{1}{2^k} + \frac{1}{(k+2)!}$ for all $x \in \pi_1(B)\backslash \cup_{i=1}^n \pi_1(B_i)$.
3. $|h_B'(x)| < 3 \cdot \frac{1}{3\cdot 2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^k}$ for all $x \in \cup_{i=1}^n \pi_1(B_i)$.
4. h_B and h_B' agree with f_k and f_k', respectively, at the end points of $\pi_1(B)$.

Let
$$\mathcal{G}_{k,B} = \left\{B_{i,L}, B_{i,M}, B_{i,R} : 1 \le i \le n\right\}$$
and observe that, for every $C \in \mathcal{G}_{k,B}$, $\mathrm{sl}(C) = \frac{1}{2^{k+1}}$. Now for each B obtain such function h_B and a collection of boxes $\mathcal{G}_{k,B}$. Define $f_{k+1} : [0,1] \to [0,1]$ by

$$f_{k+1}(x) = \begin{cases} f_k(x) & \text{if } x \in [0,1]\backslash\pi_1(\cup\mathcal{G}_k) \\ h_B(x) & \text{if } x \in \pi_1(B) \text{ for some } B \in \mathcal{G}_k. \end{cases}$$

Let $\mathcal{G}_{k+1} = \cup_{B\in\mathcal{G}_k}\mathcal{G}_{k,B}$. It is easy to verify that f_{k+1} satisfies the induction hypotheses (i) - (ix). Fix k. Then

$$|f_k'(x) - f_{k+1}'(x)| \le \sup_{x\in\cup\mathcal{G}_k} |f_k'(x)| + \sup_{x\in\cup\mathcal{G}_k} |f_{k+1}'(x)| \le \frac{7}{2} \cdot \frac{1}{2^k}$$

for all $x \in [0,1]$. Since $f_k(0) = 0$, $\{f_k\}$ converges to some C^1 function f.

Next let us show that $f^{-1}(y)$ is perfect for every $y \in M$. Clearly, $f^{-1}(y) \ne \emptyset$, since $f : [0,1] \to [0,1]$ is onto. Let $x \in [0,1]$ be such that $f(x) = y$ and let $\varepsilon > 0$. We shall show that there exists x' such that $|x - x'| < \varepsilon$ and $f(x') = y$. Let k be large enough so that $\frac{2^k}{(k+1)!} < \varepsilon$. Notice that, for each $B \in \mathcal{G}_k$, $\lambda(\pi_1(B)) < \frac{2^k}{(k+1)!}$. By induction hypotheses (v) and (ix), $(x, f_k(x)) \in B_0$ for some $B_0 \in \mathcal{G}_k$. Notice that $\pi_1(B_0) \subseteq (x-\varepsilon, x+\varepsilon)$. By hypothesis (iii) there exist disjoint boxes B_1 and B_2 in $\mathcal{G}_{k+1}$

contained in B_0 such that $y \in \pi_2(B_1) \cap \pi_2(B_2)$. For each $l > k$, $f_l(\pi_1(B_1)) = \pi_2(B_1)$ and $f_l(\pi_1(B_2)) = \pi_2(B_2)$. Since $\{f_k\}$ converges to f in the C^1 norm, we have $f(\pi_1(B_1)) = \pi_2(B_1)$ and $f(\pi_1(B_2)) = \pi_2(B_2)$. There is at least one point $x' \neq x$ in $(x - \varepsilon, x + \varepsilon)$ such that $f(x') = y$. Therefore $f^{-1}(\{y\})$ is perfect. From the same argument it follows that $f^{-1}(\{y\})$ is nowhere dense.

Now we want to show that, for each $y \in [0,1] \setminus M$, $f^{-1}(\{y\})$ is finite. Let k be large enough so that $y \notin \pi_2(\cup \mathcal{G}_k)$. Then, for all $l > k$, $f^{-1}(\{y\}) = f_l^{-1}(\{y\})$. Since f_l is piecewise monotone $f_l^{-1}(\{y\})$ is finite ∎

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