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# A Nonlinear Boundary Value Problem for a Nonlinear Ordinary Differential Operator in Weighted Sobolev Spaces

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**Abstract.** We use the Galerkin and compactness method in appropriate weighted Sobolev spaces to prove the existence of a unique weak solution of the nonlinear boundary valued problem

$$\left.\begin{array}{c} -\frac{1}{x^\gamma}\frac{d}{dx}M(x,u'(x)) + f(x,u(x)) = F(x) \quad (0 < x < 1) \\ |\lim_{x\to 0_+} x^{\gamma/p}u'(x)| < +\infty \\ M(1,u'(1)) + h(u(1)) = 0 \end{array}\right\}$$

where $\gamma > 0, p \geq 2$ are given constants and $f, F, h, M$ are given functions.

**Keywords:** *Boundary value problems, ordinary differential operators, weak solutions, existence and uniqueness, Galerkin method, weighted Sobolev spaces*

**AMS subject classification:** 34B09, 34B15, 34L30

## 1. Introduction

We consider the nonlinear boundary value problem

$$\left.\begin{array}{c} -\dfrac{1}{x^\gamma}\dfrac{d}{dx}M(x,u'(x)) + f(x,u(x)) = F(x) \quad (0 < x < 1) \\ |\lim_{x\to 0_+} x^{\gamma/p}u'(x)| < +\infty \\ M(1,u'(1)) + h(u(1)) = 0 \end{array}\right\} \tag{1.1}$$

where

$\gamma > 0$ and $p \geq 2$ are given constants

$f, F, h$ are given functions

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$M : (0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and is monotonically increasing with respect to the second variable.

In the case of $\gamma = 0$ the problem

$$\left.\begin{aligned} -\frac{1}{x^\gamma}\frac{d}{dx}M(x,u'(x)) + f(x,u(x)) = F(x) \;\; (0 < x < 1) \\ u(0) = 0 \\ M(1,u'(1)) + \gamma_1 G(1)\sin u(1) = 0 \end{aligned}\right\} \tag{1.2}$$

is related to the buckling of a nonlinear elastic bar with specific weight $\gamma_0$ immersed in a fluid with specific weight $\gamma_1$ that Tucsnak [1] has constructed in the case of

$$f(x,u) - F(x) = \big[ -\lambda + (\gamma_0 - \gamma_1)g(x) - G'(1)\big] \sin u$$

where $\lambda > 0$ is a constant, $g$ and $G$ are given functions with some mechanical meaning, and $u(x)$ is the angle between the tangent of the bar in the buckled state of a point with curvilinear abscissa $x$ and vertical axis $Oy$. Then, in the case of $g =$ const and $M(x,u') = M(u')$ being monotonically increasing and sufficiently smooth Tucsnak has studied the bifurcation of integral equations equivalent to problems (1.1) and (1.2) depending on a parameter $\lambda$.

We note that problem (1.1) with $\gamma = 0$ and $u'M(x,u') \geq C_1|u'|^p$ $(p > 1, C_1 > 0)$ independent of $x$ had been considered in [2]. In [6] problem (1.1) with $p = 2$, $M(x,u') = x^\gamma u'$ with $\gamma > 0$ and the boundary condition $u'(1) + h_1 u(1) = h_2$ with given constants $h_1 > 0$ and $h_2$ has been studied. At least, in [3, 4] the nonlinear Bessel differential equation

$$-\frac{1}{x}\frac{d}{dx}M(xu'(x)) + u^2 - u = 0 \;\; (x > 0) \tag{1.3}$$

has been studied.

In this paper we use the Galerkin and compactness method in appropriate weighted Sobolev spaces to prove the existence of a unique weak solution of problem (1.1). The results obtained here generalize those of [1 - 4, 6].

## 2. Preliminary results, notations, function spaces

Put $\Omega = (0,1)$ and $p' = \frac{p}{p-1}$. We omit the definitions of the usual function spaces $C^m(\overline{\Omega})$, $L^p(\Omega)$, $H^m(\Omega)$ and $W^{m,p}(\Omega)$. We denote by $L^p_\gamma(\Omega) \equiv L^p_\gamma$ the class of all measurable functions $u$ defined on $\Omega$ for which

$$\|u\|_{p,\gamma} < \infty \qquad (1 \leq p \leq \infty) \tag{2.1}$$

where

$$\|u\|_{p,\gamma} = \left(\int_0^1 x^\gamma |u(x)|^p dx\right)^{\frac{1}{p}} \quad (1 \leq p < \infty)$$
$$\|u\|_{\infty,\gamma} = \operatorname*{ess\,sup}_{0<x<1} |u(x)|$$

and were we identify functions that are equal almost everywhere on $\Omega$. The elements of $L^p_\gamma$ are thus actually equivalence classes of measurable functions satisfying (2.1), two functions being equivalent if they are equal a.e. in $\Omega$. Then $L^p_\gamma$ is also a Banach space with respect to the norm $\|\cdot\|_{p,\gamma}$. In particular, $L^2_\gamma$ is a Hilbert space with usual scalar product $\langle u,v\rangle = \int_0^1 x^\gamma u(x)v(x)\,dx$ and norm $\|u\|_{2,\gamma} = \sqrt{\langle u,u\rangle}$. We denote by

$$W^{1,p}_\gamma(\Omega) \equiv W^{1,p}_\gamma = \left\{v \in L^p_\gamma : v' \in L^p_\gamma\right\} \qquad (1 \le p \le \infty)$$

the real Banach space with respect to the norm

$$\begin{aligned} \|v\|_{1,p,\gamma} &= \left(\|v\|^p_{p,\gamma} + \|v'\|^p_{p,\gamma}\right)^{\frac{1}{p}} \quad (1 \le p < \infty) \\ \|v\|_{1,\infty,\gamma} &= \max\left\{\|v\|_{\infty,\gamma}, \|v'\|_{\infty,\gamma}\right\} \end{aligned}$$

with derivatives in the sense of distributions [8]. In defining the function space $W^{1,p}_\gamma(\Omega)$ with weight $x^\gamma$, we can also define $W^{1,p}_\gamma(\Omega)$ as the completion of the space

$$S_1 = \left\{u \in C^1(\bar\Omega) : \|u\|_{1,p,\gamma} < \infty\right\}$$

with respect to the norm $\|\cdot\|_{1,p,\gamma}$ (see Adams [8]).

The following imbedding inequality will be used in the sequence.

**Lemma 2.1.** *For every $u \in C^1(\bar\Omega)$, $\gamma > 0$ and $p > 1$ we have*

$$\left.\begin{aligned} \|u\|^p_{p,\gamma} &\le |u(1)|^p + K_1\|u'\|^p_{p,\gamma} \\ |u(1)| &\le K_2\|u\|_{1,p,\gamma} \\ x^{\frac{\gamma}{p}}|u(x)| &\le K_3\|u\|_{1,p,\gamma} \\ \|u\|^2_{2,\gamma} &\le K_4\|u\|_{1,p,\gamma}(|u(1)|^p + \|u'\|^p_{p,\gamma})^{\frac{1}{p}} \quad (p \ge 2 - \tfrac{1}{\gamma}) \end{aligned}\right\} \tag{2.2}$$

*where*

$$\begin{aligned} K_1 &= \left(\tfrac{p-1}{\gamma}\right)^{p-1} \\ K_2 &= (\gamma + p)^{\frac{1}{p}} \\ K_3 &= \max\{2^{\frac{1}{p}}, (\gamma + 2p - 1)^{\frac{1}{p}}\} \\ K_4 &= K_3\left(\tfrac{2^{p-1}}{1+(p-1)\gamma}\right)^{\frac{1}{p}}. \end{aligned}$$

**Proof.** **(i)** Integrating by parts in the following integral, we get

$$\begin{aligned} \|u\|^p_{p,\gamma} &= \frac{|u(1)|^p}{1+\gamma} - \frac{p}{1+\gamma}\int_0^1 x^{1+\gamma}|u(x)|^{p-2}u(x)u'(x)\,dx \\ &=: \frac{|u(1)|^p}{1+\gamma} - \frac{p}{1+\gamma} I \end{aligned} \tag{2.3}$$

where by using the Hölder inequality

$$|I| = \left|\int_0^1 x^{\frac{\gamma}{p}}u'(x)\,x^{1+\frac{\gamma}{p'}}|u(x)|^{p-2}u(x)\,dx\right| \le \|u'\|_{p,\gamma}\|u\|^{p-1}_{p,\gamma}. \tag{2.4}$$

It follows that

$$(1+\gamma)\|u\|_{p,\gamma}^{p} \leq |u(1)|^{p} + p\|u'\|_{p,\gamma}\|u\|_{p,\gamma}^{p-1}.$$

Using the Hölder inequality

$$ab \leq \frac{1}{p}\varepsilon^{-p}a^{p} + \frac{1}{p'}\varepsilon^{p'}b^{p'} \qquad (\varepsilon > 0, a \geq 0, b \geq 0)$$

it follows that

$$(1+\gamma)\|u\|_{p,\gamma}^{p} \leq |u(1)|^{p} + \varepsilon^{-p}\|u'\|_{p,\gamma}^{p} + (p-1)\varepsilon^{p'}\|u\|_{p,\gamma}^{p}$$

where $(p-1)\varepsilon^{p'} = \gamma$. Hence $(2.2)_1$ is deduced.

**(ii)** Similarly, it follows from (2.3), (2.4) and the Hölder inequality with $\varepsilon = 1$ that

$$|u(1)|^{p} = (1+\gamma)\|u\|_{p,\gamma}^{p} + pI \leq (p+\gamma)\|u\|_{p,\gamma}^{p} + \|u'\|_{p,\gamma}^{p}. \tag{2.5}$$

Hence $(2.2)_2$ is deduced.

**(iii)** We have for all $x \in [0,1]$

$$\begin{aligned} x^{\gamma}|u(x)|^{p} &= |u(1)|^{p} - \int_{x}^{1} \frac{d}{dy}(y^{\gamma}|u(y)|^{p})\,dy \\ &= |u(1)|^{p} - \gamma\int_{x}^{1} y^{\gamma-1}|u(y)|^{p}dy - p\int_{x}^{1} y^{\gamma}|u(y)|^{p-2}u(y)u'(y)\,dy \end{aligned}$$

where by using the Hölder inequality the later integral in the right-hand side is estimated as

$$\left|\int_{x}^{1} y^{\gamma}|u(y)|^{p-2}u(y)u'(y)\,dy\right| \leq \|u\|_{p,\gamma}^{p-1}\|u'\|_{p,\gamma}.$$

Taking together we deduce that

$$x^{\gamma}|u(x)|^{p} \leq |u(1)|^{p} + p\|u\|_{p,\gamma}^{p-1}\|u'\|_{p,\gamma}.$$

We again use the Hölder inequality with $\varepsilon = 1$ to get from (2.5) that

$$x^{\gamma}|u(x)|^{p} \leq (p+\gamma)\|u\|_{p,\gamma}^{p} + \|u'\|_{p,\gamma}^{p} + (p-1)\|u\|_{p,\gamma}^{p} + \|u'\|_{p,\gamma}^{p}.$$

Hence $(2.2)_3$ is proved.

**(iv)** Let $p \geq 2 - \frac{1}{\gamma}$ and $p > 1$. We have from $(2.2)_3$ that

$$\|u\|_{2,\gamma}^{2} = \int_{0}^{1} x^{\frac{\gamma}{p}}|u(x)|x^{\frac{\gamma}{p'}}|u(x)|\,dx \leq K_3\|u\|_{1,p,\gamma}\int_{0}^{1} x^{\frac{\gamma}{p'}}|u(x)|\,dx. \tag{2.6}$$

On the other hand, using the Hölder inequality we obtain the inequalities

$$\begin{aligned} |u(x)|^{p} &\leq 2^{p-1}\left[|u(1)|^{p} + \left(\int_{x}^{1} |u'(y)|^{p}dy\right)^{p}\right] \\ &\leq 2^{p-1}\left[|u(1)|^{p} + (1-x)^{p-1}\int_{x}^{1} |u'(y)|^{p}dy\right] \end{aligned}$$

and

$$\left(\int_0^1 x^{\frac{\gamma}{p'}}|u(x)|\,dx\right)^p \leq \int_0^1 x^{(p-1)\gamma}|u(x)|^p dx.$$

Hence, Taking together we deduce that

$$\begin{aligned}&\left(\int_0^1 x^{\frac{\gamma}{p'}}|u(x)|\,dx\right)^p \\ &\quad\leq \frac{2^{p-1}|u(1)|^p}{1+(p-1)\gamma} + 2^{p-1}\int_0^1 x^{(p-1)\gamma}(1-x)^{p-1}dx\int_x^1 |u'(y)|^p dy.\end{aligned} \tag{2.7}$$

Inverting the variables of integration $x$ and $y$ in the last integral we estimate that integral as

$$\begin{aligned}&\int_0^1 x^{(p-1)\gamma}(1-x)^{p-1}dx\int_x^1 |u'(y)|^p dy \\ &\quad= \int_0^1 |u'(y)|^p dy\int_0^y x^{(p-1)\gamma}(1-x)^{p-1}dx \\ &\quad\leq \int_0^1 |u'(y)|^p dy\int_0^y x^{(p-1)\gamma}dx \\ &\quad\leq \frac{1}{1+(p-1)\gamma}\int_0^1 y^{1+(p-1)\gamma}|u'(y)|^p dy\end{aligned} \tag{2.8}$$

and note that $y^{1+(p-1)\gamma} \leq y^\gamma$ for all $y \in [0,1]$ and $p \geq 2-\frac{1}{\gamma}$. Then $(2.2)_4$ is deduced from (2.6) - (2.8) ∎

**Remark 1.** The results $(2.2)_{1,2}$ proves that $(|u(1)|^p + \|u'\|^p_{p,\gamma})^{\frac{1}{p}}$ and $\|u\|_{1,p,\gamma}$ are two equivalent norms on $W^{1,p}_\gamma(\Omega)$ and

$$\frac{1}{1+K_1}\|u\|^p_{1,p,\gamma} \leq |u(1)|^p + \|u'\|^p_{p,\gamma} \leq (1+K_2^p)\|u\|^p_{1,p,\gamma} \tag{2.9}$$

for all $u \in W^{1,p}_\gamma(\Omega)$.

**Lemma 2.2.** *The imbedding $W^{1,p}_\gamma(\Omega) \hookrightarrow L^2_\gamma(\Omega)$ $(p>1)$ is continuous if $p \geq 2-\frac{1}{\gamma}$, and compact if $p \geq 2$.*

**Proof.** For $p \geq 2-\frac{1}{\gamma}$ the continuity of the imbedding $W^{1,p}_\gamma(\Omega) \hookrightarrow L^2_\gamma(\Omega)$ is deduced from $(2.2)_4$ and (2.9). For $p \geq 2$ we have $W^{1,p}_\gamma(\Omega) \hookrightarrow W^{1,2}_\gamma(\Omega) \hookrightarrow L^2_\gamma(\Omega)$ and on the other hand the imbedding $W^{1,2}_\gamma(\Omega) \hookrightarrow L^2_\gamma(\Omega)$ is compact (see [5]). Hence, $W^{1,p}_\gamma(\Omega) \hookrightarrow L^2_\gamma(\Omega)$ is also compact ∎

**Remark 2.** We also note that

$$\lim_{x\to 0_+} x^{\frac{\gamma}{p}}u(x) = 0 \qquad (u \in W^{1,p}_\gamma(\Omega)) \tag{2.10}$$

(see [7: p. 128/Lemma 5.40). On the other hand, by $W^{1,p}(\varepsilon,1) \hookrightarrow C^0([\varepsilon,1])$ $(0<\varepsilon<1)$ and

$$\varepsilon^{\frac{\gamma}{p}}\|u\|_{W^{1,p}(\varepsilon,1)} \leq \|u\|_{1,p,\gamma} \qquad (u \in W^{1,p}_\gamma, 0<\varepsilon<1) \tag{2.11}$$

it follows that

$$u|_{[\varepsilon,1]} \in C^0([\varepsilon,1]) \qquad (0<\varepsilon<1). \tag{2.12}$$

From (2.10) and (2.12) we deduce that

$$x^{\frac{\gamma}{p}} u \in C^0(\bar{\Omega}) \qquad (u \in W^{1,p}_{\gamma}(\Omega)). \tag{2.13}$$

Put $H = L^2_{\gamma}(\Omega)$ and $V = W^{1,p}_{\gamma}(\Omega)$ with $p > 1$ and $p \geq 2 - \frac{1}{\gamma}$. From the result of Lemma 2.2 with $p \geq 2 - \frac{1}{\gamma}$, $V$ is continuously embedded into $H$. Furthermore, $V$ is dense in $H$ since $C^1(\overline{\Omega})$ is dense in $H$; identifying $H$ with $H'$ (the dual of $H$), we have $V \hookrightarrow H \hookrightarrow V'$. On the other hand, the notation $\langle\cdot,\cdot\rangle$ is used for the pairing between $V$ and $V'$.

## 3. Theorem on existence and uniqueness

We assume that $p \geq 2$ and formulate the hypotheses

**($M_1$)** $M : (0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition, i.e. $M(\cdot,y)$ is measurable on $(0,1]$ for every $y \in \mathbb{R}$ and $M(x,\cdot)$ is continuous on $\mathbb{R}$ for a.e. $x \in (0,1]$.

**($M_2$)** There exist a constant $C_1 > 0$ and a function $q_1 \in L^1(\Omega)$ such that $yM(x,y) \geq C_1 x^{\gamma}|y|^p - |q_1(x)|$.

**($M_3$)** There exist a constant $C_2 > 0$ and a function $q_2$ with $x^{-\frac{\gamma}{p}} q_2 \in L^{p'}(\Omega)$ and $\lim_{x\to 0_+} x^{-\frac{\gamma}{p}}|q_2(x)| < \infty$ such that $|M(x,y)| \leq C_2 x^{\gamma}|y|^{p-1} + |q_2(x)|$.

**($M_4$)** $M$ is monotonically increasing with respect to the second variable, i.e. $(M(x,y) - M(x,\tilde{y}))(y-\tilde{y}) \geq 0$ for all $y, \tilde{y} \in \mathbb{R}$ and a.e. $x \in \Omega$.

Furthermore, we formulate the hypotheses

**($F_1$)** $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition.

**($F_2$)** There exist constants $C_3 > 0$ and $1 < r < p$ and a function $q_3 \in L^1_{\gamma}(\Omega)$ such that $yf(x,y) + C_3|y|^r \geq -|q_3(x)|$ for all $y \in \mathbb{R}$ and a.e. $x \in \Omega$.

**($F_3$)** There exist a constant $C_4 > 0$ and a function $q_4 \in L^{p'}_{\gamma}(\Omega)$ such that $|f(x,y)| \leq C_4|y|^{p-1} + |q_4(x)|$ for all $y \in \mathbb{R}$ and a.e. $x \in \Omega$.

Finally, we formulate the hypothesis

**($H_1$)** For $h \in C^0(R;R)$ there exist two constants $C_5, C_5' > 0$ with $uh(u) \geq C_5|u|^p - C_5'$ for all $u \in \mathbb{R}$.

Suppose that

$$F \in V'. \tag{3.1}$$

**Remark 3.** In hypothesis ($F_2$), $r = p$ still holds if $C_3 > 0$ is sufficiently small (see Remark 6).

The weak solution of problem (1.1) is formed from the following variational

**Problem.** Find $u \in V$ such that

$$\int_0^1 M(x, u'(x))v'(x)\,dx + h(u(1))v(1) + \langle f(x, u(x)), v\rangle = \langle F, v\rangle \tag{3.2}$$

for all $v \in V$.

**Remark 4.** By (2.13), the terms $u(1)$ and $v(1)$ appearing in (3.2) are defined for every $u, v \in V$. We obtain (3.2) by formally multiplying both sides of $(1.1)_1$ by $x^\gamma v \in V$ and then integrating by parts when taking conditions $(1.1)_{2,3}$, (2.10) and hypothesis $(\mathrm{M}_3)$.

Then we have the following

**Theorem 1.** *Let $F \in V'$ and let hypotheses $(\mathrm{M}_1)$ - $(\mathrm{M}_4)$, $(\mathrm{F}_1)$ - $(\mathrm{F}_3)$ and $(\mathrm{H}_1)$ hold. Then the variational problem (3.2) has a solution. Furthermore, if $M(x,\cdot), f(x,\cdot), h$ are non-decreasing, i.e.*

$$\left.\begin{aligned} \big(M(x,y) - M(x,\tilde y)\big)(y-\tilde y) \ge 0 \\ \big(f(x,y) - f(x,\tilde y)\big)(y-\tilde y) \ge 0 \\ \big(h(y) - h(\tilde y)\big)(y-\tilde y) \ge 0 \end{aligned}\right\} \tag{3.3}$$

*for all $y, \tilde y \in \mathbb{R}$ and a.e. $x \in \Omega$ where two of the three inequalities above are strict in the case $y \ne \tilde y$, then the solution is unique.*

*On the other hand, uniqueness of the solution also holds if condition (3.3) is replaced by the hypothesis*

**$(\mathrm{A}_1)$** *There exist constants $C_6, C_7, C_8 > 0$ with $0 < C_1 < \min\{C_8, \frac{C_6}{K_1}\}$ such that*

- **(i)** $\big(M(x,y) - M(x,\tilde y)\big)(y-\tilde y) \ge C_6 x^\gamma |y-\tilde y|^p$
- **(ii)** $\big(f(x,y) - f(x,\tilde y)\big)(y-\tilde y) \ge -C_7 |y-\tilde y|^p$
- **(iii)** $\big(h(y) - h(\tilde y)\big)(y-\tilde y) \ge C_8 |y-\tilde y|^p$

*for all $y, \tilde y \in \mathbb{R}$ and a.e. $x \in \Omega$.*

**Proof.** Since $V$ is separable there exists a sequence of linear independent elements $\{w_j\}$ which is total in $V$. We find $u_m$ under the form

$$u_m = \sum_{j=1}^m c_{mj} w_j \tag{3.4}$$

where $c_{mj}$ satisfy the nonlinear equation system

$$\int_0^1 M(x, u_m'(x))w_j'(x)\,dx + h(u_m(1))w_j(1) + \langle f(x, u_m(x)), w_j\rangle = \langle F, w_j\rangle. \tag{3.5}$$

By the Brouwer lemma (see [8: p. 53/Lemma 4.3) it follows from hypotheses $(\mathrm{M}_1)$ - $(\mathrm{M}_3)$, $(\mathrm{F}_1)$ - $(\mathrm{F}_3)$ and $(\mathrm{H}_1)$ that system (3.4) - (3.5) has a solution $u_m$. Multiplying the

$j^{th}$ equation of system (3.5) by $c_{mj}$ and then adding these equations for $j = 1, 2, \ldots, m$ we have

$$\int_0^1 M(x, u'_m(x))u'_m(x)dx + h(u_m(1))u_m(1) + \langle f(x, u_m(x)), u_m \rangle = \langle F, u_m \rangle. \tag{3.6}$$

By using hypotheses ($M_2$), ($F_2$), ($H_1$) and (2.9), (3.1) we obtain

$$\begin{aligned} C_0 \|u_m\|^p_{1,p,\gamma} \leq C_3 \int_0^1 x^\gamma |u_m(x)|^r dx \\ + \|F\|_{V'} \|u_m\|_{1,p,\gamma} + C'_5 + \|q_1\|_{L^1(\Omega)} + \|q_3\|_{1,\gamma} \end{aligned} \tag{3.7}$$

where $C_0 = \frac{\min\{C_1, C_5\}}{1+K_1}$. Using the Hölder inequality

$$ab \leq \frac{1}{p}\varepsilon_1^p a^p + \frac{1}{p'}\varepsilon_1^{-p'} b^{p'} \qquad (\varepsilon_1 > 0, a \geq 0, b \geq 0$$

we get the inequality

$$\|F\|_{V'} \|u_m\|_{1,p,\gamma} \leq \frac{1}{p}\varepsilon_1^p \|u_m\|^p_{1,p,\gamma} + \frac{1}{p'}\varepsilon_1^{-p'} \|F\|^{p'}_{V'} \tag{3.8}$$

where $\frac{1}{p}\varepsilon_1^p = \frac{C_0}{4}$. We also note that $|u_m|^r \leq \frac{r}{p}\varepsilon_2^{p/r}|u_m|^p + \frac{p-r}{p\varepsilon_2^{p/p-r}}$ for all $\varepsilon_2 > 0$. Hence we have

$$C_3 \int_0^1 x^\gamma |u_m(x)|^r dx \leq C_3 \frac{r}{p}\varepsilon_2^{p/r} \|u_m\|^p_{p,\gamma} + \frac{C_3}{1+\gamma}\frac{p-r}{p\varepsilon_2^{p/p-r}} \tag{3.9}$$

where $C_3 \frac{r}{p}\varepsilon_2^{p/r} = \frac{C_0}{4}$. Combining (3.7) - (3.9) we obtain

$$\|u_m\|_{1,p,\gamma} \leq C \tag{3.10}$$

where $C$ is a constant independent of $m$. From hypothesis ($M_3$) and (3.10) it follows that

$$\|x^{-\frac{\gamma}{p}} M(x, u'_m)\|_{L^{p'}} \leq C_2 \|u'_m\|^{p-1}_{p,\gamma} + \|x^{-\gamma/p} q_2\|_{L^{p'}} \leq C. \tag{3.11}$$

On the other hand, it follows from hypothesis ($F_3$) and (3.10) that

$$\|x^{\frac{\gamma}{p'}} f(x, u_m)\|_{L^{p'}} \leq C_4 \|u_m\|^{p-1}_{p,\gamma} + \|q_4\|_{p',\gamma} \leq C \tag{3.12}$$

where $C$ is a constant independent of $m$.

By means of (3.10), (3.11) and Lemma 2.2 the sequence $\{u_m\}$ has a subsequence still denoted by $\{u_m\}$ such that

$$\left.\begin{array}{rll} u_m \to u & \text{in } V \text{ weakly} \\ u_m \to u & \text{in } H \text{ strongly and a.e. in } \Omega \\ x^{-\frac{\gamma}{p}} M(x, u'_m) \to \chi & \text{in } L^{p'} \text{ weakly} \end{array}\right\}. \tag{3.13}$$

Note that because the embedding $W^{1,p}(\varepsilon,1) \hookrightarrow C^0([\varepsilon,1])$ $(0<\varepsilon<1)$ is compact, by (2.11) and (3.10) $\{u_m\}$ has a subsequence still denoted $\{u_m\}$ such that $u_m|_{[\varepsilon,1]} \to u_{|[\varepsilon,1]}$ in $C^0([\varepsilon,1])$. Hence

$$\left.\begin{aligned} u_m(1) &\to u(1) \\ h(u_m(1)) &\to h(u(1)) \end{aligned}\right\}. \tag{3.14}$$

On the other hand, it follows from hypothesis $(F_1)$ and $(3.13)_2$ that

$$x^{\frac{\gamma}{p'}} f(x,u_m) \to x^{\frac{\gamma}{p'}} f(x,u) \qquad \text{a.e. } x \in \Omega. \tag{3.15}$$

We shall need the following lemma, the proof of which can be found in [9].

**Lemma 3.1.** *Let $Q$ be an open bounded set of $\mathbb{R}^N$ and $G, G_m \in L^q(Q)$ $(1<q<\infty)$ such that $G_m \to G$ a.e. in $\Omega$ and $\|G_m\|_{L^q(Q)} \le C$, with $C$ being a constant independent of $m$. Then $G_m \to G$ weakly in $L^q(Q)$.*

Applying Lemma 3.1 with $N=1$, $q=p'$, $Q=\Omega$, $G_m = x^{\frac{\gamma}{p'}} f(x,u_m)$ and $G = x^{\frac{\gamma}{p'}} f(x,u)$ we deduce from (3.12) and (3.15) that

$$x^{\frac{\gamma}{p'}} f(x,u_m) \to x^{\frac{\gamma}{p'}} f(x,u) \qquad \text{weakly in } L^{p'}. \tag{3.16}$$

If we pass to the limit in equation (3.5) we find without difficulty from $(3.13)_3$, $(3.14)_2$ and (3.16) that $u$ satisfies the equation

$$\int_0^1 x^{\frac{\gamma}{p}} \chi v'(x)\,dx + h(u(1))v(1) + \langle f(x,u), v\rangle = \langle F, v\rangle \tag{3.17}$$

for all $u \in V$. So we shall prove the existence of the solution of the variational problem (3.2) if we show that $\chi = x^{-\frac{\gamma}{p}} M(x,u')$. From (3.4) and (3.5) we can deduce

$$\begin{aligned} &\int_0^1 M(x,u_m'(x))u_m'(x)\,dx \\ &\quad = -h(u_m(1))u_m(1) - \langle f(x,u_m(x)), u_m\rangle + \langle F, u_m\rangle. \end{aligned} \tag{3.18}$$

By using $(3.13)_{1,2}$, (3.14), (3.16) and (3.17) and passing to the limit in (3.18) as $m \to +\infty$ we have

$$\lim_{m\to+\infty} \int_0^1 M(x,u_m'(x))u_m'(x)\,dx = \int_0^1 x^{\frac{\gamma}{p}} \chi(x)u'(x)\,dx. \tag{3.19}$$

We deduce from $(3.13)_{1,3}$ and (3.19) that

$$\begin{aligned} &\lim_{m\to+\infty} \int_0^1 \big(M(x,u_m'(x)) - M(x,\theta(x))\big)\big(u_m'(x) - \theta(x)\big)\,dx \\ &\quad = \int_0^1 \big(x^{\frac{\gamma}{p}}\chi(x) - M(x,\theta(x))\big)\big(u'(x) - \theta(x)\big)\,dx \end{aligned}$$

for all $\theta \in L^p_\gamma$. Using the monotonicity property of $M$, we obtain

$$\int_0^1 \left(x^{\frac{\gamma}{p}}\chi(x) - M(x,\theta(x))\right)\left(u'(x) - \theta(x)\right)dx \geq 0$$

for all $\theta \in L^p_\gamma$. If we choose here $\theta = u' - \lambda w$ with $\lambda > 0$ and $w \in L^p_\gamma$ and let $\lambda \to 0_+$, we easily deduce that $\chi = x^{-\frac{\gamma}{p}} M(x,u')$ and the existence proof is completed.

To prove uniqueness let $u$ and $v$ be two solutions of the variational problem (3.2). Then $w = u - v$ satisfies the equality

$$\begin{aligned}&\int_0^1 \left(M(x,u'(x)) - M(x,v'(x))\right)w'(x)\,dx \\ &+\left(h(u(1)) - h(v(1))\right)w(1) + \langle f(x,u) - f(x,v), w\rangle = 0.\end{aligned} \tag{3.20}$$

If (3.3) holds, then evidently $u = v$. If hypothesis ($A_1$) holds, by (3.20) and (2.7) we have

$$C_6\|w'\|^p_{p,\gamma} + C_8|w(1)|^p \leq C_7\|w\|^p_{p,\gamma}$$

and

$$\begin{aligned}C_6\|w'\|^p_{p,\gamma} + C_8|w(1)|^p &\geq \min\left\{C_8, \frac{C_6}{K_1}\right\}\left(K_1\|w'\|^p_{p,\gamma} + |w(1)|^p\right) \\ &\geq \min\left\{C_8, \frac{C_6}{K_1}\right\}\|w\|^p_{p,\gamma},\end{aligned}$$

respectively, and since $0 < C_7 < \min\left\{C_8, \frac{C_6}{K_1}\right\}$ we deduce that $w = 0$. Theorem 1 is proved completely ∎

**Remark 5.** In [3], corresponding to $p = 2$ and $\gamma = 1$, we have proved that the nonlinear Bessel differential equation (1.4) associated with the boundary conditions $u(0) = 1$ and $u(+\infty) = 0$ has at least one solution. Wherein, the nonlinear term $u^2 - u$ is non-monotonic. One of the solutions above is constructed from the boundary value problem (1.4) in the interval $a < x < b$ associated with the boundary condition $u(a) = 1$ and $u(b) = 0$ wherein $x_i < a < b < x_{i+1}$ and $x_i, x_{i+1}$ are two consecutive zeros of the first order Bessel function $J_0$. Formation of a counterexample for the function $f(x,u)$ not satisfying the assumption to be monotonically increasing with respect $u$ to so that the solution of (3.2) is not unique is an open problem.

**Remark 6.** Theorem 1 still holds if hypothesis ($F_2$) is replaced by the hypothesis

**($F'_2$)** There exist a constant $C_3$ with $0 < C_3 < \min\left\{C_5, \frac{C_1}{K_1}\right\}$ and a function $q_3 \in L^1_\gamma$ such that $yf(x,y) + C_3|y|^p \geq -|q_3(x)|$ for all $y \in \mathbb{R}$ and a.e. $x \in \Omega$.

In fact, from hypotheses ($M_2$), ($F_2$'), ($H_1$) and (3.1), (3.6) we can obtain the following inequality similar to (3.7)

$$\begin{aligned}&C_1\|u'_m\|^p_{p,\gamma} + C_5|u_m(1)|^p \\ &\quad\leq C_3\|u_m\|^p_{p,\gamma} + \|F\|_{V'}\|u_m\|_{1,p,\gamma} + \|q_1\|_{L^1(\Omega)} + \|q_3\|_{1,\gamma} + C'_5.\end{aligned}$$

Choosing $C_3^*$ such that $0 < C_3 < C_3^* < \min\left\{C_5, \frac{C_1}{K_1}\right\}$ it follows from $(2.2)_{1,3}$ that

$$\left(1 - \frac{C_3^*}{C_3}\right) \frac{\min\{C_1, C_5\}}{1 + K_1} \|u_m\|_{1,p,\gamma}^p \le \|F\|_{V'} \|u_m\|_{1,p,\gamma} + \|q_1\|_{L^1(\Omega)} + \|q_3\|_{1,\gamma} + C_5'.$$

Hence, we obtain (3.10).

**Remark 7.** In Theorem 1 hypotheses ($M_2$), ($M_4$), ($F_2'$), ($H_1$) are implied by hypothesis ($A_1$). Indeed, it follows from ($A_1$) that

($\widetilde{M_2}$) $yM(x,y) \ge \widetilde{C_1} x^\gamma |y|^p - |\widetilde{q_1}(x)|$

($\widetilde{F_2}$) $yf(x,y) + \widetilde{C_3}|y|^p \ge -|\widetilde{q_3}(x)|$

($\widetilde{H_1}$) $yh(y) \ge \widetilde{C_5}|y|^p - \widetilde{C_5'}$

where

$$\left.\begin{aligned} \widetilde{C_1} &= C_6 - \frac{\varepsilon^p}{p} > 0 \\ \widetilde{C_3} &= C_7 + \frac{\varepsilon^p}{p} > 0 \\ \widetilde{C_5} &= C_8 - \frac{\varepsilon^p}{p} > 0 \\ \widetilde{C_5'} &= \frac{\varepsilon^{-p'}}{p'} |h(0)|^{p'} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} \widetilde{q_1}(x) &= \frac{\varepsilon^{-p'}}{p'} x^{-\frac{\gamma p'}{p}} |q_2(x)|^{p'} \in L^1 \\ \widetilde{q_3}(x) &= \frac{\varepsilon^{-p'}}{p'} |q_4(x)|^{p'} \in L^1_\gamma \end{aligned}\right\}.$$

From the condition $0 < C_7 < \min\left\{C_8, \frac{C_6}{K_1}\right\}$ we obtain the condition $0 < \widetilde{C_3} < \min\left\{\widetilde{C_5}, \frac{\widetilde{C_1}}{K_1}\right\}$ with $\varepsilon > 0$ sufficiently small. We then have the following

**Theorem 2.** *Let $F \in V'$ and let hypotheses* ($M_1$), ($M_3$), ($F_1$), ($F_3$), ($A_1$) *hold. Then problem* (3.2) *has a unique solution.*

**Remark 8.** Theorem 2 still holds if hypothesis ($A_1$) is implied by the following hypothesis

**($A_2$)** There exist constants $C_6, C_7, C_8$ with $0 < C_8 < \frac{1}{K_2^p} \min\{C_6, C_7\}$ such that, for all $y, \tilde{y} \in \mathbb{R}$ and a.e. $x \in \Omega$.

**(i)** $\big(M(x,y) - M(x,\tilde{y})\big)(y - \tilde{y}) \ge C_6 x^\gamma |y - \tilde{y}|^p$

**(ii)** $\big(f(x,y) - f(x,\tilde{y})\big)(y - \tilde{y}) \ge C_7 |y - \tilde{y}|^p$

**(iii)** $\big(h(y) - h(\tilde{y})\big)(y - \tilde{y}) \ge -C_8 |y - \tilde{y}|^p$.

In fact, from (3.1), (3.6) and hypotheses ($A_2$), ($M_1$), ($M_3$), ($F_1$), ($F_3$) we obtain

$$\begin{aligned} &\min\{\widetilde{C_1}, \widetilde{C_3}\} \|u_m\|_{1,p,\gamma}^p \\ &\quad \le \left(\widetilde{C_5} K_2^p + \frac{\varepsilon^p}{p}\right) \|u_m\|_{1,p,\gamma}^p + \frac{\varepsilon^{-p'}}{p'} \|F\|_{V'}^{p'} + \|\widetilde{q_1}\|_{L^1(\Omega)} + \|\widetilde{q_3}\|_{1,\gamma} + \widetilde{C_5'} \end{aligned}$$

for all $\varepsilon > 0$ where

$$\left.\begin{aligned} \widetilde{C_1} &= C_6 - \frac{\varepsilon^p}{p} \\ \widetilde{C_3} &= C_7 - \frac{\varepsilon^p}{p} \\ \widetilde{C_5} &= C_8 + \frac{\varepsilon^p}{p} \\ \widetilde{C_5'} &= \frac{\varepsilon^{-p'}}{p'}|h(0)|^{p'} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} \widetilde{q_1}(x) &= \frac{\varepsilon^{-p'}}{p'} x^{-\frac{\gamma p'}{p}} |q_2(x)|^{p'} \\ \widetilde{q_3}(x) &= \frac{\varepsilon^{-p'}}{p'} |q_4(x)|^{p'} \end{aligned}\right\}.$$

It follows from the condition $0 < C_8 < \frac{1}{K_2^p}\min\{C_6, C_7\}$ that there exists $\varepsilon > 0$ such that $\min\{\widetilde{C_1}, \widetilde{C_3}\} > \widetilde{C}_5 K_2^p + \frac{\varepsilon^p}{p}$. Hence we obtain that $\|u_m\|_{1,p,\gamma} \leq C$ where $C$ is a constant independent of $m$. We then have the following

**Theorem 3.** *Let* (3.1) *and let hypotheses* ($A_2$), ($M_1$), ($M_3$), ($F_1$), ($F_3$) *hold. Then problem* (3.2) *has a unique solution.*

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