A New Approach for Describing Electromagnetic Wave Propagation in Inhomogeneous Media

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Abstract. We propose a new approach for obtaining approximate solutions of Maxwell's equations in inhomogeneous media. This work is based on the application of quaternionic analysis technique and consists of some approximate diagonalization of Maxwell's equations. They are reduced to a pair of quaternionic equations which under some additional conditions can be solved exactly.

Keywords: *Inhomogeneous media, quaternionic analysis, Maxwell's equations*

AMS subject classification: 30 G 35, 78 A 40, 78 M 35

1. Introduction

The application of methods of hypercomplex analysis and in particular of quaternionic analysis to Maxwell's equations has more than a century of history, starting from the work of J. C. Maxwell himself. There exists a reformulation of these equations in vacuum in quaternionic terms (see, e.g., [2, 5, 10 - 12, 19, 20]) which allows some fundamental physical laws to be rewritten in a space-saving form.

Some integral representations for the solutions of Maxwell's equations in homogeneous media were obtained in [9: Section 4.5] and [8, 13, 15, 16] using methods of quaternionic analysis. In [13] and then [15, 16] a method based on the diagonalization of Maxwell's equations in isotropic and homogeneous media was proposed and implemented for obtaining new integral representations for different electromagnetic quantities and for solving a class of boundary value problems.

In the present work we use quaternionic analysis methods for studying electromagnetic propagation in inhomogeneous media. We show that under some additional assumptions about the electromagnetic characteristics of the medium the Maxwell equations can be diagonalized and can be reduced to a pair of quaternionic equations of first order. In [14] a method for solving the resulting quaternionic equations for some class of coefficients was proposed. In particular, when the coefficient depends only on one variable the method allows us not only to obtain exact solutions of the quaternionic

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equation but also to obtain the fundamental solution and to construct the corresponding right inverse operator. In other words this case can be completely solved and here we consider it in detail.

Besides this introduction, our work consists of four sections. Section 2 contains a very brief description of the notations used throughout the article. In Section 3 we rewrite the Maxwell equations in quaternionic "almost diagonalized" form. In Section 4 we consider one-dimensional Maxwell's equations in a slowly changing medium. The slow changing of the medium characteristics gives us the possibility "completely" to diagonalize the Maxwell equations, and as was mentioned above in the case of one-dimensional equations the corresponding quaternionic system can be completely solved. We give its solution and obtain the corresponding electromagnetic field. Moreover, in order to justify the neglect of some small terms due to the slow changing of the medium we estimate the norm of the corresponding integral operator. In Section 5 we give some conclusions and outline some possible continuations of this work to participate in which the interested reader is cordially invited.

2. Preliminaries

We will consider the propagation of time-harmonic (= monochromatic) electromagnetic fields in an isotropic, inhomogeneous medium. The field vectors $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$ are represented as

$$\overrightarrow{\mathbf{E}}(\vec{r}, t) = \operatorname{Re}(\overrightarrow{E}(\vec{r})e^{i\omega t})$$

$$\overrightarrow{\mathbf{H}}(\vec{r}, t) = \operatorname{Re}(\overrightarrow{H}(\vec{r})e^{i\omega t})$$

where ω denotes the frequency which can be a complex number (the circular frequency), and Maxwell's equations for the complex amplitudes $\overrightarrow{E}$ and $\overrightarrow{H}$ have the form

$$\operatorname{rot} \overrightarrow{E}(\vec{r}) = -i\omega\mu(\vec{r})\overrightarrow{H}(\vec{r}) \tag{1}$$

$$\operatorname{rot} \overrightarrow{H}(\vec{r}) = i\omega\varepsilon(\vec{r})\overrightarrow{E}(\vec{r}). \tag{2}$$

Here $\vec{r} = (x, y, z)^T$, μ is the complex permeability of the medium and ε is the complex permittivity.

For the analysis of (1) - (2) we will need the algebra of complex quaternions $\mathbb{H}(\mathbb{C})$. The elements of $\mathbb{H}(\mathbb{C})$ are represented in the form $\rho = \sum_{k=0}^{3} \rho_k i_k$ where $\rho_k \in \mathbb{C}$, i_0 is the unit and i_k $(k = 1, 2, 3)$ are standard quaternionic imaginary units: $i_k^2 = -1$ $(k = 1, 2, 3)$, $i_1 i_2 = -i_2 i_1 = i_3$, $i_2 i_3 = -i_3 i_2 = i_1$ and $i_3 i_1 = -i_1 i_3 = i_2$. Note that by definition the complex imaginary unit i commutes with i_k $(k = 0, ..., 3)$. We will use also the vector representation of complex quaternions. Any $\rho \in \mathbb{H}(\mathbb{C})$ can be represented in the form $\rho = \operatorname{Sc}(\rho) + \operatorname{Vec}(\rho)$, where $\operatorname{Sc}(\rho) = \rho_0$ and $\operatorname{Vec}(\rho) = \sum_{k=1}^{3} \rho_k i_k$. Complex quaternions of the form $\rho = \operatorname{Vec}(\rho)$ are called purely vectorial, and we identify them with vectors from $\mathbb{C}^3$: $\vec{\rho} = \operatorname{Vec}(\rho)$. The complex quaternion $\overline{\rho} := \operatorname{Sc}(\rho) - \operatorname{Vec}(\rho) = \rho_0 - \vec{\rho}$ is called conjugate to ρ.

We denote by $\mathfrak{S}$ the set of zero divisors from $\mathbb{H}(\mathbb{C})$. Note that

$$\rho \in \mathfrak{S} \quad \Longleftrightarrow \quad \rho\overline{\rho} = 0 \quad \Longleftrightarrow \quad \rho^2 = 2\rho_0\rho \quad \Longleftrightarrow \quad \rho_0^2 = (\vec{\rho})^2 \tag{3}$$

(see [16: p. 28]). As usual zero is not included to $\mathfrak{S}$.

We will consider $\mathbb{H}(\mathbb{C})$-valued functions given in some domain $\Omega \subset \mathbb{R}^3$ which may coincide with the whole space $\mathbb{R}^3$. On the set $C^1(\Omega; \mathbb{H}(\mathbb{C}))$ the well-known Moisil-Theodoresco operator is defined by the expression

$$D = i_1 \frac{\partial}{\partial x} + i_2 \frac{\partial}{\partial y} + i_3 \frac{\partial}{\partial z}.$$

It was introduced in [17, 18] and studied in hundreds of works (see, e.g., [3, 4, 7 - 9, 16] and many others). Let f be a function from $C^1(\Omega; \mathbb{H}(\mathbb{C}))$. The expression Df can be rewritten in the form

$$Df = -\operatorname{div}\vec{f} + \operatorname{grad}f_0 + \operatorname{rot}\vec{f} \tag{4}$$

where the differential operators are defined as usual. For instance,

$$\operatorname{grad}f_0 = i_1 \frac{\partial}{\partial x} f_0 + i_2 \frac{\partial}{\partial y} f_0 + i_3 \frac{\partial}{\partial z} f_0.$$

The right-hand side in (4) would have no sense in vector calculus, but here it simply means the following equalities for the scalar part and the vector part of the quaternion Df:

$$\operatorname{Sc}(Df) = -\operatorname{div}\vec{f}$$
$$\operatorname{Vec}(Df) = \operatorname{grad}f_0 + \operatorname{rot}\vec{f}.$$

Thus, the quaternionic equation $Df = 0$ is equivalent to the system

$$\left.\begin{array}{r} \operatorname{div}\vec{f} = 0 \\ \operatorname{grad}f_0 + \operatorname{rot}\vec{f} = 0 \end{array}\right\}.$$

For the operator D an analogue of the Leibniz rule holds (see, e.g., [8: p. 24] or [16: p. 63]). Let $f, g \in C^1(\Omega; \mathbb{H}(\mathbb{C}))$. Then

$$D[f \cdot g] = D[f] \cdot g + \bar{f} \cdot D[g] + 2(\operatorname{Sc}(fD))[g]$$

where

$$(\operatorname{Sc}(fD))[g] = -\left(f_1 \frac{\partial}{\partial x} + f_2 \frac{\partial}{\partial y} + f_3 \frac{\partial}{\partial z}\right)g.$$

Note that if $f \equiv f_0$ $(\operatorname{Vec}(f) \equiv 0)$, then

$$D[f_0 \cdot g] = \operatorname{grad}f_0 \cdot g + f_0 \cdot D[g]. \tag{5}$$

3. Relationship between Maxwell's equations and quaternionic operators

For the sake of simplicity we consider the case of a medium with constant permeability μ. The case of μ depending on spatial coordinates can be studied by analogy and requires only some more quite obvious calculations. Thus, we have the Maxwell equations in the form

$$\operatorname{rot}\overrightarrow{E}(\vec{r}) = -i\omega\mu\overrightarrow{H}(\vec{r}) \tag{6}$$

$$\operatorname{rot}\overrightarrow{H}(\vec{r}) = i\omega\varepsilon(\vec{r})\overrightarrow{E}(\vec{r}). \tag{7}$$

Applying the divergence to (6) - (7) we obtain the additional pair of equations

$$\left.\begin{array}{l} \operatorname{div}\overrightarrow{H}(\vec{r}) = 0 \\[2mm] \operatorname{div}\overrightarrow{E}(\vec{r}) = -\left\langle \dfrac{\operatorname{grad}\varepsilon(\vec{r})}{\varepsilon(\vec{r})}, \overrightarrow{E}(\vec{r}) \right\rangle \end{array}\right\} . \tag{8}$$

Let us introduce the auxiliary notations

$$\overrightarrow{\mathcal{E}}(\vec{r}) = \frac{1}{i\omega\varepsilon(\vec{r})}\overrightarrow{E}(\vec{r})$$

$$\overrightarrow{\mathcal{H}}(\vec{r}) = \frac{1}{\omega}\sqrt{\frac{\mu}{\varepsilon^3(\vec{r})}}\overrightarrow{H}(\vec{r}).$$

We will need also the other pair of vectors

$$\left.\begin{array}{l} \overrightarrow{\Phi} = \overrightarrow{\mathcal{E}} + \overrightarrow{\mathcal{H}} \\[2mm] \overrightarrow{\Psi} = -\overrightarrow{\mathcal{E}} + \overrightarrow{\mathcal{H}} \end{array}\right\} .$$

Note that in the mks system, as $\overrightarrow{E}$ and $\overrightarrow{H}$ are measured in V/m and A/m, respectively, these new magnitudes both are measured in V^2/A.

Applying the operator D to $\overrightarrow{\Phi}$ and using (5) we obtain the equalities

$$\begin{aligned} D\overrightarrow{\Phi} &= D\left(\frac{1}{i\omega\varepsilon}\overrightarrow{E} + \frac{1}{\omega}\sqrt{\frac{\mu}{\varepsilon^3}}\overrightarrow{H} \right) \\[2mm] &= -\frac{\operatorname{grad}\varepsilon}{i\omega\varepsilon^2}\cdot\overrightarrow{E} + \frac{1}{i\omega\varepsilon}D\overrightarrow{E} - \frac{3\sqrt{\mu}}{2\omega}\frac{\operatorname{grad}\varepsilon}{\varepsilon^{5/2}}\cdot\overrightarrow{H} + \frac{1}{\omega}\sqrt{\frac{\mu}{\varepsilon^3}}D\overrightarrow{H}. \end{aligned} \tag{9}$$

From (6) - (8) we have

$$D\overrightarrow{E} = \left\langle \frac{\operatorname{grad}\varepsilon}{\varepsilon}, \overrightarrow{E} \right\rangle - i\omega\mu\overrightarrow{H}$$

$$D\overrightarrow{H} = i\omega\varepsilon\overrightarrow{E}.$$

Thus, continuing (9) we obtain

$$
\begin{aligned}
D\overrightarrow{\Phi} &= -\frac{\operatorname{grad}\varepsilon}{i\omega\varepsilon^2}\cdot\overrightarrow{E} + \Big\langle\frac{\operatorname{grad}\varepsilon}{i\omega\varepsilon^2},\overrightarrow{E}\Big\rangle - \frac{\mu}{\varepsilon}\overrightarrow{H} - \frac{3\sqrt{\mu}}{2\omega}\frac{\operatorname{grad}\varepsilon}{\varepsilon^{5/2}}\cdot\overrightarrow{H} + i\sqrt{\frac{\mu}{\varepsilon}}\overrightarrow{E} \\
&= -\alpha\Big(\frac{1}{i\omega\varepsilon}\overrightarrow{E} + \frac{1}{\omega}\sqrt{\frac{\mu}{\varepsilon^3}}\overrightarrow{H} - \frac{i\operatorname{grad}\varepsilon}{\omega^2\mu^{1/2}\varepsilon^{5/2}}\cdot\overrightarrow{E} \\
&\quad + \frac{i}{\omega^2\mu^{1/2}}\Big\langle\frac{\operatorname{grad}\varepsilon}{\varepsilon^{5/2}},\overrightarrow{E}\Big\rangle + \frac{3}{2\omega^2}\frac{\operatorname{grad}\varepsilon}{\varepsilon^3}\cdot\overrightarrow{H}\Big)
\end{aligned}
$$

where α is the wave number: $\alpha = \omega\sqrt{\varepsilon\mu}$.

Finally, for $\overrightarrow{\Phi}$ we obtain the equation

$$
D\overrightarrow{\Phi} + \alpha\overrightarrow{\Phi} = -\alpha(\vec{g}\cdot\vec{\mathcal{E}} - \langle\vec{g},\vec{\mathcal{E}}\rangle + \tfrac{3}{2}\vec{g}\cdot\overrightarrow{\mathcal{H}}) \tag{10}
$$

where $\vec{g} = \frac{\operatorname{grad}\varepsilon}{\omega\mu^{1/2}\varepsilon^{3/2}}$. In a similar way the equation for $\overrightarrow{\Psi}$ is obtained as

$$
D\overrightarrow{\Psi} - \alpha\overrightarrow{\Psi} = \alpha(\vec{g}\cdot\vec{\mathcal{E}} - \langle\vec{g},\vec{\mathcal{E}}\rangle - \tfrac{3}{2}\vec{g}\cdot\overrightarrow{\mathcal{H}}). \tag{11}
$$

Let us notice that the quaternionic equation

$$
Df(\vec{r}) + \alpha(\vec{r})f(\vec{r}) = 0
$$

under some additional conditions over α can be solved exactly [14], and in the case of a slowly changing medium the expressions on the right-hand side in (10) and (11) can be neglected. We will use these facts in the next section for solving Maxwell's equations in the one-dimensional situation.

4. One-dimensional electromagnetic waves

In this section we will consider the propagation process depending only on x. Maxwell's equations (6) - (7) take the form $E_1 = H_1 = 0$ and

$$
\frac{\partial E_3}{\partial x} = i\omega\mu H_2, \quad \frac{\partial H_3}{\partial x} = -i\omega\varepsilon E_2, \quad \frac{\partial E_2}{\partial x} = -i\omega\mu H_3, \quad \frac{\partial H_2}{\partial x} = i\omega\varepsilon E_3.
$$

Let us assume that ε is a smooth complex function and

$$
\frac{\partial\varepsilon}{\partial x} = 0 \qquad \text{for } x \in (-\infty, 0)\cup(d, \infty)
$$

where d is a positive constant. As before μ is constant and we will suppose that the medium is changing slowly, that is the dimensionless expression $\frac{|v'(x)|}{\omega}$ is a very small number (usually it is supposed to be much less than one; see, e.g., [1]). Here v denotes the propagation speed.

We remind the relation $v = \frac{1}{\sqrt{\varepsilon\mu}}$ between this magnitude and the electromagnetic characteristics of the medium. Thus, $v' = -\frac{\varepsilon'}{2\mu^{1/2}\varepsilon^{3/2}}$ and the above mentioned condition on the slow changing medium reads as

$$\left| \frac{\varepsilon'}{\omega\mu^{1/2}\varepsilon^{3/2}} \right| \ll 2.$$

The dimensionless expression on the left-hand side is precisely $|\vec{g}|$, and assuming that it is sufficiently small we can neglect the expressions on the right-hand sides of (10) - (11). Then we have the equations

$$\left(i_1 \frac{\partial}{\partial x} + \alpha(x) \right) \vec{\Phi}(x) = 0 \tag{12}$$

$$\left(i_1 \frac{\partial}{\partial x} - \alpha(x) \right) \vec{\Psi}(x) = 0. \tag{13}$$

for $\vec{\Phi}$ and $\vec{\Psi}$.

Let us consider the equation

$$\left(i_1 \frac{\partial}{\partial x} + \alpha(x) \right) f(x) = 0 \tag{14}$$

where f is a complex quaternionic function. The solution of this equation was obtained in [14] and has the form

$$f(x) = \frac{1}{2}((1 + ii_1)e^{-i\mathcal{A}} \cdot A + (1 - ii_1)e^{i\mathcal{A}} \cdot B) \tag{15}$$

where $\mathcal{A}$ is an antiderivative of α, A and B are arbitrary constant complex quaternions. The complex quaternions $(1 \pm ii_1)$ represent the simplest example of zero divisors. Note that the solutions of (12) - (13) must be purely vectorial. Thus, we have the two additional conditions

$$\mathrm{Sc}((1 + ii_1)A) = \mathrm{Sc}((1 - ii_1)B) = 0$$

which are equivalent to the equalities

$$\left. \begin{array}{l} A_0 = iA_1 \\ B_0 = -iB_1 \end{array} \right\}.$$

Under these conditions we obtain

$$\left. \begin{array}{l} (1 + ii_1)A = a^+(i_2 + ii_3) \\ (1 - ii_1)B = b^+(i_2 - ii_3) \end{array} \right\}$$

where $a^+ = A_2 - iA_3$ and $b^+ = B_2 + iB_3$. Thus, we obtain the solution of (12) as

$$\vec{\Phi}(x) = \frac{1}{2}\left(a^+ e^{-i\mathcal{A}}(i_2 + ii_3) + b^+ e^{i\mathcal{A}}(i_2 - ii_3) \right). \tag{16}$$

In a similar way we obtain the solution of (13) as

$$\vec{\Psi}(x) = \frac{1}{2}\left(a^- e^{i\mathcal{A}}(i_2 + ii_3) + b^- e^{-i\mathcal{A}}(i_2 - ii_3) \right). \tag{17}$$

We proved the following assertion.

Proposition 1. *Let $a^{\pm}$ and $b^{\pm}$ be arbitrary complex numbers and $\mathcal{A}$ an antiderivative of α. Then the functions (16) and (17) are solutions of (12) and (13), respectively.*

Now, we obtain the representations for vectors of the electromagnetic field as

$$
\begin{aligned}
\overrightarrow{E}(x) &= \frac{i\omega\varepsilon(x)}{2}\left(\overrightarrow{\Phi}(x) - \overrightarrow{\Psi}(x)\right) \\
&= \frac{i\omega\varepsilon(x)}{4}\left(\left(a^{+}e^{-i\mathcal{A}(x)} - a^{-}e^{i\mathcal{A}(x)}\right)(i_2 + ii_3)\right. \\
&\quad \left. + \left(b^{+}e^{i\mathcal{A}(x)} - b^{-}e^{-i\mathcal{A}(x)}\right)(i_2 - ii_3)\right)
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
\overrightarrow{H}(x) &= \frac{\omega\varepsilon^{3/2}(x)}{2\mu^{1/2}}\left(\overrightarrow{\Phi}(x) + \overrightarrow{\Psi}(x)\right) \\
&= \frac{\omega\varepsilon^{3/2}(x)}{4\mu^{1/2}}\left(\left(a^{+}e^{-i\mathcal{A}(x)} + a^{-}e^{i\mathcal{A}(x)}\right)(i_2 + ii_3)\right. \\
&\quad \left. + \left(b^{+}e^{i\mathcal{A}(x)} + b^{-}e^{-i\mathcal{A}(x)}\right)(i_2 - ii_3)\right).
\end{aligned}
\tag{19}
$$

The results of [14] also allow us to construct a fundamental solution of the operator $i_1\frac{\partial}{\partial x} + \alpha(x)$ and therefore to obtain its right inverse operator. Namely, the distribution

$$
U(x) = \frac{1}{2i}(1 - \operatorname{sgn} x \cdot ii_1)e^{\operatorname{sgn} x \cdot i\mathcal{A}(x)}
$$

satisfies the equation

$$
\left(i_1\frac{\partial}{\partial x} + \alpha(x)\right)U(x) = \delta(x).
$$

Then the integral operator

$$
lf(x) = \int_{-\infty}^{\infty} U(x - \xi)f(\xi)\,d\xi
$$

is the right inverse for the operator $i_1\frac{\partial}{\partial x} + \alpha(x)$ in a suitable functional space (for example, in the Sobolev space W_2^1). Thus, the general solution of the inhomogeneous equation

$$
\left(i_1\frac{\partial}{\partial x} + \alpha(x)\right)p(x) = h(x)
\tag{20}
$$

has the form

$$
p(x) = f(x) + l[h](x),
$$

where f is defined by (15).

Note that in the one-dimensional situation equation (10) is reduced to (20) if $p = \overrightarrow{\Phi}$ and $h = -\alpha\left(\vec{g}\cdot\overrightarrow{\mathcal{E}} - \langle\vec{g},\overrightarrow{\mathcal{E}}\rangle + \frac{3}{2}\vec{g}\cdot\overrightarrow{\mathcal{H}}\right)$.

We want to estimate the norm of l in order to justify the neglect of terms on the right-hand side of (10). Obviously, a similar analysis can be done for (11). Let us notice

that under the accepted condition of finiteness of the support of $\varepsilon'(x)$ we also have that $\bar{g}$ is different from zero on the finite interval $[0, d]$. Thus, we are interested in some estimates for the operator

$$\widehat{l}h(x) = \int_0^d U(x - \xi)h(\xi)\,d\xi.$$

For this purpose we will use the following kind of module of a complex quaternion q:

$$|q|^2 := |\operatorname{Re} q|^2 + |\operatorname{Im} q|^2 = \sum_{k=0}^3 (|\operatorname{Re} q_k|^2 + |\operatorname{Im} q_k|^2)$$

where $\operatorname{Re} q_k$ and $\operatorname{Im} q_k$ are obviously the real and the imaginary parts of the component q_k. If $\operatorname{Im} q = 0$, then $|q|$ coincides with the usual quaternionic module and $|q|^2 = q \cdot \bar{q}$, but in general this is not true.

Let us prove the following important proposition.

Proposition 2. *Let p and q be complex quaternions. Then*

$$|p \cdot q| \le \sqrt{2}|p| \cdot |q|.$$

Proof. We denote $a = \operatorname{Re} p$, $b = \operatorname{Im} p$, $c = \operatorname{Re} q$ and $d = \operatorname{Im} q$. Then

$$\begin{aligned}
|p \cdot q|^2 &= |(a + ib)(c + id)|^2 \\
&= |ac - bd + i(bc + ad)|^2 \\
&= |ac - bd|^2 + |bc + ad|^2 \\
&\le 2(|ac|^2 + |bd|^2 + |bc|^2 + |ad|^2) \\
&= 2(|a|^2 + |b|^2)(|c|^2 + |d|^2) \\
&= 2|p|^2 \cdot |q|^2
\end{aligned}$$

and the statement is proved ∎

Using this proposition we obtain the estimate

$$|\widehat{l}h(x)| = \left| \int_0^d U(x - \xi)h(\xi)\,d\xi \right| \le \sqrt{2} \int_0^d |U(x - \xi)| \cdot |h(\xi)|\,d\xi.$$

It is easy to see that if $\operatorname{Im} \alpha \equiv 0$, then $|U(x - \xi)| = 1/\sqrt{2}$. In this case

$$|\widehat{l}h(x)| \le \int_0^d |h(\xi)|d\xi \le d \max_{\xi \in [0, d]} |h(\xi)|.$$

This inequality practically gives us the required estimate of the operator $\widehat{l}$ in the C-norm.

Remark 3. Of course, it would be desirable to have another kind of estimate which could give us a sufficient statement about when using the method some a priori given error would not be exceeded, but this is a general and completely open problem in the application of different asymptotic methods like, e.g., the Wentzel-Kramers-Brillouin (WKB) method: that the precision and the justification of the neglection of some small terms depends on the exact solution which is impossible to find. In such a situation the estimates like the one obtained here are the best results which in principle can be proved and give us the possibility to estimate the relative smallness of the neglected part compared with the main terms.

5. Conclusions

We proposed a new approach for obtaining asymptotic solutions of Maxwell's equations in inhomogeneous media. The Maxwell equations are reduced to a pair of quaternionic equations. In the case of a slowly changing medium the resulting quaternionic equations for some classes of coefficients α can be completely solved. Here we discussed only the situation when the coefficients depend on one variable. Although this situation is very important in a great number of applications, it is not unique such that permits a complete solution. In [14] a class of coefficients for which such a solution is obtained was described. Thus, the class of media for which this technique works is much larger than only a stratified medium .

Let us notice that the approach proposed in this article is essentially different from the asymptotic methods applied, for instance, for slowly changing stratified media (see, e.g., [6]). We reduce Maxwell's equations to a pair of quaternionic equations and on this stage neglect some small terms, thereby loosing the precision. But the resulting quaternionic equations we solve exactly in difference to other methods in which the precision is lost in both stages.

Finally, this article is only a beginning of the study of this new technique and its applications. We plan to use it in some concrete engineering problems like, e.g., the propagation of ionospheric waves and at the same time to enlarge the class of media for which the resulting quaternionic equations can be solved exactly.

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