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On a System of Functional Equations in a Multi-Dimensional Domain

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Abstract. We study the system of functional equations

$$f_i(x) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_{ijk}[x, f_j(S_{ijk}(x))] + g_i(x) \qquad (1 \le i \le n)$$

for $x \in \Omega_i$ where Ω_i are compact or non-compact domains of $\mathbb{R}^p$, $g_i : \Omega_i \to R$, $S_{ijk} : \Omega_i \to \Omega_j$, $a_{ijk} : \Omega_i \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $f_i : \Omega_i \to \mathbb{R}$ are unknown functions. The paper consists of two mains parts. In the first part we give some results on existence, uniqueness and stability of the solutions of such systems and study sufficient conditions to obtain quadratic convergence. In the second part we obtain the Maclaurin expansion and approximation of solution in the case that a_{ijk} are linear and S_{ijk} are affine functions.

Keywords: *Systems of functional equations, Maclaurin expansion, convergence in square mean*
AMS subject classification: 39B72

1. Introduction

We consider the system of functional equations

$$f_i(x) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_{ijk}[x, f_j(S_{ijk}(x))] + g_i(x) \qquad (1 \le i \le n) \tag{1.1}$$

for $x \in \Omega_i$ where

Ω_i are compact or non-compact domains of $\mathbb{R}^p$
$g_i : \Omega_i \to \mathbb{R}$, $S_{ijk} : \Omega_i \to \Omega_j$, $a_{ijk} : \Omega_i \times \mathbb{R} \to \mathbb{R}$ are given continuous functions
$f_i : \Omega_i \to \mathbb{R}$ are unknown functions.

In [1], system (1.1) is studied with $p = 1$, $\Omega_i = [-b, b]$, $m = n = 2$, S_{ijk} binomials of first degree and

$$a_{ijk}(x, y) = \tilde{\alpha}_{ijk} y \tag{1.2}$$

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where $\tilde{\alpha}_{ijk}$ are real constants. The solution is approximated by a uniformly convergent recurrent sequence, and it is stable with respect to the functions g_i. In [2 - 4] existence and uniqueness of the solution of the functional equation

$$f(x) = a(x, f(S(x)))$$

in the functional space $BC[a,b]$ are studied. In [5] we have studied a special case of system (1.1) with $p = 1$ and $\Omega_i = \Omega = [-b, b]$ or Ω an unbounded interval of $\mathbb{R}$. By using the Banach fixed point theorem we have obtained existence, uniqueness and also stability of the solution of system (1.1) with respect to the functions g_i. In the case of a_{ijk} like in (1.2), S_{ijk} being binomials of first degree, $g \in C^r(\Omega; \mathbb{R}^n)$ and $\Omega = [-b, b]$ we have obtained a Maclaurin expansion of the solution of system (1.1) until the order r. Furthermore, if g_i are polynomials of degree r, then the solution of system (1.1) is also a polynomial of degree r.

In this paper, by using the Banach fixed point theorem, we obtain existence, uniqueness and stability of the solution of system (1.1) with respect to the functions g_i where Ω_i are compact or non-compact domains of $\mathbb{R}^p$. In the case of $\Omega_i = \Omega = \{x \in \mathbb{R}^p : \sum_{l=1}^{p} |x_l| \leq r\}$, a_{ijk} like in (1.2), S_{ijk} being affine functions and $g \in C^q(\Omega; \mathbb{R}^n)$ we obtain a Maclaurin expansion up to q of the solution of the linear system

$$f_i(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \tilde{\alpha}_{ijk} f_j(S_{ijk}(x)) + g_i(x) \qquad (1 \leq i \leq n) \tag{1.3}$$

for $x \in \Omega$. Moreover, if g_i are polynomials of degree not greater than q, the solution f of system (1.3) is also such a polynomial. Afterwards, if g_i are continuous functions, the solution f of system (1.3) is approximated by a uniformly convergent polynomials sequence. In the later part, we give a sufficient condition for the quadratic convergence of the system of functional equations. This result is a generalization of [1 - 5].

2. Notations, function spaces

A point in $\mathbb{R}^p$ is denoted by $x = (x_1, \ldots, x_p)$. We call $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}_+^p$ a p-multi-index and denote by x^α the monomial $x_1^{\alpha_1} \cdots x_p^{\alpha_p}$ which has degree $|\alpha| = \sum_{i=1}^{p} \alpha_i$. Similarly, if $D_j = \frac{\partial}{\partial x_j}$ for $1 \leq j \leq p$, then

$$D^\alpha = D_1^{\alpha_1} \cdots D_p^{\alpha_p} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_p^{\alpha_p}}$$

denotes a differential operator of order $|\alpha|$. We also denote $\alpha! = \alpha_1! \cdots \alpha_p!$ With Ω_i a compact subset of $\mathbb{R}^p$, we denote by $X_i = C(\Omega_i; \mathbb{R})$ the Banach space of functions $f_i : \Omega_i \to \mathbb{R}$ continuous on Ω_i with respect to the norm

$$\|f_i\|_{X_i} = \sup_{x \in \Omega_i} |f_i(x)|. \tag{2.1}$$

When $\Omega_i \subset \mathbb{R}^p$ is a non-compact domain, we denote by $X_i = C_b(\Omega_i; \mathbb{R})$ the Banach space of continuous functions $f_i : \Omega_i \to \mathbb{R}$, bounded on Ω_i with respect to the norm (2.1). We also denote by $X = X_1 \times \ldots \times X_n$ the Banach space of functions $f = (f_1, \ldots, f_n) \in X$ with respect to the norm

$$\|f\|_X = \sum_{i=1}^{n} \|f_i\|_{X_i}.$$

We note that, if $\Omega_i \subset \mathbb{R}^p$ is open, the functions in $C(\Omega_i; \mathbb{R})$ need not to be bounded on Ω_i. If $f_i \in C(\Omega_i; \mathbb{R})$ is bounded and uniformly continuous on Ω_i, then it possesses a unique, bounded, continuous extension to the closure $\overline{\Omega_i}$ of Ω_i. Hence we define the vector space $C(\overline{\Omega_i}; \mathbb{R})$ to consist of all those functions $f_i \in C(\Omega_i; \mathbb{R})$ for which f_i is bounded and uniformly continuous on Ω_i. This is a Banach space with norm given by (2.1).

Similarly, for any non-negative integer m we put

$$C^m(\Omega_i; \mathbb{R}) = \left\{ f_i \in C(\Omega_i; \mathbb{R}) : D^\alpha f_i \in C(\Omega_i; \mathbb{R}) \ \ (|\alpha| \le m) \right\}$$

for $\Omega_i \subset \mathbb{R}^p$ a domain in $\mathbb{R}^p$, and

$$C^m(\overline{\Omega_i}; \mathbb{R}) = \left\{ f_i \in C(\overline{\Omega_i}; \mathbb{R}) : D^\alpha f_i \in C(\overline{\Omega_i}; \mathbb{R}) \ \ (|\alpha| \le m) \right\}$$

for $\Omega_i \subset \mathbb{R}^p$ an open set in $\mathbb{R}^p$. The space $C^m(\overline{\Omega_i}; \mathbb{R})$ is also of Banach type with respect to the norm

$$\|f_i\|_{C^m(\overline{\Omega}_i; \mathbb{R})} = \max_{|\alpha| \le m} \sup_{x \in \Omega_i} |D^\alpha f_i(x)|.$$

We write system (1.1) in the form of an operational equation in X

$$f = Tf \tag{2.2}$$

where $f = (f_1, \ldots, f_n)$ and $Tf = ((Tf)_1, \ldots, (Tf)_n)$ with

$$(Tf)_i(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ijk}[x, f_j(S_{ijk}(x))] + g_i(x) \qquad (1 \le i \le n)$$

for $x \in \Omega_i$.

3. Theorems on existence, uniqueness and stability

We make the following hypotheses:

(H$_1$) $S_{ijk} : \Omega_i \to \Omega_j$ are continuous.

(H$_2$) $g \in X$.

(H$_3$) $a_{ijk} : \Omega_i \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist $\tilde{\alpha}_{ijk} : \Omega_i \to \mathbb{R}$ bounded and non-negative such that:

- **(i)** $|a_{ijk}(x, y) - a_{ijk}(x, \tilde{y})| \le \tilde{\alpha}_{ijk}(x)|y - \tilde{y}|$ for all $y, \tilde{y} \in \mathbb{R}$ and $x \in \Omega_i$.
- **(ii)** $\sigma = \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \le j \le m} \sup_{x \in \Omega_i} \tilde{\alpha}_{ijk}(x) < 1$.
- **(iii)** $a_{ijk}(\cdot, 0) \in X_i$ (this condition will be omitted if Ω_i is compact in $\mathbb{R}^p$).

Then we have the following

Theorem 3.1. *Let hypotheses (H_1)–(H_3) hold. Then there exists a unique function $f \in X$ such that $f = Tf$. Moreover, f is stable with respect to g in X.*

Proof. It is evident that $Tf \in X$ for all $f \in X$. Considering $f, \tilde{f} \in X$ we easily verify by hypothesis (H_3) that $\|Tf - T\tilde{f}\|_X \leq \sigma\|f - \tilde{f}\|_X$. Then using the Banach fixed point theorem we have the existence of a unique $f \in X$ such that $f = Tf$. Now, let $f, \tilde{f} \in X$ be two solutions of equation (2.2) corresponding to $g, \tilde{g} \in X$, respectively. By an analogous evaluation, we have

$$\|f - \tilde{f}\|_X \leq \frac{1}{1-\sigma}\|g - \tilde{g}\|_X. \tag{3.1}$$

Hence f is stable with respect to g in X ∎

Remark 3.1. In Theorem 3.1 with $p = 1$, let $\Omega_i = \Omega = [a, b]$ $(1 \leq i \leq n)$ or Ω an unbounded interval of $\mathbb{R}$ and hypothesis $(H_3)/(ii)$ replaced by

$$\sum_{i,j=1}^{n} \sum_{k=1}^{m} \sup_{x \in \Omega} \tilde{\alpha}_{ijk}(x) < 1.$$

Then we obtain the result in the paper [5].

Remark 3.2. Theorem 3.1 gives a consecutive approximate algorithm. $f^{(v)} = Tf^{(v-1)}$ $(v \in \mathbb{N}, f^{(0)} \in X$ given). The sequence $\{f^{(v)}\}$ converges in X to the solution f of equation (2.2) and we have the error estimation

$$\|f^{(v)} - f\|_X \leq \|Tf^{(0)} - f^{(0)}\|_X \frac{\sigma^v}{1-\sigma}$$

for all $v \in N$.

In the case of $\Omega_i \subset \mathbb{R}^p$ $(1 \leq i \leq n)$ there exists a bijective $\tau_i : \Omega \to \Omega_i$ such that τ_i, τ_i^{-1} are continuous and system (1.1) is equivalent to the system

$$\hat{f}_i(t) = \sum_{j=1}^{n} \sum_{k=1}^{m} \hat{a}_{ijk}[t, \hat{f}_j(\hat{S}_{ijk}(t))] + \hat{g}_i(t) \qquad (1 \leq i \leq n)$$

for $t \in \Omega$ where

$$\begin{aligned} \hat{S}_{ijk} &= \tau_j^{-1} \circ S_{ijk} \circ \tau_i \\ \hat{a}_{ijk}(t, y) &= a_{ijk}(\tau_i(t), y) \quad (t \in \Omega; y \in \mathbb{R}) \\ \hat{g}_i &= g_i \circ \tau_i, \;\; \hat{f}_i = f_i \circ \tau_i. \end{aligned}$$

Thus, we can suppose that all unknown functions f_i of system (1.1) have the same domain of definition, i.e. $\Omega_i = \Omega$ for all $1 \leq i \leq n$.

Then we use the functional space X as follows: With $\Omega \subset \mathbb{R}^p$ compact, we denote by $X = C(\Omega; \mathbb{R}^n)$ the Banach space of functions $f = (f_1, \dots, f_n) : \Omega \to \mathbb{R}^n$ continuous on Ω with respect to the norm

$$\|f\|_X = \sup_{x \in \Omega} \sum_{i=1}^{n} |f_i(x)|.$$

When $\Omega \subset \mathbb{R}^p$ is non-compact, we denote by $X = C_b(\Omega;\mathbb{R}^n)$ the Banach space of functions $f : \Omega \to \mathbb{R}^n$ continuous, bounded on Ω with respect to the norm $\|\cdot\|_X$ above.

We formulate the following hypotheses:

($\mathbf{H}_1'$) $S_{ijk} : \Omega \to \Omega$ are continuous.

($\mathbf{H}_2'$) $g \in X$.

($\mathbf{H}_3'$) $a_{ijk} : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous and there exists $\tilde{\alpha}_{ijk} : \Omega \to \mathbb{R}$ bounded and non-negative such that

  (i) $|a_{ijk}(x,y) - a_{ijk}(x,\tilde{y})| \leq \tilde{\alpha}_{ijk}(x)|y - \tilde{y}|$ for all $y, \tilde{y} \in \mathbb{R}$ and $x \in \Omega$.

  (ii) $\sigma = \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1\leq j\leq n} \sup_{x\in\Omega} \tilde{\alpha}_{ijk}(x) < 1$.

  (iii) $a_{ijk}(\cdot, 0) \in C_b(\Omega;\mathbb{R})$ (this ondition will be omitted if Ω is compact in $\mathbb{R}^p$).

Then we have the following

Theorem 3.2. *Let hypotheses (H_1')–(H_3') hold. Then there exists a unique function $f = (f_1,\ldots,f_n) \in X$ being solution of the system*

$$f_i(x) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_{ijk}[x, f_j(S_{ijk}(x))] + g_i(x) \qquad (1 \leq i \leq n)$$

for $x \in \Omega$. Moreover, the solution is stable with respect to g in X.

Proof. Theorem 3.2 can be proved in a manner similar to Theorem 3.1 and we omit the details ∎

Remark 3.3. The result in [5] is a special case of Theorem 3.2 with $p = 1$.

Consider now the case of a_{ijk} of form (1.2) and formulate the hypothesis

($\mathbf{H}_3''$) $\sigma_0 = \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1\leq j\leq n} |\tilde{\alpha}_{ijk}| < 1$.

Then we have the following

Theorem 3.3. *Let hypotheses (H_1'), (H_2'), (H_3'') hold. Then there exists a unique function $f = (f_1,\ldots,f_n) \in X$ being solution of the system*

$$f_i(x) = \sum_{j=1}^{n}\sum_{k=1}^{m} \tilde{\alpha}_{ijk} f_j(S_{ijk}(x)) + g_i(x) \qquad (1 \leq i \leq n) \tag{3.2}$$

for $x \in \Omega$. Moreover, the solution of this system is stable with respect to g in X.

Proof. We apply Theorem 3.2 for $a_{ijk}(x,y) = \tilde{\alpha}_{ijk}y$. Then $\tilde{\alpha}_{ijk} = |\tilde{\alpha}_{ijk}|$ in hypothesis (H_3')/(i) and $\sigma = \sigma_0 < 1$ by hypotheses (H_3')(ii) and H_3'' ∎

Remark 3.4. Let S_{ijk} be affine functions, i.e.

$$S_{ijk}(x) = B^{ijk}x + c^{ijk} \tag{3.3}$$

where $c^{ijk} \in \mathbb{R}^p$ and $B^{ijk} = \left(b_{\mu\nu}^{ijk}\right)_{\mu,\nu=1}^{p}$ is a matrix of order p. Let

$$\Omega = \overline{B_r(0)} = \left\{x \in \mathbb{R}^p : \|x\|_1 \leq r\right\}$$

where $\|x\|_1 = \sum_{l=1}^{p} |x_l|$. Suppose that the matrices B^{ijk} and vectors c^{ijk} satisfy the hypotheses

$$(H_1'')\quad \begin{cases} \|B^{ijk}\|_1 = \max\limits_{1\le v\le p} \sum\limits_{\mu=1}^{p} |b_{\mu v}^{ijk}| < 1 \\ \max\limits_{\substack{1\le i,j\le n\\ 1\le k\le m}} \dfrac{\|c^{ijk}\|_1}{1-\|B^{ijk}\|_1} \le r. \end{cases}$$

Then hypothesis (H_1') holds and we have the following

Theorem 3.4. *Let $\Omega = \overline{B_r(0)}$, let hypothesis (H_3'') hold and let $S_{ijk}(x) = B^{ijk}x + c^{ijk}$ where the matrices B^{ijk} and vectors c^{ijk} satisfy hypothesis (H_1''). Then for each $g \in X$ there exists a unique function $f = (f_1, \ldots, f_n) \in X$ being solution of system* (3.2). *Moreover, the solution is stable with respect to g in X.*

Remark 3.5.

(i) The corresponding results in [1] and [5] are special cases of Theorem 3.4 with $m = n = 2$, $p = 1$, $\Omega = [-b, b]$ and $p = 1$, respectively.

(ii) Theorem 3.4 is still true for $\Omega = \mathbb{R}^p$ and $X = C(\mathbb{R}^p; \mathbb{R}^n)$. In this case the matrices B^{ijk} and vectors c^{ijk} need not satisfy hypothesis (H_1'').

4. Maclaurin expansion of the solution

Now we consider $\Omega = \overline{B_r(0)}$ and real numbers $\tilde{\alpha}_{ijk}$, matrices B^{ijk} and vectors c^{ijk} as in Theorem 3.4. Let $f \in C^1(\Omega; \mathbb{R}^n)$ be the unique solution of system (3.2) - (3.3) corresponding to $g \in C^1(\Omega; \mathbb{R}^n)$. Differentiating two members of (3.2) with respect to the variable x_μ $(1 \le \mu \le p)$ we obtain

$$D_\mu f_i(x) = \sum_{j=1}^{n}\sum_{k=1}^{m} \tilde{\alpha}_{ijk} \sum_{v=1}^{p} b_{v\mu}^{ijk} D_v f_j(S_{ijk}(x)) + D_\mu g_i(x)$$

for $i = 1, \ldots, n$, $\mu = 1, \ldots, p$ and $x \in \Omega$. Put

$$F_i^\mu = D_\mu f_i$$
$$F = \left(F_1^1, \ldots, F_1^p, F_2^1, \ldots, F_2^p, \ldots, F_n^1, \ldots, F_n^p\right).$$

Then $F \in (C(\Omega; \mathbb{R}))^{np} = X^{(1)}$ is the solution of the system of functional equations

$$F_i^\mu(x) = \sum_{j=1}^{n}\sum_{k=1}^{m} \tilde{\alpha}_{ijk} \sum_{v=1}^{p} b_{v\mu}^{ijk} F_j^v(S_{ijk}(x)) + D_\mu g_i(x). \tag{4.1}$$

Rewrite system (4.1) in the form of operational equation in $X^{(1)}$

$$F = TF$$

where

$$TF = \Big((TF)_1^1, \ldots, (TF)_1^p, (TF)_2^1, \ldots, (TF)_2^p, \ldots, (TF)_n^1, \ldots, (TF)_n^p\Big)$$

and

$$(TF)_i^\mu(x) = \sum_{j=1}^{n}\sum_{k=1}^{m} \tilde{\alpha}_{ijk} \sum_{v=1}^{p} b_{v\mu}^{ijk} F_j^v(S_{ijk}(x)) + D_\mu g_i(x).$$

We note that $X^{(1)}$ is a Banach space with respect to the norm

$$\|F\|_{X^{(1)}} = \max_{1\le\mu\le p} \sum_{j=1}^{n} \sup_{x\in\Omega} |F_j^\mu(x)|.$$

We can easily check that $T : X^{(1)} \to X^{(1)}$ and

$$\|TF - T\tilde{F}\|_{X^{(1)}} \le \sigma^{(1)} \|F - \tilde{F}\|_{X^{(1)}}$$

for all $F, \tilde{F} \in X^{(1)}$, with

$$\sigma^{(1)} = \sum_{i=1}^{n}\sum_{k=1}^{m} \max_{1\le j\le n} |\tilde{\alpha}_{ijk}| \|B^{ijk}\|_1.$$

It follows from hypotheses (H_1'') and (H_3'') that $\sigma^{(1)} \le \sigma_0 < 1$. Then using the Banach fixed point theorem, there exists a unique function $F \in X^{(1)}$ being solution of system (4.1). Moreover, from the uniqueness, this solution is also "the 1-order derivative" of f, i.e.

$$F_i^\mu = D_\mu f_i \tag{4.2}$$

for all $1 \le i \le n$ and $\mu = 1, \ldots, p$.

Thus we have the following

Theorem 4.1. *Suppose that $\Omega = \overline{B_r(0)}$, that the real numbers $\tilde{\alpha}_{ijk}$, matrices B^{ijk} and vectors c^{ijk} are as in Theorem 3.4, and let $g \in C^1(\Omega;\mathbb{R}^n)$. Then there exist $f \in C^1(\Omega;\mathbb{R}^n)$ and $F \in X^{(1)}$ being the unique solutions of systems (3.2) – (3.3) and (4.1), respectively. Furthermore, we have also (4.2).*

Similarly, let $f \in C^q(\Omega; R^n)$ be the solution of system (3.2) - (3.3) corresponding to $g \in C^q(\Omega; R^n)$. Differentiating two members of (3.2) in all variables until the order q, we obtain

$$\begin{aligned} D^\gamma f_i(x) &= D^\gamma g_i(x) \\ &\quad + \sum_{j=1}^{n}\sum_{k=1}^{m} \tilde{\alpha}_{ijk} \sum_{\substack{\alpha_s\in\mathbb{Z}_+^p, |\alpha_s|=\gamma_s \\ s=1,\ldots,p}} \gamma! \frac{(b_1^{ijk})^{\alpha_1}}{\alpha_1!} \cdots \frac{(b_p^{ijk})^{\alpha_p}}{\alpha_p!} D^{\alpha_1+\ldots+\alpha_p} f_j(S_{ijk}(x)) \end{aligned}$$

for all $x \in \Omega$, $\gamma = (\gamma_1, \ldots, \gamma_p) \in \mathbb{Z}_+^p$ with $|\gamma| = q$ and $i = 1, \ldots, n$ where we denote

$$\left.\begin{aligned} D^\gamma &= D_1^{\gamma_1} \cdots D_p^{\gamma_p} \\ \gamma! &= \gamma_1! \cdots \gamma_p! \\ \alpha_s &= (\alpha_{1s}, \ldots, \alpha_{ps}) \in \mathbb{Z}_+^p \\ |\alpha_s| &= |\alpha_{1s}| + \ldots + |\alpha_{ps}| \\ \alpha_s! &= \alpha_{1s}! \cdots \alpha_{ps}! \\ (b_s^{ijk})^{\alpha_s} &= (b_{1s}^{ijk})^{\alpha_{1s}} \cdots (b_{ps}^{ijk})^{\alpha_{ps}} \end{aligned}\right\}.$$

For every p-multi-index $\gamma = (\gamma_1, \ldots, \gamma_p)$ such that $|\gamma| = q$ we put

$$F_i^\gamma = F_i^{(\gamma_1 \cdots \gamma_p)} = D^\gamma f_i = D_1^{\gamma_1} \cdots D_p^{\gamma_p} f_i.$$

Then the functions F_i^γ are solutions of the following problem:

($\mathbf{S_q}$) Find $F_i^\gamma \in C(\Omega; \mathbb{R})$ $(i = 1, \ldots, n;\ |\gamma| = q)$ such that

$$F_i^\gamma(x) = D^\gamma g_i(x) + \sum_{j=1}^{n} \sum_{k=1}^{m} \tilde{\alpha}_{ijk} \sum_{\substack{\alpha_s \in \mathbb{Z}_+^p, |\alpha_s| = \gamma_s \\ s=1,\ldots,p}} \gamma! \frac{(b_1^{ijk})^{\alpha_1}}{\alpha_1!} \cdots \frac{(b_p^{ijk})^{\alpha_p}}{\alpha_p!} F_j^{\alpha_1 + \ldots + \alpha_p}(S_{ijk}(x)) \quad (4.3)$$

for all $x \in \Omega$.

On the other hand, if we put

$$\Gamma_q = \{\gamma \in \mathbb{Z}_+^p : |\gamma| = q\},$$

then the number of elements of Γ_q is given by $N = \operatorname{card}(\Gamma_q) = \frac{(p+q-1)!}{q!(p-1)!}$. We rewrite the set Γ_q as $\Gamma_q = \{\gamma^1, \ldots, \gamma^N\}$, put

$$F = \left(F_1^{\gamma^1}, \ldots, F_1^{\gamma^N}, F_2^{\gamma^1}, \ldots, F_2^{\gamma^N}, \ldots, F_n^{\gamma^1}, \ldots, F_n^{\gamma^N}\right)$$

and rewrite system (4.3) in the form of the operational equation

$$F = UF \qquad \text{in } X^{(q)} = (C(\Omega; \mathbb{R}))^{nN} \quad (4.4)$$

where

$$UF = \left((UF)_1^{\gamma^1}, \ldots, (UF)_1^{\gamma^N}, (UF)_2^{\gamma^1}, \ldots, (UF)_2^{\gamma^N}, \ldots, (UF)_n^{\gamma^1}, \ldots, (UF)_n^{\gamma^N}\right)$$

and

$$(UF)_i^\gamma(x) = D^\gamma g_i(x) + \sum_{j=1}^{n} \sum_{k=1}^{m} \tilde{\alpha}_{ijk} \sum_{\substack{\alpha_s \in \Gamma_{\gamma_s} \\ s=1,\ldots,p}} \gamma! \frac{(b_1^{ijk})^{\alpha_1}}{\alpha_1!} \cdots \frac{(b_p^{ijk})^{\alpha_p}}{\alpha_p!} F_j^{\alpha_1 + \ldots + \alpha_p}(S_{ijk}(x))$$

with $1 \leq i \leq n$ and $\gamma = (\gamma_1, \ldots, \gamma_p) \in \Gamma_q$. We note that $X^{(q)}$ is a Banach space with respect to the norm

$$\|F\|_{X^{(q)}} = \max_{\gamma\in\Gamma_q} \sum_{i=1}^{n} \sup_{x\in\Omega} |F_i^\gamma(x)|.$$

Hence, we can easily verify that $U : X^{(q)} \to X^{(q)}$ and

$$\|UF - U\tilde{F}\|_{X^{(q)}} \leq \sigma^{(q)} \|F - \tilde{F}\|_{X^{(q)}}$$

for all $F, \tilde{F} \in X^{(q)}$ where

$$\sigma^{(q)} = \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1\leq j\leq n} |\tilde{\alpha}_{ijk}| \, \|B^{ijk}\|_1^q \leq \sigma_0 < 1$$

by hypothesis (H_3''). Hence, there exists a unique function $F \in X^{(q)}$ being solution of system (4.4). i.e. system (4.3). Furthermore, from the uniqueness, this solution is also "the q-order derivative" of f ,i.e.

$$F_i^\gamma = D^\gamma f_i \qquad (1 \leq i \leq n;\ \gamma \in \Gamma_q). \tag{4.5}$$

Then we have the following

Theorem 4.2. *Suppose that $\Omega = \overline{B_r(0)}$, that the real numbers $\tilde{\alpha}_{ijk}$, matrices B^{ijk} and vectors c^{ijk} are as in Theorem 3.4, and let $g \in C^q(\Omega; \mathbb{R}^n)$. Then there exist $f \in C^q(\Omega; \mathbb{R}^n)$ and $F \in X^{(q)}$ being the unique solutions of systems (3.2) – (3.3) and (4.3), respectively. Furthermore, we have also (4.5).*

Remark 4.1. In the case of $\Omega = \mathbb{R}^p$ we suppose additionally that the real numbers $\tilde{\alpha}_{ijk}$ and the matrices B^{ijk} satisfy the condition

$$\max_{0\leq s\leq q} \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1\leq j\leq n} |\tilde{\alpha}_{ijk}| \, \|B^{ijk}\|_1^s < 1 \tag{4.6}$$

(note that B^{ijk} and c^{ijk} need not to satisfy conditions hypothesis (H_1'')). Then, if

$$g \in C_b^q(R^p; R^n) = \left\{ g \in C_b(R^p; R^n) : \ D^\gamma g_i \in C_b(R^p; R^n) \ \ (|\gamma| \leq q;\ 1 \leq i \leq n) \right\}$$

the conclusion of Theorem 4.2 is still true, where the functional spaces $C^q(\Omega; \mathbb{R}^n)$ and $X^{(q)}$ appearing in Theorem 4.2 are replaced by $C_b^q(R^p; \mathbb{R}^n)$ and $(C_b(R^p; \mathbb{R}^n))^{nN}$, respectively, where $N = \frac{(p+q-1)!}{q!(p-1)!}$. The proof of this result is the same as that of Theorem 4.2.

Now we return to the same case of $\Omega = \overline{B_r(0)}$. Suppose that $f \in C^q(\Omega; \mathbb{R}^n)$ is the unique solution of system (3.2) - (3.3) corresponding to $g \in C^q(\Omega; \mathbb{R}^n)$. For each $\tilde{q} = 1, \ldots, q$ we have F_i^γ $(\gamma \in \Gamma_{\tilde{q}};\ 1 \leq i \leq n)$ as in Theorem 4.2 corresponding to $q = \tilde{q}$. Then from the Maclaurin formula we have

$$f_i(x) = \sum_{|\gamma|\leq q-1} \frac{1}{\gamma!} D^\gamma f_i(0) x^\gamma + q \int_0^1 (1-t)^{q-1} \sum_{|\gamma|=q} \frac{1}{\gamma!} D^\gamma f_i(tx) x^\gamma dt$$

for $1 \le i \le n$. On the other hand, we have

$$F_i^\gamma = \begin{cases} f_i & \text{if } |\gamma| = 0 \\ D^\gamma f_i & \text{if } 1 \le |\gamma| \le q \end{cases} \qquad (1 \le i \le n). \tag{4.7}$$

It follows that

$$f_i(x) = \sum_{|\gamma| \le q-1} \frac{1}{\gamma!} F_i^\gamma(0) x^\gamma + q \int_0^1 (1-t)^{q-1} \sum_{|\gamma|=q} \frac{1}{\gamma!} F_i^\gamma(tx) x^\gamma dt \tag{4.8}$$

for $1 \le i \le n$.

Inversely, suppose that a function $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_n) \in C^q(\Omega; \mathbb{R}^n)$ is given by the formula

$$\tilde{f}_i(x) = \sum_{|\gamma| \le q-1} \frac{1}{\gamma!} F_i^\gamma(0) x^\gamma + q \int_0^1 (1-t)^{q-1} \sum_{|\gamma|=q} \frac{1}{\gamma!} F_i^\gamma(tx) x^\gamma dt$$

for $1 \le i \le n$. Then from (4.7) and (4.8) we have

$$\tilde{f}_i(x) = \sum_{|\gamma| \le q-1} \frac{1}{\gamma!} D^\gamma f_i(0) x^\gamma + q \int_0^1 (1-t)^{q-1} \sum_{|\gamma|=q} \frac{1}{\gamma!} D^\gamma f_i(tx) x^\gamma dt = f_i(x)$$

for all $x \in \Omega$. Therefore $\tilde{f}$ is a solution of system (3.2) - (3.3).

Then we have the following

Theorem 4.3. *Under the hypotheses of Theorem 3.4, let $g \in C^q(\Omega; \mathbb{R}^n)$. Then the solution $f \in C^q(\Omega; \mathbb{R}^n)$ of system (3.2) – (3.3) is represented by (4.8) where $F_i^\gamma \in C(\Omega; \mathbb{R})$ $(1 \le i \le n;\ \gamma \in \Gamma_{\tilde{q}}$ with $0 < \tilde{q} \le q)$ is the unique solution of problem $(S_{\tilde{q}})$. Inversely, every function $f \in C^q(\Omega; \mathbb{R}^n)$ represented by (4.8) is a solution of system (3.2) – (3.3).*

Remark 4.2. In the case of $\Omega = \mathbb{R}^p$, and real numbers $\tilde{\alpha}_{ijk}$ and matrices B^{ijk} satisfying condition (4.6), if $g \in C_b^q(\mathbb{R}^p; \mathbb{R}^n)$, the conclusion of Theorem 4.3 is still true, where the functional spaces $C(\Omega; \mathbb{R})$ and $C^q(\Omega; \mathbb{R}^n)$ appearing in Theorem there are replaced by $C_b(\mathbb{R}^p; \mathbb{R})$ and $C_b^q(\mathbb{R}^p; \mathbb{R}^n)$, respectively.

Returning to the case of $\Omega = \overline{B_r(0)}$ we have the following

Corollary 4.1. *If $g_1, \dots, g_n$ are polynomials of degree not greater than $q-1$, the solution f of system (3.2) – (3.3) is also a sequence of such polynomials.*

Proof. We have $D^\gamma g_i(x) = 0$ for $x \in \Omega$, $|\gamma| \ge q$ and $1 \le i \le n$. Then $F_i^\gamma = 0$ $(|\gamma| = q;\ 1 \le i \le n)$ is the unique solution of system (4.3). Applying (4.8) we obtain $f_i(x) = \sum_{|\gamma| \le q-1} \frac{1}{\gamma!} F_i^\gamma(0) x^\gamma$ and the statement is proved ∎

Theorem 4.4. *Under the hypotheses of Theorem 3.4, suppose that $f \in C^q(\Omega;\mathbb{R}^n)$ is the solution of system (3.2) – (3.3) corresponding to $g \in C^q(\Omega;\mathbb{R}^n)$ and $\tilde{f} = (\tilde{f}_1,\dots,\tilde{f}_n)$ is a sequence of polynomials of degree not greater than $q-1$ and satisfying system (3.2) – (3.3) corresponding to $\tilde{g} = (\tilde{g}_1,\dots,\tilde{g}_n)$ where*

$$\tilde{g}_i(x) = \sum_{|\gamma|\le q-1} \frac{1}{\gamma!} D^\gamma g_i(0) x^\gamma \qquad (i = 1,\dots,n).$$

Then

$$\|f - \tilde{f}\|_X \le \frac{r^q}{1-\sigma_0} \sum_{|\gamma|=q} \frac{1}{\gamma!} \|D^\gamma g\|_X. \tag{4.9}$$

Proof. We have the Maclaurin expansion of g_i $(1 \le i \le n)$ until the order q

$$g_i(x) = \tilde{g}_i(x) + q \int_0^1 (1-t)^{q-1} \sum_{|\gamma|=q} \frac{1}{\gamma!} D^\gamma g_i(tx) x^\gamma dt. \tag{4.10}$$

Applying estimate (3.1) with $\sigma = \sigma_0$ we have

$$\|f - \tilde{f}\|_X \le \frac{1}{1-\sigma_0} \|g - \tilde{g}\|_X. \tag{4.11}$$

From (4.10) we have

$$\begin{aligned} \|g - \tilde{g}\|_X &= \sup_{x\in\Omega} \sum_{i=1}^n |g_i(x) - \tilde{g}_i(x)| \\ &\le q \int_0^1 (1-t)^{q-1} dt \sum_{|\gamma|=q} \frac{1}{\gamma!} \sup_{x\in\Omega} \sum_{i=1}^n |D^\gamma g_i(tx)|\, |x^\gamma|. \end{aligned}$$

We note that $\sup_{x\in\Omega} \sum_{i=1}^n |D^\gamma g_i(tx)| \le \|D^\gamma g\|_X$ for all $t \in [0,1]$ and $|x^\gamma| \le \|x\|_1^{|\gamma|} \le r^q$ for all $x \in \Omega$ and $|\gamma| = q$. Hence we obtain (4.9) from (4.11) ∎

Corollary 4.2. *Under the hypotheses of Theorem 3.4, if for $g \in C^\infty(\Omega;\mathbb{R}^n)$ there exists $d > 0$ such that*

$$\|D^\gamma g\|_X \le d^{|\gamma|} \qquad (\gamma \in \mathbb{Z}_+^p), \tag{4.12}$$

f is a solution of system (3.2) – (3.3) corresponding to g and $\tilde{f}^{[q]}$ is a sequence of polynomials of degree not greater than $q-1$ satisfying system (3.2) – (3.3) corresponding to $\tilde{g}$ as in Theorem 4.4, then

$$\lim_{q\to+\infty} \|f - \tilde{f}^{[q]}\|_X = 0.$$

Moreover, we have the estimates

$$\|f - \tilde{f}^{[q]}\|_X \le \frac{1}{1-\sigma_0} \frac{(dpr)^q}{q!} \qquad (q \in \mathbb{N}). \tag{4.13}$$

Proof. It follows from (4.9) and (4.12) that

$$\|f - \tilde{f}^{[q]}\|_X \le \frac{(rd)^q}{1-\sigma_0} \sum_{|\gamma|=q} \frac{1}{\gamma!}. \tag{4.14}$$

Using the equality $\left(\sum_{i=1}^p x_i\right)^q = \sum_{|\gamma|=q} \frac{q!}{\gamma!} x^\gamma$ $(x \in \mathbb{R}^p)$ with $x_1 = \dots = x_p = 1$, it follows that $\sum_{|\gamma|=q} \frac{1}{\gamma!} = \frac{p^q}{q!}$. Hence, we obtain (4.13) from (4.14) ∎

Corollary 4.3. *Under the hypotheses of Theorem 3.4, let $g \in C(\Omega; \mathbb{R}^n)$ and f be the solution of system (3.2) – (3.3) corresponding to g. Then there exists a sequence $\tilde{f}^{[q]} = (\tilde{f}_1^{[q]}, \dots, \tilde{f}_n^{[q]})$ of polynomials of degree not greater than $q-1$ such that*

$$\lim_{q \to +\infty} \|f - \tilde{f}^{[q]}\|_X = 0.$$

Proof. By the Weierstrass theorem, each function g_i is approximated by a sequence of polynomials $P_i^{[q]}$ converging uniformly to g_i when the degree $q-1 \to +\infty$. Hence, $P^{[q]} = (P_1^{[q]}, \dots, P_n^{[q]})$ converges in $C(\Omega; \mathbb{R}^n)$ to g when $q \to +\infty$. Let $\tilde{f}^{[q]}$ be a polynomial solution of system (3.2) - (3.3) corresponding to $g = P^{[q]}$. By estimate (3.1) with $\sigma = \sigma_0$, $\tilde{f} = \tilde{f}^{[q]}$ and $\tilde{g} = P^{[q]}$ we have

$$\|\tilde{f}^{[q]} - f\|_X \leq \frac{1}{1-\sigma_0} \|P^{[q]} - g\| \to 0 \tag{4.15}$$

as $q \to +\infty$ ∎

Remark 4.3. The results of Theorem 4.1 - 4.4 generalize those in [5].

Example 1. Consider the linear system with $m = 1$ and $n = p = 2$

$$f_i(x) = \sum_{j=1}^{2} \alpha_{ij} f_j(\beta_{ij} x) + g_i(x) \qquad (i = 1, 2) \tag{4.16}$$

for

$$x \in \Omega = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \|x\|_1 = |x_1| + |x_2| \leq 1 \right\}$$

where α_{ij} and β_{ij} are given real given numbers satisfying

$$\left. \begin{array}{r} |\beta_{ij}| \leq 1 \\ \displaystyle\sum_{i=1}^{2} \max_{1 \leq j \leq 2} |\alpha_{ij}| < 1 \end{array} \right\}.$$

(i) Let $g_i(x) = \sum_{|\gamma| \leq r} d_{i\gamma} x^\gamma$ $(i = 1, 2)$ be polynominals of degree not greater than r. It follows from Corollary 4.1 that the solution f of system (4.16) consists of polynomials of the same type. Put $f_i(x) = \sum_{|\gamma| \leq r} c_{i\gamma} x^\gamma$ $(i = 1, 2)$. Substituting f_i into (4.16) we obtain $(c_{1\gamma}, c_{2\gamma})$ that is the solution of a linear system

$$c_{i\gamma} - \sum_{j=1}^{2} \alpha_{ij} \beta_{ij}^{|\gamma|} c_{j\gamma} = d_{i\gamma} \qquad (i = 1, 2;\ |\gamma| \leq r).$$

Hence

$$\left. \begin{array}{l} c_{1\gamma} = \dfrac{(1 - \alpha_{22}\beta_{22}^{|\gamma|}) d_{1\gamma} + \alpha_{12}\beta_{12}^{|\gamma|} d_{2\gamma}}{(1 - \alpha_{11}\beta_{11}^{|\gamma|})(1 - \alpha_{22}\beta_{22}^{|\gamma|}) - \alpha_{12}\alpha_{21}\beta_{12}^{|\gamma|}\beta_{21}^{|\gamma|}} \\ c_{2\gamma} = \dfrac{\alpha_{21}\beta_{21}^{|\gamma|} d_{1\gamma} + (1 - \alpha_{11}\beta_{11}^{|\gamma|}) d_{2\gamma}}{(1 - \alpha_{11}\beta_{11}^{|\gamma|})(1 - \alpha_{22}\beta_{22}^{|\gamma|}) - \alpha_{12}\alpha_{21}\beta_{12}^{|\gamma|}\beta_{21}^{|\gamma|}} \end{array} \right\} \qquad (|\gamma| \leq r).$$

(ii) Let $g \in C^q(\Omega; R^2)$. It follows from Theorem 4.4 that

$$\|f - \tilde{f}\|_X \le \frac{1}{1-\sigma_0} \sum_{|\gamma|=q} \frac{1}{\gamma!}\|D^\gamma g\|_X$$

where $\sigma_0 = \sum_{i=1}^{2} \max_{1\le j\le 2} |\alpha_{ij}|$,

$$\tilde{f}_i(x) = \sum_{|\gamma|\le q-1} c_{i\gamma} x^\gamma \qquad (i = 1, 2)$$

and

$$\left.\begin{aligned} c_{1\gamma} &= \frac{(1-\alpha_{22}\beta_{22}^{|\gamma|})\frac{1}{\gamma!}D^\gamma g_1(0) + \alpha_{12}\beta_{12}^{|\gamma|}\frac{1}{\gamma!}D^\gamma g_2(0)}{(1-\alpha_{11}\beta_{11}^{|\gamma|})(1-\alpha_{22}\beta_{22}^{|\gamma|}) - \alpha_{12}\alpha_{21}\beta_{12}^{|\gamma|}\beta_{21}^{|\gamma|}} \\ c_{2\gamma} &= \frac{\alpha_{21}\beta_{21}^{|\gamma|}\frac{1}{\gamma!}D^\gamma g_1(0) + (1-\alpha_{11}\beta_{11}^{|\gamma|})\frac{1}{\gamma!}D^\gamma g_2(0)}{(1-\alpha_{11}\beta_{11}^{|\gamma|})(1-\alpha_{22}\beta_{22}^{|\gamma|}) - \alpha_{12}\alpha_{21}\beta_{12}^{|\gamma|}\beta_{21}^{|\gamma|}} \end{aligned}\right\} \qquad (|\gamma| \le q-1).$$

(iii) Let $g = (g_1, g_2)$ with $g_i(x) = (1 - \frac{x_1+x_2}{1+i})^{-1}$ $(i = 1, 2;\ x = (x_1, x_2) \in \Omega)$. We rewrite g_i as

$$g_i(x) = \sum_{j=0}^{\infty}\left(\frac{x_1+x_2}{1+i}\right)^j = \sum_{|\gamma|\le q-1}\frac{1}{\gamma!}\frac{|\gamma|!}{(1+i)^{|\gamma|}}x^\gamma + \sum_{j\ge q}\left(\frac{x_1+x_2}{1+i}\right)^j.$$

Putting

$$P_i^{[q]}(x) = \sum_{|\gamma|\le q-1}\frac{1}{\gamma!}\frac{|\gamma|!}{(1+i)^{|\gamma|}}x^\gamma$$

we have

$$|g_i(x) - P_i^{[q]}(x)| = \left|\sum_{j\ge q}\left(\frac{x_1+x_2}{1+i}\right)^j\right| \le \sum_{j\ge q}\frac{1}{(1+i)^j} = \frac{1}{i(1+i)^{q-1}} \tag{4.17}$$

for $x \in \Omega$. Hence $P_i^{[q]} \to g_i$ uniformly on Ω as $q \to +\infty$. Applying inequalities (4.15) and (4.17) we obtain

$$\|\tilde{f}^{[q]} - f\|_X \le \frac{1}{1-\sigma_0}\|P^{[q]} - g\|_X \le \frac{1}{1-\sigma_0}\left(\frac{1}{2^{q-1}} + \frac{1}{2.3^{q-1}}\right) \to +\infty$$

as $q \to +\infty$ where $\tilde{f}^{[q]} = (\tilde{f}_1^{[q]}, \tilde{f}_2^{[q]})$ with $\tilde{f}_i^{[q]}(x) = \sum_{|\gamma|\le q-1} c_{i\gamma}x^\gamma$ $(i = 1, 2)$ and the coefficients $c_{i\gamma}$ are calculated by the same formulas as in the case (i) with $D^\gamma g_i(0) = \frac{|\gamma|!}{(1+i)^{|\gamma|}}$ $(|\gamma| \le q-1)$.

5. The second order algorithm

In this section, we consider the algorithm for system (1.1)

$$\begin{aligned} f_i^{(v)}(x) = \sum_{j=1}^{n}\sum_{k=1}^{m} \Big\{ & a_{ijk}[x, f_j^{(v-1)}(S_{ijk}(x))] \\ & + \frac{\partial a_{ijk}}{\partial y}\left[x, f_j^{(v-1)}(S_{ijk}(x))\right]\left[f_j^{(v)}(S_{ijk}(x)) - f_j^{(v-1)}(S_{ijk}(x))\right]\Big\} \\ & + g_i(x) \end{aligned} \tag{5.1}$$

for $x \in \Omega_i$, $1 \le i \le n$ and $v \ge 1$ where $f^{(0)} = (f_1^{(0)}, \ldots, f_n^{(0)}) \in X$ is given. Rewrite (5.1) as linear system of functional equations

$$f_i^{(v)}(x) = \sum_{j=1}^{n}\sum_{k=1}^{m} \alpha_{ijk}^{(v)}(x) f_j^{(v)}(S_{ijk}(x)) + g_i^{(v)}(x) \tag{5.2}$$

where

$$\alpha_{ijk}^{(v)}(x) = \frac{\partial a_{ijk}}{\partial y}\left[x, f_j^{(v-1)}(S_{ijk}(x))\right] \tag{5.3}$$

and

$$g_i^{(v)}(x) = g_i(x) + \sum_{j=1}^{n}\sum_{k=1}^{m} \left\{ a_{ijk}\left[x, f_j^{(v-1)}(S_{ijk}(x))\right] - \alpha_{ijk}^{(v)}(x) f_j^{(v-1)}(S_{ijk}(x)) \right\}. \tag{5.4}$$

Thus we have the following

Theorem 5.1. *Let hypotheses $(H_1) - (H_2)$ hold and suppose $a_{ijk} \in C(\Omega_i \times \mathbb{R}; \mathbb{R})$ are such that*

$$\left.\begin{aligned} & \frac{\partial a_{ijk}}{\partial y} \in C(\Omega_i \times \mathbb{R}; \mathbb{R}) \\ a_{ijk}, & \frac{\partial a_{ijk}}{\partial y} \in C_b(\Omega_i \times [-M, M]; \mathbb{R}) \quad \forall\, M > 0 \end{aligned}\right\} \tag{5.5}$$

where condition $(5.5)_2$ will be omitted if Ω_i is compact in $\mathbb{R}^p$. If $f^{(v-1)} \in X$ satisfies

$$\sum_{i=1}^{n}\sum_{k=1}^{m} \max_{1 \le j \le n} \sup_{x \in \Omega_i} |\alpha_{ijk}^{(v)}(x)| < 1,$$

there exists a unique function $f^{(v)} \in X$ being solution of system (5.2), (5.4).

Proof. Apply Theorem 3.1 for $a_{ijk}(x, y) = \alpha_{ijk}^{(v)}(x) y$, $g_i(x) = g_i^{(v)}(x)$ and $\tilde{a}_{ijk} = |\alpha_{ijk}^{(v)}|$ ∎

We make the following hypotheses:

(A_1) $a_{ijk} \in C(\Omega_i \times \mathbb{R}; \mathbb{R})$ satisfy

(i) $\frac{\partial a_{ijk}}{\partial y}, \frac{\partial^2 a_{ijk}}{\partial y^2} \in C(\Omega_i \times \mathbb{R}; \mathbb{R})$.

(ii) $a_{ijk}, \frac{\partial a_{ijk}}{\partial y}, \frac{\partial^2 a_{ijk}}{\partial y^2} \in C_b(\Omega_i \times [-M, M]; \mathbb{R})$ for all $M > 0$ (this condition will be omitted if Ω_i is compact of $\mathbb{R}^p$).

(A_2) There exists a constant $M > 0$ such that

$$\|g\|_X + \sum_{i,j=1}^{n} \sum_{k=1}^{m} A_{ijk}^{(0)}(M) + 2M\sigma_M \leq M \tag{5.6}$$

where

$$A_{ijk}^{(0)}(M) = \sup_{x \in \Omega_i, |y| \leq M} |a_{ijk}(x, y)|$$

$$A_{ijk}^{(1)}(M) = \sup_{x \in \Omega_i, |y| \leq M} \left| \frac{\partial a_{ijk}}{\partial y}(x, y) \right|$$

$$\sigma_M = \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \leq j \leq n} A_{ijk}^{(1)}(M).$$

Thus we have the following

Theorem 5.2. *Let hypotheses* $(H_1) - (H_2)$ *and* $(A_1) - (A_2)$ *hold, let* f *be the solution of system* (1.1) *and the sequence* $\{f^{(v)}\}$ *be defined by algorithm* (5.1).

(i) *If* $\|f^{(0)}\|_X \leq M$, *then*

$$\|f^{(v)} - f\|_X \leq \tilde{\sigma}_M \|f^{(v-1)} - f\|_X^2 \qquad (v \geq 1) \tag{5.7}$$

where

$$\tilde{\sigma}_M = \frac{1}{2(1 - \sigma_M)} \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \leq j \leq n} A_{ijk}^{(2)}(M) \tag{5.8}$$

and

$$A_{ijk}^{(2)}(M) = \sup_{x \in \Omega_i, |y| \leq M} \left| \frac{\partial^2 a_{ijk}}{\partial y^2}(x, y) \right|.$$

(ii) *If choosing the first term* $f^{(0)}$ *sufficiently near* f *such that* $\tilde{\sigma}_M \|f^{(0} - f\|_X < 1$, *then the sequence* $\{f^{(v)}\}$ *converges quadratically to* f *and satisfies the error estimation*

$$\|f^{(v)} - f\|_X \leq \frac{1}{\tilde{\sigma}_M} (\tilde{\sigma}_M \|f^{(0)} - f\|_X)^{2^v} \qquad (v \geq 1). \tag{5.9}$$

Proof. First we will verify that if $\|f^{(0)}\|_X \leq M$, then

$$\|f^{(v)}\|_X \leq M \qquad (v \in \mathbb{N}). \tag{5.10}$$

Indeed, supposing

$$\|f^{(v-1)}\|_X \leq M \tag{5.11}$$

it follows from (5.2) and (5.11) that $\|f^{(v)}\|_X \leq \sigma_M \|f^{(v)}\|_X + \|g^{(v)}\|_X$. Note that (5.6) implies $0 < \sigma_M \leq \frac{1}{2}$, hence we obtain

$$\|f^{(v)}\|_X \leq \frac{1}{1-\sigma_M} \|g^{(v)}\|_X. \tag{5.12}$$

On the other hand, from (5.3) - (5.4) we obtain

$$\|g^{(v)}\|_X \leq \|g\|_X + \sum_{i,j=1}^{n} \sum_{k=1}^{m} A_{ijk}^{(0)}(M) + M\sigma_M. \tag{5.13}$$

From estimations (5.6) and (5.12) - (5.13) we obtain (5.10).

Now we shall estimate $\|f - f^{(v)}\|_X$. Putting $e^{(v)} = f - f^{(v)}$ we obtain from (1.1) and (5.1) the system

$$\begin{aligned} e_i^{(v)}(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \Big\{ & a_{ijk}[x, f_j(S_{ijk}(x))] - a_{ijk}[a, f_j^{(v-1)}(S_{ijk}(x))] \\ & + \frac{\partial a_{ijk}}{\partial y}[x, f_j^{(v-1)}(S_{ijk}(x))]\,[f_j^{(v)}(S_{ijk}(x)) - f_j^{(v-1)}(S_{ijk}(x))] \Big\}. \end{aligned} \tag{5.14}$$

Using Taylor's expansion of the function $a_{ijk}[x, f_j]$ about the point $(x, f_j^{(v-1)})$ up to order two, we obtain

$$\begin{aligned} & a_{ijk}[a, f_j] - a_{ijk}[x, f_j^{(v-1)}] \\ & = \frac{\partial a_{ijk}}{\partial y}[x, f_j^{(v-1)}](f_j - f_j^{(v-1)}) + \frac{1}{2!} \frac{\partial^2 a_{ijk}}{\partial y^2}[x, \lambda_j^{(v)}](f_j - f_j^{(v-1)})^2 \end{aligned} \tag{5.15}$$

where

$$\lambda_j^{(v)} = f_j^{(v-1)} + \theta_j^{(v)} e_j^{(v-1)} \qquad (0 < \theta_j^{(v)} < 1).$$

Substituting (5.15) into (5.14) where the arguments of f_j, $f_j^{(v-1)}$, $\lambda_j^{(v)}$ appearing in (5.15) are replaced by S_{ijk} we obtain

$$\begin{aligned} e_i^{(v)}(x) = & \sum_{j=1}^{n} \sum_{k=1}^{m} \alpha_{ijk}^{(v)}(x) e_j^{(v)}(S_{ijk}(x)) \\ & + \frac{1}{2!} \sum_{j=1}^{n} \sum_{k=1}^{m} \frac{\partial^2 a_{ijk}}{\partial y^2}[x, \lambda_j^{(v)}(S_{ijk}(x))]\, |e_j^{(v-1)}(S_{ijk}(x))|^2. \end{aligned}$$

From here and (5.11) we deduce that

$$\|e^{(v)}\|_X \leq \sigma_M \|e^{(v)}\|_X + \frac{1}{2!} \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \leq j \leq n} A_{ijk}^{(2)}(M) \sum_{j=1}^{n} \|e_j^{(v-1)}\|_{X_j}^2. \tag{5.16}$$

We note that

$$\sum_{j=1}^{n} \|e_j^{(v-1)}\|_{X_j}^2 \leq \left(\sum_{j=1}^{n} \|e_j^{(v-1)}\|_{X_j} \right)^2. \tag{5.17}$$

Hence we obtain (5.7) by (5.8), (5.16) and (5.17). Finally, we deduce easily (5.9) from (5.7) ∎

Example 2. Consider the nonlinear system with $m = 1$ and $n = p = 2$

$$f_i(x) = \sum_{j=1}^{2} \alpha_{ij} f_j^2(\beta_{ij} x) + g_i(x) \qquad (i = 1, 2) \tag{5.18}$$

for

$$x \in \Omega = \Omega_i = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \|x\|_1 = |x_1| + |x_2| \leq 1 \right\}$$

where

$$g_i(x) = \|x\|_1^i - \sum_{j=1}^{2} \alpha_{ij} (\beta_{ij} \|x\|_1)^{2j}$$

and α_{ij}, β_{ij} are given real numbers satisfying

$$\left.\begin{array}{r} \alpha_{ij} \leq 0 \\ |\beta_{ij}| \leq 1 \\ 4\sum_{i=1}^{2}\left(\sum_{j=1}^{2}|\alpha_{ij}| + 4\max_{1\leq j\leq 2}|\alpha_{ij}|\right)\left(2 - \sum_{i=1}^{2}\sum_{j=1}^{2}\alpha_{ij}(\beta_{ij})^{2j}\right) < 1 \end{array}\right\}.$$

The functions $a_{ij}(x, y) = a_{ij}(y) = \alpha_{ij} y^2$ and $S_{ij}(x) = \beta_{ij} x$, $g_i(x)$ satisfy hypotheses $(H_1) - (H_2)$ and $(A_1) - (A_2)$ where in (A_2) the constant $M > 0$ is chosen as

$$\frac{1 - \sqrt{1 - 4\gamma_0 \|g\|_X}}{2\gamma_0} \leq M \leq \frac{1 + \sqrt{1 - 4\gamma_0 \|g\|_X}}{2\gamma_0}$$

with

$$\left.\begin{array}{l} \gamma_0 = \sum_{i=1}^{2}\left(\sum_{j=1}^{2}|\alpha_{ij}| + 4\max_{1\leq j\leq 2}|\alpha_{ij}|\right) \\ \|g\|_X = 2 - \sum_{i=1}^{2}\sum_{j=1}^{2}\alpha_{ij}(\beta_{ij})^{2j} \end{array}\right\}.$$

The exact solution of system (5.18) is $f_i(x) = (\|x\|_1)^i$ $(i = 1, 2)$. The second order algorithm for system (5.18) is

$$f_i^{(v)}(x) = 2\sum_{j=1}^{2} \alpha_{ij} f_j^{(v-1)}(\beta_{ij} x) f_j^{(v)}(\beta_{ij} x) - \sum_{j=1}^{2} \alpha_{ij} (f_j^{(v-1)}(\beta_{ij} x))^2 + g_i(x)$$

for $x \in \Omega$, $i = 1, 2$ and $v \geq 1$. If we choose the initial iterative step $f^{(0)} = (f_1^{(0)}, f_2^{(0)})$ such that $\|f^{(0)}\|_X \leq M$ and

$$\tilde{\sigma}_M \|f^{(0)} - f\|_X < 1 \qquad \text{with} \quad \begin{cases} \tilde{\sigma}_M = \frac{\sigma_0}{1 - 2M\sigma_0} \\ \sigma_0 = \sum_{i=1}^{2} \max_{1\leq j\leq 2} |\alpha_{ij}| \end{cases}$$

then we have

$$\|f^{(v)} - f\|_X \leq \frac{1}{\tilde{\sigma}_M} \left(\tilde{\sigma}_M \|f^{(0)} - f\|_X\right)^{2^v} \qquad (v \geq 1).$$

Chooseing $f^{(0)}$ note that the sequence $\{g^{(\mu)}\}$ is defined by

$$g_i^{(\mu)}(x) = \sum_{j=1}^{2} \alpha_{ij}(g_j^{(\mu-1)}(\beta_{ij}x))^2 + g_i(x)$$

for $x \in \Omega$, $i = 1,2$ and $\mu \geq 1$ where $g^{(0)} = (g_1^{(0)}, g_2^0) \equiv (0,0)$ which is $\|g^{(\mu)}\|_X \leq M \quad (\mu \geq 1)$, $g^{(\mu)} \to f \in X$ as $\mu \to +\infty$ and $\|g^{(\mu)} - f\|_X \leq \frac{M}{1-\gamma_1}\gamma_1^\mu \qquad (\mu \geq 1)$ where $\gamma_1 = 2M\sigma_0 < M\gamma_0 < 1$. Choose μ_0 sufficient large such that $\tilde{\sigma}_M\|g^{(\mu_0)} - f\|_X < 1$. Then we take $f^{(0)} = g^{(\mu_0)}$.

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