On a System of Functional Equations in a Multi-Dimensional Domain

N. **T. Long and** N. **H. Nghia**

Abstract. We study the system of functional equations

N. T. Long and N. H. Nghia
udy the system of functional equations

$$
f_i(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ijk}[x, f_j(S_{ijk}(x))] + g_i(x) \qquad (1 \le i \le n)
$$

for $x \in \Omega_i$ where Ω_i are compact or non-compact domains of \mathbb{R}^p , $g_i: \Omega_i \to R$, $S_{ijk}: \Omega_i \to \Omega_j$, a_{ijk} : $\Omega_i \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and f_i : $\Omega_i \to \mathbb{R}$ are unknown functions. The paper consists of two mains parts. In the first part we give some results on existence, uniqueness and stability of the solutions of such systems and study sufficient conditions to obtain quadratic convergence. In the second part we obtain the Maclaurin expansion and approximation of solution in the case that a_{ijk} are linear and S_{ijk} are affine functions. from two mains
lity of the
vergence.
lition in the
of function: 3
n
 $\lim_{n \to \infty} \sum_{j=1}^{n} \sum_{k=1}^{m}$

Keywords: *Systems of functional equations, Maclaurin expansion, convergence in square mean* AMS subject classification: 39B72

1. Introduction

We consider the system of functional equations

stability of the solutions of such systems and study sufficient conditions to
\nc convergence. In the second part we obtain the Maclaurin expansion and
\nof solution in the case that
$$
a_{ijk}
$$
 are linear and S_{ijk} are affine functions.
\n
$$
stems of functional equations, Maclaurin expansion, convergence in square mean\nclassification: 39B72
$$
\n
\n**ction**\n
\n
$$
f_i(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ijk}[x, f_j(S_{ijk}(x))] + g_i(x) \qquad (1 \le i \le n) \qquad (1.1)
$$
\n
\n
$$
f_i(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ijk}[x, f_j(S_{ijk}(x))] + g_i(x) \qquad (1 \le i \le n) \qquad (1.1)
$$
\n
\nwhere
\n
$$
R, S_{ijk}: \Omega_i \to \Omega_j, a_{ijk}: \Omega_i \times \mathbb{R} \to \mathbb{R}
$$
 are given continuous functions
\n
$$
\mathbb{R}
$$
 are unknown functions.
\n(1.1) is studied with $p = 1, \Omega_i = [-b, b], m = n = 2, S_{ijk}$ binomials of
\n
$$
a_{ijk}(x, y) = \tilde{\alpha}_{ijk}y \qquad (1.2)
$$
\nLong: Polytechnic Univ. of HoChiMinh City, 268 Ly Thuong Kiet Str., Dist.
\nCity, Vietnam; Longnt@netnam2.org.vn

for $x \in \Omega_i$, where

 Ω_i are compact or non-compact domains of \mathbb{R}^p

 $g_i: \Omega_i \to \mathbb{R}, S_{ijk}: \Omega_i \to \Omega_j, a_{ijk}: \Omega_i \times \mathbb{R} \to \mathbb{R}$ are given continuous functions

 f_i : $\Omega_i \rightarrow \mathbb{R}$ are unknown functions.

In [1], system (1.1) is studied with $p = 1$, $\Omega_i = [-b, b]$, $m = n = 2$, S_{ijk} binomials of first degree and

$$
a_{ijk}(x,y) = \tilde{\alpha}_{ijk}y \tag{1.2}
$$

Nguyen Thanh Long: Polytechnic Univ. of HoChiMinh City, 268 Ly 'l'huong Kiet Str., Dist. 10, HoChiMinh City, Vietnam; Longnt@netnam2.org.vn

Nguyen Hoi Nghia: Vietnam Nat. Univ. of HoChiMinh City, Dept. Math. & Comp. Sci., 227 Nguyen Van Cu Str., Dist. 5, HoChiMinh City, Vietnam; nhnghia©vnuhcrn.edu.vn

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where $\tilde{\alpha}_{ijk}$ are real constants. The solution is approximated by a uniformly convergent Figure 1.1, the solution of the solution is approximated by a uniformly convergent
recurrent sequence, and it is stable with respect to the functions g_i . In [2 - 4] existence
and uniqueness of the solution of the functi and uniqueness of the solution of the functional equation

$$
f(x) = a(x, f(S(x)))
$$

in the functional space $BC[a, b]$ are studied. In [5] we have studied a special case of system (1.1) with $p = 1$ and $\Omega_i = \Omega = [-b, b]$ or Ω an unbounded interval of R. By using the Banach fixed point theorem we have obtained existence, uniqueness and also stability of the solution of system (1.1) with respect to the functions g_i . In the case of *0 j_{ik}* like in (1.2), S_{ijk} being binomials of first degree, $g \in C^{r}(\Omega; \mathbb{R}^{n})$ and $\Omega = [-b, b]$ we have obtained a Maclaurin expansion of the solution of system (1.1) until the order *r.* Furthermore, if g_i are polynomials of degree r, then the solution of system (1.1) is also a polynomial of degree r . In unbounded interval of \mathbb{R} . By

d existence, uniqueness and also

b the functions g_i . In the case of
 $\in C^r(\Omega; \mathbb{R}^n)$ and $\Omega = [-b, b]$ we

of system (1.1) until the order r .

e solution of system (1.1) is als

In this paper, by using the Banach fixed point theorem, we obtain existence, uniqueness and stability of the solution of system (1.1) with respect to the functions g_i where Ω_i are compact or non-compact domains of \mathbb{R}^p . In the case of $\Omega_i = \Omega = \{x \in \mathbb{R}^p :$ $\sum_{i=1}^p |x_i| \leq r$, a_{ijk} like in (1.2), S_{ijk} being affine functions and $g \in C^q(\Omega; \mathbb{R}^n)$ we obtain a Maclaurin expansion up to *q* of the solution of the linear system *a*₁ *a*₁₃ *a*₁₃ *a*₁₃ *a*₁₃ *a*₁₃ *a*₁₃ *a*₁₃ *a*₁₃ *a*₁₃ *a*₁ *a*

$$
f_i(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \tilde{\alpha}_{ijk} f_j(S_{ijk}(x)) + g_i(x) \qquad (1 \le i \le n)
$$
 (1.3)

for $x \in \Omega$. Moreover, if g_i are polynomials of degree not greater than q_i , the solution f of system (1.3) is also such a polynomial. Afterwards, if g_i are continuous functions, the solution *f* of system (1.3) is approximated by a uniformly convergent polynomials sequence. In the later part, we give a sufficient condition for the quadratic convergence of the system of functional equations. This result is a generalization of $[1 - 5]$.

2. Notations, function spaces

A point in \mathbb{R}^p is denoted by $x = (x_1, \ldots, x_p)$. We call $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}_+^p$ a p-multi-**2.** Notations, function spaces
A point in \mathbb{R}^p is denoted by $x = (x_1, \ldots, x_p)$. We call $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}_+^p$ a p-multi-
index and denote by x^{α} the monomial $x_1^{\alpha_1} \cdots x_p^{\alpha_p}$ which has degree $|\alpha| = \$ Similarly, if $D_j = \frac{\partial}{\partial x_j}$ for $1 \leq j \leq p$, then *.*, x_p). We call $\alpha = (d \alpha)^{2\alpha}$
 a^{α_1} *...* $x_p^{\alpha_p}$ which *h*
 en
 D_p $p = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_p^{\alpha_p}}$ aces
 $\begin{align*}\n a_1, \ldots, a_n \\
 b_2, \text{ then} \\
 b_3, \text{ then} \\
 \text{order } |c_1|\n \end{align*}$ **paces**
 x_1, \ldots, x_p). We call $\alpha = (\alpha_1, \ldots, \alpha_p)$
 nomial $x_1^{\alpha_1} \cdots x_p^{\alpha_p}$ which has deg
 p , then
 $p_1^{\alpha_1} \cdots p_p^{\alpha_p} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_p^{\alpha_p}}$

order $|\alpha|$. We also denote $\alpha! =$

the by $X_i = C(\Omega_i; \mathbb{$

$$
D^{\alpha} = D_1^{\alpha_1} \cdots D_p^{\alpha_p} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_p^{\alpha_p}}
$$

denotes a differential operator of order $|\alpha|$. We also denote $\alpha! = \alpha_1! \cdots \alpha_p!$ With Ω_i a compact subset of \mathbb{R}^p , we denote by $X_i = C(\Omega_i; \mathbb{R})$ the Banach space of functions $f_i : \Omega_i \to \mathbb{R}$ continuous on Ω_i with respect to the norm

$$
||f_i||_{X_i} = \sup_{x \in \Omega_i} |f_i(x)|. \tag{2.1}
$$

When $\Omega_i \subset \mathbb{R}^p$ is a non-compact domain, we denote by $X_i = C_b(\Omega_i; \mathbb{R})$ the Banach space of continuous functions $f_i : \Omega_i \to \mathbb{R}$, bounded on Ω_i with respect to the norm (2.1). We

also denote by $X = X_1 \times \ldots \times X_n$ the Banach space of functions $f = (f_1, \ldots, f_n) \in X$ with respect to the norm

On a System
\nn the Banach space
\n
$$
||f||_X = \sum_{i=1}^n ||f_i||_{X_i}.
$$

We note that, if $\Omega_i \subset \mathbb{R}^p$ is open, the functions in $C(\Omega_i; \mathbb{R})$ need not to be bounded on Ω_i . If $f_i \in C(\Omega_i; \mathbb{R})$ is bounded and uniformly continuous on Ω_i , then it possesses a unique, bounded, continuous extension to the closure $\overline{\Omega_i}$ of Ω_i . Hence we define the vector space $C(\overline{\Omega_i}; \mathbb{R})$ to consist of all those functions $f_i \in C(\Omega_i; \mathbb{R})$ for which f_i is bounded and uniformly continuous on Ω_i . This is a Banach space with norm given by $(2.1).$

Similarly, for any non-negative integer *m* we put

$$
C^m(\Omega_i; \mathbb{R}) = \left\{ f_i \in C(\Omega_i; \mathbb{R}) : D^{\alpha} f_i \in C(\Omega_i; \mathbb{R}) \ \left(|\alpha| \leq m \right) \right\} \qquad \dots \qquad \dots
$$

for $\Omega_i \subset \mathbb{R}^p$ a domain in \mathbb{R}^p , and

$$
C^m(\overline{\Omega_i}; \mathbb{R}) = \left\{ f_i \in C(\overline{\Omega_i}; \mathbb{R}) : D^{\alpha} f_i \in C(\overline{\Omega_i}; \mathbb{R}) \ \left(|\alpha| \leq m \right) \right\}
$$

for $\Omega_i\subset\mathbb{R}^p$ an open set in $\mathbb{R}^p.$ The space $C^m(\overline{\Omega_i};\mathbb{R})$ is also of Banach type with respect to the norm $\begin{aligned} \n\mathbb{R}^p. \quad & \text{The space} \ \|\mathbb{R}^p. \quad & \text{The space} \\\\ \n\|\hat{I}_i\|_{C^m(\overline{\Omega}_i;\mathbb{R})}. \end{aligned}$ **f**): $D^{\alpha} f_i \in C(\overline{\Omega_i}; \mathbb{R}) \left(|\alpha| \leq m \right)$
 $\in C^m(\overline{\Omega_i}; \mathbb{R})$ is also of Banach type with respect
 $\lim_{|\alpha| \leq m} \sup_{x \in \Omega_i} |D^{\alpha} f_i(x)|$.

operational equation in X
 $f = Tf$ (2.2)
 $\lim_{|\alpha| \leq m} (Tf)_n$ with

$$
||f_i||_{C^m(\overline{\Omega}_i; \mathbb{R})} = \max_{|\alpha| \leq m} \sup_{x \in \Omega_i} |D^{\alpha} f_i(x)|.
$$

We write system (1.1) in the form of an operational equation in X

$$
f = Tf \tag{2.2}
$$

where $f = (f_1, ..., f_n)$ and $Tf = ((Tf)_1, ..., (Tf)_n)$ with

$$
\text{Tr }\Omega_i \subset \mathbb{R}^p \text{ an open set in } \mathbb{R}^p. \text{ The space } C^m(\overline{\Omega_i}; \mathbb{R}) \text{ is also of Banach type we}
$$
\n
$$
\|f_i\|_{C^m(\overline{\Omega_i}; \mathbb{R})} = \max_{|\alpha| \le m} \sup_{x \in \Omega_i} |D^{\alpha} f_i(x)|.
$$
\n
$$
\text{We write system (1.1) in the form of an operational equation in } X
$$
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$$
f = Tf
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\n
$$
\text{here } f = (f_1, \ldots, f_n) \text{ and } Tf = ((Tf)_1, \ldots, (Tf)_n) \text{ with}
$$
\n
$$
(Tf)_i(x) = \sum_{j=1}^n \sum_{k=1}^m a_{ijk}[x, f_j(S_{ijk}(x))] + g_i(x) \qquad (1 \le i \le n)
$$
\n
$$
\text{or } x \in \Omega_i.
$$
\n
$$
\text{Theorems on existence, uniqueness and stability}
$$
\n
$$
\text{We make the following hypotheses:}
$$
\n
$$
\text{(H}_1) \quad S_{ijk} : \Omega_i \to \Omega_j \text{ are continuous.}
$$
\n
$$
\text{(H}_2) \quad g \in X.
$$
\n
$$
\text{(H}_3) \quad g_{ijk} : \Omega_i \to \mathbb{R} \text{ are continuous and there exist } \tilde{\alpha}_{ijk} : \Omega_i \to \mathbb{R} \text{ be}
$$

for $x \in \Omega_i$.

3. Theorems on existence, uniqueness and stability

We make the following hypotheses:

-
- (H_2) $q \in X$.
- (H₃) $a_{ijk}: \Omega_i \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist $\tilde{a}_{ijk}: \Omega_i \to \mathbb{R}$ bounded and non-negative such that: *(i) Iajk(^x ,y) — aijk(^x , ^y) (iii)* $a_{ijk}: \Omega_i \times \mathbb{R} \to \mathbb{R}$ are continuous and there exis
non-negative such that:
(i) $|a_{ijk}(x,y) - a_{ijk}(x,\tilde{y})| \leq \tilde{\alpha}_{ijk}(x)|y - \tilde{y}|$ for all y.
(ii) $\sigma = \sum_{i=1}^{n} \sum_{k=1}^{n} \max_{1 \leq j \leq m} \sup_{x \in \Omega_i} \tilde{\alpha}_{ijk}(x) < 1$.
iii)
	- (i) $|a_{ijk}(x,y) a_{ijk}(x,\tilde{y})| \leq \tilde{\alpha}_{ijk}(x)|y \tilde{y}|$ for all $y, \tilde{y} \in \mathbb{R}$ and $x \in \Omega_i$.
	-
	- (iii) $a_{ijk}(\cdot,0) \in X_i$ (this condition will be omitted if Ω_i is compact in *RP*).

Then we have the following

Theorem 3.1. Let hypotheses $(H_1) - (H_3)$ hold. Then there exists a unique function $f \in X$ such that $f = Tf$. Moreover, f is stable with respect to g in X.

Proof. It is evident that $Tf \in X$ for all $f \in X$. Considering $f, \tilde{f} \in X$ we easily verify by hypothesis (H_3) that $||Tf - T\tilde{f}||_X \le \sigma ||f - \tilde{f}||_X$. Then using the Banach fixed point theorem we have the existence of a unique $f \in X$ such that $f = Tf$. Now, let $f, \tilde{f} \in X$ be two solutions of equation (2.2) corresponding to $g, \tilde{g} \in X$, respectively. By an analogous evaluation, we have *liftuality* $\int f(x,y) \, dy \, dy$ *liftuality f is stable with respect to g in X*
 liftuality f is stable with respect to g in X
 l $||Tf - T\tilde{f}||_X \leq \sigma ||f - \tilde{f}||_X$. Then using sistence of a unique $f \in X$ such that *f* **1020 a** *M*. T. tong and *N*. H. Nghia
 Theorem 3.1. Let hypotheses $(H_1) - (H_3) hold$. Then there exists $f \in X$ usch that $f = Tf$. Moreover, f is stable with respect to g in X.
 Proof. It is ender that $T f \in X$ for al

$$
||f - \tilde{f}||_X \le \frac{1}{1 - \sigma} ||g - \tilde{g}||_X. \tag{3.1}
$$

Hence f is stable with respect to g in $X \blacksquare$

Remark 3.1. In Theorem 3.1 with $p = 1$, let $\Omega_i = \Omega = [a, b]$ $(1 \le i \le n)$ or Ω an unbounded interval of $\mathbb R$ and hypothesis $(H_3)/(ii)$ replaced by

$$
\sum_{i,j=1}^n \sum_{k=1}^m \sup_{\mathbf{z}\in\Omega} \tilde{\alpha}_{ijk}(\mathbf{x}) < 1.
$$

Then we obtain the result in the paper [5].

Remark 3.2. Theorem 3.1 gives a consecutive approximate algorithm. $f^{(v)} =$ **Remark 3.2.** Theorem 3.1 gives a consecutive approximate algorithm $f^{(v)} = Tf^{(v-1)}$ $(v \in \mathbb{N}, f^{(0)} \in X$ given). The sequence $\{f^{(v)}\}$ converges in X to the solution *f* of equation (2.2) and we have the error estimation *X* given). The

und we have the
 $||f^{(v)} - f||_X \le$

$$
||f^{(v)} - f||_X \le ||Tf^{(0)} - f^{(0)}||_X \frac{\sigma^v}{1 - \sigma}
$$

for all $v \in N$.

In the case of $\Omega_i \subset \mathbb{R}^p$ $(1 \leq i \leq n)$ there exists a bijective $\tau_i : \Omega \to \Omega_i$ such that τ_i, τ_i^{-1} are continuous and system (1.1) is equivalent to the system

$$
\hat{f}_i(t) = \sum_{j=1}^n \sum_{k=1}^m \hat{a}_{ijk}[t, \hat{f}_j(\hat{S}_{ijk}(t))] + \hat{g}_i(t) \qquad (1 \le i \le n)
$$

for $t \in \Omega$ where

$$
\hat{S}_{ijk} = \tau_j^{-1} \circ S_{ijk} \circ \tau_i
$$

\n
$$
\hat{a}_{ijk}(t, y) = a_{ijk}(\tau_i(t), y) \quad (t \in \Omega; y \in \mathbb{R})
$$

\n
$$
\hat{g}_i = g_i \circ \tau_i, \ \hat{f}_i = f_i \circ \tau_i.
$$

Thus, we can suppose that all unknown functions f_i of system (1.1) have the same domain of definition, i.e. $\Omega_i = \Omega$ for all $1 \leq i \leq n$.

Then we use the functional space X as follows: With $\Omega \subset \mathbb{R}^p$ compact, we denote I field we use the functional space X as follows: With $\Omega \subset \mathbb{R}^p$ compact, we denote by $X = C(\Omega; \mathbb{R}^n)$ the Banach space of functions $f = (f_1, \ldots, f_n) : \Omega \to \mathbb{R}^n$ continuous on Ω with respect to the norm $||f||_$ on Ω with respect to the norm

$$
||f||_X = \sup_{x \in \Omega} \sum_{i=1}^n |f_i(x)|.
$$

When $\Omega \subset \mathbb{R}^p$ is non-compact, we denote by $X = C_b(\Omega; \mathbb{R}^n)$ the Banach space of functions $f: \Omega \to \mathbb{R}^n$ continuous, bounded on Ω with respect to the norm $\|\cdot\|_X$ above.

We formulate the following hypotheses:

- **(H'**₁) S_{ijk} : $\Omega \rightarrow \Omega$ are continuous.
- (H'_2) $g \in X$.
- (H¹₃) a_{ijk} : $\Omega \times \mathbb{R} \to \mathbb{R}$ are continuous and there exists $\tilde{\alpha}_{ijk}$: $\Omega \to \mathbb{R}$ bounded and non-negative such that

(i) $|a_{ijk}(x,y) a_{ijk}(x,\tilde{y})| \leq \tilde{\alpha}_{ijk}(x)|y \tilde{y}|$ for all $y, \tilde{y} \in \mathbb{R}$ and $x \in \Omega$.
 non-negative such that We formulate the following hypotheses:
 $\binom{1}{1}$ $S_{ijk} : \Omega \to \Omega$ are continuous.
 $\binom{1}{2}$ $g \in X$.
 $\binom{1}{3}$ $a_{ijk} : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous and there exists $\tilde{\alpha}_{ijk} : \Omega \to \mathbb{R}$ bounded

non-negative such t
	- (i) $|a_{ijk}(x,y)-a_{ijk}(x,\tilde{y})|\leq \tilde{\alpha}_{ijk}(x)|y-\tilde{y}|$ for all $y,\tilde{y}\in\mathbb{R}$ and $x\in\Omega$.
	-
	-

Then we have-the following

Theorem 3.2. Let hypotheses $(H'_1) - (H'_3)$ hold. Then there exists a unique function $f = (f_1, \ldots, f_n) \in X$ being solution of the system (H's) **a0** *=*

$$
l \rightarrow Ω
$$
 are continuous.
\n
$$
l \rightarrow Ω
$$
 are continuous.
\n
$$
l \rightarrow \mathbb{R}
$$
 are continuous and there exists $\tilde{\alpha}_{ijk} : Ω \rightarrow \mathbb{R}$
\ngative such that
\n
$$
r, y) - a_{ijk}(x, \tilde{y})| \leq \tilde{\alpha}_{ijk}(x)|y - \tilde{y}|
$$
 for all $y, \tilde{y} \in \mathbb{R}$ and $x \in \Omega$
\n
$$
l = 1 \sum_{k=1}^n \sum_{k=1}^n \max_{1 \leq j \leq n} \sup_{x \in \Omega} \tilde{\alpha}_{ijk}(x) < 1.
$$

\n
$$
l = 0 \in C_b(\Omega; \mathbb{R}) \text{ (this condition will be omitted if } Ω \text{ is compa}
$$

\nthe following
\n3.2. Let hypothesis $(H'_1) - (H'_3)$ hold. Then there exists a u
\n
$$
l = \sum_{j=1}^n \sum_{k=1}^m a_{ijk}[x, f_j(S_{ijk}(x))] + g_i(x) \qquad (1 \leq i \leq n)
$$

\n
$$
f_i(x) = \sum_{j=1}^n \sum_{k=1}^m a_{ijk}[x, f_j(S_{ijk}(x))] + g_i(x) \qquad (1 \leq i \leq n)
$$

\nreover, the solution is stable with respect to g in X.

for $x \in \Omega$. Moreover, the solution is stable with respect to g in X.

Proof. Theorem 3.2 can be proved in a manner similar to Theorem 3.1 and we **omit the details U**

Remark 3.3. The result in [5] is a special case of Theorem 3.2 with $p = 1$.

Consider now the case of a_{ijk} of form (1.2) and formulate the hypothesis

 (H''_3) $\sigma_0 = \sum_{i=1}^n \sum_{k=1}^m \max_{1 \leq j \leq n} |\tilde{\alpha}_{ijk}| < 1.$

Then we have the following

Theorem 3.3. Let hypotheses (H'_1) , (H'_2) , (H''_3) hold. Then there exists a unique function $f = (f_1, \ldots, f_n) \in X$ *being solution of the system*

orem 3.2 can be proved in a manner similar to Theorem 3.1 and we

\n3. The result in [5] is a special case of Theorem 3.2 with
$$
p = 1
$$
.

\nwhere a_{ijk} is a special case of Theorem 3.2 with $p = 1$.

\nwhere $a_{ijk} = \sum_{k=1}^{m} \max_{1 \leq j \leq n} |\tilde{\alpha}_{ijk}| < 1$.

\nwhere follows:

\n3. Let $hypo$ the series $(H_1'), (H_2'), (H_3')$ hold. Then there exists a unique $..., f_n \in X$ being solution of the system.

\n
$$
f_i(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \tilde{\alpha}_{ijk} f_j(S_{ijk}(x)) + g_i(x) \qquad (1 \leq i \leq n)
$$

\n(3.2)

\nwhere, the solution of this system is stable with respect to g in X .

\napply Theorem 3.2 for $a_{ijk}(x, y) = \tilde{\alpha}_{ijk} y$. Then $\tilde{\alpha}_{ijk} = |\tilde{\alpha}_{ijk}|$ in hypothesis (H_3') and $\sigma = \sigma_0 < 1$ by hypotheses (H_3') (ii) and $H_3'' \blacksquare$

\n4. Let S_{ijk} be affine functions, i.e., $S_{ijk}(x) = B^{ijk}x + c^{ijk}$

\n(3.3)

\nand $B^{ijk} = (b_{ji}^{\dagger} b)^p_{\mu,\nu=1}$ is a matrix of order p . Let

for $x \in \Omega$. Moreover, the solution of this system is stable with respect to g in X.

Proof. We apply Theorem 3.2 for $a_{ijk}(x, y) = \tilde{\alpha}_{ijk}y$. Then $\tilde{\alpha}_{ijk} = |\tilde{\alpha}_{ijk}|$ in hypothesis $(H_3')/(i)$ and $\sigma = \sigma_0 < 1$ by hypotheses (H_3') (ii) and H_3''

Remark 3.4. Let S_{ijk} be affine functions, i.e.

$$
S_{ijk}(x) = B^{ijk}x + c^{ijk} \tag{3.3}
$$

pothesis $(H_3')/(i)$ and $\sigma = \sigma_0 < 1$ by hypothesis (H_3)/(i) and $\sigma = \sigma_0 < 1$ by hypothesis $S_{ijk}(x)$
where $c^{ijk} \in \mathbb{R}^p$ and $B^{ijk} = (b_{\mu\nu}^{ijk})_{\mu,\nu=1}^p$ is **a** matrix **of** order p. Let $S_{ijk}(x)$
 $\binom{ijk}{\mu}$
 $\binom{p}{\mu, \nu=1}$

$$
\Omega = \overline{B_r(0)} = \left\{ x \in \mathbb{R}^p : ||x||_1 \le r \right\}
$$

where $||x||_1 = \sum_{l=1}^p |x_l|$. Suppose that the matrices B^{ijk} and vectors c^{ijk} satisfy the hypotheses

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\nhere
$$
||x||_1 = \sum_{l=1}^p |x_l|
$$
. Suppose that t
\nypotheses
\n
$$
(H_1'')\n\begin{cases}\n||B^{ijk}||_1 = \max_{1 \le v \le p} \sum_{\mu=1}^p |b_{\mu v}^{ijk}| < 1 \\
\max_{1 \le i, j \le n} \frac{||c^{ijk}||_1}{1 - ||B^{ijk}||_1} \le r.\n\end{cases}
$$
\nthen hypothesis (H_1') holds and we have

Then hypothesis (H'_1) holds and we have the following

Theorem 3.4. Let $\Omega = \overline{B_r(0)}$, let hypothesis (H''_3) hold and let $S_{ijk}(x) = B^{ijk}x + C_j$ *and vectors cI)k satisfy hypothesis (Hr). Then for each* $g \in X$ there exists a unique function $f = (f_1, \ldots, f_n) \in X$ being solution of system *(3.2). Moreover, the solution* **is** *stable with respect to g in X.*

Remark 3.5.

(i) The corresponding results in [1] and [5] are special cases of Theorem 3.4 with $m = n = 2$, $p = 1$, $\Omega = [-b, b]$ and $p = 1$, respectively.

(ii) Theorem 3.4 is still true for $\Omega = \mathbb{R}^p$ and $X = C(\mathbb{R}^p; \mathbb{R}^n)$. In this case the matrices B^{ijk} and vectors c^{ijk} need not satisfy hypothesis (H''_i) .

4. Maclaurin expansion of the solution

Now we consider $\Omega = \overline{B_r(0)}$ and real numbers $\tilde{\alpha}_{ijk}$, matrices B^{ijk} and vectors c^{ijk} as in Theorem 3.4. Let $f \in C^1(\Omega;\mathbb{R}^n)$ be the unique solution of system (3.2) - (3.3) corresponding to $g \in C^1(\Omega;\mathbb{R}^n)$. Differentiating two members of (3.2) with respect to the variable x_{μ} $(1 \leq \mu \leq p)$ we obtain pontaing results in [1] and [5] are special cases of T
 $\Omega = [-b, b]$ and $p = 1$, respectively.

3.4 is still true for $\Omega = \mathbb{R}^p$ and $X = C(\mathbb{R}^p; \mathbb{R}^n)$.

vectors c^{ijk} need not satisfy hypothesis (H''_1) .
 expansi

$$
D_{\mu}f_i(x) = \sum_{j=1}^n \sum_{k=1}^m \tilde{\alpha}_{ijk} \sum_{v=1}^p b_{v\mu}^{ijk} D_v f_j(S_{ijk}(x)) + D_{\mu}g_i(x)
$$

for $i=1,\ldots,n, \mu=1,\ldots,p$ and $x \in \Omega$. Put

$$
F_i^{\mu} = D_{\mu} f_i
$$

\n
$$
F = (F_1^1, \dots, F_1^p, F_2^1, \dots, F_2^p, \dots, F_n^1, \dots, F_n^p).
$$

\n
$$
F_i^{\mu} = X^{(1)}
$$
 is the solution of the system of function
\n
$$
F^{\mu}(x) = \sum_{i=1}^n \sum_{j=1}^m \tilde{\alpha}_{ijk} \sum_{j=1}^p b_{ij\mu}^{ijk} F_j^{\nu}(S_{ijk}(x)) + D_{\mu} g_i(x)
$$

Then $F \in (C(\Omega;\mathbb{R}))^{np} = X^{(1)}$ is the solution of the system of functional equations

$$
D_{\mu}f_{i}(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \tilde{\alpha}_{ijk} \sum_{v=1}^{p} b_{v\mu}^{ijk} D_{v}f_{j}(S_{ijk}(x)) + D_{\mu}g_{i}(x)
$$

\n
$$
\iota = 1, ..., p \text{ and } x \in \Omega. \text{ Put}
$$

\n
$$
F_{i}^{\mu} = D_{\mu}f_{i}
$$

\n
$$
F = (F_{1}^{1}, ..., F_{1}^{p}, F_{2}^{1}, ..., F_{2}^{p}, ..., F_{n}^{1}, ..., F_{n}^{p}).
$$

\nR))^{np} = X⁽¹⁾ is the solution of the system of functional equations
\n
$$
F_{i}^{\mu}(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \tilde{\alpha}_{ijk} \sum_{v=1}^{p} b_{v\mu}^{ijk} F_{j}^{v}(S_{ijk}(x)) + D_{\mu}g_{i}(x).
$$

\n(4.1)
\n4.1) in the form of operational equation in X⁽¹⁾

Rewrite system (4.1) in the form of operational equation in $X^{(1)}$

 $F = TF$

where

$$
TF = \left((TF)_1^1, \ldots, (TF)_1^p, (TF)_2^1, \ldots, (TF)_2^p, \ldots, (TF)_n^1, \ldots, (TF)_n^p \right)
$$

and

On a System of Functional Equ-
\nwhere
\n
$$
TF = \left((TF)_1^1, \dots, (TF)_1^p, (TF)_2^1, \dots, (TF)_2^p, \dots, (TF)_n^1, \dots, (
$$
\nand
\n
$$
(TF)_i^{\mu}(x) = \sum_{j=1}^n \sum_{k=1}^m \tilde{\alpha}_{ijk} \sum_{v=1}^p b_{v\mu}^{ijk} F_j^v(S_{ijk}(x)) + D_{\mu}g_i(x).
$$
\nWe note that $X^{(1)}$ is a Banach space with respect to the norm
\n
$$
||F||_{X^{(1)}} = \max_{1 \leq \mu \leq p} \sum_{j=1}^n \sup_{x \in \Omega} |F_j^{\mu}(x)|.
$$
\nWe can easily check that $T : X^{(1)} \to X^{(1)}$ and
\n
$$
||TF - T\tilde{F}||_{X^{(1)}} \leq \sigma^{(1)} ||F - \tilde{F}||_{X^{(1)}}
$$
\nfor all $F, \tilde{F} \in X^{(1)}$, with

$$
= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\alpha_{ijk}}{v_{ij} \mu} \sum_{\mu=1}^{\infty} b_{ijk}^{\nu} F_j^{\nu}(S_{ijk}(x))
$$

each space with respect to the no

$$
||F||_{X^{(1)}} = \max_{1 \le \mu \le p} \sum_{j=1}^{n} \sup_{x \in \Omega} |F_j^{\mu}(x)|.
$$

We can easily check that $T: X^{(1)} \to X^{(1)}$ and

$$
||TF - T\tilde{F}||_{X^{(1)}} \le \sigma^{(1)}||F - \tilde{F}||_{X^{(1)}}
$$

for all $F, \tilde{F} \in X^{(1)}$, with

$$
||F||_X(t) - \max_{1 \le \mu \le p} \sum_{j=1}^{\infty} \sup_{x \in \Omega} |F_j(x)|.
$$

$$
T: X^{(1)} \to X^{(1)}
$$
 and

$$
||TF - T\tilde{F}||_{X^{(1)}} \le \sigma^{(1)} ||F - \tilde{F}||_{X^{(1)}}
$$

$$
\sigma^{(1)} = \sum_{i=1}^n \sum_{k=1}^m \max_{1 \le j \le n} |\tilde{\alpha}_{ijk}| ||B^{ijk}||_1.
$$

$$
(H_i'')
$$
 and (H_i'') that $\sigma^{(1)} \le \sigma_0 \le$

It follows from hypotheses (H''_1) and (H''_3) that $\sigma^{(1)} \leq \sigma_0 < 1$. Then using the Banach fixed point theorem, there exists a unique function $F \in X^{(1)}$ being solution of system (4.1). Moreover, from the uniqueness, this solution is also "the 1-order derivative" of *f,* i.e. $\leq \sigma^{(1)} \|F - \tilde{F}\|_{X}$
 $\max_{1 \leq j \leq n} |\tilde{\alpha}_{ijk}| \|B^{ijk}\|_1$
 \Rightarrow that $\sigma^{(1)} \leq \sigma_0 <$
 σ_1 function $F \in X^k$
 σ_2 solution is also
 σ_1
 σ_2 σ_3

$$
F_i^{\mu} = D_{\mu} f_i \tag{4.2}
$$

for all $1 \leq i \leq n$ and $\mu = 1, \ldots, p$.

Thus we have the following

Theorem 4.1. *Suppose that* $\Omega = \overline{B_r(0)}$, that the real numbers $\tilde{\alpha}_{ijk}$, matrices B^{ijk} and vectors c^{ijk} are as in Theorem 3.4, and let $g \in C^1(\Omega; \mathbb{R}^n)$. Then there exist $f \in C^1(\Omega; \mathbb{R}^n)$ and $F \in X^{(1)}$ and vectors c^{ijk} are as in Theorem 3.4, and let $g \in C^1(\Omega; \mathbb{R}^n)$. Then there exist $f \in C^1(\Omega; \mathbb{R}^n)$ and $F \in \mathbb{R}^{(1)}$. *F*^{μ} = $D_{\mu} f_i$ (4.2)

for all $1 \le i \le n$ and $\mu = 1, ..., p$.

Thus we have the following
 C'frace in the unique solutions of systems (3.2) - (3.3) and (4.1),
 *C*¹(Ω _i **R**ⁿ) and $F \in X^{(1)}$ being the unique sol *respectively. Furthermore, we have also (4.2).* all $1 \leq i \leq n$ and $\mu = 1, ..., n$
 Thus we have the following
 Theorem 4.1. Suppose that
 D vectors c^{ijk} are as in Theor
 $(\Omega; \mathbb{R}^n)$ and $F \in X^{(1)}$ being t

pectively. Furthermore, we had

Similarly, let $f \in C^q(\$

Similarly, let $f \in C^q(\Omega; R^n)$ be the solution of system (3.2) - (3.3) corresponding to we obtain

$$
g \in C^{q}(\Omega; R^{n}).
$$
 Differentiating two members of (3.2) in all variables until the order q , we obtain
\n
$$
D^{\gamma} f_i(x) = D^{\gamma} g_i(x)
$$
\n
$$
+ \sum_{j=1}^{n} \sum_{k=1}^{m} \tilde{\alpha}_{ijk} \sum_{\alpha_{j} \in \mathbb{Z}_{p}^{p}, |\alpha_{j}| = \gamma, \atop \alpha_{j} \in \mathbb{Z}_{p}^{p},
$$

for all $x \in \Omega$, $\gamma = (\gamma_1, \ldots, \gamma_p) \in \mathbb{Z}_+^p$ with $|\gamma| = q$ and $i = 1, \ldots, n$ where we denote

$$
D^{\gamma} = D_1^{\gamma_1} \cdots D_p^{\gamma_p}
$$

\n
$$
\gamma! = \gamma_1! \cdots \gamma_p!
$$

\n
$$
\alpha_s = (\alpha_{1s}, \ldots, \alpha_{ps}) \in \mathbb{Z}_+^p
$$

\n
$$
|\alpha_s| = |\alpha_{1s}| + \ldots + |\alpha_{ps}|
$$

\n
$$
\alpha_s! = \alpha_{1s}! \cdots \alpha_{ps}!
$$

\n
$$
(\dot{b}_s^{ijk})^{\alpha_r} = (\dot{b}_{1s}^{ijk})^{\alpha_{1r}} \cdots (\dot{b}_{ps}^{ijk})^{\alpha_{ps}}
$$

\n
$$
= (\gamma_1, \ldots, \gamma_p) \text{ such that } |\gamma| = q \text{ we } p
$$

\n
$$
= F_i^{(\gamma_1 \ldots \gamma_p)} = D^{\gamma} f_i = D_1^{\gamma_1} \cdots D_p^{\gamma_p} f_i.
$$

\nsolutions of the following problem:

For every *p*-multi-index $\gamma = (\gamma_1, \ldots, \gamma_p)$ such that $|\gamma| = q$ we put

$$
F_i^{\gamma} = F_i^{(\gamma_1 \dots \gamma_p)} = D^{\gamma} f_i = D_1^{\gamma_1} \cdots D_p^{\gamma_p} f_i.
$$

Then the functions F_i^{γ} are solutions of the following problem:

 $(\mathbf{S}_{\mathbf{q}})$ Find $F_i^{\gamma} \in C(\Omega;\mathbb{R})$ $(i = 1, \ldots, n; |\gamma| = q)$ such that

$$
|\alpha_{s}| = |\alpha_{1s}| + \ldots + |\alpha_{ps}|
$$

\n
$$
\alpha_{s}! = \alpha_{1s}! \cdots \alpha_{p1}!
$$

\n
$$
(\delta_{j}^{ijk})^{\alpha_{j}} = (\delta_{1j}^{ijk})^{\alpha_{1i}} \cdots (\delta_{p_{j}}^{ijk})^{\alpha_{pj}}
$$

\nFor every *p*-multi-index $\gamma = (\gamma_{1}, \ldots, \gamma_{p})$ such that $|\gamma| = q$ we put
\n
$$
F_{i}^{\gamma} = F_{i}^{(\gamma_{1} \ldots \gamma_{p})} = D^{\gamma} f_{i} = D_{1}^{\gamma_{1}} \cdots D_{p}^{\gamma_{p}} f_{i}.
$$

\nThen the functions F_{i}^{γ} are solutions of the following problem:
\n(S_q) Find $F_{i}^{\gamma} \in C(\Omega; \mathbb{R})$ $(i = 1, \ldots, n; |\gamma| = q)$ such that
\n
$$
F_{i}^{\gamma}(x) = D^{\gamma} g_{i}(x)
$$

\n
$$
+ \sum_{j=1}^{n} \sum_{k=1}^{m} \tilde{\alpha}_{ijk} \sum_{\substack{e \in \mathbb{Z}_{j}^{p}, |\alpha_{i}| = \gamma_{i} \\ e = 1, \ldots, p}} \gamma! \frac{(b_{1}^{ijk})^{\alpha_{1}}}{\alpha_{1}!} \cdots \frac{(b_{p}^{ijk})^{\alpha_{p}}}{\alpha_{p}!} F_{j}^{\alpha_{1} + \ldots + \alpha_{p}}(S_{ijk}(x))
$$
(4.3)
\nfor all $x \in \Omega$.
\nOn the other hand, if we put
\n
$$
\Gamma_{q} = \{\gamma \in \mathbb{Z}_{+}^{p}: |\gamma| = q\},
$$

\nthen the number of elements of Γ_{q} is given by $N = \text{card}(\Gamma_{q}) = \frac{(p+q-1)!}{q!(p-1)!}$. We rewrite
\nthe set Γ_{q} as $\Gamma_{q} = \{\gamma^{1}, \ldots, \gamma^{N}\},$ put
\n
$$
F = (F_{1}^{\gamma^{1}}, \ldots, F_{1}^{\gamma^{N}}, F_{2}^{\gamma^{1}}, \ldots, F_{2
$$

for all $x \in \Omega$.

On the other hand, if we put

$$
\Gamma_q = \{ \gamma \in \mathbb{Z}_+^p : |\gamma| = q \},\
$$

then the number of elements of Γ_q is given by $N = \text{card}(\Gamma_q) = \frac{(p+q-1)!}{q!(p-1)!}$. We rewrite the set Γ_q as $\Gamma_q = {\gamma^1, \ldots, \gamma^N}$, put *F*_q = { $\gamma \in \mathbb{Z}_+^p$: $|\gamma| = q$ },

ents of Γ_q is given by $N = \text{card}(\Gamma_q) = \frac{(p+q-1)}{q!(p-1)}$
 \ldots, γ^N }, put
 $\ldots, F_1^{\gamma^N}, F_2^{\gamma^1}, \ldots, F_2^{\gamma^N}, \ldots, F_n^{\gamma^1}, \ldots, F_n^{\gamma^N}$

In the form of the operational equation
 F

$$
F = \left(F_1^{\gamma^1}, \ldots, F_1^{\gamma^N}, F_2^{\gamma^1}, \ldots, F_2^{\gamma^N}, \ldots, F_n^{\gamma^1}, \ldots, F_n^{\gamma^N}\right)
$$

and rewrite system (4.3) in the form of the operational equation

$$
F = UF \qquad \text{in } X^{(q)} = (C(\Omega; \mathbb{R}))^{nN} \tag{4.4}
$$

where

$$
UF=\left((UF)_1^{\gamma^1},\ldots,(UF)_1^{\gamma^N},(UF)_2^{\gamma^1},\ldots,(UF)_2^{\gamma^N},\ldots,(UF)_n^{\gamma^1},\ldots,(UF)_n^{\gamma^N}\right)
$$

and

$$
F = UF \quad \text{in } X^{(q)} = (C(\Omega; \mathbb{R}))^{nN}
$$
\n
$$
IF = ((UF)_1^{\gamma^1}, \dots, (UF)_1^{\gamma^N}, (UF)_2^{\gamma^1}, \dots, (UF)_2^{\gamma^N}, \dots, (UF)_n^{\gamma^1}, \dots, (UF)_n^{\gamma^N})
$$
\n
$$
(UF)_i^{\gamma}(x) = D^{\gamma}g_i(x)
$$
\n
$$
+ \sum_{j=1}^{n} \sum_{k=1}^{m} \tilde{\alpha}_{ijk} \sum_{\substack{\alpha_1 \in \Gamma_{\gamma_k} \\ \alpha_2 \in \Gamma_{\gamma_k}}} \gamma! \frac{(b_1^{ijk})^{\alpha_1}}{\alpha_1!} \cdots \frac{(b_p^{ijk})^{\alpha_p}}{\alpha_p!} F_j^{\alpha_1 + \dots + \alpha_p}(S_{ijk}(x))
$$

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with $1 \le i \le n$ and $\gamma = (\gamma_1, \ldots, \gamma_p) \in \Gamma_q$. We note that $X^{(q)}$ is a Banach space with respect to the norm respect to the norm

On a System of
$$
F_1
$$
,..., γ_p) $\in \Gamma_q$. We note that Σ
\n
$$
||F||_{X^{(q)}} = \max_{\gamma \in \Gamma_q} \sum_{i=1}^n \sup_{z \in \Omega} |F_i^{\gamma}(x)|.
$$
\nthat $U : X^{(q)} \to X^{(q)}$ and
\n
$$
UF - U\tilde{F}||_{X^{(q)}} \leq \sigma^{(q)} ||F - \tilde{F}||_X
$$

Hence, we can easily verify that $U : X^{(q)} \to X^{(q)}$ and

$$
||UF - U\tilde{F}||_{X^{(q)}} \leq \sigma^{(q)}||F - \tilde{F}||_{X^{(q)}}
$$

for all $F, \tilde{F} \in X^{(q)}$ where

$$
\mathcal{F} \in \mathbb{F}_q
$$
\n
$$
\text{Verify that } U: X^{(q)} \to X^{(q)} \text{ and}
$$
\n
$$
\|UF - U\tilde{F}\|_{X^{(q)}} \le \sigma^{(q)} \|F - \tilde{F}\|_{X^{(q)}}
$$
\n
$$
\text{here}
$$
\n
$$
\sigma^{(q)} = \sum_{i=1}^n \sum_{k=1}^m \max_{1 \le j \le n} |\tilde{\alpha}_{ijk}| \|B^{ijk}\|_1^q \le \sigma_0 < 1
$$

by hypothesis (H''_3) . Hence, there exists a unique function $F \in X^{(q)}$ being solution of system (4.4). i.e. system (4.3). Furthermore, from the uniqueness, this solution is also "the q-order derivative" of *f ,i.e.* $=\sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \leq j \leq n} |\tilde{\alpha}_{ijk}| \|B^{ijk}\|_{1}^{q} \leq \sigma_{0} < 1$
 First F₁ Exists a unique function $F \in X^{(q)}$ *being solution of (4.3). Furthermore, from the uniqueness, this solution is also* f *, i.e.
 F_{i}^{\gamma*

$$
F_i^{\gamma} = D^{\gamma} f_i \qquad (1 \le i \le n; \, \gamma \in \Gamma_q). \tag{4.5}
$$

Then we have the following

Theorem 4.2. Suppose that $\Omega = \overline{B_r(0)}$, that the real numbers $\tilde{\alpha}_{ijk}$, matrices B^{ijk} and vectors c^{ijk} are as in Theorem 3.4, and let $q \in C^q(\Omega; \mathbb{R}^n)$. Then there exist $f \in$ $C^q(\Omega; \mathbb{R}^n)$ and $F \in X^{(q)}$ being the unique solutions of systems $(3.2) - (3.3)$ and (4.3) , *respectively. Furthermore, we have also (4.5).* max max *I'ijkI lI ^B II <* 1 (4.6)

Remark 4.1. In the case of $\Omega = \mathbb{R}^p$ we suppose additionally that the real numbers $\tilde{\alpha}_{ijk}$ and the matrices B^{ijk} satisfy the condition

$$
\max_{0 \le s \le q} \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \le j \le n} |\tilde{\alpha}_{ijk}| \|B^{ijk}\|_{1}^{s} < 1
$$
\n(4.6)

(note that B^{ijk} and c^{ijk} need not to satisfy conditions hypothesis (H''_1)). Then, if

$$
g\in C_b^q(R^p;R^n)=\left\{g\in C_b(R^p;R^n):D^\gamma g_i\in C_b(R^p;R^n)\ \left(|\gamma|\leq q;\,1\leq i\leq n\right)\right\}
$$

the conclusion of Theorem 4.2 is still true, where the functional spaces $C^q(\Omega; \mathbb{R}^n)$ and $X^{(q)}$ appearing in Theorem 4.2 are replaced by $C_b^q(R^p; \mathbb{R}^n)$ and $(C_b(R^p; \mathbb{R}^n))^{nN}$, respectively, where $N = \frac{(p+q-1)!}{q!(p-1)!}$. The proof of this result is the same as that of Theorem 4.2.

Now we return to the same case of $\Omega = \overline{B_r(0)}$. Suppose that $f \in C^q(\Omega; \mathbb{R}^n)$ is the unique solution of system (3.2) - (3.3) corresponding to $g \in C^{q}(\Omega; \mathbb{R}^{n})$. For each $\tilde{q} = 1, \ldots, q$ we have F_i^{γ} $(\gamma \in \Gamma_{\tilde{q}}; 1 \leq i \leq n)$ as in Theorem 4.2 corresponding to $q = \tilde{q}$. Then from the Maclaurin formula we have sion of Theorem 4.2 is still true, where the functional spaces $C^{\gamma}(\Omega)$
aring in Theorem 4.2 are replaced by $C^q_b(R^p; \mathbb{R}^n)$ and $(C_b(R^p; \mathbb{R}^n)$
where $N = \frac{(p+q-1)!}{q!(p-1)!}$. The proof of this result is the same as $\frac{q-1}{p-1}$. The pr
 e same case of
 χ $\gamma \in \Gamma_{\vec{q}}$; $1 \leq i$

formula we h
 $\frac{1}{\gamma!} D^{\gamma} f_i(0) x^{\gamma}$

$$
f_i(x) = \sum_{|\gamma| \le q-1} \frac{1}{\gamma!} D^{\gamma} f_i(0) x^{\gamma} + q \int_0^1 (1-t)^{q-1} \sum_{|\gamma| = q} \frac{1}{\gamma!} D^{\gamma} f_i(tx) x^{\gamma} dt
$$

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for $1 \leq i \leq n$. On the other hand, we have

and N. H. Nghia
ne other hand, we have

$$
F_i^{\gamma} = \begin{cases} f_i & \text{if } |\gamma| = 0 \\ D^{\gamma} f_i & \text{if } 1 \le |\gamma| \le q \end{cases} \qquad (1 \le i \le n).
$$
 (4.7)

It follows that

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\nn. On the other hand, we have

\n
$$
F_i^{\gamma} = \begin{cases} f_i & \text{if } |\gamma| = 0 \\ D^{\gamma} f_i & \text{if } 1 \leq |\gamma| \leq q \end{cases} \qquad (1 \leq i \leq n). \tag{4.7}
$$
\n1. (4.7)

\n1. (4.8)

\n
$$
f_i(x) = \sum_{|\gamma| \leq q-1} \frac{1}{\gamma!} F_i^{\gamma}(0) x^{\gamma} + q \int_0^1 (1-t)^{q-1} \sum_{|\gamma| = q} \frac{1}{\gamma!} F_i^{\gamma}(tx) x^{\gamma} dt \qquad (4.8)
$$
\nn. (5.1)

\n1. (4.8)

\n2. (4.1)

\n3. (4.1)

\n4. (4.2)

\n5. (4.3)

\n6. (4.4)

\n7. (4.5)

\n8. (4.5)

\n9. (4.7)

\n10. (4.8)

\n11. (4.9)

\n12. (4.1)

\n13. (4.1)

\n14. (4.2)

\n15. (4.1)

\n16. (4.2)

\n17. (4.3)

\n18. (4.4)

\n19. (4.5)

\n19. (4.7)

\n11. (4.8)

\n11. (4.9)

\n12. (4.1)

\n13. (4.1)

\n14. (4.2)

\n15. (4.1)

\n16. (4.1)

\n17. (4.1)

\n18. (4.2)

\n19. (4.1)

\n11. (4.2)

\n11. (4.3)

\n12. (4.2)

\n13. (4.3)

\n14. (4.4)

\n15. (4.3)

\n16. (4.4)

\n17. (4.5)

\n18. (4.5)

\n19.

for $1 \leq i \leq n$.

Inversely, suppose that a function $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n) \in C^q(\Omega; \mathbb{R}^n)$ is given by the formula

$$
\tilde{f}_i(x) = \sum_{|\gamma| \le q-1} \frac{1}{\gamma!} F_i^{\gamma}(0) x^{\gamma} + q \int_0^1 (1-t)^{q-1} \sum_{|\gamma| = q} \frac{1}{\gamma!} F_i^{\gamma}(tx) x^{\gamma} dt
$$

for $1 \leq i \leq n$. Then from (4.7) and (4.8) we have

inversely, suppose that a function
$$
f = (f_1, \ldots, f_n) \in C^q(\Omega; \mathbb{R}^n)
$$
 is given by:

\nand

\n
$$
\tilde{f}_i(x) = \sum_{|\gamma| \leq q-1} \frac{1}{\gamma!} F_i^{\gamma}(0) x^{\gamma} + q \int_0^1 (1-t)^{q-1} \sum_{|\gamma| = q} \frac{1}{\gamma!} F_i^{\gamma}(tx) x^{\gamma} dt
$$
\n
$$
\leq i \leq n.
$$
\nThen from (4.7) and (4.8) we have

\n
$$
\tilde{f}_i(x) = \sum_{|\gamma| \leq q-1} \frac{1}{\gamma!} D^{\gamma} f_i(0) x^{\gamma} + q \int_0^1 (1-t)^{q-1} \sum_{|\gamma| = q} \frac{1}{\gamma!} D^{\gamma} f_i(tx) x^{\gamma} dt = f_i(x)
$$
\n
$$
\lim_{|\gamma| \leq q-1} \int_0^1 (1-t)^{q-1} \sum_{|\gamma| = q} \frac{1}{\gamma!} D^{\gamma} f_i(tx) x^{\gamma} dt = f_i(x)
$$

for all $x \in \Omega$. Therefore \tilde{f} is a solution of system (3.2) - (3.3).

Then we have the following

Theorem 4.3. Under the hypotheses of Theorem 3.4, let $g \in C^q(\Omega; \mathbb{R}^n)$. Then *the solution* $f \in C^{q}(\Omega; \mathbb{R}^{n})$ *of system* $(3.2) - (3.3)$ is represented by (4.8) where $F_i^{\gamma} \in$ $C(\Omega;\mathbb{R})$ $(1 \leq i \leq n; \gamma \in \Gamma_{\tilde{q}}$ with $0 < \tilde{q} \leq q)$ is the unique solution of problem $(S_{\tilde{q}})$. *Inversely, every function* $f \in C^{q}(\Omega; \mathbb{R}^{n})$ *represented by* (4.8) is a solution of system *(3.2) - (3.3).*

Remark 4.2. In the case of $\Omega = \mathbb{R}^p$, and real numbers $\tilde{\alpha}_{ijk}$ and matrices B^{ijk} satisfying condition (4.6), if $g \in C_b^q(\mathbb{R}^p; \mathbb{R}^n)$, the conclusion of Theorem 4.3 is still true, where the functional spaces $C(\Omega; \mathbb{R})$ and $C^q(\Omega; \mathbb{R}^n)$ appearing in Theorem there are where the functional spaces $C(\Omega; \mathbb{R})$ and $C^q(\Omega; \mathbb{R}^n)$ appearing in Theorem there are replaced by $C_b(\mathbb{R}^p;\mathbb{R})$ and $C_b^q(\mathbb{R}^p;\mathbb{R}^n)$, respectively.

Returning to the case of $\Omega = \overline{B_r(0)}$ we have the following

Corollary 4.1. *If* g_1, \ldots, g_n are polynomials of degree not greater than $q-1$, the solution f of system $(3.2) - (3.3)$ is also a sequence of such polynomials.

Proof. We have $D^{\gamma}g_i(x) = 0$ for $x \in \Omega$, $|\gamma| \ge q$ and $1 \le i \le n$. Then $F_i^{\gamma} =$ 0 $(|\gamma| = q; 1 \le i \le n)$ is the unique solution of system (4.3). Applying (4.8) we obtain $f_i(x) = \sum_{|\gamma| \leq q-1} \frac{1}{\gamma!} F_i^{\gamma}(0) x^{\gamma}$ and the statement is proved **I**

Theorem 4.4. Under the hypotheses of Theorem 3.4, suppose that $f \in C^{q}(\Omega; \mathbb{R}^{n})$ is *the solution of system* $(3.2) - (3.3)$ corresponding to $g \in C^q(\Omega; \mathbb{R}^n)$ and $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)$ is a sequence of polynomials of degree not greater than $q-1$ and satisfying system $(3.2) - (3.3)$ corresponding to $\tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_n)$ where On a System of Functional
 r the hypotheses of Theorem 3.4, suppose t

(i) – (3.3) corresponding to $g \in C^q(\Omega; \mathbb{R}^n)$
 ials of degree not greater than $q - 1$ *a:*
 g to $\tilde{g} = (\tilde{g}_1, ..., \tilde{g}_n)$ where
 $= \sum_{|\gamma| \$ Or
 IIf the hypotheses of $\{2\}$ — (3.3) corresponds
 IIIs of degree not
 III σ \bar{g} = $(\tilde{g}_1, \ldots, \tilde{g}_n)$
 $= \sum_{|\gamma| \leq q-1} \frac{1}{\gamma!} D^{\gamma} g_i(0)$
 $||f - \tilde{f}||_X \leq \frac{r^q}{1-\sigma_0}$

Maclaurin expansion stem of Functional Equations 1027
 em 3.4, suppose that $f \in C^q(\Omega; \mathbb{R}^n)$ is
 $o \ g \in C^q(\Omega; \mathbb{R}^n)$ and $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)$
 er than $q - 1$ and satisfying system
 re
 $(i = 1, \ldots, n).$
 $\frac{1}{\gamma!} ||D^{\gamma} g||_$ *C*^q(*D*_{*i*} Rⁿ) and $\tilde{f} = (\tilde{f}_1, ..., \tilde{f}_n)$
 D^q(*D*_{*i*} Rⁿ) and $\tilde{f} = (\tilde{f}_1, ..., \tilde{f}_n)$
 in $q - 1$ and satisfying system
 $= 1, ..., n$).
 γ ^q \parallel x. (4.9)
 $\leq i \leq n$) until the order q
 $D^{\gamma}g_i(tx)x^{\gamma$

$$
\tilde{g}_i(x) = \sum_{|\gamma| \leq q-1} \frac{1}{\gamma!} D^{\gamma} g_i(0) x^{\gamma} \qquad (i = 1, \ldots, n).
$$

Then

ng to
$$
\tilde{g} = (\tilde{g}_1, ..., \tilde{g}_n)
$$
 where
\n
$$
= \sum_{|\gamma| \leq q-1} \frac{1}{\gamma!} D^{\gamma} g_i(0) x^{\gamma} \qquad (i = 1, ..., n).
$$
\n
$$
||f - \tilde{f}||_X \leq \frac{r^q}{1 - \sigma_0} \sum_{|\gamma| = q} \frac{1}{\gamma!} ||D^{\gamma} g||_X. \qquad (4.9)
$$
\nMaclaurin expansion of g_i $(1 \leq i \leq n)$ until the order q
\n $\tilde{g}_i(x) + q \int_0^1 (1 - t)^{q-1} \sum_{|\gamma| = q} \frac{1}{\gamma!} D^{\gamma} g_i(tx) x^{\gamma} dt. \qquad (4.10)$
\nwith $\sigma = \sigma_0$ we have
\n
$$
||f - \tilde{f}||_X \leq \frac{1}{1 - \sigma_0} ||g - \tilde{g}||_X. \qquad (4.11)
$$

Proof. We have the Maclaurin expansion of g_i $(1 \leq i \leq n)$ until the order q

have the Maclaurin expansion of
$$
g_i
$$
 $(1 \le i \le n)$ until the order q
\n
$$
g_i(x) = \tilde{g}_i(x) + q \int_0^1 (1-t)^{q-1} \sum_{\substack{|\gamma| = q}} \frac{1}{\gamma!} D^{\gamma} g_i(tx) x^{\gamma} dt.
$$
\n(4.10)
\ne (3.1) with $\sigma = \sigma_0$ we have
\n
$$
||f - \tilde{f}||_X \le \frac{1}{1 - \sigma_0} ||g - \tilde{g}||_X.
$$
\n(4.11)
\n
$$
|A \tilde{f}|_X = \frac{1}{1 - \sigma_0} ||g - \tilde{g}||_X.
$$

Applying estimate (3.1) with $\sigma = \sigma_0$ we have

$$
||f - \tilde{f}||_X \le \frac{1}{1 - \sigma_0} ||g - \tilde{g}||_X. \tag{4.11}
$$

From *(4.10)* we have

Applying estimate (3.1) with
$$
\sigma = \sigma_0
$$
 we have
\n
$$
||f - \tilde{f}||_X \le \frac{1}{1 - \sigma_0} ||g - \tilde{g}||_X.
$$
\n(4.11)
\nFrom (4.10) we have
\n
$$
||g - \tilde{g}||_X = \sup_{x \in \Omega} \sum_{i=1}^n |g_i(x) - \tilde{g}_i(x)|
$$
\n
$$
\le q \int_0^1 (1 - t)^{q-1} dt \sum_{|\gamma| = q} \frac{1}{\gamma!} \sup_{x \in \Omega} \sum_{i=1}^n |D^{\gamma}g_i(tx)| |x^{\gamma}|.
$$
\nWe note that $\sup_{x \in \Omega} \sum_{i=1}^n |D^{\gamma}g_i(tx)| \le ||D^{\gamma}g||_X$ for all $t \in [0, 1]$ and $|x^{\gamma}| \le ||x||_1^{|\gamma|} \le r^q$
\nfor all $x \in \Omega$ and $|\gamma| = q$. Hence we obtain (4.9) from (4.11) \blacksquare
\nCorollary 4.2. Under the hypotheses of Theorem 3.4, if for $g \in C^{\infty}(\Omega; \mathbb{R}^n)$ there
\nexists $d > 0$ such that
\n
$$
||D^{\gamma}g||_X \le d^{|\gamma|} \qquad (\gamma \in \mathbb{Z}_+^p), \qquad (4.12)
$$
\nf is a solution of system (3.2) - (3.3) corresponding to g and $\tilde{f}^{[q]}$ is a sequence of
\npolynomials of degree not greater than $q - 1$ satisfying system (3.2) - (3.3) corresponding
\nto \tilde{g} as in \tilde{g} there at t then $q - 1$ satisfying system (3.2) - (3.3) corresponding

for all $x \in \Omega$ and $|\gamma| = q$. Hence we obtain (4.9) from (4.11) $\left|\gamma\right| = q$
 $\left|\gamma g_i(tx)\right| \leq \left|\left|D^\gamma g\right|\right| \geq q$
 nce we obtain (4.9)
 he hypotheses of Th
 $\left|\left|D^\gamma g\right|\right| \geq \left|d^{\left|\gamma\right|}$
 $\left|\left|D^\gamma g\right|\right| \geq (3.3)$ *correspe*

Corollary 4.2. Under the hypotheses of Theorem 3.4, if for $q \in C^{\infty}(\Omega; \mathbb{R}^n)$ there *exists d > 0 such that*

$$
||D^{\gamma}g||_X \le d^{|\gamma|} \qquad (\gamma \in \mathbb{Z}_+^p), \tag{4.12}
$$

f is a solution of system $(3.2) - (3.3)$ corresponding to g and $\tilde{f}^{[q]}$ is a sequence of *polynomials of degree not greater than q - 1 satisfying system (3.2) - (3.3) corresponding to* \tilde{g} *as in Theorem 4.4, then*
 $\lim_{q \to +\infty} ||f - \tilde{f}^{[q]}||_X = 0.$
 Moreover, we have the estimates
 $||f - \tilde{f}^{[q]}||_X \leq \$ *to j as in Theorem 4.4, then li* - Hence we obtained the hypoth
 $||D^{\gamma}g||_X$
 $em (3.2) - (3.3)$

of greater than
 lim
 d - $\lim_{q \to +\infty}$

estimates
 $||f - \tilde{f}^{[q]}||_X \leq \lim_{q \to +\infty}$
 $||f - \tilde{f}^{[q]}||_X$ *(d)* \blacksquare (d) \blacksquare
 (d) \blacksquare
 (d) \blacksquare
 (d) **1)** \blacksquare
 (d) (d) \blacksquare
 (d) (d) \blacksquare
 (g) \blacksquare *(d) (d) (d)*
 (g) \blacksquare *(d) (d) (d)* corresponding to g and $f^{(4)}$ is a sequence of

1 satisfying system (3.2) – (3.3) corresponding
 $-\tilde{f}^{[q]} \|_X = 0.$
 $\frac{1}{\sigma_0} \frac{(dp\tau)^q}{q!}$ ($q \in \mathbb{N}$). (4.13)

(2) that
 $\leq \frac{(rd)^q}{1-\sigma_0} \sum_{|\gamma|=q} \frac{1}{\gamma!}$ (4.14)

$$
\lim_{q \to +\infty} \|f - \tilde{f}^{[q]}\|_X = 0.
$$

Moreover, we have the estimates

estimates
\n
$$
||f - \tilde{f}^{[q]}||_X \le \frac{1}{1 - \sigma_0} \frac{(dpr)^q}{q!} \qquad (q \in \mathbb{N}).
$$
\n(4.13)

Proof. It follows from (4.9) and (4.12) that
\n
$$
||f - \tilde{f}^{[q]}||_X \leq \frac{(rd)^q}{1 - \sigma_0} \sum_{|\gamma| = q} \frac{1}{\gamma!}.
$$
\n(4.14)
\n
$$
||f - \tilde{f}^{[q]}||_X \leq \frac{(rd)^q}{1 - \sigma_0} \sum_{|\gamma| = q} \frac{1}{\gamma!}.
$$
\n(4.14)
\n
$$
||f - \tilde{f}^{[q]}||_X \leq \frac{(rd)^q}{1 - \sigma_0} \sum_{|\gamma| = q} \frac{1}{\gamma!}.
$$
\n(4.15)
\n
$$
||f - \tilde{f}^{[q]}||_X \leq \frac{(rd)^q}{1 - \sigma_0} \sum_{|\gamma| = q} \frac{1}{\gamma!}.
$$

Moreover, we have the estimates
 $||f - \tilde{f}^{[q]}||_X$
 Proof. It follows from (4.9) an
 $||f - \tilde{f}^{[q]}||_X$

Using the equality $\left(\sum_{i=1}^p x_i\right)^q =$

follows that $\sum_{|\gamma|=q} \frac{1}{\gamma!} = \frac{p^q}{q!}$. Hence polynomials of aegrec not greater than $q-1$ satisfying system (3.2)-
to \tilde{g} as in Theorem 4.4, then
 $\lim_{q \to +\infty} ||f - \tilde{f}^{[q]}||_X = 0.$
Moreover, we have the estimates
 $||f - \tilde{f}^{[q]}||_X \le \frac{1}{1-\sigma_0} \frac{(dpr)^q}{q!}$ $(q \in \math$

Corollary 4.3. *Under the hypotheses of Theorem 3.4, let* $g \in C(\Omega; \mathbb{R}^n)$ *and f be the solution of system (3.2) - (3.3) corresponding to g. Then there exists a sequence* $\tilde{f}^{[q]} = (\tilde{f}^{[q]}_1, \ldots, \tilde{f}^{[q]}_n)$ of polynomials of degree not greater than $q-1$ such that

$$
\lim_{q \to +\infty} \|f - \tilde{f}^{[q]}\|_X = 0.
$$

Proof. By the Weierstrass theorem, each function *gj* is approximated by a sequence of polynomials $P_i^{[q]}$ converging uniformly to g_i when the degree $q - 1 \rightarrow +\infty$. Hence, $P^{[q]} = (P_1^{[q]}, \ldots, P_n^{[q]})$ converges in $C(\Omega; \mathbb{R}^n)$ to *g* when $q \to +\infty$. Let $\tilde{f}^{[q]}$ be a polynomial solution of system (3.2) - (3.3) corresponding to $g = P^{\{q\}}$. By estimate (3.1) with $= \sigma_0$, $\tilde{f} = \tilde{f}^{[q]}$ and $\tilde{g} = P^{[q]}$ we have $\lim_{q \to +\infty} \|f - \tilde{f}\|$
strass theorem, each
rging uniformly to
verges in $C(\Omega; \mathbb{R}^n)$
3.2) - (3.3) correspo
 $P^{[q]}$ we have
 $\|\tilde{f}^{[q]} - f\|_{X} \leq \frac{1}{1 - \epsilon}$ *heorem* 3.4, let $g \in C(\Omega; \mathbb{R}^n)$ and f be
 ng to g . Then there exists a sequence
 t greater than $q - 1$ such that
 $x = 0$.

action g_i is approximated by a sequence

when the degree $q - 1 \rightarrow +\infty$. Hence,
 g for extractional substraints interesting the distribution of the degree $q - 1 \rightarrow +\infty$. Hence, converges in $C(\Omega; \mathbb{R}^n)$ to g when $q \rightarrow +\infty$. Let $\tilde{f}^{[q]}$ be a polynom (3.2) - (3.3) corresponding to $g = P^{[q]}$. By est

$$
\|\tilde{f}^{[q]} - f\|_{X} \le \frac{1}{1 - \sigma_0} \|P^{[q]} - g\| \to 0
$$
\n(4.15)

as $q \to +\infty$

Remark 4.3. The results of Theorem *4.1 - 4.4* generalize those in [5].

Example 1. Consider the linear system with $m = 1$ and $n = p = 2$

$$
f_i(x) = \sum_{j=1}^{2} \alpha_{ij} f_j(\beta_{ij} x) + g_i(x) \qquad (i = 1, 2)
$$
 (4.16)

for

$$
f_i(x) = \sum_{j=1} \alpha_{ij} f_j(\beta_{ij} x) + g_i(x) \qquad (i = 1, 2)
$$

$$
x \in \Omega = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : ||x||_1 = |x_1| + |x_2| \le 1 \right\}
$$

where α_{ij} and β_{ij} are given real given numbers satisfying

$$
|\beta_{ij}| \le 1
$$

$$
\sum_{i=1}^{2} \max_{1 \le j \le 2} |\alpha_{ij}| < 1
$$

(i) Let $g_i(x) = \sum_{|\gamma| \le r} d_{i\gamma} x^{\gamma}$
 i r. It follows from Corollary 4. $(i = 1, 2)$ be polynominals of degree not greater than r . It follows from Corollary 4.1 that the solution f of system (4.16) consists of (i) Let $g_i(x) = \sum_{|\gamma| \le r} d_{i\gamma} x^{\gamma}$ ($i = 1, 2$) be polynomithan r. It follows from Corollary 4.1 that the solution *f* of polynomials of the same type. Put $f_i(x) = \sum_{|\gamma| \le r} c_{i\gamma} x^{\gamma}$ into (4.16) we obtain (c_1, c_2) th $(i = 1, 2)$. Substituting f_i into (4.16) we obtain $(c_{1\gamma}, c_{2\gamma})$ that is the solution of a linear system *composity if* $|\beta_{ij}| \le 1$
 $\sum_{i=1}^{2} \max_{1 \le j \le 2} |\alpha_{ij}| < 1$
 $\sum_{|\gamma| \le r} d_{i\gamma} x^{\gamma}$ (*i* = 1,2) be polynominals of

rom Corollary 4.1 that the solution *f* of systes

same type. Put $f_i(x) = \sum_{|\gamma| \le r} c_{i\gamma} x^{\gamma}$ (*i* = 1 (a) $\frac{\gamma}{\gamma} \left[\frac{c}{r} \frac{d_{i\gamma}r^{\gamma}}{dt} \right]$ (*i* = 1, 2) be polynomina

Corollary 4.1 that the solution f of is
 γ , $c_{2\gamma}$) that is the solution of a linear
 $\sum_{j=1}^{2} \alpha_{ij} \beta_{ij}^{|\gamma|} c_{j\gamma} = d_{i\gamma}$ (*i* = 1, 2; | $\$

$$
c_{i\gamma} - \sum_{j=1}^2 \alpha_{ij} \beta_{ij}^{|\gamma|} c_{j\gamma} = d_{i\gamma} \qquad (i = 1, 2; |\gamma| \le r).
$$

Hence

$$
i=1
$$
\n
$$
g_i(x) = \sum_{|\gamma| \le r} d_{i\gamma} x^{\gamma} \quad (i = 1, 2) \text{ be polynomials of degree no follows from Corollary 4.1 that the solution } f \text{ of system (4.16) or less than the value of } f \text{ of the same type. Put } f_i(x) = \sum_{|\gamma| \le r} c_{i\gamma} x^{\gamma} \quad (i = 1, 2). \text{ Substitute obtain } (c_{1\gamma}, c_{2\gamma}) \text{ that is the solution of a linear system}
$$
\n
$$
c_{i\gamma} - \sum_{j=1}^{2} \alpha_{ij} \beta_{ij}^{|\gamma|} c_{j\gamma} = d_{i\gamma} \qquad (i = 1, 2; |\gamma| \le r).
$$
\n
$$
c_{1\gamma} = \frac{(1 - \alpha_{22} \beta_{22}^{|\gamma|}) d_{1\gamma} + \alpha_{12} \beta_{12}^{|\gamma|} d_{2\gamma}}{(1 - \alpha_{11} \beta_{11}^{|\gamma|})(1 - \alpha_{22} \beta_{22}^{|\gamma|}) - \alpha_{12} \alpha_{21} \beta_{12}^{|\gamma|} \beta_{21}^{|\gamma|}}
$$
\n
$$
c_{2\gamma} = \frac{\alpha_{21} \beta_{21}^{|\gamma|} d_{1\gamma} + (1 - \alpha_{11} \beta_{11}^{|\gamma|}) d_{2\gamma}}{(1 - \alpha_{11} \beta_{11}^{|\gamma|})(1 - \alpha_{22} \beta_{22}^{|\gamma|}) - \alpha_{12} \alpha_{21} \beta_{12}^{|\gamma|} \beta_{21}^{|\gamma|}}
$$
\n
$$
(|\gamma| \le r).
$$

(ii) Let $g \in C^q(\Omega; R^2)$. It follows from Theorem 4.4 that

On a System of Fur
\n(ii) Let
$$
g \in C^q(\Omega; R^2)
$$
. It follows from Theorem 4.4 that
\n
$$
||f - \tilde{f}||_X \le \frac{1}{1 - \sigma_0} \sum_{|\gamma| = q} \frac{1}{\gamma!} ||D^{\gamma}g||_X
$$
\nwhere $\sigma_0 = \sum_{i=1}^2 \max_{1 \le j \le 2} |\alpha_{ij}|$,
\n
$$
\tilde{f}_i(x) = \sum_{|\gamma| \le q-1} c_{i\gamma} x^{\gamma} \qquad (i = 1, 2)
$$

where $\sigma_0 = \sum_{i=1}^2 \max_{1 \leq j \leq 2} |\alpha_{ij}|$,

$$
\tilde{f}_i(x) = \sum_{|\gamma| \le q-1} c_{i\gamma} x^{\gamma} \qquad (i = 1, 2)
$$

and

$$
i \quad \text{or} \quad \frac{1}{|\gamma|} = q^{-1}.
$$
\n
$$
j = \sum_{i=1}^{2} \max_{1 \leq j \leq 2} |\alpha_{ij}|,
$$
\n
$$
\tilde{f}_i(x) = \sum_{|\gamma| \leq q-1} c_{i\gamma} x^{\gamma} \quad (i = 1, 2)
$$
\n
$$
c_{1\gamma} = \frac{(1 - \alpha_{22}\beta_{22}^{|\gamma|})\frac{1}{\gamma!}D^{\gamma}g_1(0) + \alpha_{12}\beta_{12}^{|\gamma|}\frac{1}{\gamma!}D^{\gamma}g_2(0)}{(1 - \alpha_{11}\beta_{11}^{|\gamma|})(1 - \alpha_{22}\beta_{22}^{|\gamma|}) - \alpha_{12}\alpha_{21}\beta_{12}^{|\gamma|}\beta_{21}^{|\gamma|}}
$$
\n
$$
c_{2\gamma} = \frac{\alpha_{21}\beta_{21}^{|\gamma|}\frac{1}{\gamma!}D^{\gamma}g_1(0) + (1 - \alpha_{11}\beta_{11}^{|\gamma|})\frac{1}{\gamma!}D^{\gamma}g_2(0)}{(1 - \alpha_{11}\beta_{11}^{|\gamma|})(1 - \alpha_{22}\beta_{22}^{|\gamma|}) - \alpha_{12}\alpha_{21}\beta_{12}^{|\gamma|}\beta_{21}^{|\gamma|}}
$$
\n
$$
\text{Let } g = (g_1, g_2) \text{ with } g_i(x) = (1 - \frac{z_1 + z_2}{1 + i})^{-1} \quad (i = 1, 2; \ x = (x_1, x_2) \in \Omega).
$$
\n
$$
g_i(x) = \sum_{j=0}^{\infty} \left(\frac{x_1 + x_2}{1 + i}\right)^j = \sum_{|\gamma| \leq q-1} \frac{1}{\gamma!} \frac{|\gamma|!}{(1 + i)^{|\gamma|}} x^{\gamma} + \sum_{j \geq q} \left(\frac{x_1 + x_2}{1 + i}\right)^j.
$$

(iii) Let $g = (g_1, g_2)$ with $g_i(x) = (1 - \frac{x_1 + x_2}{1+i})^{-1}$ $(i = 1, 2; x = (x_1, x_2) \in \Omega)$. We rewrite g_i as

$$
c_{2\gamma} = \frac{\alpha_{21}\beta_{21}^{|\gamma|} \frac{1}{\gamma!}D^{\gamma}g_{1}(0) + (1 - \alpha_{11}\beta_{11}^{|\gamma|}) \frac{1}{\gamma!}D^{\gamma}g_{2}(0)}{(1 - \alpha_{11}\beta_{11}^{|\gamma|})(1 - \alpha_{22}\beta_{22}^{|\gamma|}) - \alpha_{12}\alpha_{21}\beta_{12}^{|\gamma|}\beta_{21}^{|\gamma|}})
$$

Let $g = (g_{1}, g_{2})$ with $g_{i}(x) = (1 - \frac{x_{1} + x_{2}}{1 + i})^{-1}$ $(i = 1, 2; x = (x_{1}, x_{2}) \in$

$$
g_{i}(x) = \sum_{j=0}^{\infty} \left(\frac{x_{1} + x_{2}}{1 + i}\right)^{j} = \sum_{|\gamma| \le q-1} \frac{1}{\gamma!} \frac{|\gamma|!}{(1 + i)^{|\gamma|}} x^{\gamma} + \sum_{j \ge q} \left(\frac{x_{1} + x_{2}}{1 + i}\right)^{j}.
$$

$$
P_{i}^{[q]}(x) = \sum_{|\gamma| \le q-1} \frac{1}{\gamma!} \frac{|\gamma|!}{(1 + i)^{|\gamma|}} x^{\gamma}
$$

Putting

$$
P_i^{[q]}(x) = \sum_{|\gamma| \le q-1} \frac{1}{\gamma!} \frac{|\gamma|!}{(1+i)^{|\gamma|}} x^{\gamma}
$$

we have

Putting
\n
$$
P_i^{[q]}(x) = \sum_{|\gamma| \le q-1} \frac{1}{\gamma!} \frac{|\gamma|!}{(1+i)^{|\gamma|}} x^{\gamma}
$$
\nwe have
\n
$$
|g_i(x) - P_i^{[q]}(x)| = \left| \sum_{j \ge q} \left(\frac{x_1 + x_2}{1+i} \right)^j \right| \le \sum_{j \ge q} \frac{1}{(1+i)^j} = \frac{1}{i(1+i)^{q-1}} \qquad (4.17)
$$
\nfor $x \in \Omega$. Hence $P_i^{[q]} \to g_i$ uniformly on Ω as $q \to +\infty$. Applying inequalities (4.15)
\nand (4.17) we obtain
\n
$$
||\tilde{f}^{[q]} - f||_X \le \frac{1}{1-\sigma_0} ||P^{[q]} - g||_X \le \frac{1}{1-\sigma_0} \left(\frac{1}{2^{q-1}} + \frac{1}{2 \cdot 3^{q-1}} \right) \to +\infty
$$
\nas $q \to +\infty$ where $\tilde{f}^{[q]} = (\tilde{f}_1^{[q]}, \tilde{f}_2^{[q]})$ with $\tilde{f}_i^{[q]}(x) = \sum_{|\gamma| \le q-1} c_{i\gamma} x^{\gamma}$ $(i = 1, 2)$ and the

$$
\|\tilde{f}^{[q]} - f\|_{X} \le \frac{1}{1 - \sigma_0} \|P^{[q]} - g\|_{X} \le \frac{1}{1 - \sigma_0} \left(\frac{1}{2^{q-1}} + \frac{1}{2 \cdot 3^{q-1}}\right) \to +\infty
$$

for $x \in \Omega$. Hence $P_i^{[q]} \to g_i$ uniformly on Ω as $q \to +\infty$. Applying is and (4.17) we obtain $\|\tilde{f}^{[q]} - f\|_X \leq \frac{1}{1 - \sigma_0} \|P^{[q]} - g\|_X \leq \frac{1}{1 - \sigma_0} \left(\frac{1}{2^{q-1}} + \frac{1}{2 \cdot 3^{q-1}}\right)$
as $q \to +\infty$ where $\tilde{f}^{[q]}$ $\rightarrow +\infty$ where $\tilde{f}^{[q]} = (\tilde{f}_1^{[q]}, \tilde{f}_2^{[q]})$ with $\tilde{f}_i^{[q]}(x) = \sum_{|\gamma| \le q-1} c_{i\gamma} x^{\gamma}$ $(i = 1, 2)$ and the cients $c_{i\gamma}$ are calculated by the same formulas as in the case (i) with $D^{\gamma}g_i(0) = |\gamma| \le q - 1$. coefficients $c_{i\gamma}$ are calculated by the same formulas as in the case (i) with $D^{\gamma}g_i(0) = \frac{|\gamma|!}{(1+i)^{|\gamma|}}$ ($|\gamma| \leq q - 1$). Let $P_i^{[q]} \to g_i$ uniformly on Ω as $q \to +\infty$. Applying inequalition
 $\|f\|_X \leq \frac{1}{1-\sigma_0} \|P^{[q]} - g\|_X \leq \frac{1}{1-\sigma_0} \left(\frac{1}{2^{q-1}} + \frac{1}{2 \cdot 3^{q-1}}\right) \to +\infty$

here $\tilde{f}^{[q]} = (\tilde{f}_1^{[q]}, \tilde{f}_2^{[q]})$ with $\tilde{f}_i^{[q]}($

5. The second order algorithm

In this section, we consider the algorithm for system (1.1)

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\n5. The second order algorithm
\nIn this section, we consider the algorithm for system (1.1)
\n
$$
f_i^{(v)}(x) = \sum_{j=1}^n \sum_{k=1}^m \left\{ a_{ijk} [x, f_j^{(v-1)}(S_{ijk}(x))] + \frac{\partial a_{ijk}}{\partial y} [x, f_j^{(v-1)}(S_{ijk}(x))] [f_j^{(v)}(S_{ijk}(x)) - f_j^{(v-1)}(S_{ijk}(x))] \right\}
$$
\n
$$
+ g_i(x)
$$
\nfor $x \in \Omega_i$, $1 \le i \le n$ and $v \ge 1$ where $f^{(0)} = (f_1^{(0)}, \ldots, f_n^{(0)}) \in X$ is given. Rewrite
\n(5.1) as linear system of functional equations
\n
$$
f_i^{(v)}(x) = \sum_{j=1}^n \sum_{k=1}^m \alpha_{ijk}^{(v)}(x) f_j^{(v)}(S_{ijk}(x)) + g_i^{(v)}(x)
$$
\nwhere
\n
$$
\alpha_{ijk}^{(v)}(x) = \frac{\partial a_{ijk}}{\partial y} [x, f_j^{(v-1)}(S_{ijk}(x))]
$$
\n(5.2)
\nand

and $v \ge 1$ where $f^{(0)}$

functional equations
 $(x) = \sum_{i=1}^{n} \sum_{i=1}^{m} \alpha_{ijk}^{(v)}(x)$

$$
+ g_i(x)
$$
\nfor $x \in \Omega_i$, $1 \leq i \leq n$ and $v \geq 1$ where $f^{(0)} = (f_1^{(0)}, \ldots, f_n^{(0)}) \in X$ is given. Rewrite (5.1) as linear system of functional equations

\n
$$
f_i^{(v)}(x) = \sum_{j=1}^n \sum_{k=1}^m \alpha_{ijk}^{(v)}(x) f_j^{(v)}(S_{ijk}(x)) + g_i^{(v)}(x) \tag{5.2}
$$
\nwhere

\n
$$
\alpha_{ijk}^{(v)}(x) = \frac{\partial a_{ijk}}{\partial y} [x, f_j^{(v-1)}(S_{ijk}(x))]
$$
\nand

\n
$$
(5.3)
$$

where

$$
\alpha_{ijk}^{(v)}(x) = \frac{\partial a_{ijk}}{\partial y} \left[x, f_j^{(v-1)}(S_{ijk}(x)) \right] \tag{5.3}
$$

and

$$
f_i^{(v)}(x) = \sum_{j=1}^n \sum_{k=1}^m \alpha_{ijk}^{(v)}(x) f_j^{(v)}(S_{ijk}(x)) + g_i^{(v)}(x)
$$
(5.2)
here

$$
\alpha_{ijk}^{(v)}(x) = \frac{\partial a_{ijk}}{\partial y} [x, f_j^{(v-1)}(S_{ijk}(x))]
$$
(5.3)
d

$$
g_i^{(v)}(x) = g_i(x) + \sum_{j=1}^n \sum_{k=1}^m \left\{ a_{ijk} [x, f_j^{(v-1)}(S_{ijk}(x))] - \alpha_{ijk}^{(v)}(x) f_j^{(v-1)}(S_{ijk}(x)) \right\}.
$$
(5.4)

Thus we have the following

Theorem 5.1. Let hypotheses $(H_1) - (H_2)$ hold and suppose $a_{ijk} \in C(\Omega_i \times \mathbb{R}; \mathbb{R})$ *are such that*

$$
\sum_{j=1}^{n} \sum_{k=1}^{m} \left\{ a_{ijk} \left[x, f_j^{(v-1)}(S_{ijk}(x)) \right] - \alpha_{ijk}^{(v)}(x) f_j^{(v-1)}(S_{ijk}(x)) \right\}. \quad (5.4)
$$
\nllowing

\nLet hypothesis $(H_1) - (H_2)$ hold and suppose $a_{ijk} \in C(\Omega_i \times \mathbb{R}; \mathbb{R})$

\n
$$
\frac{\partial a_{ijk}}{\partial y} \in C(\Omega_i \times \mathbb{R}; \mathbb{R})
$$
\n
$$
a_{ijk}, \frac{\partial a_{ijk}}{\partial y} \in C_b(\Omega_i \times [-M, M]; \mathbb{R}) \quad \forall M > 0
$$
\n)

\nwill be omitted if Ω_i is compact in \mathbb{R}^p . If $f^{(v-1)} \in X$ satisfies

\n
$$
\sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \le j \le n} \sup_{x \in \Omega_i} |\alpha_{ijk}^{(v)}(x)| < 1,
$$
\nfunction $f^{(v)} \in X$ being solution of system (5.2) (5.4)

where condition $(5.5)_2$ *will be omitted if* Ω_i *is compact in* \mathbb{R}^p . *If* $f^{(v-1)} \in X$ *satisfies*

$$
\sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \leq j \leq n} \sup_{x \in \Omega_i} |\alpha_{ijk}^{(v)}(x)| < 1,
$$

there exists a unique function $f^{(v)} \in X$ *being solution of system* (5.2), (5.4).

Proof. Apply Theorem 3.1 for $a_{ijk}(x,y) = \alpha_{ijk}^{(v)}(x)y$, $g_i(x) = g_i^{(v)}(x)$ and $\tilde{a}_{ijk} =$ $|\alpha_{ijk}^{(v)}|$ \blacksquare

We make the following hypotheses:

- (A_1) $a_{ijk} \in C(\Omega_i \times \mathbb{R}; \mathbb{R})$ satisfy
	- (i) $\frac{\partial a_{ijk}}{\partial y}, \frac{\partial^2 a_{ijk}}{\partial y^2} \in C(\Omega_i \times \mathbb{R}; \mathbb{R}).$
- **(iii)** $a_{ijk} \in C(\Omega_i \times \mathbb{R}; \mathbb{R})$ satisfy
 (i) $\frac{\partial a_{ijk}}{\partial y}$, $\frac{\partial^2 a_{ijk}}{\partial y^2} \in C(\Omega_i \times \mathbb{R}; \mathbb{R})$.
 (ii) a_{ijk} , $\frac{\partial^2 a_{ijk}}{\partial y}$, $\frac{\partial^2 a_{ijk}}{\partial y^2} \in C_b(\Omega_i \times [-M, M]; \mathbb{R})$ for all $M > 0$ (this condition will be omi omitted if Ω_i is compact of \mathbb{R}^p).
- (A_2) There exists a constant $M > 0$ such that

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\ng hypothesis:
\nR) satisfy
\n
$$
Q_i \times R; R
$$
\n
$$
P_i \times R; R
$$
\n
$$
E C_b(\Omega_i \times [-M, M]; R) \text{ for all } M > 0 \text{ (this condition will be\nimpact of } R^p).
$$
\n
$$
\text{stant } M > 0 \text{ such that}
$$
\n
$$
||g||_X + \sum_{i,j=1}^n \sum_{k=1}^m A_{ijk}^{(0)}(M) + 2M\sigma_M \le M \qquad (5.6)
$$
\n
$$
A_{ijk}^{(0)}(M) = \sup_{x \in \Omega_i, |y| \le M} |a_{ijk}(x, y)|
$$

where

$$
A_{ijk}^{(0)}(M) = \sup_{x \in \Omega_i, |y| \le M} |a_{ijk}(x, y)|
$$

$$
A_{ijk}^{(1)}(M) = \sup_{x \in \Omega_i, |y| \le M} \left| \frac{\partial a_{ijk}}{\partial y}(x, y) \right|
$$

$$
\sigma_M = \sum_{i=1}^n \sum_{k=1}^m \max_{1 \le j \le n} A_{ijk}^{(1)}(M).
$$

Thus we have the following

Theorem 5.2. Let hypotheses $(H_1) - (H_2)$ and $(A_1) - (A_2)$ hold, let f be the
tion of system (1.1) and the sequence $\{f^{(v)}\}$ be defined by algorithm (5.1).
(i) If $||f^{(0)}||_X \leq M$, then *solution of system* (1.1) and the sequence ${f^{(v)}}$ be defined by algorithm (5.1). *f(V) - flix aMIif') - fII* (x, y)
 (x, y)
 $(-A_2) hold, let f be the
by algorithm (5.1).$
 $(v \ge 1)$ (5.7)

$$
||f^{(v)} - f||_X \le \tilde{\sigma}_M ||f^{(v-1)} - f||_X^2 \qquad (v \ge 1)
$$
 (5.7)

where

$$
\sigma_M = \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \leq j \leq n} A_{ijk}^{(1)}(M).
$$

\n
$$
\text{We have the following}
$$
\n
$$
\text{Theorem 5.2.} Let \text{ hypotheses } (H_1) - (H_2) \text{ and } (A_1) - (A_2) \text{ hold, let } f \text{ be the}
$$
\n
$$
\text{the non of system (1.1) and the sequence } \{f^{(v)}\} \text{ be defined by algorithm (5.1)}.
$$
\n
$$
\text{(i) } |f||f^{(0)}||x \leq M, \text{ then}
$$
\n
$$
||f^{(v)} - f||x \leq \tilde{\sigma}_M||f^{(v-1)} - f||_X^2 \qquad (v \geq 1) \qquad (5.7)
$$
\n
$$
\tilde{\sigma}_M = \frac{1}{2(1 - \sigma_M)} \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \leq j \leq n} A_{ijk}^{(2)}(M) \qquad (5.8)
$$
\n
$$
A_{ijk}^{(2)}(M) = \sup_{x \in \Omega_i, |y| \leq M} \left| \frac{\partial^2 a_{ijk}}{\partial y^2}(x, y) \right|.
$$
\n
$$
\text{(ii) If choosing the first term } f^{(0)} \text{ sufficiently near } f \text{ such that } \tilde{\sigma}_M ||f^{(0)} - f||_X < 1,
$$
\n
$$
\text{the sequence } \{f^{(v)}\} \text{ converges quadratically to } f \text{ and satisfies the error estimation}
$$
\n
$$
||f^{(v)} - f||_X \leq \frac{1}{\tilde{\sigma}_M} (\tilde{\sigma}_M || f^{(0)} - f ||_X)^2 \qquad (v \geq 1).
$$
\n
$$
\text{Proof. First we will verify that if } ||f^{(0)}||_X \leq M, \text{ then}
$$
\n
$$
||f^{(v)}||_X \leq M \qquad (v \in \mathbb{N}). \qquad (5.10)
$$
\n
$$
\text{end, supposing}
$$
\n
$$
||f^{(v-1)}||_X \leq M \qquad (0.5.11)
$$

and

$$
A_{ijk}^{(2)}(M) = \sup_{x \in \Omega_i, |y| \le M} \left| \frac{\partial^2 a_{ijk}}{\partial y^2}(x, y) \right|.
$$

then the sequence $\{f^{(v)}\}$ converges quadratically to f and satisfies the error estimation

$$
A_{ijk}^{(2)}(M) = \sup_{x \in \Omega_i, |y| \le M} \left| \frac{\partial^2 a_{ijk}}{\partial y^2} (x, y) \right|.
$$

\ng the first term $f^{(0)}$ sufficiently near f such that $\tilde{\sigma}_M ||f^{(0)} - f||_X < 1$,
\n $\{f^{(v)}\}$ converges quadratically to f and satisfies the error estimation
\n $||f^{(v)} - f||_X \le \frac{1}{\tilde{\sigma}_M} (\tilde{\sigma}_M ||f^{(0)} - f||_X)^{2^v} \qquad (v \ge 1).$ (5.9)
\nwe will verify that if $||f^{(0)}||_X \le M$, then
\n $||f^{(v)}||_X \le M \qquad (v \in \mathbb{N}).$ (5.10)
\n $||f^{(v-1)}||_X \le M$ (5.11)

Proof. First we will verify that if $||f^{(0)}||_X \leq M$, then

$$
||f^{(v)}||_X \le M \qquad (v \in \mathbb{N}). \tag{5.10}
$$

Indeed, supposing

$$
||f^{(v-1)}||_X \le M \tag{5.11}
$$

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it follows from (5.2) and (5.11) that $||f^{(v)}||_X \le \sigma_M ||f^{(v)}||_X + ||g^{(v)}||_X$. Note that (5.6)
implies $0 < \sigma_M \le \frac{1}{2}$, hence we obtain implies $0 < \sigma_M \leq \frac{1}{2}$, hence we obtain

Nghia
\n) that
$$
||f^{(v)}||_X \le \sigma_M ||f^{(v)}||_X + ||g^{(v)}||_X
$$
. Note that (5.6)
\n: obtain
\n $||f^{(v)}||_X \le \frac{1}{1 - \sigma_M} ||g^{(v)}||_X$. (5.12)
\n- (5.4) we obtain

On the other hand, from (5.3) - (5.4) we obtain

nd N. H. Nghia
\nand (5.11) that
$$
||f^{(v)}||_X \le \sigma_M ||f^{(v)}||_X + ||g^{(v)}||_X
$$
. Note that (5.6)
\nhence we obtain
\n
$$
||f^{(v)}||_X \le \frac{1}{1 - \sigma_M} ||g^{(v)}||_X.
$$
\n(5.12)
\ncom (5.3) - (5.4) we obtain
\n
$$
||g^{(v)}||_X \le ||g||_X + \sum_{i,j=1}^n \sum_{k=1}^m A_{ijk}^{(0)}(M) + M \sigma_M.
$$
\n(5.13)
\n5) and (5.12) - (5.13) we obtain (5.10).
\nimate $||f - f^{(v)}||_Y$. Putting $e^{(v)} = f - f^{(v)}$ we obtain from (1.1)

From estimations (5.6) and (5.12) - (5.13) we obtain (5.10) .

the other hand, from (5.3) - (5.4) we obtain
 $||g^{(v)}||_X \le ||g||_X + \sum_{i,j=1}^n \sum_{k=1}^m A_{ijk}^{(0)}(\lambda)$

in estimations (5.6) and (5.12) - (5.13) we obtain (8.12)

Now we shall estimate $||f - f^{(v)}||_X$. Putting $e^{(v)}$

(5.1) the s $f(f(x)) = f - f^{(v)}$ we obtain from (1.1) and (5.1) the system

On the other hand, from (5.3) - (5.4) we obtain
\n
$$
||g^{(v)}||_X \le ||g||_X + \sum_{i,j=1}^n \sum_{k=1}^m A_{ijk}^{(0)}(M) + M\sigma_M.
$$
\n(5.13)
\nFrom estimations (5.6) and (5.12) - (5.13) we obtain (5.10).
\nNow we shall estimate $||f - f^{(v)}||_X$. Putting $e^{(v)} = f - f^{(v)}$ we obtain from (1.1)
\nand (5.1) the system
\n
$$
e_i^{(v)}(x) = \sum_{j=1}^n \sum_{k=1}^m \left\{ a_{ijk}[x, f_j(S_{ijk}(x))] - a_{ijk}[a, f_j^{(v-1)}(S_{ijk}(x))] + \frac{\partial a_{ijk}}{\partial y}[x, f_j^{(v-1)}(S_{ijk}(x))] \right\}.
$$
\n(5.14)
\nUsing Taylor's expansion of the function $a_{ijk}[x, f_j]$ about the point $(x, f_j^{(v-1)})$ up to
\norder two, we obtain
\n
$$
a_{ijk}[a, f_j] - a_{ijk}[x, f_j^{(v-1)}]
$$
\n
$$
= \frac{\partial a_{ijk}}{\partial y}[x, f_j^{(v-1)}](f_j - f_j^{(v-1)}) + \frac{1}{2!} \frac{\partial^2 a_{ijk}}{\partial y^2}[x, \lambda_j^{(v)}](f_j - f_j^{(v-1)})^2
$$
\n(5.15)
\nwhere
\n
$$
\lambda_j^{(v)} = f_j^{(v-1)} + \theta_j^{(v)}e_j^{(v-1)}
$$
\n(0 $< \theta_j^{(v)} < 1$).

order two, we obtain

$$
+\frac{\partial a_{ijk}}{\partial y}[x, f_j^{(v-1)}(S_{ijk}(x))] [f_j^{(v)}(S_{ijk}(x)) - f_j^{(v-1)}(S_{ijk}(x))] \Big\}.
$$

Using Taylor's expansion of the function $a_{ijk}[x, f_j]$ about the point $(x, f_j^{(v-1)})$ up to order two, we obtain

$$
a_{ijk}[a, f_j] - a_{ijk}[x, f_j^{(v-1)}]
$$

$$
= \frac{\partial a_{ijk}}{\partial y}[x, f_j^{(v-1)}](f_j - f_j^{(v-1)}) + \frac{1}{2!} \frac{\partial^2 a_{ijk}}{\partial y^2}[x, \lambda_j^{(v)}](f_j - f_j^{(v-1)})^2
$$
(5.15)
where

$$
\lambda_j^{(v)} = f_j^{(v-1)} + \theta_j^{(v)} e_j^{(v-1)}
$$
 $(0 < \theta_j^{(v)} < 1).$
Substituting (5.15) into (5.14) where the arguments of f_j , $f_j^{(v-1)}$, $\lambda_j^{(v)}$ appearing in

where

$$
\lambda_j^{(v)} = f_j^{(v-1)} + \theta_j^{(v)} e_j^{(v-1)} \qquad (0 < \theta_j^{(v)} < 1).
$$

 $f_j - j$
1).
 $(v-1)$
i , $\lambda_i^{(v)}$ appearing in (5.15) are replaced by S_{ijk} we obtain $\lambda_j^2 = J_j^2 + \theta$

ng (5.15) into (5.14) where

replaced by S_{ijk} we obtain
 $S_{ijk}^{(v)}(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ijk}^{(v)}(x) e_j^{(v)}(x)$

$$
k_{t}[a, f_{j}] - a_{ijk}[x, f_{j}] \qquad (5.15)
$$
\n
$$
= \frac{\partial a_{ijk}}{\partial y}[x, f_{j}^{(v-1)}](f_{j} - f_{j}^{(v-1)}) + \frac{1}{2!} \frac{\partial^{2} a_{ijk}}{\partial y^{2}}[x, \lambda_{j}^{(v)}](f_{j} - f_{j}^{(v-1)})^{2} \qquad (5.15)
$$
\n
$$
\lambda_{j}^{(v)} = f_{j}^{(v-1)} + \theta_{j}^{(v)} e_{j}^{(v-1)} \qquad (0 < \theta_{j}^{(v)} < 1).
$$
\nting (5.15) into (5.14) where the arguments of f_{j} , $f_{j}^{(v-1)}$, $\lambda_{j}^{(v)}$ appearing in
\ne replaced by S_{ijk} we obtain

\n
$$
e_{i}^{(v)}(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \alpha_{ijk}^{(v)}(x) e_{j}^{(v)}(S_{ijk}(x))
$$
\n
$$
+ \frac{1}{2!} \sum_{j=1}^{n} \sum_{k=1}^{m} \frac{\partial^{2} a_{ijk}}{\partial y^{2}}[x, \lambda_{j}^{(v)}(S_{ijk}(x))] |e_{j}^{(v-1)}(S_{ijk}(x))|^{2}.
$$
\nre and (5.11) we deduce that

\n
$$
||e^{(v)}||_{X} \leq \sigma_{M} ||e^{(v)}||_{X} + \frac{1}{2!} \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \leq j \leq n} A_{ijk}^{(2)}(M) \sum_{j=1}^{n} ||e_{j}^{(v-1)}||_{X_{j}}^{2}.
$$
\n(5.16)

\nthat

\n
$$
\sum_{j=1}^{n} ||e_{j}^{(v-1)}||_{X_{j}}^{2} \leq \left(\sum_{j=1}^{n} ||e_{j}^{(v-1)}||_{X_{j}}\right)^{2}.
$$
\n(5.17)

\nre obtain (5.7) by (5.8), (5.16) and (5.17). Finally, we deduce easily (5.9) from

From here and (5.11) we deduce that

$$
+\frac{1}{2!} \sum_{j=1}^{n} \sum_{k=1}^{m} \frac{\partial^{2} a_{ijk}}{\partial y^{2}} [x, \lambda_{j}^{(v)}(S_{ijk}(x))] |e_{j}^{(v-1)}(S_{ijk}(x))|^{2}.
$$

From here and (5.11) we deduce that

$$
||e^{(v)}||_{X} \le \sigma_{M} ||e^{(v)}||_{X} + \frac{1}{2!} \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \le j \le n} A_{ijk}^{(2)}(M) \sum_{j=1}^{n} ||e_{j}^{(v-1)}||_{X_{j}}^{2}.
$$

$$
(5.16)
$$
We note that

$$
\sum_{j=1}^{n} ||e_{j}^{(v-1)}||_{X_{j}}^{2} \le \left(\sum_{j=1}^{n} ||e_{j}^{(v-1)}||_{X_{j}}\right)^{2}.
$$

$$
(5.17)
$$
Hence we obtain (5.7) by (5.8), (5.16) and (5.17). Finally, we deduce easily (5.9) from

We note that **2**

$$
\|x\|_{X} + \frac{1}{2!} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \max_{1 \leq j \leq n} A_{ijk}^{(2)}(M) \sum_{j=1}^{\infty} \|e_j^{(v-1)}\|_{X_j}^2.
$$
 (5.16)

$$
\sum_{j=1}^n \|e_j^{(v-1)}\|_{X_j}^2 \leq \left(\sum_{j=1}^n \|e_j^{(v-1)}\|_{X_j}\right)^2.
$$
 (5.17)

 (5.7)

Example 2. Consider the nonlinear system with $m = 1$ and $n = p = 2$

On a System of Functional Equations
\n2. Consider the nonlinear system with
$$
m = 1
$$
 and $n = p = 2$
\n
$$
f_i(x) = \sum_{j=1}^{2} \alpha_{ij} f_j^2(\beta_{ij} x) + g_i(x) \qquad (i = 1, 2)
$$
\n
$$
x \in \Omega = \Omega_i = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : ||x||_1 = |x_1| + |x_2| \le 1 \right\}
$$
\n(5.18)

for

$$
x \in \Omega = \Omega_i = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : ||x||_1 = |x_1| + |x_2| \le 1 \right\}
$$

where

$$
z = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : ||x||_1 = |x_1| \right\}
$$

$$
g_i(x) = ||x||_1^i - \sum_{j=1}^2 \alpha_{ij} (\beta_{ij} ||x||_1)^{2j}
$$

and α_{ij}, β_{ij} are given real numbers satisfying

$$
x \in \Omega = \Omega_i = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : ||x||_1 = |x_1| + |x_2| \le 1 \right\}
$$

$$
g_i(x) = ||x||_1^i - \sum_{j=1}^2 \alpha_{ij}(\beta_{ij}||x||_1)^{2j}
$$

are given real numbers satisfying

$$
\alpha_{ij} \le 0
$$

$$
|\beta_{ij}| \le 1
$$

$$
4 \sum_{i=1}^2 \left(\sum_{j=1}^2 |\alpha_{ij}| + 4 \max_{1 \le j \le 2} |\alpha_{ij}| \right) \left(2 - \sum_{i=1}^2 \sum_{j=1}^2 \alpha_{ij}(\beta_{ij})^{2j} \right) < 1
$$

$$
\left| \sum_{j=1}^2 |\alpha_{ij}| + 4 \max_{1 \le j \le 2} |\alpha_{ij}| \right) \left(2 - \sum_{i=1}^2 \sum_{j=1}^2 \alpha_{ij}(\beta_{ij})^{2j} \right) < 1
$$

$$
\left| \sum_{j=1}^2 |\alpha_{ij}| (x, y) = a_{ij}(y) = \alpha_{ij} y^2 \text{ and } S_{ij}(x) = \beta_{ij} x, g_i(x) \text{ satisfy}
$$

$$
\left| \sum_{j=1}^2 \alpha_{ij} \right| \left(\sum_{j=1}^2 |\alpha_{ij}| + 4 \max_{j=1} |\alpha_{ij}| \right) \right|
$$

$$
\gamma_0 = \sum_{j=1}^2 \left(\sum_{j=1}^2 |\alpha_{ij}| + 4 \max_{j=1} |\alpha_{ij}| \right)
$$

The functions $a_{ij}(x, y) = a_{ij}(y) = \alpha_{ij} y^2$ and $S_{ij}(x) = \beta_{ij} x$, $g_i(x)$ satisfy hypotheses $(H_1) - (H_2)$ and $(A_1) - (A_2)$ where in (A_2) the constant $M > 0$ is chosen as

$$
\frac{1 - \sqrt{1 - 4\gamma_0 \|g\|_X}}{2\gamma_0} \le M \le \frac{1 + \sqrt{1 - 4\gamma_0 \|g\|_X}}{2\gamma_0}
$$

with

$$
\frac{1 - \sqrt{1 - 4\gamma_0 \|g\|_X}}{2\gamma_0} \le M \le \frac{1 + \sqrt{1 - 4\gamma_0 \|g\|_1}}{2\gamma_0}
$$

$$
\gamma_0 = \sum_{i=1}^2 \left(\sum_{j=1}^2 |\alpha_{ij}| + 4 \max_{1 \le j \le 2} |\alpha_{ij}| \right)
$$

$$
\|g\|_X = 2 - \sum_{i=1}^2 \sum_{j=1}^2 \alpha_{ij} (\beta_{ij})^{2j}
$$

of system (5.18) is $f_i(x) = (\|x\|_1)^i$ $(i = 1, 2)$

$$
\sum_{j=1}^2 \alpha_{ij} f_j^{(v-1)}(\beta_{ij} x) f_j^{(v)}(\beta_{ij} x) - \sum_{j=1}^2 \alpha_{ij} (f_j^{(v-1)}(\beta_{ij} x))^{2j}
$$

and $v \ge 1$. If we choose the initial iterative $s \le M$ and

algorithm for system (5.18) is

$$
\gamma_0 = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |\alpha_{ij}| + 4 \max_{1 \le j \le 2} |\alpha_{ij}| \right)
$$

$$
||g||_X = 2 - \sum_{i=1}^2 \sum_{j=1}^2 \alpha_{ij} (\beta_{ij})^{2j}
$$

The exact solution of system (5.18) is $f_i(x) = (||x||_1)^i$ $(i = 1, 2)$. The second order
algorithm for system (5.18) is
$$
f_i^{(v)}(x) = 2 \sum_{j=1}^2 \alpha_{ij} f_j^{(v-1)}(\beta_{ij} x) f_j^{(v)}(\beta_{ij} x) - \sum_{j=1}^2 \alpha_{ij} (f_j^{(v-1)}(\beta_{ij} x))^2 + g_i(x)
$$

for $x \in \Omega$, $i = 1,2$ and $v \ge 1$. If we choose the initial iterative step $f^{(0)} =$ $\begin{pmatrix} 0 \ 1 \end{pmatrix}, f_2^{\text{(0)}}$ such that $||f^{(0)}||_X \leq M$ and

$$
= 2 \sum_{j=1} \alpha_{ij} f_j^{(v-1)}(\beta_{ij} x) f_j^{(v)}(\beta_{ij} x) - \sum_{j=1} \alpha_{ij} (f_j^{(v-1)}(\beta_{ij} x))^2
$$

\n= 1, 2 and $v \ge 1$. If we choose the initial iterative step $f^{(0)}$
\n $||x| \le M$ and
\n $\tilde{\sigma}_M || f^{(0)} - f ||_{X} < 1$ with $\begin{cases} \tilde{\sigma}_M = \frac{\sigma_0}{1 - 2M\sigma_0} \\ \sigma_0 = \sum_{i=1}^2 \max_{1 \le j \le 2} |\alpha_{ij}| \end{cases}$

then we have

$$
\|f^{(v)} - f\|_{X} \le \frac{1}{\tilde{\sigma}_M} \left(\tilde{\sigma}_M \|f^{(0)} - f\|_{X}\right)^{2^v} \qquad (v \ge 1).
$$

Chooseing $f^{(0)}$ note that the sequence $\{g^{(\mu)}\}$ is defined by

$$
- f||x \le \frac{1}{\tilde{\sigma}_M} \left(\tilde{\sigma}_M || f^{(0)} - f ||_X \right)^2 \qquad (i)
$$

at the sequence $\{g^{(\mu)}\}$ is defined by

$$
g_i^{(\mu)}(x) = \sum_{j=1}^2 \alpha_{ij} (g_j^{(\mu-1)}(\beta_{ij}x))^2 + g_i(x)
$$

 $g_i^{(\mu)}(x) = \sum_{j=1}^2 \alpha_{ij} (g_j^{(\mu-1)}(\beta_{ij}x))^2 + g_i(x)$
for $x \in \Omega$, $i = 1, 2$ and $\mu \ge 1$ where $g^{(0)} = (g_1^{(0)}, g_2^0) \equiv (0,0)$ which is $||g^{(\mu)}||_X$
M ($\mu > 1$) $g^{(\mu)} \rightarrow f \in X$ as $\mu \rightarrow +\infty$ and $||g^{(\mu)} - f||_X \le \frac{M}{\mu} \gamma^{\mu}$ ($\$ $g_i^{(\mu)}(x) = \sum_{j=1} \alpha_{ij} (g_j^{(\mu-1)}(\beta_{ij}x))^2 + g_i(x)$
for $x \in \Omega$, $i = 1, 2$ and $\mu \ge 1$ where $g^{(0)} = (g_1^{(0)}, g_2^0) \equiv (0, 0)$
M $(\mu \ge 1)$, $g^{(\mu)} \to f \in X$ as $\mu \to +\infty$ and $||g^{(\mu)} - f||_X \le \frac{M}{1-\gamma_1}$
 $\gamma_1 = 2M\sigma_0 < M\gamma_0 < 1$. Ch that the sequence $\{g^{(\mu)}\}\$ is defined by
 $g_i^{(\mu)}(x) = \sum_{j=1}^2 \alpha_{ij} (g_j^{(\mu-1)}(\beta_{ij}x))^2 + g_i(x)$

and $\mu \ge 1$ where $g^{(0)} = (g_1^{(0)}, g_2^0) \equiv (0,0)$ which is $||g^{(\mu)}||_X \le$
 $f \in X$ as $\mu \to +\infty$ and $||g^{(\mu)} - f||_X \le \frac{M}{1-\gamma_1} \gamma_$ $\gamma_1 = 2M\sigma_0 < M\gamma_0 < 1$. Choose μ_0 sufficient large such that $\tilde{\sigma}_M ||g^{(\mu_0)} - f||_X < 1$. Then we take $f^{(0)} = g^{(\mu_0)}$.

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