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Univalent Functions with Range Restrictions

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Abstract. Let Σ be the class of functions $f(z) = z + a_0 + a_{-1}z^{-1} + \cdots$ analytic and univalent in $|z| > 1$. In this paper we investigate the problem to maximize $\Re a_{-1}$ in two subclasses of Σ: (i) the class of all functions $f \in \Sigma$ which omit two given values $\pm w_1$ $(0 < |w_1| < 2)$ and (ii) the class of all functions $f \in \Sigma$ with $a_0 = 0$ which map onto regions of prescribed width $b_f = b$ $(0 < b < 4)$ in the direction of the imaginary axis. We solve these problems by applying a variational method to a coefficient problem in two subclasses of univalent Bieberbach-Eilenberg functions which are equivalent to these problems.

Keywords: *Conformal mappings, coefficient problems, width of domains, Bieberbach-Eilenberg functions*

AMS subject classification: 30C45, 30C70, 30C85

1. Introduction and results

Let us denote by Σ the usual class of functions $f = f(z)$ analytic and univalent in $|z| > 1$ which have series development about infinity beginning

$$f(z) = z + a_0 + a_{-1}z^{-1} + \cdots . \tag{1}$$

Let b_f denote the width of the smallest parallel strip parallel to the real axis containing the set E_f of all points omitted by f. Let Σ_b denote the subclass of Σ consisting of all functions f for which $b_f \geq b$ for given $b \in (0,4)$. We denote by Σ_b^0 the class of functions $f \in \Sigma_b$ with the normalization $a_0 = 0$.

Our main purpose is to treat the extremal problem

$$\max_{f \in \Sigma_b^0} \Re a_{-1}. \tag{2}$$

Clearly, for the limiting cases $b = 0$ and $b = 4$ problem (2) has the uniquely determined solutions $f(z) = z + \frac{1}{z}$ and $f(z) = z - \frac{1}{z}$, respectively, as simple consequence of the area theorem.

It is evident that there must exist a solution of problem (2). Its existence can be shown by usual compactness and kernel convergence arguments of conformal mapping theory.

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To motivate extremal problem (2), let us mention its application in connection with the well-known process of iterated horizontal slit-mappings of simply-connected domains for the approximation of the horizontal slits-mapping of a multiply-connected domain due to H. Grötzsch [4] and G. M. Goluzin [3]: Let f^{-1} denote the inverse function of $f \in \Sigma_b^0$. Consider the function

$$f^{-1}(w) + \frac{1}{f^{-1}(w)} = w + c_{-1}w^{-1} + \cdots \qquad (c_{-1} = 1 - a_{-1})$$

which maps $f(\{|z| > 1\})$ into the exterior of the horizontal slit $[-2,2]$. Extremal problem (2) is equivalent to

$$\min_{f \in \Sigma_b^0} \Re c_{-1}. \tag{3}$$

A lower estimate of the minimum in (3) depending on b and the logarithmic capacity R (= 1) of E_f was derived in [4: p. 33] and more explicitly in [1: p. 237/Formula (4.37)] to prove the convergence of the iteration process mentioned above.

Let us now keep two points $\pm w_1$ $(0 < |w_1| < 2)$ fixed, where $0 < \arg w_1 \leq \frac{\pi}{2}$ without loss of generality. We denote by $\Sigma(\pm w_1)$ the class of all functions $f \in \Sigma$ which omit the points $\pm w_1$. To solve problem (2), first we shall treat the problem

$$\max_{f \in \Sigma(\pm w_1)} \Re a_{-1} \tag{4}$$

which has an interest of its own. The existence of such a maximum is immediate by standard arguments of conformal mapping theory. We investigate problem (4) by applying a special variational method to a coefficient problem in a subclass of univalent Bieberbach-Eilenberg functions which is equivalent to (4).

Clearly, every extremal function of (2) is (up to translation and reflection on the real axis under which $\Re a_{-1}$ and b_f are invariant) also an extremal function of (4) for a suitable point $w_1 = u + i\frac{b}{2}$ $(u \geq 0)$ which we shall characterize by an additional variation preserving the class Σ_b.

Our purpose is to prove the following theorems.

Theorem 1. *Let $f(z) = z + a_0 + a_{-1}z^{-1} \cdots$ be an extremal function of (4). Then $w = f(z)$ defines w as a univalent function of $\zeta = z + \frac{1}{z}$ which satisfies the differential equation*

$$\frac{w^2 - w_0^2}{w^2 - w_1^2}\, dw^2 = \frac{\zeta^2 - \mu^2}{\zeta^2 - 4}\, d\zeta^2 \tag{5}$$

in the exterior of the line segment $[-2,2]$ with some complex constant $w_0 \neq \pm w_1$ and some constant $\mu \geq 2$. These unknown constants satisfy the three relations

$$\int_0^{\pi/2} \Im \left[(w_0^2 - w_1^2 \sin^2\theta)^{\frac{1}{2}}\right] d\theta = 0 \tag{6}$$

$$\int_0^{\pi/2} \Re \left[(w_0^2 - w_1^2 \sin^2\theta)^{\frac{1}{2}}\right] d\theta = \begin{cases} \int_0^{\pi/2} (\mu^2 - 4\sin^2\theta)^{\frac{1}{2}}\, d\theta & \text{if } \mu > 2 \\ \leq 2 & \text{if } \mu = 2 \end{cases} \tag{7}$$

$$\int_0^{\pi/2} \Im \left[\frac{w_0^2 \cos^2\theta}{(w_1^2 - w_0^2 \sin^2\theta)^{\frac{1}{2}}}\right] d\theta = -\int_2^{\mu} \left(\frac{\mu^2 - \tau^2}{\tau^2 - 4}\right)^{\frac{1}{2}} d\tau \tag{8}$$

by a suitable choice of the branch of the square root, and we have

$$a_{-1} = \tfrac{1}{2}(\mu^2 - 2 + w_1^2 - w_0^2). \tag{9}$$

Further, E_f is made up of finitely many trajectory arcs of the quadratic differential

$$\frac{w^2 - w_0^2}{w^2 - w_1^2}\, dw^2 \tag{10}$$

and E_f does not separate the plane.

Case I: $\mu > 2$. The zeros $\pm w_0$ of (10) are image points of $w = f(z)$ corresponding to $\pm\mu = z + \frac{1}{z}$, and E_f consists of $\pm w_1$ and a single analytic curve through 0 connecting $\pm w_1$, which is symmetric with respect to the origin and homotopic (in the complex plane punctured at $\pm w_0$ and $\pm w_1$) to the line segment $(-w_1, w_1)$. The extremal function f is uniquely determined. Case I certainly occurs if $|w_1|$ is sufficiently close to 2 and $\varepsilon < \Im w_1 < 2 - \varepsilon$ for arbitrarily given $\varepsilon \in (0,1)$.

Case II: $\mu = 2$. E_f consists of the points $\pm w_0$, $\pm w_1$ and the union C_f of at most three analytic curves which is symmetric to the origin and connects $-w_1, -w_0, w_0$ with w_1, plus some extra slit (if any) joining into $\pm w_0$, respectively (see Fig. 1). The extremal function f is uniquely determined if and only if equality in (7) holds, that is, E_f has no extra slits. Case II certainly occurs if $\Re w_1$ is sufficiently close to 0 and $\varepsilon < \Im w_1 < 2-\varepsilon$ for arbitrarily given $\varepsilon \in (0,1)$.

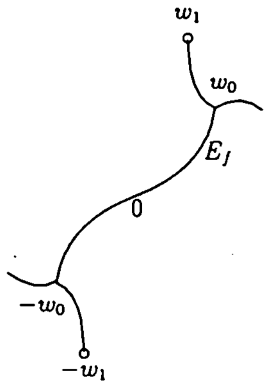

Figure 1 ($\mu = 2$)

Remark 1. Evidently, for the limiting case $|w_1| = 2$, the class $\Sigma(\pm w_1)$ consists only of the function $f(z) = z + \frac{w_1}{\bar{w}_1} z^{-1}$ which maps $|z| > 1$ onto the exterior of the line segment $[-w_1, w_1]$. Evidently, this function is extremal for (4) with $\Re a_{-1} = 1 - \frac{1}{2}(\Im w_1)^2$ and may be taken as solution of (5) in the limiting case $w_0 \to \infty$ as $\mu \to \infty$.

Let us emphasize the case of purely imaginary w_1 which allows an explicit solution of (4) in

Theorem 2. *Let $c \in (0,2)$ be given and $f \in \Sigma(\pm ic)$, $f(z) = z + a_0 + a_{-1}z^{-1} + \cdots$. Then*

$$\Re a_{-1} \leq 1 - \tfrac{1}{2}c^2. \tag{11}$$

Equality in (11) holds if and only if f maps $|z| > 1$ onto the exterior of a cross which consists of the line segment $[-ic, ic]$ and some segment on the real axis. There exists infinitely many such mappings.

Remark 2. There is a close connection to the Garabedian-Schiffer inequality of Grunsky-type derived in [2] which involves values omitted by $f \in \Sigma(\pm w_1)$. By setting $u = -\nu = w_1, m = 1, \lambda_0 = 0$ and $\lambda_1 = i$, the Garabedian-Schiffer inequality (26) in [8: p. 108/ Theorem 4.5] yields

$$\Re a_{-1} \leq 1 + \Re(\tfrac{1}{2}w_1^2). \tag{12}$$

Applying [8: p. 115/Theorem 4.6], equality in (12) holds for a function $f \in \Sigma(\pm w_1)$ if and only if E_f consists of trajectory arcs of the quadratic differential (10), where $w_0 = 0$. From this and (6) we conclude $w_1 = ic$ and hence, the sharp inequality (11) follows together with the equality assertion. Thus inequality (12) is best possible if $w_0 = 0$. We remark that Grunsky-type inequalities are sharp only if the numerator of the quadratic differential has no odd order zero. For estimates related to (12), see [8: p. 117/Theorem 4.7].

In order to integrate the differential equation (5), we have to know the parameters w_0 and μ. Conversely, the given points $\pm w_1$ and if $\mu = 2$ the zeros $\pm w_0$ of (10) must lie on the extremal continuum E_f which consists of trajectory arcs of (10). The condition that the trajectory arcs through $\pm w_1$ and possibly $\pm w_0$ hang together to form one single continuum is a very restrictive condition on the unknown parameters in (5) and leads to the very implicit set of relations (6) - (8). The analysis of the quadratic differential (10) and relations (6) and (8) enables us to get estimates of the zeros $\pm w_0$ of (10) and the extremal continuum E_f in

Theorem 3. *Let $0 < |w_1| < 2, 0 < \arg w_1 < \frac{\pi}{2}$ and $f(z) = z + a_0 + a_{-1}z^{-1} \cdots$ be an extremal function of (4). Then in all cases ($\mu \geq 2$)*

$$w_0 \in \mathcal{Z} = \left\{ w \;\middle|\; \begin{array}{l} 0 < \arg w < \min(\frac{1}{4}\pi, \arg w_1) \\ -\frac{1}{2}\pi < \arg(w - w_1) \\ \arg(w_1^2 - w^2) < \pi \end{array} \right\} \tag{13}$$

where $\mathcal{Z}$ is an unbounded domain in the first quadrant whose boundary is formed by two or three line segments and one segment of an hyperbola,

$$E_f \subset \mathcal{D} \cup (-\mathcal{D}) \tag{14}$$

except for the points 0 and $\pm w_1$ where

$$\mathcal{D} = \left\{ w \;\middle|\; 0 < \arg w < \arg w_1 \text{ and } \arg(w_1^2 - w^2) < \pi \right\} \supset \mathcal{Z}$$

and we have the inequality

$$1 - \tfrac{1}{2}(\Im w_1)^2 < \Re a_{-1} < 1 + \tfrac{1}{2}\Re(w_1^2). \tag{15}$$

In particular, if $\mu = 2$, then

$$w_0 \in \mathcal{Z}^* = \left\{ w \;\middle|\; \begin{array}{l} \frac{1}{2}\arg w_1 < \arg w < \min(\frac{1}{2}\pi, \arg w_1) \\ -\frac{1}{2}\pi < \arg(w - w_1) \\ \arg(w_1^2 - w^2) < \frac{1}{2}\pi + \arg w_1 \end{array} \right\} \tag{16}$$

where $\mathcal{Z}^* \subset \mathcal{Z}$ is a bounded domain whose boundary is formed by two or three line segments and one segment of an hyperbola (see Fig. 2).

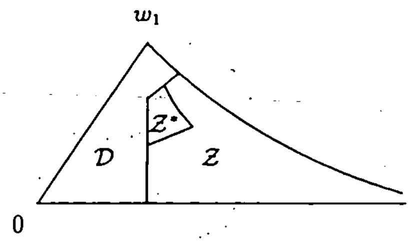

Figure 2 ($\mathcal{Z}^* \subset \mathcal{Z} \subset \mathcal{D}$)

Remark 3. Inequality (15) is asymptotically sharp for $\Re w_1 \to 0$. In view of Cases I and II of Theorem 1, estimates (13), (14) and (16) are also asymptotically sharp in the following sense: Let $\varepsilon \in (0,1)$ be given arbitrarily. If $\varepsilon < \Im w_1 < 2 - \varepsilon$ and $|w_1| \to 2$, then $\mu \to \infty, |w_0| \to \infty$ and, by (13), $\arg w_0 \to 0$. If w_1 is close by $ic(0 \leq c \leq 2)$, then by (14) the extremal continuum E_f is (in the sense of the Euclidian metric) close to the cross consisting of the line segment $[-ic, ic]$ and the real axis. By Theorem 2 and (13), we have $w_0 = 0$ if and only if w_1 is purely imaginary. In particular, if $\Re w_1 > 0$ and $w_1 \to ic$ $(\varepsilon \leq c \leq 2 - \varepsilon)$, then by (16) $|w_0| \to 0$ and $\arg w_0 \to \frac{1}{4}\pi$.

Theorem 4. *Let $f(z) = z + a_{-1}z^{-1} + \cdots$ be an extremal function of (2). Then $b_f = b$, and f is (up to a translation and reflection on the real axis) a solution of (4) for some*

$$w_1 = u + i\tfrac{1}{2}b \qquad \left(0 < u < (4 - \tfrac{1}{4}b^2)^{\frac{1}{2}}\right) \tag{17}$$

and $w = f(z)$ defines w as a univalent function of $\zeta = z + \frac{1}{z}$ which satisfies the differential equation (5) in the exterior of the line segment $[-2, 2]$ where

$$w_0^2 = w_1^2 - ivw_1 \qquad (0 < v < b) \tag{18}$$

and $\mu > 2$. The unknown real constants u, v, μ satisfy equations (6) – (8) and we have

$$\Re a_{-1} = \tfrac{1}{2}(\mu^2 - 2 - \tfrac{1}{2}vb). \tag{19}$$

Furthermore, E_f is a single analytic curve not passing through $\pm w_0$ which is symmetric to the origin and joins $\pm w_1$ perpendicular to the lines $\Im w = \pm\frac{1}{2}b$, respectively (see Figure 3), and

$$E_f \subset \mathcal{T} \cup (-\mathcal{T}) \tag{20}$$

except for the points $0, \pm w_1$ *where* $\mathcal{T}$ *is the open triangle with vertices* $0, \Re w_1, w_1$.

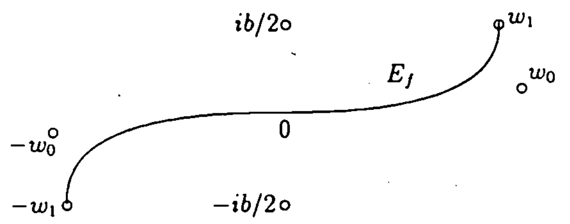

Figure 3 ($\mu > 2$ and $b_f = b$)

Remark 4. A simple calculation shows that, in view of (17) and (18), for given $u \in (0, \sqrt{4 - \frac{1}{2}b^2})$ the left-hand side of (6) is a continuously decreasing function of $v > 0$ changing sign once. Therefore, for given $w_1 = u + i\frac{1}{2}b$ the points $\pm w_0$ satisfying (18) are uniquely determined by (6) such that v and by (7) and (19) also μ and $\Re a_{-1}$ may be considered as functions of u. The unicity of a solution of problem (2) remains an open problem, because it is not clear whether $\Re a_{-1}$ can have several maxima.

2. Proofs

Throughout this section let us denote by $d\Omega^2$ the quadratic differential (10).

Proof of Theorem 1. A function $F = F(Z)$ is called a *Bieberbach – Eilenberg* function if it is analytic and univalent in $U = \{Z : |Z| < 1\}$, so that it has a series development about the origin beginning

$$F(Z) = b_1 Z + b_2 Z^2 + b_3 Z^3 + \cdots \tag{21}$$

and is such that $F(Z_1)\,F(Z_2) \neq 1$ for any $Z_1, Z_2 \in U$. Observe that each such function omits the values ± 1. We will denote this class of Bieberbach-Eilenberg functions by $\mathcal{E}$. It is well known that $|b_1| \leq 1$ for all $F \in \mathcal{E}$, with equality if and only if $F(Z) = e^{i\alpha} Z$, α real.

In view of the following, the crucial point is the observation that the function

$$w = \tfrac{1}{2} w_1 (W + \tfrac{1}{W}) \tag{22}$$

has a single-valued inverse in the exterior of a continuum passing through $\pm w_1$. By substitutions (22) and $Z = \frac{1}{z}$, each function $W = F(Z) \in \mathcal{E}$ having series development (21) with $b_1 = \frac{w_1}{2}$ defines a function $w = f(z) \in \Sigma(\pm w_1)$ having series development (1) and conversely, where

$$\begin{aligned} a_0 &= -\frac{b_2}{b_1} \\ a_{-1} &= \left(\frac{b_2}{b_1}\right)^2 - \frac{b_3}{b_1} + b_1^2. \end{aligned} \tag{23}$$

Therefore, problem (4) is equivalent to maximize

$$\Re\left[\left(\frac{b_2}{b_1}\right)^2 - \frac{b_3}{b_1} + b_1^2\right] \tag{24}$$

over all functions $F \in \mathcal{E}$ with prescribed first coefficient $b_1 = \frac{w_1}{2}$. We shall solve problem (24) by applying a standard technique for constrained variation within the class $\mathcal{E}$ based on the implicit function theorem (see [5] for a more detailed discussion).

Let $F(Z) = b_1 Z + \cdots \in \mathcal{E}$, $Z_1, Z_2 \in U$ and $\varepsilon_1, \varepsilon_2$ be any complex numbers. Using [6: p. 8/Theorem 2.2] (after small modification), for all sufficiently small $\varepsilon = \max(|\varepsilon_1|, |\varepsilon_2|)$ there exists a function $F^*(Z) = b_1^* Z + \cdots \in \mathcal{E}$ such that

$$F^*(Z) = F(Z) + \sum_{n=1}^{2} \mathcal{F}(Z, Z_n, \varepsilon_n, F) + o(\varepsilon) \tag{25}$$

where

$$\begin{aligned}\mathcal{F}(Z, Z_n, \varepsilon_n, F) &= \varepsilon_n\left(\frac{F(Z)}{F(Z)-F(Z_n)} - \frac{F(Z)^2}{1-\overline{F(Z_n)}F(Z)}\right)\\ &\quad - \varepsilon_n\left(\frac{F(Z_n)}{Z_nF'(Z_n)^2}\right)\frac{ZF'(Z)}{Z-Z_n} + \overline{\varepsilon_n\left(\frac{F(Z_n)}{Z_nF'(Z_n)^2}\right)}\frac{Z^2F'(Z)}{1-\overline{Z_n}Z}\end{aligned}$$

and

$$b_1^* = b_1 + \sum_{n=1}^{2} \varepsilon_n B(Z_n, F) + o(\varepsilon) \tag{26}$$

where

$$B(Z,F) = \frac{b_1}{F(Z)}\left[\left(\frac{F(Z)}{ZF'(Z)}\right)^2 - 1\right].$$

Suppose $B(Z,F) \equiv 0$ for an extremal F. Then $F(Z) \equiv \frac{1}{2}w_1 Z$ and the function $f(z) = z + \left(\frac{w_1}{2}\right)^2\frac{1}{z} \in \Sigma(\pm w_1)$ corresponding with F were an extremal function of (4) which maps $|z| > 1$ onto the exterior of an ellipse with focal points at $\pm w_1$. A suitable rotational variation applied to f leads to a contradiction to the extremality of f. Thus $B(Z,F) \not\equiv 0$, and the method of *Lagrange* multiplier is applicable to (24): Define

$$\Psi(F) = \lambda b_1 + \left(\frac{b_2}{b_1}\right)^2 - \frac{b_3}{b_1} + b_1^2 \tag{27}$$

for $F \in \mathcal{E}$ where λ is a complex constant. Let $F(Z)$ be extremal for (24). Then

$$\Re\Psi(F^*) \le \Re\Psi(F) \tag{28}$$

for some complex Lagrange multiplier λ and all (nearby) functions $F^* \in \mathcal{E}$. Here, "nearby" is in the sense of convergence on compact subsets of U. Applying [6: p.

12/Theorem 4.1], after a calculation we therefore get: The extremal F satisfies the differential equation

$$\left[b_1^2\left(W+\frac{1}{W}\right)^2-4b_1^2-\lambda b_1\right]\frac{dW^2}{W^2}=\left[\left(Z+\frac{1}{Z}\right)^2-2-2\Psi(F)+\lambda b_1\right]\frac{dZ^2}{Z^2} \tag{29}$$

where the right-hand side is real and non-negative for $|Z|=1$. By the substitutions $Z=\frac{1}{z}$ and (22), $W=F(Z)$ defines a function $w=f(z)\in\Sigma(\pm w_1)$ which maps $|z|>1$ onto the exterior of a continuum E_f having no interior (see [6: p. 13/Theorem 4.2]). Since $w^2-w_1^2=(\frac{w_1}{2})^2(W-\frac{1}{W})^2$, we have $\frac{dW^2}{W^2}=\frac{dw^2}{w^2-w_1^2}$, and hence from (29), $w=f(z)$ defines w as a univalent function of $\zeta=z+\frac{1}{z}$ satisfying the differential equation (5) where, in view of (23), (27) and $b_1=\frac{w_1}{2}$,

$$w_0^2=w_1^2+\tfrac{1}{2}\lambda w_1 \tag{30}$$

and

$$\mu^2=2+2a_{-1}+\tfrac{1}{2}\lambda w_1. \tag{31}$$

Because the right-hand side of (5) is real and non-negative for $\zeta\in[-2,2]$, it follows that $\mu^2\geq 4$. Combining (30) and (31), we get (9). The continuum E_f is the image of $[-2,2]$ under the mapping $\zeta\to w$ defined by $\zeta=z+\frac{1}{z}$ and $w=f(z)$.

We now analyze the continuum E_f more carefully. If $\mu>2$, then the right-hand side of (5) has simple zeros at $\pm\mu$ and hence the continuum E_f will not contain the critical points $\pm w_0$. If $\mu=2$, then E_f must contain the points $\pm w_0$. We see that E_f contains the points $\pm w_1$ and may or may not contain the points $\pm w_0$. Except for these points it must consist of analytic arcs which are trajectories of $d\Omega^2$. Clearly, $w_0^2\neq w_1^2$. Thus $d\Omega^2$ has three or four finite critical points. At the simple poles $\pm w_1$, exactly one trajectory leaves. At $\pm w_0$ exactly three or four leave at equal angles depending on whether $w_0\neq 0$ or $w_0=0$. Obviously, $d\Omega^2$ is invariant under reflection with respect to the origin. Thus $0\in E_f$ and E_f consists of either:

($\mu>2$) the points $\pm w_1$ and a single analytic curve not passing through $\pm w_0$, which is symmetric to the origin

or:

($\mu=2$) the points $\pm w_0,\pm w_1$ and the union C_f of at most three analytic curves, which is symmetric to the origin and connects the points $-w_1,-w_0,w_0$ with w_1, plus possibly one analytic slit leaving at $\pm w_0$.

To derive relations (6) - (8) we observe first that $\int d\Omega$ over any segment of E_f is real and any path can be altered homotopically in the complex plane punctured at the four points $\pm w_0$ and $\pm w_1$ without changing the integral of $d\Omega$.

Next we show that if J is the line segment $[-w_1,w_1]$, then either $\int d\Omega$ over J is real, or there are four disjoint subintervalls of J for which $\int d\Omega$ is real. First, suppose $\mu>2$. Then either E_f is homotopic (in the complex plane punctured at $\pm w_0$ and $\pm w_1$) to J, or the three trajectories leaving $\pm w_0$ must cross the line segment $(0,\pm w_1)$. The above assertion therefore holds. On the other hand, if $\mu=2$, then E_f is homotopic to J or else the segments of E_f from 0 to $\pm w_0$ and from $\pm w_0$ to $\pm w_1$, together with a third trajectory from $\pm w_0$ to some point of $(0, \pm w_1)$, make up two paths homotopic to disjoint segments of the line segment $(0, \pm w_1)$. Again the conclusion follows.

Let $I = [w_1 \sin\theta_1, w_1 \sin\theta_2]$ be any segment of the line segment $[0, w_1]$. Then $\int_I d\Omega = \int_{\theta_1}^{\theta_2} (w_0^2 - w_1^2 \sin^2\theta)^{\frac{1}{2}} d\theta$. This is the weighted mean value of complex numbers on a segment of an hyperbola. Two such disjoint integrals could be real only if w_0^2 and w_1^2 both are real, that is, w_0 is either real or purely imaginary and w_1 is purely imaginary. Analyzing the qualitative nature of the trajectories of $d\Omega^2$ we conclude $w_0 = 0$, and therefore E_f consists of the segment $[-w_1, w_1]$ on the imaginary axis plus some segment on the real axis. Otherwise, if $w_0 \neq 0$ were real, then the only trajectory connecting the points $\pm w_1$ is the line segment $[-w_1, w_1]$ having capacity < 1 which is a contradiction. On the other hand, if $w_0 \neq 0$ were purely imaginary, then the only trajectory from $\pm w_1$ is the segment from $\pm w_1$ to $\pm w_0$ on the imaginary axis. Moreover, the real axis is also a trajectory. Hence, no bounded E_f can satisfy the requirements, and this case cannot occur.

Therefore, we have shown $\Im \int_J d\Omega = 0$ or, equivalently, (6).

This is only one relation among three real unknowns. Suppose $\mu > 2$. Then from (5) the integral of $d\Omega$ around E_f in the w-plane will equal the integral around $[-2, 2]$ in the ζ-plane. In view of the fact that E_f is symmetric to the origin and homotopic to the line segment $[-w_1, w_1]$, we therefore get relation (7), where equality holds. Suppose $\mu = 2$. Then E_f passes through $\pm w_0$ and equation (5) simplifies to $d\Omega = d\zeta$ in the exterior of the segment $[-2, 2]$. This segment has "open ends" at ± 2, i.e. $d\Omega$ changes sign if we change directions at those points. The integral of $d\Omega$ around $[-2, 2]$, starting at -2, will be the same as the integral of $d\Omega$ around E_f in the w-plane, starting at the "open end" which will be $-w_0$ or the tip of the possibly "extra" slit joining into $-w_0$. That is, we therefore must have

$$\int_0^{\pi/2} \Re\left[(w_0^2 - w_1^2 \sin^2\theta)^{\frac{1}{2}}\right] d\theta + I_+ + I_- = 2$$

where $I_- \geq 0$ and $I_+ \geq 0$ are the integrals of $d\Omega$ along E_f from the tip of a possibly "extra" slit to $-w_0$ and from w_0 to the tip of another possibly "extra" slit, respectively. This implies the inequality in (7) with equality only if the "extra" slits are of zero length.

A third relation is obtained from (5) with the help of the observation that $\zeta = \pm\mu$ corresponds to $w = \pm w_0$, respectively. Thus, the integral of $d\Omega$ in the ζ-plane along the line segment $l = [2, \mu]$ must equal the integral of $d\Omega$ in the w-plane along L, the image of l by the mapping $\zeta \to w$. Sincs l lies along an orthogonal trajectory, L must be an orthogonal trajectory of (10) from some point $\neq \pm w_1$ of E_f to w_0. This integral is purely imaginary, while the integral along any part of E_f is real. Hence

$$\Im \int_S d\Omega = -\int_2^{\mu} \left(\frac{\mu^2 - \tau^2}{\tau^2 - 4}\right)^{\frac{1}{2}} d\tau \tag{32}$$

where by Cauchy's integral theorem we may integrate along any analytic curve S which connects the point 0 or w_1 of E_f with w_0. Choosing S as the line segments $[0, w_0]$, relation (32) is equivalent to (8).

By using the General Coefficient Theorem [8: p. 246/Theorem 8.12], we see that if $\mu > 2$, or $\mu = 2$ and E_f has no extra slits, then the extremal function is uniquely determined, and if $\mu = 2$ and E_f has some extra slit, then there are infinitely many extremal function and at least one for which E_f is symmetric to the origin.

It only remains to show that Cases I and II can occur actually. For proving this we use a continuity argument.

First we consider the Case I. Assuming that Case I would not occur if $|w_1|$ is close to 2 and $\varepsilon < \Im w_1 < 2-\varepsilon$ for arbitrarily given $\varepsilon \in (0,1)$. Then there exists a sequence of points $w_{1,n}$ with $|w_{1,n}| \to 2$ for which the corresponding sequence of extremal functions $f_n \in \Sigma(\pm w_{1,n})$ of (4) (or a suitable subsequence of it) converge locally uniformly in $|z| > 1$ to a limit function $f \in \Sigma$ fulfilling $b_f > 0$. Moreover, every function $w = f_n(z)$ defines w as a univalent function of $\zeta = z + \frac{1}{z}$ which satisfies the differential equation

$$\frac{w^2 - w_{0,n}^2}{w^2 - w_{1,n}^2}\, dw^2 = d\zeta^2 \tag{33}$$

in the exterior of the line segment $[-2,2]$ with some complex constants $w_{0,n} \in E_{f_n}, w_{0,n}^2 \neq w_{1,n}^2$. We may assume $w_{0,n} \to w_0$. From (33) we infer that the limit function $w = f(z)$ defines w as a univalent function of $\zeta = z+\frac{1}{z}$ which must satisfy the differential equation

$$\frac{w^2 - w_0^2}{w^2 - w_1^2}\, dw^2 = d\zeta^2 \tag{34}$$

in the exterior of the line segment $[-2,2]$ with $0 < \arg w_1 < \frac{\pi}{2}, |w_1| = 2$ and, since $b_f > 0$ we have $w_0 \neq \pm w_1$. Therefore $\pm w_0, \pm w_1 \in E_f$ and E_f must be a trajectory of (10). Because of $|w_1| = 2$ and $\pm w_1 \in E_f$, the continuum E_f must have diameter 4 and hence, the limit function f maps $|z| > 1$ onto the exterior of the line segment $E_f = [-w_1, w_1]$. This leads to a contradiction because $[-w_1, w_1]$ for $0 < \arg w_1 < \frac{\pi}{2}$ is no trajectory of (10).

Finally, we consider the remaining Case II. Assuming that Case II would not occur if $\Re w_1$ is close to 0 and $\varepsilon < \Im w_1 < 2 - \varepsilon$ for arbitrarily given $\varepsilon \in (0,1)$. Then there exists a sequence of points $w_{1,n} \to w_1 = ic$ $(0 < c < 2)$ for which the sequence of extremal functions $f_n \in \Sigma(\pm w_{1,n})$ of (4) (or a suitable subsequence of it) converge locally uniformly in $|z| > 1$ to a limit function $f \in \Sigma$ fulfilling $b_f > 0$. Moreover, every function $w = f_n(z)$ defines w as a univalent function of $\zeta = z + \frac{1}{z}$ which satisfies the differential equation

$$\frac{w^2 - w_{0,n}^2}{w^2 - w_{1,n}^2}\, dw^2 = \frac{\zeta^2 - \mu_n^2}{\zeta^2 - 4}\, d\zeta^2 \tag{35}$$

in the exterior of the line segment $[-2,2]$ with $\mu_n > 2$ and some complex constants $w_{0,n} \notin E_{f_n}, w_{0,n}^2 \neq w_{1,n}^2$. Besides, in view of (7), we must have

$$\int_0^{\pi/2} \Re\left[(w_{0,n}^2 - w_{1,n}^2 \sin^2\theta)^{\frac{1}{2}}\right] d\theta > 2 \tag{36}$$

for all n. There are only two possible cases to distinguish: either both sequences $\{w_{0,n}\}$ and $\{\mu_n\}$ are bounded, where we may assume $w_{0,n} \to w_0$ and $\mu_n \to \mu$, or unbounded.

From (35) we infer that the limit function $w = f(z)$ defines w as a univalent function of $\zeta = z + \frac{1}{z}$ which must satisfy either the differential equation (5) in the exterior of the line segment $[-2, 2]$ with $\mu \geq 2$, $w_1 = ic \in E_f$ $(0 < c < 2)$ and $w_0 \neq \pm w_1$, or

$$\frac{A dw^2}{w^2 - w_1^2} = \frac{d\zeta^2}{\zeta^2 - 4} \tag{37}$$

in the exterior of the line segment $[-2, 2]$. Because of $f'(\infty) = 1$ the constant A in (37) is equal to 1. The second case may be excluded. Indeed, integreting (37) we easily see that E_f is the line segment $[-ic, ic]$ having capacity $\frac{c}{2} < 1$ which is a contradiction. Hence, the first case must occur. The limit function f maps $|z| > 1$ onto the exterior of a cross which consists of the line segment $[-ic, ic]$ and some segment on the real axis (see the proof of Theorem 2). In particular, we have $w_0 = 0$ and

$$\int_0^{\pi/2} \Re\left[(w_0^2 - w_1^2 \sin^2\theta)^{\frac{1}{2}}\right] d\theta = c < 2. \tag{38}$$

Letting n tend to infinity, from (36) we get a contradiction to (38). This completes the proof of Theorem 1 ∎

Proof of Theorem 2. Let $w_1 = ic$ $(0 < c < 2)$ and f be an extremal function of (4). Because of (6) we must have

$$\int_0^{\pi/2} \Im\left[(w_0^2 + c^2 \sin^2\theta)^{\frac{1}{2}}\right] d\theta = 0. \tag{39}$$

Suppose $\Im(w_0^2) > 0$ (< 0). Clearly, the quantity $[w_0^2 + c^2 \sin^2\theta]$ has the constant positive (negative) imaginary part $\Im(w_0^2)$. The square root of this quantity would lie in the first (fourth) quadrant. Hence, the imaginary part of $[w_0^2 + c^2 \sin^2\theta]^{\frac{1}{2}}$ could never change sign and the integral average in (39) could never be zero. Thus, w_0^2 must be real. From this we can conclude $w_0 = 0$ by analyzing the qualitative nature of the trajectories of (10) which we have already accomplished in the proof of Theorem 1. Therefore, f maps $|z| > 1$ onto the exterior of a cross which consists of the line segment $[-ic, ic]$ and some segment on the real axis. As the Case II of Theorem 1, all such mappings are extremal functions. The analytic representation of such (odd) one is given by (55) and (56), and from (57) we get (11). This proves Theorem 2 ∎

By (5), the trajectories of (10) are the images of the trajectories of the quadratic differential

$$\frac{\zeta^2 - \mu^2}{\zeta^2 - 4} d\zeta^2 \tag{40}$$

through that function

$$w = g(\zeta) = \zeta + c_0 + c_{-1}\zeta^{-1} + \cdots \tag{41}$$

defined by $w = f(z)$ and $\zeta = z + \frac{1}{z}$. Obviously, the trajectories of (40) are symmetric with respect to the real and imaginary axis. If $\mu > 2$, then the line segments $(-\infty, -\mu), (-2, 2), (\mu, \infty)$, some analytic curve Γ which lies in $\Im z > 0$ and connects the points $\pm\mu$, and $-\Gamma$ are critical trajectories of (40). Hence, the complement of the union of all these critical trajectories and all critical points of (40) and (10), respectively, is composed of one ring domain and two end domains. Evidently, if $\mu = 2$, then the trajectories of (40) are lines parallel to the real axis and hence, the complement of the union of all these critical trajectories and all critical points of (40) and (10), respectively, is composed of two end domains. These domains are covered exactly once by a certain family of trajectories. In the case of end domains these trajectories are open Jordan arcs which tangent to horizontal lines near infinity, while in the case of a ring domain these trajectories are closed Jordan curves.

Let $\zeta = g^{-1}(w)$ be the inverse function of (41). Through $\zeta = g^{-1}(w)$ the continuum E_f is mapped onto the exterior of the line segment $[-2, 2]$. Because this parallel slit mapping is uniquely determined and we may assume that E_f is symmetric to the origin, we can conclude that $\zeta = g^{-1}(w)$ is an odd function. Hence, $c_0 = 0$ and the critical trajectories $g((\mu, \infty))$ and $g((-\infty, -\mu))$ of (10) are tangent to the positive and negative real axis at infinity, respectively.

Remark 5. The following is evident on topological grounds: If $\mu > 2$, then on every analytic curve γ connecting any two points of $E_f = g([-2, 2])$ there exists at least one intermediate point on γ at which some trajectory of (10) is tangent to γ. This assertion is also true, if γ tends from any point of $g(\pm\Gamma)$ and is tangent to the real axis at infinity. On the other hand, if $\mu = 2$, then on every analytic curve γ connecting any two points of the union U of all critical trajectories of (10) there exists at least one intermediate point on γ at which some trajectory of (10) is tangent to γ. This assertion is also true, if γ tends from any point of U and is tangent to the real axis at infinity.

Proof of Theorem 3. First we shall prove estimates (13) and (16). Since $0, w_1 \in E_f$, as Remark 5 there is at least one point τw_1 $(0 < \tau < 1)$ on the line segment $\gamma = (0, w_1)$ at which some trajectory of (10) is tangent to γ. This implies by (10) that the quantity $w_0^2 - \tau^2 w_1^2$ is real positive; that is,

$$\left.\begin{array}{l} \Im(w_0^2) = \tau^2 \Im(w_1^2) \\ \Re(w_0^2) > \tau^2 \Re(w_1^2) \end{array}\right\}. \tag{42}$$

By assumption, $\Im(w_1^2) > 0$ and hence, from the equation in (42) we get

$$0 < \Im(w_0^2) = \tau^2 \Im(w_1^2) < \Im(w_1^2) \tag{43}$$

and, combining the two relations in (42),

$$\cot(2 \arg w_0) = \frac{\Re(w_0^2)}{\Im(w_0^2)} > \frac{\Re(w_1^2)}{\Im(w_1^2)} = \cot(2 \arg w_1). \tag{44}$$

Therefore, from (43) and (44) we conclude

$$w_0 \in \mathcal{D} = \Big\{ w : 0 < \arg w < \arg w_1 \text{ and } \arg(w_1^2 - w^2) < \pi \Big\}. \tag{45}$$

To continue the estimate of w_0, we next use the fact that, in view of (32), the imaginary part of $\int_S d\Omega$ is negative or equal to zero according as $\mu > 2$ or $\mu = 2$, where we may integrate along any analytic curve S that connects the point 0 or w_1 of E_f with w_0. To do this we have to estimate the sign of $\Im \int_S d\Omega$ on $\mathcal{D}$.

First let S be the line segment $[0, w_0]$, $w_0 \in \mathcal{D}$. The imaginary part of $\int_S d\Omega$ becomes a function Φ of w_0 for which

$$\frac{\partial \Phi}{\partial |w_0|} = \Im \int_0^{w_0} \frac{\frac{w_0^2}{|w_0|}}{[(w_0^2 - w^2)(w_1^2 - w^2)]^{\frac{1}{2}}} dw \tag{46}$$

holds. The argument of the differential $d\omega$ in (46) is equal to $2\arg w_0 - \frac{1}{2}\arg(w_1^2 - w^2)$ where in view of (45) we have

$$2\arg w_0 - \tfrac{1}{2}\pi < \arg d\omega < 2\arg w_0 - \arg w_1 . \tag{47}$$

If $w_0 \in \mathcal{D}$ and $\arg w_0 \geq \frac{\pi}{4}$, then (47) implies $\frac{\partial \Phi}{\partial |w_0|} > 0$ and therefore, since $\Phi(0) = 0$, we have $\Phi(w_0) > 0$. Hence, in every case ($\mu \geq 2$) the zero $w_0 \in \mathcal{D}$ of (10) must satisfy

$$\arg w_0 < \tfrac{1}{4}\pi . \tag{48}$$

On the other hand, if $w_0 \in \mathcal{D}$ and $\arg w_0 \leq \frac{1}{2}\arg w_1$, then (47) implies $\frac{\partial \Phi}{\partial |w_0|} < 0$ and therefore, since $\Phi(0) = 0$, we have $\Phi(w_0) < 0$. Hence, if the zero $w_0 \in \mathcal{D}$ of (10) is a branching point of E_f, then it must satisfy

$$\tfrac{1}{2}\arg w_1 < \arg w_0 < \tfrac{1}{4}\pi . \tag{49}$$

Next let now S be the line segment $[w_1, w_0]$, $w_0 \in \mathcal{D}$. The imaginary part of $\int_S d\Omega$ becomes a function Ψ of w_0 for which

$$\frac{\partial \Psi}{\partial |w_0|} = \Im \int_{w_1}^{w_0} \frac{\frac{w_0^2}{|w_0|}}{[(w_0^2 - w^2)(w_1^2 - w^2)]^{\frac{1}{2}}} dw \tag{50}$$

holds. In order to show that the derivative in (50) is negative we shall estimate the argument of the differential $d\omega$ in (50). In view of (45) and $w \in S$, we have

$$\arg d\omega = \arg w_0 - \tfrac{1}{2}[\arg(w + w_0) + \arg(w + w_1)] + \arg w_0 - \tfrac{1}{2}\pi$$

where $-\pi < \arg d\omega < 0$. This implies $\frac{\partial \Psi}{\partial |w_0|} < 0$ $(w_0 \in \mathcal{D})$ and hence, Ψ is a decreasing function of $|w_0|$ for fixed $\arg w_0 \in (0, \arg w_1)$.

Now we estimate the sign of $\Psi(w_0) \equiv \Im \int_{w_1}^{w_0} d\Omega$ in $\mathcal{D}$. Let $\arg(w_0 - w_1) = -\frac{\pi}{2}$. In view of $w \in S$, we have

$$\arg d\Omega = -\pi + \tfrac{1}{2}[\arg(w + w_0) + \arg(w + w_1)]$$

where $-2\pi < \arg d\Omega < -\pi$. This implies

$$\Psi(w_0) > 0 \qquad (w_0 \in \mathcal{D}, \arg(w_0 - w_1) = -\tfrac{1}{2}\pi). \tag{51}$$

Let now $w_0 \in \mathcal{D}$ be a point on that segment of the hyperbola $w = (w_1^2 - ivw_1)^{\frac{1}{2}}$ $(v > 0)$ which tends from w_1. For the estimate of the sign of $\Psi(w_0) \equiv \Im \int_{w_1}^{w_0} d\Omega$ it is more convenient to integrate from w_1 to w_0 along this segment of the hyperbola. Since

$$\begin{aligned}\arg(w_0^2 - w^2) &= \arg w_1 - \tfrac{1}{2}\pi\\ \arg(w_1^2 - w^2) &= \arg w_1 + \tfrac{1}{2}\pi\\ \arg dw &= -\tfrac{1}{2}\pi - \tfrac{1}{2}\arg\left(1 - i\tfrac{v}{w_1}\right) \quad (v > 0)\end{aligned}$$

along this path of integration, we have

$$\arg d\Omega = -\pi - \tfrac{1}{2}\arg\left(1 - i\tfrac{v}{w_1}\right) \qquad (v > 0)$$

where $-\pi < \arg d\Omega < 0$. This implies

$$\Psi(w_0) < 0 \qquad (w_0 \in \mathcal{D}, \arg(w_1^2 - w_0^2) = \tfrac{1}{2}\pi + \arg w_1). \tag{52}$$

As we saw, Ψ is a decreasing function of $|w_0|$ for fixed $\arg w_0 \in (0, \arg w_1)$. Hence, from (51) and (52) we conclude that if

$$w_0 \in \mathcal{D}, \arg(w_0 - w_1) \leq -\tfrac{1}{2}\pi \qquad \text{and} \qquad w_0 \in \mathcal{D}, \arg(w_1^2 - w_0^2) \geq \tfrac{1}{2}\pi + \arg w_1,$$

then $\Psi(w_0) > 0$ and $\Psi(w_0) < 0$, respectively. Hence the zero $w_0 \in \mathcal{D}$ of (10) must satisfy $-\frac{\pi}{2} < \arg(w_0 - w_1)$. This together with (48) gives (13). Besides, if the the zero $w_0 \in \mathcal{D}$ of (10) is a branching point of E_f, then it must satisfy $-\frac{\pi}{2} < \arg(w_0 - w_1)$ and $\arg(w_1^2 - w_0^2) < \frac{\pi}{2} + \arg w_1$. This together with (49) gives (16).

Next we shall prove estimate (14). First, from (10) we conclude that E_f intersects the real axis at the origin under the angle $\alpha(0) = \arg w_1 - \arg w_0$. By (13), it follows $0 < \alpha(0) < \arg w_1$. Hence E_f and $\mathcal{D}$ have common points.

Suppose the assertion in (14) is false. Then E_f and the boundary of $\mathcal{D} \cup (-\mathcal{D})$ must have at least one common point $\hat{w} \neq 0, \pm w_1$. By symmetry, there are only three cases to consider:

Case 1. Suppose $\hat{w} = \tau_0 w_1$ $(0 < \tau_0 < 1)$. Then, as Remark 5, there are at least two points $\tau_1 w_1$ and $\tau_2 w_1$ $(0 < \tau_1 < \tau_0 < \tau_2 < 1)$ at which some trajectories of (10) are tangent to the line segment $\gamma = (0, w_1)$. But this contradicts the fact that the equation in (42) has exactly one solution τ on $(0, 1)$.

Case 2. Suppose $\hat{w} = \tau_0$ $(\tau_0 > 0)$. Then, as Remark 5, there are at least two points τ_1 and τ_2 $(0 < \tau_1 < \tau_0 < \tau_2)$ at which some trajectories of (10) are tangent to the half line $\gamma : w = \tau$ $(\tau > 0)$. This implies by (10) that the quantities $[(\tau_i^2 - w_0^2)(\tau_i^2 - \overline{w_1}^2)]$ $(i = 1, 2)$ are real positive. Therefore, its imaginary parts must be equal to zero or, equivalently,

$$\Im(w_0^2 \overline{w_1}^2) + \tau_i^2 \Im(w_1^2 - w_0^2) = 0 \qquad (i = 1, 2). \tag{53}$$

From here it follows that $\Im(w_0^2 \overline{w_1}^2) = \Im(w_1^2 - w_0^2) = 0$ in contradiction to (13).

Case 3. Suppose $\hat{w} = (w_1^2 + \tau_0)^{\frac{1}{2}}$ $(\tau_0 > 0)$ be a point on that segment of the hyperbola $\gamma : w = (w_1^2 + \tau)^{\frac{1}{2}}$ $(\tau > 0)$ which tends from w_1. Then, as Remark 5, there are at least two points $(w_1^2+\tau_1)^{\frac{1}{2}}$ and $(w_1^2+\tau_2)^{\frac{1}{2}}$ $(0<\tau_1<\tau_0<\tau_2)$ on γ at which some trajectories of (10) are tangent to γ. This implies by (10) that the quantities $[(w_1^2-w_0^2+\tau_i)(\overline{w_1}^2+\tau_i)]$ $(i=1,2)$ are real positive. Therefore, its imaginary parts must be equal to zero or, equivalently,

$$\Im(w_0^2\overline{w_1}^2)+\tau_i\Im(w_0^2)=0 \qquad (i=1,2). \tag{54}$$

From here it follows that $\Im(w_0^2\overline{w_1}^2)=\Im(w_0^2)=0$ in contradiction to (13).

As we have just seen, all three cases lead to a contradiction. Thus (14) must hold.

It only remains to prove inequality (15). Let $0<|w_1|<2$. We shall first construct a suitable slit mapping $g\in\Sigma(\pm w_1)$ as follows: The function $w=g(z)$ $(|z|>1)$ given by

$$w=\frac{w_1}{2}\left(\frac{|w_1|}{w_1}\zeta+\frac{w_1}{|w_1|}\frac{1}{\zeta}\right) \tag{55}$$

and

$$z+\frac{1}{z}=\frac{|w_1|}{2}\left(\zeta+\frac{1}{\zeta}\right) \tag{56}$$

has a series development $z+b_{-1}z^{-1}+\cdots$ about infinity where

$$b_{-1}=1+\tfrac{1}{4}(w_1^2-|w_1|^2) \tag{57}$$

and maps $|z|>1$ onto a domain whose exterior consists of the line segment $[-w_1,w_1]$ plus two analytic slits s and $-s$. In particular, if $w_1=ic$ $(0<c<2)$, then $s\cup(-s)$ is a segment on the real axis, such that g maps $|z|>1$ onto the exterior of a cross.

Let $0<\arg w_1<\frac{\pi}{2}, |w_1|<2$ and $f(z)=z+a_0+a_{-1}z^{-1}\cdots$ be an extremal function of (4). Then by (57) we have $\Re b_{-1}=1-\frac{1}{2}(\Im w_1)^2\le\Re a_{-1}$. Suppose $\Re b_{-1}=\Re a_{-1}$. Then g is also an extremal function of (4) in contradiction to (14). Thus the left inequality in (15) must hold. The right inequality in (15) has already been established in Remark 2. This completes the proof of Theorem 3 ∎

Proof of Theorem 4. First we show that $b_f=b$ in the extremal case. Suppose $b_f>b$. Then, by applying the method of interior, rotational and slit variation to f, we get $f(z)=z+\frac{1}{z}$ where $b_f=0$ which is a contradiction. Furthermore, every extremal function of (2) is (up to translation and reflection on the real axis) also an extremal function of (4) for a suitable point

$$w_1=u+i\tfrac{1}{2}b \qquad \left(0\le u\le(4-\tfrac{1}{4}b^2)^{\frac{1}{2}}\right)$$

which we shall characterize by an additional variation preserving the class Σ_b. In view of Remark 1 and Theorem 2, from the left inequality in (15) we can conclude that (17) must hold.

Define

$$\Phi(F)=\left(\frac{b_2}{b_1}\right)^2-\frac{b_3}{b_1}+b_1^2$$

for $F(Z) = b_1 Z + b_2 Z^2 + b_3 Z^3 \cdots \in \mathcal{E}$. As we have seen at the beginning of the proof of Theorem 1, problem (2) is equivalent to maximize

$$\Re\Phi(F) \tag{58}$$

over the class of all functions $F \in \mathcal{E}$ such that $\Im b_1 = \Im(\frac{1}{2}w_1) = \frac{1}{4}b$.

In the following we use variations (25) and (26) to construct variations which preserve this subclass. Let $F \in \mathcal{E}$ be an extremal function of (58), and $Z_1, Z_2 \in U$ and $\varepsilon_1, \varepsilon_2$ be any complex numbers. Choose $Z_2 \in U$ such that $B(Z_2, F) \neq 0$. Then, for sufficiently small $\varepsilon = \max(|\varepsilon_1|, |\varepsilon_2|)$ there exists a function $F^*(Z) = b_1^* Z + \cdots \in \mathcal{E}$ with the asymptotic development (25), where ε_2 can be chosen as a continuously differentiable function of ε_1 such that

$$b_1^* = b_1 + \Re\varepsilon_1. \tag{59}$$

This last step is based on the implicit function theorem and is a standard technique for constrained variation. Combining (25), (26) and (59) to eliminate ε_2 in (25) and taking into account that F is a solution of (29), after a calculation we obtain from (25)

$$\Re\Phi(F^*) = \Re\Phi(F) - (\Re\varepsilon_1)(\Re\lambda) + o(\varepsilon) \tag{60}$$

where λ is the Lagrange multiplier involved in (29). Because of the extremality of F, we have $\Re\Phi(F^*) \leq \Re\Phi(F)$, and (60) gives $\Re\lambda = 0$. In view of (30) and (13), this implies (18).

From (10) and (18) we can conclude that the extremal E_f joins into $\pm w_1$ under the angle $[\arg(\pm w_1) - \arg(w_1^2 - w_0^2)] = \mp\frac{1}{2}\pi$. In view of the last equation (with the upper sign) and Theorem 3, we see that (16) and therefore $\mu = 2$ cannot hold. Thus $\mu > 2$.

By Theorem 1, the unknown real constants u, v in (17),(18) and μ satisfy equations (6) - (8). Inserting (17), (18) into (9), we get (19).

It only remains to prove (20). Suppose the assertion in (20) is false. Then E_f and the boundary of $\mathcal{T} \cup (-\mathcal{T})$ must have at least one common point $\hat{w} \neq 0, \pm w_1$. In view of (14) and by symmetry, there is only to consider the case $\hat{w} = w_1 - i\tau_0$ $(0 < \tau_0 < \frac{1}{2}b)$. As Remark 5, there is at least one point $w = w_1 - i\tau$ $(0 < \tau < \tau_0)$ at which some trajectory of (10) is tangent to the line segment $\gamma = [w_1, \hat{w}]$. This implies by (10) that the quantity $[(w^2 - w_0^2)(\overline{w}^2 - \overline{w_1}^2)]$ is real negative for $w = w_1 - i\tau$ $(0 < \tau < \tau_0)$. Therefore, its imaginary part must be equal to zero or, in view of (17) and (18), equivalently $uv\tau^2 = 0$. But this is obviously a contradiction. Thus (20) must hold and Theorem 4 is proved completely ∎

As a final remark, we observe that the estimates of the extremal continuum E_f given in Theorems 3 and 4 can be refined on. Moreover, the considerations could be extended to investigate further geometric properties, as convexity and curvature of the extremal E_f. In view of that, we refer to the paper [7] in which the author investigates a continuum containing three given points with minimal transfinite diameter.

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