Viscosity Solutions
of Nonlinear Systems of
Degenerated Elliptic Equations of Second Order

Liu Wei-an and Chen Hua

Abstract. In this paper, we discuss the viscosity solutions of Dirichlet problem for weakly coupled systems of fully nonlinear second order degenerated elliptic equations. We prove the existence, uniqueness and continuity of solutions by Perron's method combined with the technique of coupled solutions. Our results generalize those in [9] for the case of general quasi-monotonic systems.

Keywords: *viscosity solutions, systems of partial differential equations, fully nonlinear degenerated elliptic equations, Perron's method, coupled solution*

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1. Introduction

In this paper we are concerned with the boundary problem of weakly coupled systems of fully nonlinear second order degenerated elliptic equations of the form

$$F_i\big(x; u_j(x); u_i(x), Du_i(x), D^2u_i(x)\big) = 0 \quad (x \in \Omega) \ \Big\} \quad \big({}^{i,j=1,2}_{i \neq j}\big) \qquad (1.1)$$
$$u_i = 0 \quad (x \in \partial\Omega) \ \Big\}$$

where

- $u = (u_1, u_2) : \Omega \to \mathbb{R}^2$ is the unknown function
- $F = (F_1, F_2) : \Omega \times \mathbb{R}^2 \times \mathbb{R}^n \times S(n) \to \mathbb{R}^2$ is a given locally bounded function
- $S(n)$ is the set of real symmetric matrices of order n
- $Du_i = \big(\frac{\partial u_i}{\partial x_k}\big) \ (1 \leq k \leq n)$ is the gradient of u_i with respect to x
- $D^2u_i = \big[\frac{\partial^2 u_i}{\partial x_k \partial x_l}\big] \ (1 \leq k, l \leq n)$ is the Hessian matrix of u_i
- Ω is a bounded open subset of $\mathbb{R}^n$.

The function u_i is called the *principal argument* of F_i, while u_j $(j \neq i)$ is the *coupled argument* of F_i. Further, the function F_i is said to be *quasi-increasing* (or *decreasing*) if

it is increasing (or decreasing) with respect to the coupled argument u_j. Finally, system (1) is said to be

- *quasi-increasing* (or *decreasing*) if both functions F_1 and F_2 are quasi-increasing (or decreasing);
- *mixed quasi-monotonic* if one of the functions F_1 or F_2 is quasi-increasing and the other function is quasi-decreasing (see [19, 20, 23]).

In 1983 Crandall and Lions [4] introduced the concept of viscosity solution, which is regarded as a kind of weak solution for first order fully nonlinear partial differential equations. They also established global uniqueness and existence of viscosity solutions, and showed that a classical solution is always a viscosity solution. Later, in P.L. Lions [17], the result has been extended to second order problems. Furthermore, from [1, 2, 7, 8, 11], the theory of viscosity solutions has been completely developed and provided a powerful tool to deal with fully nonlinear partial differential equations. However, when we wish to study systems, even those with very special coupling structure, there are only a few results in this direction. In fact, if we regard a viscosity solution as a kind of comparison method, for scalar equations, the image of F is in a total-ordered space which can be compared easily; but the image of $(F_1, \ldots, F_m)$ for a system is in a partial-ordered space in which trichotomy law is invalid. So it is more difficult to deal with systems, e.g. in order to use the maximum principle method, quasi-decreasing assumptions have been introduced in [5, 9, 10, 12, 13]. However, the comparison does not hold for sub-solutions and super-solutions in the classical definition of non-quasi-decreasing systems.

In this paper, we introduce first a new definition of sub- and super-solutions for quasi-increasing systems or mixed-quasi-monotone systems making use of the technique of coupled solutions (the idea of couple solutions comes from [6, 20, 23]). Then, by Perron's method, existence and uniqueness of the viscosity solution for system (1) are proved, in which the system may be assumed to be quasi-increasing or mixed-quasi-monotone. We also prove that the viscosity solution of system (1) is continuous even if (F_1, F_2) is not continuous, thanks to the definition of sub- and super-solutions by locally bounded functions which comes from [9]. For non-quasi-monotone systems, we prove the same results by use of a fixed point theorem. Our results generalize those of Ishii and Koike [9] to the case of more general quasi-monotonic systems. Actually, in [9] they only discussed the case of quasi-decreasing systems (i.e. "quasi-monotone" in term of [9]).

For convenience, we discuss systems of two equations only. However, our results can be easily generalized to systems of arbitrary m equations by introducing some appropriate notations used, for example, in [5, 9] and [19: Section 6.1].

In Section 2, we shall first give some results about scalar equations which will be used later in dealing with systems. In sections 3 and 4, we apply Perron's method, combined with technique of coupled solutions, to study quasi-increasing and mixed quasi-monotone systems, respectively. In Section 5, we prove comparison theorems for quasi-increasing and mixed quasi-monotone systems. In Section 6, we prove existence, uniqueness and continuity of viscosity solutions for non-quasi-monotonic systems by Banach fixed point theorem. Finally, in Section 7, some examples are presented to show that our results are non-trivial extensions of those in Ishii and Koike [9].

2. Some results for scalar equations

For a function $v : \bar{\Omega} \to \mathbb{R}$, its upper and lower envelopes are defined by

$$
\begin{aligned}
v^*(x) &= \limsup_{\varepsilon \to 0} \left\{ v(y) : |x - y| < \varepsilon \text{ for } y \in \bar{\Omega} \right\} \\
v_*(x) &= \liminf_{\varepsilon \to 0} \left\{ v(y) : |x - y| < \varepsilon \text{ for } y \in \bar{\Omega} \right\}
\end{aligned}
\qquad (x \in \bar{\Omega}),
$$

respectively. It is clear from the definition that v^* and v_* are upper and lower semi-continuous on $\bar{\Omega}$, respectively, with values in $\mathbb{R} \cup \{\pm\infty\}$ and $v_* \leq v \leq v^*$ in $\bar{\Omega}$. By $\mathrm{USC}(\Omega)$ and $\mathrm{LSC}(\Omega)$ we denote the set of all upper and lower semi-continuous functions on Ω, respectively. It can be easily seen that if v is upper or lower semi-continuous at $x \in \Omega$, then $v(x) = v^*(x)$ or $v(x) = v_*(x)$, respectively.

For an upper or lower semi-continuous function $v : \bar{\Omega} \to \mathbb{R}$, its *super-jet* $J_\Omega^{2,+} v$ or *sub-jet* $J_\Omega^{2,-} v$ of second order at x on Ω is defined by

$$
J_\Omega^{2,+} v(x) = \left\{ (p, X) \in \mathbb{R}^n \times \mathbf{S}(n) \,\middle|\, v(x+h) \leq v(x) + \langle p, h \rangle + \tfrac{1}{2} \langle Xh, h \rangle + o(|h|^2) \, (h \to 0) \right\}
$$

and

$$
J_\Omega^{2,-} v(x) = \left\{ (p, X) \in \mathbb{R}^n \times \mathbf{S}(n) \,\middle|\, v(x+h) \geq v(x) + \langle p, h \rangle + \tfrac{1}{2} \langle Xh, h \rangle + o(|h|^2) \, (h \to 0) \right\},
$$

respectively. The "closure" of $J_\Omega^{2,+}$ is defined by

$$
\bar{J}_\Omega^{2,+} v(x) = \left\{ (p, X) \in \mathbb{R}^n \times \mathbf{S}(n) \,\middle|\,
\begin{array}{l}
\exists \, (x^k, p^k, X^k) \in \Omega \times \mathbb{R}^n \times \mathbf{S}(n) \\
\text{such that } (p^k, X^k) \in J^{2,+} v(x) \text{ and} \\
(x^k, v(x^k), p^k, X^k) \to (x, v(x), p, X) \, (k \to \infty)
\end{array}
\right\}.
$$

Similarly we can define the closure $\bar{J}_\Omega^{2,-}$ of $J_\Omega^{2,-}$.

We consider the Dirichlet problem for fully nonlinear second order partial differential equations

$$
\left.
\begin{aligned}
f(x, u(x), Du, D^2 u) &= 0 \quad \text{in } \Omega \\
u(x) &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\right\}
\qquad (2.1)
$$

where $u : \Omega \to \mathbb{R}$ is the unknown function, $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbf{S}(n) \to \mathbb{R}$ is a given function which is locally bounded, the others are the same as mentioned in Section 1.

The function f is said to be *proper* if

$$
f^*(x, r, p, X) \leq f_*(x, s, p, Y) \qquad \text{whenever } r \leq s \text{ and } Y \leq X
$$

where $r, s \in \mathbb{R}$, $x, p \in \mathbb{R}^n$, $X, Y \in \mathbf{S}(n)$, and $\mathbf{S}(n)$ is equipped with its usual order (see [3]).

Definition 2.1. Suppose that f is proper. A locally bounded function u is called a *viscosity sub-* or *super-solution* of $f = 0$ in Ω if for all $x \in \Omega$

$$f_*(x, u^*(x), p, X) \leq 0 \qquad \text{for all } (p, X) \in J_\Omega^{2,+} u^*(x)$$

or

$$f^*(x, u_*(x), q, Y) \geq 0 \qquad \text{for all } (q, Y) \in J_\Omega^{2,-} u_*(x),$$

respectively. A locally bounded function u is called a *viscosity solution* of $f = 0$ in Ω if it is both a viscosity sub- and super-solution of $f = 0$ in Ω.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be locally compact, $v \in \text{USC}(\Omega)$ (or $\hat{v} \in \text{LSC}(\Omega)$), $z \in \Omega$ and $(p, X) \in J_\Omega^{2,+} v(z)$ (or $(p, X) \in J_\Omega^{2,-} \hat{v}(z)$). Suppose also that $u^{(k)}$ is a sequence of upper or lower semi-continuous functions on Ω such that:*

   (i)   *There exists $x^{(k)} \in \Omega$ such that $(x^{(k)}, u^{(k)}(x^{(k)})) \to (z, v(z))$ (or $(z, \hat{v}(z))$).*

   (ii) *If $z^{(k)} \in \Omega$ and $z^{(k)} \to x \in \Omega$, then*

$$\limsup_{k \to \infty} u^{(k)}(z^{(k)}) \leq v(x) \qquad \left(or \ \liminf_{k \to \infty} u^{(k)}(z^{(k)}) \geq \hat{v}(x) \right).$$

Then there exists

$$\hat{x}^{(k)} \in \Omega \quad and \quad (p^{(k)}, X^{(k)}) \in J_\Omega^{2,+} u^{(k)}(\hat{x}^{(k)}) \quad \left(or \ (p^{(k)}, X^{(k)}) \in J_\Omega^{2,-} u^{(k)}(\hat{x}^{(k)}) \right)$$

such that

$$\left(\hat{x}^{(k)}, u^{(k)}(\hat{x}^{(k)}), p^{(k)}, X^{(k)} \right) \to (z, v(z), p, X) \ \left(or \ (z, \hat{v}(z), p, X) \right).$$

As applications of Proposition 2.2, the following results can be easily proved.

Proposition 2.3. *Suppose that f is proper. A locally bounded function u is a viscosity sub- or super-solution of $f = 0$ in Ω if and only if for all $x \in \Omega$*

$$f_*(x, u^*(x), p, X) \leq 0 \qquad for \ all \ (p, X) \in \bar{J}_\Omega^{2,+} u^*(x)$$

or

$$f^*(x, u_*(x), q, Y) \geq 0 \qquad for \ all \ (q, Y) \in \bar{J}_\Omega^{2,-} u_*(x).$$

Lemma 2.4. *Suppose that $\Omega \subset \mathbb{R}^n$ is locally compact and $f \in \text{LSC}(\Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n))$. Let*

$$S = \left\{ u \mid u \text{ is a viscosity sub-solution of } f = 0 \right\}$$
$$\bar{u} = \sup \left\{ u^* \mid u \in S \right\} \tag{2.2}$$

and assume that $\bar{u}(x) < \infty$ for $x \in \Omega$. Then $\bar{u}$ is a viscosity sub-solution of $f = 0$ in Ω.

Lemma 2.5. *Suppose that $\Omega \subset \mathbb{R}^n$ is locally compact and $f \in \text{USC}(\Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n))$. Let*

$$P = \left\{ u \mid u \text{ is a viscocity super-solution of } f = 0 \right\}$$
$$\bar{v} = \inf \left\{ u_* \mid u \in P \right\} \tag{2.3}$$

and assume that $-\bar{v}(x) < \infty$ for $x \in \Omega$. Then $\bar{v}$ is a viscosity super-solution of $f = 0$ in Ω.

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ be open and u be a viscosity sub- (or super-solution) of $f = 0$ in Ω. If u fails to be a super-solution (or sub-solution) at some point $\hat{x}$, then for any small $r > 0$ there is another viscosity sub-solution (or super-solution) w of $f = 0$ in Ω satisfying*

$$
\left.
\begin{aligned}
& w(x) \geq u(x) \ (x \in \Omega), \quad \sup_{\Omega}(w - u) > 0 \\
& \big(or \ w(x) \leq u(x) \ (x \in \Omega), \quad \sup_{\Omega}(w - u) < 0\big) \\
& w(x) = u(x) \ (x \in \Omega, |x - \hat{x}| \geq r)
\end{aligned}
\right\} .
$$

In fact, we can follow the ideas in [3, 9] to prove Proposition 2.3 and Lemma 2.4 directly. The proof of Lemma 2.6 is similar to those of corresponding results in [3], and Lemma 2.5 may be easily proved if we choose $v = \hat{v}$ in conditions (i) and (ii) of Proposition 2.2.

Using the lemmas above, it can be proved that there exists a viscosity solution to problem (2.1).

Theorem 2.7. *Let comparison hold for problem (2.1), i.e. if U is a sub- and V a super-solution of that problem, then $U^*(x) \leq V_*(x)$ for $x \in \bar{\Omega}$. Suppose that there exists a sub-solution U and a super-solution V of problem (2.1) as well satisfying the boundary condition $U_*(x) = V^*(x) = 0$ for $x \in \partial\Omega$. Then $\bar{u} \equiv \bar{v} \in C(\bar{\Omega})$, as defined by (2.2) and (2.3), is a unique viscosity solution of problem (2.1).*

Proposition 2.8 (Comparison). *Let Ω be a bounded open subset of $\mathbb{R}^n$, f be proper, locally bounded and satisfying the following conditions:*

(i) *There exists $\gamma > 0$ such that*

$$
\gamma(r - s) \leq f_*(x, r, p, X) - f^*(x, s, p, X)
$$

for $r \geq s$ and $(x, p, X) \in \bar{\Omega} \times \mathbb{R}^n \times S(n)$.

(ii) *There exists a continuous function $\omega : (0, \infty) \to [0, \infty)$ that satisfies $\omega(0^+) = 0$ such that*

$$
f^*\big(y, r, \alpha(x - y), Y\big) - f_*\big(x, r, \alpha(x - y), X\big) \leq \omega\big(\alpha|x - y|^2 + |x - y|\big)
$$

whenever $x, y \in \Omega$, $r \in \mathbb{R}$ and $X, Y \in S(n)$ which satisfy

$$
-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} . \tag{2.4}
$$

Let u and v be viscosity sub- and super-solutions of $f = 0$ in Ω, respectively, and $u^* \leq v_*$ on $\partial\Omega$. Then $u^* \leq v_*$ on $\bar{\Omega}$.

Lemma 2.9 (see [3: Theorem 3.1]). *Let Ω be a subset of $\mathbb{R}^n$, $u \in \text{USC}(\Omega)$ and $v \in \text{LSC}(\Omega)$, and*

$$
M_\alpha = \sup_{\Omega} \big(u(x) - v(y) - \tfrac{\alpha}{2}|x - y|^2\big) \qquad (\alpha > 0).
$$

Let $M_\alpha < \infty$ for large α and (x_α, y_α) be such that

$$\lim_{\alpha \to \infty} \left(M_\alpha - \left(u(x_\alpha) - v(y_\alpha) - \tfrac{\alpha}{2}|x_\alpha - y_\alpha|^2 \right) \right) = 0.$$

Then:

   (i) $\lim_{\alpha \to \infty} \alpha |x_\alpha - y_\alpha|^2 = 0$.

   (ii) $\lim_{\alpha \to \infty} M_\alpha = u(\hat{x}) - v(\hat{x}) = \sup_\Omega (u(x) - v(x))$ *whenever* $\hat{x} \in \Omega$ *is a limit point of x_α as $\alpha \to \infty$.*

Lemma 2.10 (see [3]). *Let Ω be a locally compact subset of $\mathbb{R}^n$, $u \in \mathrm{USC}(\Omega)$ and $v \in \mathrm{LSC}(\Omega)$. Suppose that $(\hat{x}, \hat{y})$ is a local maximum of the function*

$$u(x) - v(y) - \tfrac{\alpha}{2}|x - y|^2$$

relative to Ω. Then for each $\varepsilon > 0$ there exist $X, Y \in S(n)$ satisfing (2.4) and

$$\left. \begin{array}{l} (\alpha(\hat{x} - \hat{y}), X) \in \bar{J}^{2,+}_\Omega u(\hat{x}) \\ (\alpha(\hat{x} - \hat{y}), Y) \in \bar{J}^{2,-}_\Omega v(\hat{y}) \end{array} \right\}.$$

3. Viscosity solutions of quasi-increasing systems

We denote

$$\left. \begin{array}{l} u = (u_1, u_2) \\ v = (v_1, v_2) \end{array} \right\} \quad \text{and} \quad \left. \begin{array}{l} U = (U_1, U_2) \\ V = (V_1, V_2) \end{array} \right\}$$

where u_i, v_i $(i = 1, 2)$ are functions from Ω into $\mathbb{R}$. The notation $u \leq v$ means $u_i \leq v_i$ $(i = 1, 2)$. Also, denote

$$u^* = (u_1^*, u_2^*)$$
$$u_* = (u_{1*}, u_{2*}).$$

The pair (F_1, F_2) is said to be *proper* if both

$$F_1(x; , u_2; u_1, p_1, X_1)$$
$$F_2(x; u_1; u_2, p_2, X_2)$$

are proper with respect to the principal argument.

Definition 3.1. Let

$$\Big(F_1(x; , u_2; u_1, p_1, X_1), F_2(x; u_1; u_2, p_2, X_2) \Big)$$

be proper on $\Omega \times \mathbb{R}^2 \times \mathbb{R}^n \times S(n)$ and quasi-increasing. We call the pair (U, V) a *couple of viscosity sub- and super-solutions* of the Dirichlet problem (1.1) if U and V are locally bounded in Ω and such that

$$F_{i*}(x; V_{j*}; U_i^*, p_i, X_i) \leq 0 \qquad ((p_i, X_i) \in J^{2,+}_\Omega U_i^*(x))$$
$$F_i^*(x; U_j^*; V_{i*}, q_i, Y_i) \geq 0 \qquad ((q_1, Y_1) \in J^{2,-}_\Omega V_{1*}(x))$$

for all $x \in \Omega$ and $i, j = 1, 2$ with $j \neq i$ and

$$U_i^* \leq 0 \leq V_{i*}$$

for all $x \in \partial\Omega$ and $i = 1, 2$. And U and V are called *viscosity sub-solution* and *super-solution* of problem (1.1), respectively.

Definition 3.2. Let

$$\left(F_1(x;, u_2; u_1, p_1, X_1), F_2(x; u_1; u_2, p_2, X_2) \right)$$

be proper on $\Omega \times \mathbb{R}^2 \times \mathbb{R}^n \times \mathbf{S}(n)$ and quasi-increasing. The pair (U, V) is called a *viscosity coupled solution* of the Dirichlet problem (1.1) if (U, V) and (V, U) are both couples of viscosity sub- and super-solutions of (1.1).

Definition 3.3. Let F_1 and F_2 be proper and quasi-increasing. The function u is called a *viscosity solution* of problem (1.1) if (u, u) is a viscosity coupled solution of (1.1).

We define

$$S = S_1 \times S_2 = \left\{ U = (U_1, U_2) \,\middle|\, \begin{array}{l} \exists\, V \text{ such that } (U, V) \text{ is a couple of} \\ \text{viscosity sub- and super-solutions of (1.1)} \end{array} \right\} \quad (3.1)$$

$$P = P_1 \times P_2 = \left\{ V = (V_1, V_2) \,\middle|\, \begin{array}{l} \exists\, U \text{ such that } (U, V) \text{ is a couple of} \\ \text{viscosity sub- and super-solutions of (1.1)} \end{array} \right\} \quad (3.2)$$

and

$$\left. \begin{array}{l} \bar{u} = (\bar{u}_1, \bar{u}_2) \\ \bar{v} = (\bar{v}_1, \bar{v}_2) \end{array} \right\} \quad \text{where} \quad \left. \begin{array}{l} \bar{u}_i(x) = \sup\left\{ U_i(x) \,\middle|\, U \in S \right\} \\ \bar{v}_i(x) = \inf\left\{ V_i(x) \,\middle|\, V \in P \right\} \end{array} \right\} \quad (i = 1, 2). \quad (3.3)$$

Lemma 3.4. *Assume that F_1 and F_2 are proper and quasi-increasing, and the sets S, P are non-empty. If*

$$\left. \begin{array}{l} \bar{u}_i^*(x) < +\infty \\ \bar{v}_{i*}(x) > -\infty \end{array} \right\} \quad (x \in \bar{\Omega};\, i = 1, 2),$$

then $\bar{u} \in S$ and $\bar{v} \in P$, i.e. $(\bar{u}, \bar{v})$ is a couple of viscosity sub- and super-solutions of problem (1.1).

Proof. Suppose that $z \in \Omega$ and $(p_1, X_1) \in J_\Omega^{2,+} \bar{u}_1^*(z)$. From the definition of $\bar{u}^*$, it is known that there is a sequence $\{u_1^{(k)}\} \subset S_1$ (we may suppose that all components of the sequence are upper semi-continuous) and a sequence of points $\{x^{(k)}\} \subset \Omega$ such that $(x^{(k)}, u_1^{(k)}) \to (z, \bar{u}_1^*)$. By virtue of upper semi-continuity of $\bar{u}_1^*(z)$, we have

$$\limsup_{k \to \infty} u_1^{(k)}(y^{(k)}) \leq \bar{u}_1^*(z)$$

for every sequence $(y^{(k)}, u_1^{(k)}(y^{(k)}))$. From Proposition 2.2, there exist

$$\hat{x}^{(k)} \in \Omega \quad \text{and} \quad (p_1^{(k)}, X_1^{(k)}) \in J_\Omega^{2,+} u_1^{(k)}(\hat{x}^{(k)})$$

such that

$$(\hat{x}^{(k)}, u_1^{(k)}(\hat{x}^{(k)}), p_1^{(k)}, X_1^{(k)}) \to (z, \bar{u}_1^*(z), p_1, X_1).$$

From the definitions of $\bar{u}_1$ and $u_1^{(k)}$, there is $\bar{v}_2$ related $\bar{u}_1$ and $v_2^{(k)}$ related $u_1^{(k)}$. By virtue of lower semi-continuity of $\bar{v}_{2*}$, we know that

$$\liminf_{k \to \infty} v_2^{(k)}(\hat{x}^{(k)}) \geq \bar{v}_{2*}(z).$$

Due to the quasi-monotonicity and semi-continuity of F_1,

$$F_{1*}\left(z; \bar{v}_{2*}(z); \bar{u}_1^*(z), p_1, X_1\right) \leq \liminf_{k \to \infty} F_{1*}\left(\hat{x}^{(k)}; v_2^{(k)}(\hat{x}^{(k)}); u_1^{(k)}(\hat{x}^{(k)}), p_1^{(k)}, X_1^{(k)}\right)$$
$$\leq 0.$$

Similarly, for $(q_2, Y_2) \in J_\Omega^{2,-} \bar{v}_{2*}(z)$, there exists a sequence

$$\left\{(y^{(j)}, v_2^{(j)}, q_2^{(j)}, Y_2^{(j)})\right\} \subset \Omega \times P_2 \times \mathbb{R}^n \times \mathbf{S}(n)$$

such that

$$F_2^*\left(z; \bar{u}_1^*(z); \bar{v}_{2*}(z), q_2, Y_2\right) \geq \limsup_{j \to \infty} F_2^*\left(y^{(j)}; u_1^{(j)}(y^{(j)}); v_2^{(j)}(y^{(j)}), q_2^{(j)}, Y_2^{(j)}\right)$$
$$\geq 0.$$

In the same way, we can also prove that, for all $z \in \Omega$,

$$F_{2*}(z; \bar{v}_{1*}(z); \bar{u}_2^*(z), p_2, X_2) \leq 0 \quad ((p_2, X_2) \in J_\Omega^{2,+} \bar{u}_2^*(z))$$
$$F_1^*(z; \bar{u}_2^*(z); \bar{v}_{1*}(z), q_1, Y_1) \geq 0 \quad ((q_1, Y_1) \in J_\Omega^{2,-} \bar{v}_{1*}(z)).$$

The proof is completed ∎

Lemma 3.5. *Let F_1 and F_2 be proper and quasi-increasing. Assume that the sets S, P are both non-empty. If $(u, v) \in S \times P$ while $(v, u) \notin S \times P$, then there is another $(U, V) \in S \times P$ such that $U \geq u$, $V \leq v$ and $(U, V) \neq (u, v)$.*

Proof. If $(u, v) \in S \times P$ while u_1 is not a viscosity super-solution of (3.1), then we know from Lemma 2.6 [3: Lemma 4.4] that there exists another viscosity sub-solution of (3.1) $\hat{u}_1$ such that $\hat{u}_1(x) \geq u_1(x)$ and $\hat{u}_1(x) \neq u_1(x)$. Since F_2 is quasi-increasing,

$$F_2^*\left(x; \hat{u}_1^*; v_{2*}, q_2, Y_2\right) \geq F_2^*\left(x; u_1^*; v_{2*}, q_2, Y_2\right) \geq 0$$

for all $x \in \Omega$ and $(q_2, Y_2) \in J_\Omega^{2,-} v_{2*}(x)$. Thus, there exist $U = (\hat{u}_1, u_2)$ and $V = v$ such that Lemma 3.5 holds. The other situations can be proved similarly ∎

Theorem 3.6. *Suppose that F_1 and F_2 are both proper and quasi-increasing, and the sets S and P are both non-empty. If*

$$\left.\begin{array}{l} \bar{u}_i^*(x) < \infty \\ \bar{v}_{i*}(x) > -\infty \end{array}\right\} \quad (x \in \bar{\Omega};\ i = 1, 2),$$

then $\bar{u} \in S$ and $\bar{v} \in P$, i.e. $(\bar{u}, \bar{v})$ is a viscosity coupled solution.

Proof. From Lemma 3.4, we know that $(\bar{u}, \bar{v}) \in S \times P$. If $(\bar{v}, \bar{u}) \notin S \times P$, we know from Lemma 3.5 that there exists another $(U, V) \in S \times P$ such that $\bar{u}^* \leq U$ and $\bar{v}_* \geq V$, and $(\bar{u}^*, \bar{v}_*) \neq (U, V)$. This is a contradiction to the definition of $(\bar{u}, \bar{v})$. Hence, $(\bar{v}, \bar{u}) \in S \times P$ ∎

However, a viscosity coupled solution is not a real viscosity solution. Furthermore, we have the following theorem.

Theorem 3.7. *Under the conditions of Theorem 3.6, if the comparison for viscosity sub- and super-solutions holds, i.e.*

$$u^*(x) \leq v_*(x) \quad (x \in \Omega) \qquad \text{whenever } (u, v) \in S \times P,$$

then $\bar{u} \equiv \bar{v} \in C(\Omega)$ is a unique viscosity solution of problem (1.1).

Proof. From the comparison we know that S is bounded above and every $v \in P$ is an upper bound of S. Also, P is bounded below and every $u \in S$ is a lower bound of P. From the definition of viscosity coupled solution, we know that $(\bar{u}, \bar{v}) \in S \times P$ and $(\bar{v}, \bar{u}) \in S \times P$. We have the inclusions

$$(\bar{u}, \bar{v}) \in S \times P \quad \Longrightarrow \quad \bar{u} \leq \bar{u}^* \leq \bar{v}_* \leq \bar{v}$$
$$(\bar{v}, \bar{u}) \in S \times P \quad \Longrightarrow \quad \bar{v} \leq \bar{v}^* \leq \bar{u}_* \leq \bar{u}.$$

This implies that $\bar{u} \equiv \bar{v} \in C(\Omega)$. The uniqueness can be proved in the same way ∎

4. Viscosity solutions of mixed quasi-monotone systems

First, we introduce the sub- and super-solutions of the system.

Definition 4.1. Let

$$\Big(F_1(x; , u_2; u_1, p_1, X_1),\ F_2(x; u_1; u_2, p_2, X_2)\Big)$$

be proper on $\Omega \times \mathbb{R}^2 \times \mathbb{R}^n \times \mathbf{S}(n)$ and be mixed quasi-monotonic (we might always suppose as well that F_1 is quasi-decreasing and F_2 quasi-increasing when we talk about a mixed quasi-monotone system). Then

$$\left.\begin{array}{l} U(x) = \big(U_1(x), U_2(x)\big) \\ V(x) = \big(V_1(x), V_2(x)\big) \end{array}\right\}$$

are called a *couple of viscosity sub- and super-solutions* of the Dirichlet problem (1.1) if they are locally bounded in Ω and

$$F_{1*}(x; U_2^*; U_1^*, p_1, X_1) \leq 0 \qquad \forall\, (p_1, X_1) \in J_\Omega^{2,+} U_1^*(x)$$

$$F_{2*}(x; V_{1*}; U_2^*, p_2, X_2) \leq 0 \qquad \forall\, (p_2, X_2) \in J_\Omega^{2,+} U_2^*(x)$$

$$F_1^*(x; V_{2*}; V_{1*}, q_1, Y_1) \geq 0 \qquad \forall\, (q_1, Y_1) \in J_\Omega^{2,-} V_{1*}(x)$$

$$F_2^*(x; U_1^*; V_{2*}, q_2, Y_2) \geq 0 \qquad \forall\, (q_2, y_2) \in J_\Omega^{2,-} V_{2*}(x)$$

for all $x \in \Omega$ and

$$U_i^* \leq 0 \leq V_{i*} \quad (i = 1, 2)$$

for all $x \in \partial\Omega$.

Similarly to Definitions 3.2 and 3.3, we can define *viscosity coupled solutions* and *viscosity solutions of mixed quasi-monotonic systems*, as well as S, P, $\bar{u}_i$ and $\bar{v}_i$ (see (3.1) - (3.3)).

Lemma 4.2. Assume that (F_1, F_2) is proper and mixed quasi-monotone, and the sets S, P are non-empty. If

$$\left.\begin{array}{l} \bar{u}_i^*(x) < +\infty \\ \bar{v}_{i*}(x) > -\infty \end{array}\right\} \quad (x \in \bar{\Omega};\ i = 1, 2),$$

then $\bar{u} \in S$ and $\bar{v} \in P$, i.e. $(\bar{u}, \bar{v})$ is a couple of viscosity sub- and super-solutions of problem (1.1) as well.

Proof. Suppose that $z \in \Omega$ and $(q_1, Y_1) \in J_\Omega^{2,-} \bar{v}_{1*}(z)$. From the definition of $\bar{v}_*$, we know that there exists a sequence $v_1^{(k)} \in P_1$ (we may as well suppose that they are lower semi-continuous) and points $x^{(k)} \in \Omega$ such that $(x^{(k)}, v_1^{(k)}) \to (z, \bar{v}_{1*})$. By virtue of lower semi-continuity of $\bar{v}_{1*}(z)$, we see that

$$\liminf_{k \to \infty} v_1^{(k)}(y^{(k)}) \geq \bar{v}_{1*}(z)$$

for every sequence $(y^{(k)}, v_1^{(k)}(y^{(k)}))$. From Proposition 2.2, there exist

$$\hat{x}^{(k)} \in \Omega \qquad \text{and} \qquad (q_1^{(k)}, Y_1^{(k)}) \in J_\Omega^{2,-} v_1^{(k)}(\hat{x}^{(k)})$$

such that

$$\left(\hat{x}^{(k)}, v_1^{(k)}(\hat{x}^{(k)}), q_1^{(k)}, Y_1^{(k)}\right) \to (z, \bar{v}_{1*}(z), q_1, Y_1).$$

From the definition of $\bar{v}_*$, it follows that there exists a sequence of lower semi-continuous functions $v_2^{(k)} \in P_2$ such that $v_2^{(k)}(\hat{x}^{(k)}) \to \bar{v}_{2*}(z)$. Thus, we know that

$$\left(\hat{x}^{(k)}, v_1^{(k)}(\hat{x}^{(k)}), v_2^{(k)}(\hat{x}^{(k)}), q_1^{(k)}, Y_1^{(k)}\right) \to \left(z, \bar{v}_{1*}(z), \bar{v}_{2*}(z), q_1, Y_1\right).$$

By using upper semi-continuity of F_1^* and the fact that $(v_1^{(k)}, v_2^{(k)}) \in P$,

$$F_1^* \left(z; \bar{v}_{2*}(z); \bar{v}_{1*}(z), q_1, Y_1 \right) \geq \limsup_{k \to \infty} F_1 \left(\hat{x}^{(k)}; v_2^{(k)}(\hat{x}^{(k)}); v_1^{(k)}(\hat{x}^{(k)}), q_1^{(k)}, Y_1^{(k)} \right)$$

$$\geq 0.$$

Similarly, we can show that, for all $x \in \Omega$,

$$F_{1*} \left(z; \bar{u}_2^*(z); \bar{u}_1^*(z), p_1, X_1 \right) \leq 0 \qquad \forall (p_1, X_1) \in J_\Omega^{2,+} \bar{u}_1^*(z)$$

$$F_{2*} \left(z; \bar{v}_{1*}(z); \bar{u}_2^*(z), p_2, X_2 \right) \leq 0 \qquad \forall (p_2, X_2) \in J_\Omega^{2,+} \bar{u}_2^*(z)$$

$$F_2^* \left(z; \bar{u}_1^*(z); \bar{v}_{2*}(z), q_2, Y_2 \right) \geq 0 \qquad \forall (q_2, Y_2) \in J_\Omega^{2,-} \bar{v}_{2*}(z).$$

Thus, we have proved that $(\bar{u}, \bar{v}) \in S \times P$ ∎

In the same way to prove Lemma 3.5, we can prove the following lemma.

Lemma 4.3. *Let (F_1, F_2) be proper and mixed quasi-monotone, and the sets S, P be both non-empty. If $(u, v) \in S \times P$ while $(v, u) \notin S \times P$, then there is another $(U, V) \in S \times P$ such that $U \geq u$, $V \leq v$ and $(U, V) \neq (u, v)$.*

As in the proof of Theorem 3.6, we can use Lemmas 4.2 and 4.3 to prove the following

Theorem 4.4. *Suppose that (F_1, F_2) is proper and mixed quasi-monotone, and the sets S and P are both non-empty. If*

$$\left. \begin{array}{l} \bar{u}_i^*(x) < \infty \\[4pt] \bar{v}_{i*}(x) > -\infty \end{array} \right\} \qquad (x \in \bar{\Omega}; i = 1, 2),$$

then $(\bar{u}, \bar{v})$ is a viscosity coupled solution of problem (1.1).

Corresponding Theorem 3.7, we have

Theorem 4.5. *Under the conditions of Theorem 4.4, if the comparison for viscosity sub- and super-solutions holds, then $\bar{u} \equiv \bar{v} \in C(\Omega)$ is the unique viscosity solution of problem (1.1).*

Remark. As for quasi-decreasing system, the same results can be obtained by use of our method. In addition, using our method, it can be shown without the comparison that problem (1.1) has two viscosity solutions $\bar{u}$ and $\bar{v}$, which are not necessarily identical and continuous. With the comparison, it can be proved that problem (1.1) has a unique continuous viscosity solution.

5. Comparison

First, we consider the following conditions (A1) - (A4).

(A1) There exist $\gamma_i > 0$ $(i = 1, 2)$ such that

$$\gamma_i(r_i - s_i) \leq F_{i*}(x; u_j; r_i, p, X) - F_i^*(x; u_j; s_i, p, X) \qquad (A1.1)$$

or

$$\gamma_i(r_i - s_i) \leq F_{i*}(x; u_j; r_i, p, X) - F_{i*}(x; u_j; s_i, p, X) \qquad (A1.2)$$

or

$$\gamma_i(r_i - s_i) \leq F_i^*(x; u_j; r_i, p, X) - F_i^*(x; u_j; s_i, p, X) \qquad (A1.3)$$

for all $r_i \geq s_i$, $(x; u_j; p, X) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbf{S}(n)$ and $i, j = 1, 2$ with $j \neq i$.

(A2) There are modular functions $\omega_i : [0, \infty] \to [0, \infty]$ $(i = 1, 2)$ that satisfy $\omega_i(0^+) = 0$ such that

$$F_i^*\big(y; u_j; r_i, \alpha(x - y), Y\big) - F_{i*}\big(x; u_j; r_i, \alpha(x - y), X\big) \leq \omega_i\big(\alpha|x - y|^2 + |x - y|\big) \quad (A2.1)$$

or

$$F_{i*}\big(y; u_j; r_i, \alpha(x - y), Y\big) - F_{i*}\big(x; u_j; r_i, \alpha(x - y), X\big) \leq \omega_i\big(\alpha|x - y|^2 + |x - y|\big) \quad (A2.2)$$

or

$$F_i^*\big(y; u_j; r_i, \alpha(x - y), Y\big) - F_i^*\big(x; u_j; r_i, \alpha(x - y), X\big) \leq \omega_i\big(\alpha|x - y|^2 + |x - y|\big) \quad (A2.3)$$

whenever $x, y \in \Omega$, $r_i, u_j \in \mathbb{R}$ and $X, Y \in \mathbf{S}(n)$ which satisfy (2.11) for $i, j = 1, 2$ with $j \neq i$.

(A3) There are constants $L_i \geq 0$ $(i = 1, 2)$ such that

$$F_i^*(x; v_j; r_i, p, X) - F_{i*}(x; u_j; r_i, p), X) \leq L_i(v_j - u_j) \qquad (A3.1)$$

or

$$F_{i*}(x; v_j; r_i, p, X) - F_{i*}(x; u_j; r_i, p, X) \leq L_i(v_j - u_j) \qquad (A3.2)$$

or

$$F_i^*(x; v_j; r_i, p, X) - F_i^*(x; u_j; r_i, p, X) \leq L_i(v_j - u_j) \qquad (A3.3)$$

whenever $v_j, u_j \in \mathbb{R}$ with $v_j \geq u_j$, $r_i \in \mathbb{R}$ $(j = 1, 2, j \neq i)$, $x \in \Omega$, $p \in \mathbb{R}^n$ and $X \in \mathbf{S}(n)$.

(A4) The constants γ_i, L_i $(i = 1, 2)$ in (A1) and (A3) satisfy $\frac{L_1 L_2}{\gamma_1 \gamma_2} < 1$.

Remark 1. It is clear that the first inequalities in (A1) - (A3) imply the other two since $F_i^* \geq F_{i*}$.

Remark 2. Similar conditions can be found in [3, 9]. Actually, (A1) means that F_{i*} or F_i^* ($i = 1, 2$) are strictly increasing with respect to the principal argument u_i, and (A2) implies that F_{i*} or F_i^* ($i = 1, 2$) are degenerated elliptic and uniformly continuous with respect to the rgument x (see [3]). (A1) and (A2) are slightly stronger than weak properness. (A3) implies F_{i*} or F_i^* ($i = 1, 2$) are Lipschitz continuous with respect to the coupled argument u_j ($j \neq i$) since they are quasi-increasing.

Remark 3. Saying that conditions (A1) - (A3) hold means one of the following combinations of three of nine inequalities in (A1) - (A3) holds:

$$
\begin{array}{lll}
(A1.1),(A2.1),(A3.1) & (A1.1),(A2.2),(A3.2) & (A1.1),(A2.3),(A3.3) \\
(A1.2),(A2.1),(A3.2) & (A1.2),(A2.2),(A3.1) & (A1.2),(A2.3),(A3.1) \\
(A1.3),(A2.1),(A3.2) & (A1.3),(A2.2),(A3.1) & (A1.3),(A2.3),(A3.1).
\end{array}
$$

The following is the main result in this section.

Theorem 5.1. *Let F_1 and F_2 be locally bounded, quasi-increasing and satisfy conditions* (A1) - (A4). *If u and v are a couple of viscosity sub- and super-solution of problem* (1.1), *then $u^* \leq v_*$ on $\bar{\Omega}$.*

Proof. If the comparison is not true, we may suppose without loss of generality that

$$
\sup_{x \in \bar{\Omega}} \left\{ u_1^*(x) - v_{1*}(x) \right\} = \delta_1 > 0. \tag{5.1}
$$

Let

$$
M(x, y) = u_1^*(x) - v_{1*}(y) - \tfrac{\alpha}{2}|x - y|^2.
$$

For a fixed α, we assume that $M(x, y)$ attains its local maximum point at $(x_\alpha, y_\alpha) \in \Omega \times \Omega$ (denoted by $M_\alpha = M(x_\alpha, y_\alpha)$). From boundary condition, $(x_\alpha, y_\alpha) \notin \partial\Omega \times \partial\Omega$. Then

$$
u_1^*(x_\alpha) - v_{1*}(y_\alpha) \geq M_\alpha \geq \delta_1 > 0. \tag{5.2}
$$

From Lemma 2.10, there exist $p = \alpha(x_\alpha - y_\alpha)$ and $X, Y \in \mathbf{S}(n)$ such that

$$
\left.
\begin{array}{l}
(p, X) \in \bar{J}_\Omega^{2,+} u_1^*(x_\alpha) \\
(p, Y) \in \bar{J}_\Omega^{2,-} v_{1*}(y_\alpha)
\end{array}
\right\}. \tag{5.3}
$$

If conditions (A1.1), (A2.3), (A3.3) hold, then (A1.1) gives

$$
\begin{aligned}
\gamma_1 &(u_1^*(x_\alpha) - v_{1*}(y_\alpha)) \\
&\leq F_{1*}\big(x_\alpha; v_{2*}(x_\alpha); u_1^*(x_\alpha), p, X\big) - F_1^*\big(x_\alpha; v_{2*}(x_\alpha); v_{1*}(y_\alpha), p, X\big) \\
&\leq F_{1*}\big(x_\alpha; v_{2*}(x_\alpha); u_1^*(x_\alpha), p, X\big) - F_1^*\big(y_\alpha; u_2^*(y_\alpha); v_{1*}(y_\alpha), p, Y\big) \\
&\quad + F_1^*\big(y_\alpha; u_2^*(y_\alpha); v_{1*}(y_\alpha), p, Y\big) - F_1^*\big(x_\alpha; u_2^*(y_\alpha); v_{1*}(y_\alpha), p, X\big) \\
&\quad + F_1^*\big(x_\alpha; u_2^*(y_\alpha); v_{1*}(y_\alpha), p, X\big) - F_1^*\big(x_\alpha; v_{2*}(x_\alpha); v_{1*}(y_\alpha), p, X\big).
\end{aligned}
$$

Since u, v is a couple of viscosity sub- and super-solution of problem (1.1),

$$F_{1*}\big(x_\alpha; v_{2*}(x_\alpha); u_1^*(x_\alpha), p, X\big) - F_1^*\big(y_\alpha; u_2^*(y_\alpha); v_{1*}(y_\alpha), p, Y\big) \le 0.$$

From (A2.3),

$$F_1^*\big(y_\alpha; u_2^*(y_\alpha); v_{1*}(y_\alpha), p, Y\big) - F_1^*\big(x_\alpha; u_2^*(y_\alpha); v_{1*}(y_\alpha), p, X\big)$$
$$\le \omega_1\big(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|\big).$$

Thus we have

$$
\begin{aligned}
0 < \gamma_1\delta_1 \\
\le \gamma_1(u_1^*(x_\alpha) - v_{1*}(y_\alpha)) \\
\le \omega_1(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|) \\
+ F_1^*\big(x_\alpha; u_2^*(y_\alpha); v_{1*}(y_\alpha), p, X\big) - F_1^*\big(x_\alpha; v_{2*}(x_\alpha); v_{1*}(y_\alpha), p, X\big).
\end{aligned}
\tag{5.4}
$$

Now we can assert that

$$\sup_{x \in \bar\Omega}\big\{u_2^*(x) - v_{2*}(x)\big\} = \delta_2 > 0. \tag{5.5}$$

Otherwise, from Lemma 2.9, we know $\alpha|x_\alpha - y_\alpha|^2 \to 0$ as $\alpha \to \infty$. Observe from upper semi-continuity of $u_2^*(x) - v_{2*}(y)$, we can deduce that, for α large enough, $u_2^*(y_\alpha) \le v_{2*}(x_\alpha)$, i.e.

$$F_1^*\big(x_\alpha; u_2^*(y_\alpha); v_{1*}(y_\alpha), p, X\big) - F_1^*\big(x_\alpha; v_{2*}(x_\alpha); v_{1*}(y_\alpha), p, X\big) \le 0$$

since F_1 is quasi-increasing. Thus

$$0 < \gamma_1\delta_1 \le \gamma_1\big(u_1^*(x_\alpha) - v_{1*}(y_\alpha)\big) \le \omega_1\big(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|\big) \to 0$$

as $\alpha \to \infty$ which is a contradiction. So (5.5) holds, and $u_2^*(y_\alpha) \ge v_{2*}(x_\alpha)$ for α large enough. Hence, we can use (A3.3) to obtain

$$
\begin{aligned}
\gamma_1\delta_1 &\le \gamma_1\big(u_1^*(x_\alpha) - v_{1*}(y_\alpha)\big) \\
&\le \omega_1\big(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|\big) + L_1\big(u_2^*(y_\alpha) - v_{2*}(x_\alpha)\big).
\end{aligned}
$$

Then we get

$$\gamma_1\delta_1 \le L_1\delta_2 \tag{5.6}$$

as $\alpha \to \infty$. Similarly, starting from (5.5), we repeat the process above to get

$$\gamma_2\delta_2 \le L_2\delta_1. \tag{5.7}$$

Then we can deduce from (5.6), (5.7) and (A4) that

$$0 < \delta_1 \le \frac{L_1}{\gamma_1}\delta_2 \le \frac{L_1 L_2}{\gamma_1 \gamma_2}\delta_1 < \delta_1$$

which is a contradiction. So the comparison is true.

The proofs for the other combinations in conditions (A1) - (A3) (see Remark 3) will be similar, so we omitted them here ∎

Remark 4. In general, the conditions (A1) and (A2) are not necessary for the comparison to be hold. For instance, the problem $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$, where although $u = (u_1, u_2)$ does not satisfy condition (A1), the comparison holds. However, (A1) and (A2) can be easily checked in general rather than to check the comparison directly.

Now we consider the comparison for mixed quasi-monotonic systems.

Theorem 5.2. *Let (F_1, F_2) be proper, mixed quasi-monotone and satisfying conditions (A1) - (A4). If (U, V) is a couple of viscosity sub- and super-solution of problem (1.1), then $u^* \le v_*$ on $\bar\Omega$.*

Proof. If the comparison is not true, similar to the proof of Theorem 5.1, we may suppose that

$$\sup_{x \in \bar\Omega} \left\{ u_1^*(\bar x) - v_{1*}(x) \right\} = \delta_1 > 0. \tag{5.8}$$

Consider the function

$$M(x, y) = u_1^*(x) - v_{1*}(y) - \tfrac{\alpha}{2}|x - y|^2.$$

We assume that (x_α, y_α) is the local maximum point of $M(x, y)$ for a fixed $\alpha > 0$. It can be easily seen that

$$u_1^*(x_\alpha) - v_{1*}(y_\alpha) \ge M_\alpha \ge \delta_1 > 0 \tag{5.9}$$

where $M_\alpha = M(x_\alpha, y_\alpha)$.

Suppose we have conditions (A1.2), (A2.1) and (A3.2). Then, since F_1 is quasi-decreasing and F_2 is quasi-increasing, we have from (A1.2)

$$
\begin{aligned}
\gamma_1 &\left(u_1^*(x_\alpha) - v_{1*}(y_\alpha) \right) \\
&\le F_{1*}\left(x_\alpha; u_2^*(x_\alpha); u_1^*(x_\alpha), p, X \right) - F_{1*}\left(x_\alpha; u_2^*(x_\alpha); v_{1*}(y_\alpha), p, X \right) \\
&\le F_{1*}\left(x_\alpha; u_2^*(x_\alpha); u_1^*(x_\alpha), p, X \right) - F_1^*\left(y_\alpha; v_{2*}(y_\alpha); v_{1*}(y_\alpha), p, Y \right) \\
&\quad + F_1^*\left(y_\alpha; v_{2*}(y_\alpha); v_{1*}(y_\alpha), p, Y \right) - F_{1*}\left(x_\alpha; v_{2*}(y_\alpha); v_{1*}(y_\alpha), p, X \right) \\
&\quad + F_{1*}\left(x_\alpha; v_{2*}(y_\alpha); v_{1*}(y_\alpha), p, X \right) - F_{1*}\left(x_\alpha; u_2^*(x_\alpha); v_{1*}(y_\alpha), p, X \right)
\end{aligned}
$$

where $(p, X) \in \bar J_\Omega^{2,+} u_1^*(x_\alpha)$ and $(p, Y) \in \bar J_\Omega^{2,-} v_{1*}(y_\alpha)$. Since u, v is a couple of viscosity sub- and super-solution of problem (1.1),

$$F_{1*}\left(x_\alpha; u_2^*(x_\alpha); u_1^*(x_\alpha), p, X \right) - F_1^*\left(y_\alpha; v_{2*}(y_\alpha); v_{1*}(y_\alpha), p, Y \right) \le 0.$$

From (A2.1),

$$
\begin{aligned}
F_1^*&\left(y_\alpha; u_2^*(y_\alpha); v_{1*}(y_\alpha), p, Y \right) - F_{1*}\left(x_\alpha; v_{2*}(y_\alpha); v_{1*}(y_\alpha), p, X \right) \\
&\le \omega_1\left(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha| \right).
\end{aligned}
$$

Thus, we have again that

$$
\begin{aligned}
0 < \gamma_1 \delta_1 \\
\le \gamma_1 \big(u_1^*(x_\alpha) - v_{1*}(y_\alpha) \big) \\
\le \omega_1 \big(\alpha |x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha| \big) \\
+ F_{1*} \big(x_\alpha; v_{2*}(y_\alpha); v_{1*}(y_\alpha), p, X \big) - F_{1*} \big(x_\alpha; u_2^*(x_\alpha); v_{1*}(y_\alpha), p, X \big).
\end{aligned}
\tag{5.10}
$$

Similar to the proof of Theorem 5.1, we can assert that

$$
\sup_{x \in \Omega} \big\{ u_2^*(x) - v_{2*}(x) \big\} = \delta_2 > 0
\tag{5.11}
$$

and that $u_2^*(x_\alpha) \ge v_{2*}(y_\alpha)$ for α large enough. Thus we have from (A3.2) that

$$
\begin{aligned}
\gamma_1 \delta_1 \le \gamma_1 \big(u_1^*(x_\alpha) - v_{1*}(y_\alpha) \big) \\
\le \omega_1 \big(\alpha |x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha| \big) + L_1 \big(u_2^*(x_\alpha) - v_{2*}(y_\alpha) \big).
\end{aligned}
$$

Letting $\alpha \to \infty$, we obtain

$$
\gamma_1 \delta_1 \le L_1 \delta_2.
\tag{5.12}
$$

It is the same we can obtain as well

$$
\gamma_2 \delta_2 \le L_2 \delta_1.
\tag{5.13}
$$

Then, again from (5.12), (5.13) and condition (A4), we conclude the contrary inequality

$$
0 < \delta_1 \le \frac{L_1 L_2}{\gamma_1 \gamma_2} \delta_1 < \delta_1.
$$

The proof is completed ∎

Conditions (A1) - (A4) will also imply that the comparison holds in the case of quasi-decreasing systems. Thus combining the results in Sections 3 and 4, we have the following obvious result.

Theorem 5.3. *Let (F_1, F_2) be proper and quasi-monotone (i.e. quasi-decreasing, increasing or mixed quasi-monotonic), and satisfy conditions (A1) - (A4). If (U, V) is a couple of viscosity sub- and super-solutions of problem (1.1), then this problem has a unique viscosity solution $u \in C(\Omega)$ satisfying $U^* \le u \le V_*$ on $\bar{\Omega}$.*

6. Viscosity solution for non-quasi-monotone systems

In this section, we prove the existence, uniqueness and continuity of viscosity solution for non-quasi-monotonic systems. First, we introduce the concept of a so-called strong couple of viscosity sub- and super-solutions in the frame of locally bounded functions, which is similar to that introduced by [9] in the frame of semi-continuous functions. However, our concept here is more succinct and convenient.

Definition 6.1. Let $U(x) = (U_1(x), U_2(x))$ and $V(x) = (V_1(x), V_2(x))$ be locally bounded. Suppose that there exists a continuous function $w(x) = (w_1(x), w_2(x))$ between $U(x)$ and $V(x)$, i.e. for all $x \in \Omega$, $U_i^*(x) \le w_i(x) \le V_{i*}(x)$ if $U_i^*(x) \le V_{i*}(x)$ and $V_{i*}(x) \le w_i(x) \le U_i^*(x)$ if $V_{i*}(x) \le U_i^*(x)$. Then (U, V) is called a *strong couple of viscosity sub- and super-solutions* of problem (1.1) if for any continuous function $w(x) = (w_1(x), w_2(x))$ which is between $U(x)$ and $V(x)$, we have

$$F_{i*}(x; w_j; U_i^*, p_i, X_i) \le 0 \le F_i^*(x; w_j; V_{i*}, q_i, Y_i)$$

for all $x \in \Omega$, $(p_i, X_i) \in J_\Omega^{2,+} U_i^*(x)$ and $(q_i, Y_i) \in J_\Omega^{2,-} V_{i*}(x)$, and

$$U_{i*}(x) \le 0 \le V_i^*(x)$$

for all $x \in \partial\Omega$ and $i, j = 1, 2$ with $j \ne i$.

Lemma 6.2. *Suppose conditions* (A1) − (A3) *hold. If* (U, V) *is a strong couple of viscosity sub- and super-solutions of problem* (1.1), *then* $U_i^*(x) \le V_{i*}(x)$ *for all* $x \in \bar{\Omega}$ *and* $i = 1, 2$.

Proof. The proof is easy. In fact, if w_2 is a given function between U_2 and V_2, then U_1 and V_1 are the viscosity sub- and super-solutions, respectively, of the boundary value problem

$$F_1(x; w_2; u, Du, D^2 u) = 0 \quad (x \in \Omega) \atop u = 0 \quad (x \in \partial\Omega) \Bigg\} . \tag{6.1}$$

Thus from Lemma 2.8 we have $U_1^*(x) \le V_{1*}(x)$ $(x \in \bar{\Omega})$. Similarly we may prove $U_2^*(x) \le V_{2*}(x)$ $(x \in \bar{\Omega})$ ∎

Theorem 6.3. *Suppose conditions* (A1) - (A4) *hold. If there exists a strong couple of viscosity sub- and super-solutions of problem* (1.1) (U, V) *which are bounded, then this problem has a unique viscosity solution* $u \in C(\Omega)$ *such that*

$$U_i^*(x) \le u_i(x) \le V_{i*}(x)$$

for all $x \in \Omega$ *and* $i = 1, 2$.

Proof. For any $u \in C(\Omega)$ we define its norm as

$$\|u\| = \sup_\Omega \big\{ |u_1(x)| + |u_2(x)| \big\}$$

and define

$$B = \Big\{ u = (u_1, u_2) \in C(\Omega) : U_i^*(x) \le u_i(x) \le V_{i*}(x) \ (x \in \bar{\Omega}; i = 1, 2) \Big\}.$$

From the assumption above, B is a non-empty, bounded and close subset of $C(\Omega)$, and $\|V_* - U^*\|$ is bounded. For a given continuous function $\nu \in B$, consider the boundary value problems

$$
\left.
\begin{aligned}
F_i(x; \nu_j; u_i, Du_i, D^2 u_i) &= \quad (x \in \Omega) \\
u_i &= \quad (x \in \partial\Omega)
\end{aligned}
\left(\begin{smallmatrix} i=1,2 \\ j \neq i \end{smallmatrix} \right)
\right\} .
\tag{6.2^i}
$$

Thus U_i and V_i are viscosity sub- and super-solutions of problem (6.2)^i, respectively. From Theorem 2.7, there exists a unique viscosity solution u_i of (6.2)^i which is continuous and satisfies $U_i^*(x) \leq u_i(x) \leq V_{i*}(x)$ for all $x \in \bar{\Omega}$ and $i = 1, 2$ which implies $u \in B$. Thus we can define an operator T on B by $u = T\nu$. Obviously, a fixed point u of T on B is a continuous viscosity solution of problem (1.1).

We can prove that T has a unique fixed point by Banach fixed point theorem. Actually, we have the following lemma.

Lemma 6.4. *Let u be a unique fixed point of T^2. Then u is a unique fixed point of T.*

Proof. Let $u = T^2 u = T(Tu)$ and $v = Tu$. Then $T^2 v = T^2(Tu) = T(T^2 u) = Tu = v$, i.e. v is a fixed point of T^2. From the uniqueness of the fixed point of T^2, we know that $u = v$ is a fixed point of T. If w is another fixed point of T, then $T^2 w = T(Tw) = Tw = w$, i.e. w is also a fixed point of T^2, hence $w = u$ ∎

Thus, we only need to prove that T^2 is a deflatable. Let $\mu, \nu \in B$ and $u^{(k)} = T^k \mu$ as well as $v^{(k)} = T^k \nu$ $(k = 1, 2)$. First, we suppose that

$$
\sup_{x \in \Omega} \left\{ v_1^{(1)}(x) - u_1^{(1)}(x) \right\} > 0.
$$

Then from Lemma 2.9, there exists a point set $(x_\alpha, y_\alpha) \in \Omega \times \Omega$ such that x_α, $y_\alpha \to \hat{x}$ as $\alpha \to \infty$ and

$$
v_1^{(1)}(x_\alpha) - u_1^{(1)}(y_\alpha) \geq v_1^{(1)}(\hat{x}) - u_1^{(1)}(\hat{x}) = \sup_{x \in \Omega} \left\{ v_1^{(1)}(x) - u_1^{(1)}(x) \right\} > 0.
$$

From Lemma 2.10, there exist $p = \alpha(x_\alpha - y_\alpha)$ and X_1^v, Y_1^u satisfying (2.4) such that

$$
\left.
\begin{aligned}
(p, X_1^v) &\in J_\Omega^{2,+} v_1^{(1)}(x_\alpha) \\
(p, Y_1^u) &\in J_\Omega^{2,-} u_1^{(1)}(y_\alpha)
\end{aligned}
\right\} .
$$

If, without loss of generality, we have conditions (A1.3), (A2.3) and (A3.1), then inequality (A1.3) tells us that

$$
\begin{aligned}
\gamma_1 \big(& v_1^{(1)}(x_\alpha) - u_1^{(1)}(y_\alpha) \big) \\
&\leq F_1^* \big(y_\alpha; \mu_2(y_\alpha); v_1^{(1)}(x_\alpha), p, Y_1^u \big) - F_1^* \big(y_\alpha; \mu_2(y_\alpha); u_1^{(1)}(y_\alpha), p, Y_1^u \big) \\
&= F_{1*} \big(x_\alpha; \nu_2(x_\alpha); v_1^{(1)}(x_\alpha), p, X_1^v \big) - F_1^* \big(y_\alpha; \mu_2(y_\alpha); u_1^{(1)}(y_\alpha), p, Y_1^u \big) \\
&\quad + F_1^* \big(y_\alpha; \mu_2(y_\alpha); v_1^{(1)}(x_\alpha), p, Y_1^u \big) - F_1^* \big(x_\alpha; \mu_2(y_\alpha); v_1^{(1)}(x_\alpha), p, X_1^v \big) \\
&\quad + F_1^* \big(x_\alpha; \mu_2(y_\alpha); v_1^{(1)}(x_\alpha), p, X_1^v \big) - F_{1*} \big(x_\alpha; \nu_2(x_\alpha); v_1^{(1)}(x_\alpha), p, X_1^v \big).
\end{aligned}
$$

Next, from inequalities (A2.3) and (A3.1) we have

$$v_1^{(1)}(x_\alpha) - u_1^{(1)}(y_\alpha) \leq \tfrac{1}{\gamma_1}\omega_1\big(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|\big) + \tfrac{L_1}{\gamma_1}\big|\mu_2(y_\alpha) - \nu_2(x_\alpha)\big|.$$

Letting $\alpha \to \infty$ we get

$$v_1^{(1)}(\hat{x}) - u_1^{(1)}(\hat{x}) \leq \tfrac{L_1}{\gamma_1}\big|\mu_2(\hat{x}) - \nu_2(\hat{x})\big|.$$

Also, starting from $\sup_{x \in \Omega}\big\{u_1^{(1)}(x) - v_1^{(1)}(x)\big\} > 0$ we can get in the same way that

$$u_1^{(1)}(\hat{x}) - v_1^{(1)}(\hat{x}) \leq \tfrac{L_1}{\gamma_1}\big|\mu_2(\hat{x}) - \nu_2(\hat{x})\big|.$$

Thus

$$\big|v_1^{(1)}(\hat{x}) - u_1^{(1)}(\hat{x})\big| \leq \tfrac{L_1}{\gamma_1}\big|\mu_2(\hat{x}) - \nu_2(\hat{x})\big|. \tag{6.3}$$

Similarly, we may deduce as well that

$$\big|v_2^{(1)}(\hat{x}) - u_2^{(1)}(\hat{x})\big| \leq \tfrac{L_2}{\gamma_2}\big|\mu_1(\hat{x}) - \nu_1(\hat{x})\big|. \tag{6.4}$$

Replacing μ and ν by $u^{(1)}$ and $v^{(1)}$ in (6.3) and (6.4), respectively, we can obtain a similar estimate

$$\big|v_i^{(2)}(\hat{x}) - u_i^{(2)}(\hat{x})\big| \leq \tfrac{L_i}{\gamma_i}\big|v_j^{(1)}(\hat{x}) - u_j^{(1)}(\hat{x})\big| \tag{6.5}$$

for $i, j = 1, 2$ with $j \neq i$. Substituting (6.3) and (6.4) into (6.5), we have

$$\big|v_i^{(2)}(\hat{x}) - u_i^{(2)}(\hat{x})\big| \leq \tfrac{L_1 L_2}{\gamma_1 \gamma_2}\big|\mu_i(\hat{x}) - \nu_i(\hat{x})\big| \qquad (i = 1, 2).$$

Thus, we obtain

$$\|v^{(2)} - u^{(2)}\| = \|T^2\nu - T^2\mu\| \leq \tfrac{L_1 L_2}{\gamma_1 \gamma_2}\|\nu - \mu\|.$$

Due to condition (A4), $\frac{L_1 L_2}{\gamma_1 \gamma_2} < 1$ which implies that T^2 is deflatable. According to the Banach fixed point theorem and Lemma 6.4, there is a unique fixed point w of T, which is the unique viscosity solution of problem (1.1). Thus Theorem 6.3 is proved ∎

Concerning the existence of a strong couple of viscosity sub- and super-solutions, we have the following

Theorem 6.5. *Let inequalities (A1.2), (A1.3), (A3.2), (A3.3) and (A.4) hold. If U and V are standard viscosity sub- and super- solutions of problem (1.1), i.e.*

$$F_{i*}\big(x; U_j^*; U_i^*, p_i, X_i\big) \leq 0 \leq F_i^*\big(x; V_{j*}; V_{i*}, q_i, Y_i\big)$$

for all $x \in \Omega$ and $(p_i, X_i) \in J_\Omega^{2,+}U_i(x), (q_i, Y_i) \in J_\Omega^{2,-}V_i(x)$, and

$$U_i^*(x) \leq 0 \leq V_{i*}(x)$$

for all $x \in \partial\Omega$ and $i, j = 1, 2$ with $j \neq i$, then there exists a strong couple of viscosity sub- and super-solutions of problem (1.1).

Proof. Let

$$
\begin{aligned}
\hat{U}(x) &= \big(\hat{U}_1(x), \hat{U}_2(x)\big) = \big(U_1^*(x) - b_1, U_2^*(x) - b_2\big) \\
\hat{V}(x) &= \big(\hat{V}_1(x), \hat{V}_2(x)\big) = \big(V_{1*}(x) + c_1, V_{2*}(x) + c_2\big)
\end{aligned}
\tag{6.6}
$$

where b_1, b_2, c_1 and c_2 are non-negative constants, which will be determined later. Clearly, $\hat{U}_i \in \text{USC}(\Omega)$ and $\hat{V}_i \in \text{LSC}(\Omega)$, and

$$
\left.
\begin{aligned}
\hat{U}_i(x) &\le U_i^*(x) \\
\hat{V}_i(x) &\ge V_{i*}(x)
\end{aligned}
\right\}
\qquad (x \in \Omega;\ i = 1, 2).
$$

From inequalities (A1.2) and (A3.2), for all $(p, X) \in \mathbb{R}^n \times \mathbf{S}(n)$ and continuous function μ between $\hat{U}$ and $\hat{V}$, we have

$$
\begin{aligned}
F_{1*}&(x; U_2^*; U_1^*, p, X) - F_{1*}(x; \mu_2; \hat{U}_1, p, X) \\
&= F_{1*}(x; U_2^*; U_1^*, p, X) - F_{1*}(x; U_2^*; \hat{U}_1, p, X) \\
&\quad + F_{1*}(x; U_2^*; \hat{U}_1, p, X) - F_{1*}(x; \mu_2; \hat{U}_1, p, X) \\
&\ge \gamma_1 b_1 - L_1 |\mu_2 - U_2^*|.
\end{aligned}
$$

and

$$
\begin{aligned}
F_{2*}&(x; U_1^*; U_2^*, p, X) - F_{2*}(x; \mu_1; \hat{U}_2, p, X) \\
&= F_{2*}(x; U_1^*; U_2^*, p, X) - F_{2*}(x; U_1^*; \hat{U}_2, p, X) \\
&\quad + F_{2*}(x; U_1^*; \hat{U}_2, p, X) - F_{2*}(x; \mu_1; \hat{U}_2, p, X) \\
&\ge \gamma_2 b_2 - L_2 |\mu_1 - U_1^*|.
\end{aligned}
$$

From inequalities (A1.3) and (A3.3), for all $(q, Y) \in \mathbb{R}^n \times \mathbf{S}(n)$ and continuous function ν between $\hat{U}$ and $\hat{V}$, we have

$$
\begin{aligned}
F_1^*&(x; \nu_2; \hat{V}_1, q, Y) - F_1^*(x; V_{2*}; V_{1*}, q, Y) \\
&= F_1^*(x; \nu_2; \hat{V}_1, q, Y) - F_1^*(x; \nu_2; V_{1*}, q, Y) \\
&\quad + F_1^*(x; \nu_2; V_{1*}, q, Y) - F_1^*(x; V_{2*}; V_{1*}, q, Y) \\
&\ge \gamma_1 c_1 - L_1 |V_{2*} - \nu_2|
\end{aligned}
$$

and

$$
\begin{aligned}
F_2^*&(x; \nu_1; \hat{V}_2, q, Y) - F_2^*(x; V_{1*}; V_{2*}, q, Y) \\
&= F_2^*(x; \nu_1; \hat{V}_2, q, Y) - F_2^*(x; \nu_1; V_{2*}, q, Y) \\
&\quad + F_2^*(x; \nu_1; V_{2*}, q, Y) - F_2^*(x; V_{1*}; V_{2*}, q, Y) \\
&\ge \gamma_2 c_2 - L_2 |V_{1*} - \nu_1|.
\end{aligned}
$$

In order to estimate $|\mu_i(x) - U_i^*(x)|$ and $|V_{i*}(x) - \nu_i(x)|$ $(i = 1, 2)$ we let

$$
e_i = \sup_{x \in \Omega} \big| V_{i*}(x) - U_i^*(x) \big| \qquad (i = 1, 2).
$$

Thus we know that

$$|\mu_i(x) - U_i^*(x)| \leq \begin{cases} b_i & \text{if } \hat{U}_i(x) \leq \mu_i(x) \leq U_i^*(x) \\ c_i + e_i & \text{otherwise} \end{cases} \quad (x \in \Omega; \; i = 1, 2).$$

In the same way, we also have

$$|V_{i*}(x) - \nu_i(x)| \leq \begin{cases} c_i & \text{if } V_{i*}(x) \leq \nu_i(x) \leq \hat{V}_i(x) \\ b_i + e_i & \text{otherwise} \end{cases} \quad (x \in \Omega; \; i = 1, 2),$$

i.e. for $\hat{U}_i(x) \leq \mu_i(x) \leq U_i^*(x)$ $(i = 1, 2)$,

$$F_{1*}(x; U_2^*; U_1^*, p, X) - F_{1*}(x; \mu_2; \hat{U}_1, p, X) \geq \gamma_1 b_1 - L_1 b_2$$
$$F_{2*}(x; U_1^*; U_2^*, p, X) - F_{2*}(x; \mu_1; \hat{U}_2, p, X) \geq \gamma_2 b_2 - L_2 b_1$$

or, otherwise,

$$F_{1*}(x; U_2^*; U_1^*, p, X) - F_{1*}(x; \mu_2; \hat{U}_1, p, X) \geq \gamma_1 b_1 - L_1(c_2 + e_2)$$
$$F_{2*}(x; U_1^*; U_2^*, p, X) - F_{2*}(x; \mu_1; \hat{U}_2, p, X) \geq \gamma_2 b_2 - L_2(c_1 + e_1).$$

Hence, on choosing

$$\frac{L_2}{\gamma_2} b_1 \leq b_2 \leq \frac{\gamma_1}{L_1} b_1 \tag{6.7}$$

and

$$\left. \begin{array}{l} b_1 \geq \frac{L_1}{\gamma_1}(c_2 + e_2) \\ b_2 \geq \frac{L_2}{\gamma_2}(c_1 + e_1) \end{array} \right\} \tag{6.8}$$

we always have

$$\left. \begin{array}{l} F_{1*}(x; U_2^*; U_1^*, p, X) - F_{1*}(x; \mu_2; \hat{U}_1, p, X) \geq 0 \\ F_{2*}(x; U_1^*; U_2^*, p, X) - F_{2*}(x; \mu_1; \hat{U}_2, p, X) \geq 0 \end{array} \right. \tag{6.9}$$

On the other hand, for $V_{i*}(x) \leq \nu_i(x) \leq \hat{V}_i(x)$ $(i = 1, 2)$ we have

$$F_1^*(x; \nu_2; \hat{V}_1, q, Y) - F_1^*(x; V_{2*}; V_{1*}, q, Y) \geq \gamma_1 c_1 - L_1 c_2$$
$$F_2^*(x; \nu_1; \hat{V}_2, q, Y) - F_2^*(x; V_{1*}; V_{2*}, q, Y) \geq \gamma_2 c_2 - L_2 c_1$$

or, otherwise,

$$F_1^*(x; \nu_2; \hat{V}_1, q, Y) - F_1^*(x; V_{2*}; V_{1*}, q, Y) \geq \gamma_1 c_1 - L_1(b_2 + e_2)$$
$$F_2^*(x; \nu_1; \hat{V}_2, q, Y) - F_2^*(x; V_{1*}; V_{2*}, q, Y) \geq \gamma_2 c_2 - L_2(b_1 + e_1).$$

Hence, if we choose

$$\frac{L_2}{\gamma_2} c_1 \leq c_2 \leq \frac{\gamma_1}{L_1} c_1 \tag{6.10}$$

and

$$c_1 \geq \frac{L_1}{\gamma_1}(b_2 + e_2) \atop c_2 \geq \frac{L_2}{\gamma_2}(b_1 + e_1) \Bigg\} , \tag{6.11}$$

then we have

$$F_1^*(x; \nu_2; \hat{V}_1, q, Y) - F_1^*(x; V_{2*}; V_{1*}, q, Y) \geq 0$$
$$F_2^*(x; \nu_1; \hat{V}_2, q, Y) - F_2^*(x; V_{1*}; V_{2*}, q, Y) \geq 0 \tag{6.12}$$

Observe that

$$J_\Omega^{2,\pm} \hat{U}_i(x) = J_\Omega^{2,\pm} U_i^*(x) \atop J_\Omega^{2,\pm} \hat{V}_i(x) = J_\Omega^{2,\pm} V_{i*}(x) \Bigg\} \qquad (x \in \Omega; \ i = 1, 2)$$

and that U and V are standard viscosity sub- and super-solutions of problem (1.1). Thus for any continuous function $\mu = (\mu_1, \mu_2)$ which is between $\hat{U}$ and $\hat{V}$ we have from (6.9) and (6.12) that, for all $x \in \Omega$,

$$F_{1*}(x; \mu_2; \hat{U}_1, p, X) \leq 0 \qquad ((p, X) \in J_\Omega^{2,+} \hat{U}_1(x))$$
$$F_{2*}(x; \mu_1; \hat{U}_2, p, X) \leq 0 \qquad ((p, X) \in J_\Omega^{2,+} \hat{U}_2(x))$$
$$F_1^*(x; \mu_2; \hat{V}_1, q, Y) \geq 0 \qquad ((q, Y) \in J_\Omega^{2,-} \hat{V}_1(x)) \tag{6.13}$$
$$F_2^*(x; \mu_1; \hat{V}_2, q, Y) \geq 0 \qquad ((q, Y) \in J_\Omega^{2,-} \hat{V}_2(x)).$$

This implies that $\hat{U}, \hat{V}$ is a strong couple of viscosity sub- and super-solutions of problem (1.1).

Next, condition (A4) can ensure the existence of constants b_1, b_2 and c_1, c_2 such that inequalities (6.7),(6.8) and (6.10),(6.11) hold simultaneously, e.g. we can choose

$$b_1 = c_1 = \frac{L_1 L_2 e_1 + \gamma_2 L_1 e_2}{\gamma_1 \gamma_2 - L_1 L_2} \atop b_2 = c_2 = \frac{L_1 L_2 e_2 + \gamma_1 L_2 e_1}{\gamma_1 \gamma_2 - L_1 L_2}. \tag{6.14}$$

The proof of Theorem 6.5 is complete ∎

7. Examples

In this section, we present two examples.

Example 1. Let

$$-\Delta u_1 + u_1 + a u_2 = 0 \atop -\Delta u_2 + u_2 - 1 = 0 \quad (x \in \Omega) \atop u_1 = 0 \atop u_2 = 0 \quad (x \in \partial\Omega) \Bigg\} . \tag{7.1}$$

System (7.1) is not quasi-decreasing ("quasi-monotone" in term of [9]) if $a > 0$ (i.e. the condition of Theorems 3.3 in [9] is not satisfied). Indeed, in this case (7.1) is

quasi-increasing. Furthermore, if $a > 1$, the condition (A.1) of [9: Theorem 4.4] is not satisfied. So, we cannot prove existence of a viscosity solution by using the result of [9] directly. On the other hand, from standard elliptic partial differential equation theory we may easily prove that system (7.1) has a unique classical solution.

However, it is easy to see that the conditions in our Sections 3 and 5 are satisfied if we choose $U = (-2,0)$ and $V = (0,1)$ for $a > 1$. Thus we can prove that system (7.1) has a unique viscosity solution $u = (u_1, u_2)$ by Theorem 5.3 here. In fact, u is the unique classical solution as well, satisfying, by the classical maximum principle, $u_2 \geq 0, u_2 \not\equiv 0$ and $u_1 \leq 0, u_1 \not\equiv 0$ on $\bar{\Omega}$.

Example 2. Consider the weakly coupled system

$$\left. \begin{array}{r} G_1(x, Du_1, D^2u_1) + au_1 + bu_2 = 0 \\ G_2(x, Du_2, D^2u_2) + cu_1 + du_2 = 0 \end{array} \right\} \quad (x \in \Omega) \atop \left. \begin{array}{r} u_1 = 0 \\ u_2 = 0 \end{array} \right\} \quad (x \in \partial\Omega) \right\} . \tag{7.2}$$

System (7.2) is quasi-decreasing if $b \leq 0$ and $c \leq 0$. It is quasi-increasing if $b \geq 0$ and $c \geq 0$, and mixed quasi-monotonic if $bc \leq 0$. Clearly, for the later two cases. we cannot use directly the result of [9: Theorem 3.3]. Also in the case of $a < b$ and $d < c$ we cannot use the result of [9: Theorem 4.4]. If $a > 0, d > 0$ and $\frac{bc}{ad} < 1$, system (7.2) satisfies conditions (A1) - (A4). Using our Theorem 5.3 (if $b \geq 0, c \geq 0$, or $bc \leq 0$) and combining with the viscosity sub- and super-solutions, we may obtain the existence of a viscosity solution.

Example 3 (switching games [10, 14, 20, 21]). Consider the problem

$$\min\left\{ \max\left\{ G_i(x, u_1, Du_1, D^2u_1), M_i(x; u_1, u_2)\right\}, N_i(x; u_1, u_2)\right\} = 0 \ (x \in \Omega) \atop u_i = \phi_i(x) \ (x \in \partial\Omega;\ i = 1, 2) \tag{7.3}$$

where G_i are proper and ϕ_i are continuous. Let G_i satisfy conditions (A1) and (A2) with constants γ_i, M_i and N_i satisfy condition (A3) with constants L_i, and γ_i and L_i satisfy condition (A4). Then the existence and uniqueness of viscosity solutions for system (7.3) can be arrived at by our results, no matter whether (7.3) is quasi-monotone or not.

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