Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 19 (2000), No. 4, 1075–1086

Some Properties of Legendre Functions and Related Applications

Chie-Ping Chu

Abstract. Some properties of Legendre functions in an asymmetric interval (with respect to zero) with zero boundary values are obtained through variational methods. There are given some applications to the monotonicity and estimates of the first Dirichlet eigenvalue for moving bands on S^2.

Keywords: *Peak and shape of Legendre functions, monotonicity and estimates of eigenvalues*
AMS subject classification: 34B30, 34C11, 35P15, 49R05

1. Introduction

Legendre functions are a classical topic, still their behavior over the interval between two consecutive zeros does not seem to have been thoroughly investigated. For $h \in (0,1)$ and a such that $-\frac{1}{2}h \le a < 1-h$ we consider the Dirichlet eigenvalue problem for the Legendre equation in $(a, a+h)$

$$\left.\begin{aligned} \frac{d}{dz}\left[(1-z^2)\frac{dF}{dz}\right] + \lambda F(z) = 0 \\ F(a) = F(a+h) = 0 \end{aligned}\right\}. \tag{1}$$

Let $u_a > 0$ denote the first eigenfunction of problem (1) corresponding to the first eigenvalue $\lambda = \lambda_a > 0$. Then according to the variational principle (see [1:p.104])

$$\lambda_a = \inf_{F \in H_a} \frac{\int_a^{a+h} (1-z^2)[F'(z)]^2 dz}{\int_a^{a+h} F^2(z)\, dz}$$

where $H_a = W_0^{1,2}(a, a+h)$ is the standard Sobolev space. With some variational method, first we estimate the position of the "peak" of u_a as well as get some quantitative results about the shape of u_a (see Theorem 1). Also, we prove that λ_a is a decreasing function of a (see Theorem 2) which leads to some monotonicity property of the distance between two consecutive zeros of any Legendre function (see Corollary 1).

As a type of special functions, Legendre functions are useful in applications. Among those we mention for example [8] where it is proved by differentiation through spherical

Chie-Ping Chu: Soochow University, Dept. Math., Shihlin, Taipei, Taiwan 11102.R.O.C.
chieping@math.scu.edu.tw

ISSN 0232-2064 / $ 2.50

coordinates that among all equal-area spherical bands the first Dirichlet eigenvalue decreases as the band moves toward the north pole. If we treat these equal-area spherical bands through cylindrical coordinates (we think that for equal-area spherical bands it is natural to use this coordinate system), then the Legendre equation would appear. It follows that we could obtain not only the monotonicity of those first eigenvalues as a corollary of Theorem 2, but we also give some estimates for those smallest eigenvalues. In particular, we could locate (sharp in certain cases) the first Dirichlet eigenvalue of the equal-area spherical cap (which is the band containing the north pole, see Section 3) between $n(n+1)$ and $(n+1)(n+2)$ with positive integer n depending only on the area (see Theorem 3). It turns out that we can get a universal lower bound depending only on the area of the above mentioned first eigenvalues. Moreover, by the same technique, we could also get similar results for the first Dirichlet eigenvalues on general surfaces of revolution (Section 4).

2. Main results

Denote the (unique) point in $[a, a+h]$ which attains the maximum value of $u_a(z)$ by z_0. Then first we will give some information about the position of z_0 and the shape of u_a:

Theorem 1.

(i) *$a + \frac{1}{2}h < z_0 \le M$ for $a \in (-\frac{1}{2}h, 1-h)$ where $M \doteq \left[\frac{a^\alpha + (a+h)^\alpha}{2}\right]^{1/\alpha}$ for $a \in [0, 1-h)$ and $M = \frac{1}{2}^{1/\alpha}(a+h)$ for $a \in (-\frac{1}{2}h, 0)$, with $\alpha > 1$ satisfying $(a+h)^2 \le \frac{\alpha-1}{\alpha+1}$.*

(ii) *For $a \in [-\frac{1}{2}h, 1+h)$, $u_a(z) \le u_a(z^*)$ for $z \in [a, a+\frac{1}{2}h]$ where z^* is the reflection point of z with respect to $z = a + \frac{1}{2}h$.*

(iii) *For $a \in \frac{1}{2}h, 0)$, $u'(z) > |u'(-z)|$ for $z \in (a, 0)$.*

(iv) *For $a \in [-\frac{1}{2}h, 1-h)$, $u_a'(a) \le -u_a'(a+h) = |u_a'(a+h)|$.*

Proof. Fix $a \in [-\frac{1}{2}h, 1-h)$. For convenience, we write λ for λ_a and $u(z)$ for $u_a(z)$ here. Consider $w(z) = u(-z)$ for $z \in [-a-h, -a]$. Then

$$\begin{aligned} &[(1-z^2)w']'(z) + \lambda w(z) = 0 \quad \text{in } (-a-h, -a) \\ &w(-a-h) = w(-a) = 0. \end{aligned} \tag{2}$$

We will complete the proof of Theorem 1 through the folling Lemmas 1 - 5.

Lemma 1. *For $a \in [-\frac{1}{2}h, 1-h)$, $u'(z) > 0$ in (a, z_0), $u'(z) < 0$ in $(z_0, a+h)$ and $u''(z) < 0$ in $(a, a+h)$.*

Proof. Since $[(1-z^2)u']'(z) = -\lambda u(z) < 0$, we know that $(1-z^2)u'(z)$ is strictly decreasing in $(a, a+h)$ which implies that $u'(z) > 0$ in (a, z_0) and $u'(z) < 0$ in $(z_0, a+h)$ by the fact that $u'(z_0) = 0$. To prove that $u''(z) < 0$ in $(a, a+h)$, first we consider the case $a < 0$. We see that in $(a, 0)$ the increasing of $1 - z^2$ implies that $u'(z)$ must be decreasing, hence $u''(z) < 0$ in $(a, 0)$. Similarly we could know that $w''(z) < 0$ in $(-a-h, 0)$ from (2) and it follows immediately that $u''(z) < 0$ in $(0, a+h)$. Besides, a direct computation shows that $u''(0) = -\lambda u(0) < 0$. In summary, if $a < 0$, we have $u''(z) < 0$ in $(a, a+h)$. As for the case $a \ge 0$, taking account of the increasing of $1 - z^2$ in $(-a-h, -a)$ we get that $w'(z)$ is decreasing hence $w''(z) < 0$ in $(-a-h, -a)$. It follows that $u''(z) < 0$ in $(a, a+h)$ ∎

Lemma 2. *$z_0 > a + \frac{1}{2}h$ for $a \in (-\frac{1}{2}h, 1-h)$.*

Proof. First we reflect $u(z)$ with respect to the line $z = a + \frac{1}{2}h$. Let $z^* = 2(a + \frac{1}{2}h) - z$ be the reflection point of z with respect to $z = a + \frac{1}{2}h$ and define $t(z) = u(z^*)$ for $z \in [a, a+h]$. Then

$$[1-(z^*)^2]t''(z) + 2z^*t'(z) + \lambda t(z) = 0 \tag{3}$$

or

$$\frac{d}{dz}\left\{[1-(z^*)^2]\frac{dt}{dz}\right\} + \lambda t(z) = 0 \tag{3'}$$

in $(a, a+h)$ with $t(a) = t(a+h) = 0$. Now that $u(z)$ satisfies (1) would imply

$$\int_a^{a+\frac{1}{2}h}(1-z^2)[u'(z)]^2dz - [1-(a+\tfrac{1}{2}h)^2]u'(a+\tfrac{1}{2}h)\,u(a+\tfrac{1}{2}h)$$

$$= \lambda\int_a^{a+\frac{1}{2}h} u^2(z)dz$$

while (3') and $t(a+h) = 0$ imply

$$\int_{a+\frac{1}{2}h}^{a+h}[1-(z^*)^2]\,[t'(z)]^2dz + [1-(a+\tfrac{1}{2}h)^2]t'(a+\tfrac{1}{2}h)t(a+\tfrac{1}{2}h)$$

$$= \lambda\int_{a+\frac{1}{2}h}^{a+h} t^2(z)\,dz.$$

Suppose $z_0 \le a + \frac{1}{2}h$. Then $u'(a+\frac{1}{2}h) \le 0$ and accordingly $t'(a + \frac{1}{2}h \ge 0$. It follows that

$$\int_a^{a+\frac{1}{2}h}(1-z^2)[u'(z)]^2dz \le \lambda\int_a^{a+\frac{1}{2}h} u^2(z)\,dz \tag{4}$$

and

$$\int_{a+\frac{1}{2}h}^{a+h}(1-z^2)[t'(z)]^2dz < \int_{a+\frac{1}{2}h}^{a+h}[1-(z^*)^2]\,[t'(z)]^2dz \le \lambda\int_{a+\frac{1}{2}h}^{a+h} t^2(z)\,dz. \tag{5}$$

Since $u(a+\frac{1}{2}h) = t(a+\frac{1}{2}h)$, so the function k defined by $k(z) = u(z)$ for $z \in [a, a+\frac{1}{2}h]$ and $k(z) = t(z)$ for $z \in [a+\frac{1}{2}h, a+h]$ is in H_a. But from (4) - (5) we get

$$\frac{\int_a^{a+h}(1-z^2)[k'(z)]^2dz}{\int_a^{a+h}k^2(z)\,dz} < \lambda$$

which contradicts the variational principle. Hence $z_0 > a + \frac{1}{2}h$ ∎

Lemma 3. *When $a \geq 0$, then $z_0 \leq \frac{\sqrt[\alpha]{a^\alpha+(a+h)^\alpha}}{\sqrt[\alpha]{2}}$ and when $a < 0$, then $z_0 \leq \frac{a+h}{\sqrt[\alpha]{2}}$ where $\alpha > 1$ satisfies $(a+h)^2 \leq \frac{\alpha-1}{\alpha+1}$.*

Proof. First we consider the case $a \geq 0$. For $z \in (a, a+h)$ let $x = z^\alpha$ where $\alpha > 1$ satisfies $(a+h)^2 \leq \frac{\alpha-1}{\alpha+1}$. Then for $x \in (a^\alpha, (a+h)^\alpha)$, from (1), $u(z(x))$ would satisfy

$$\alpha^2 x^{2-(2/\alpha)}(1-x^{2/\alpha})u''(x) + \alpha x[(\alpha-1)x^{-(2/\alpha)} - (\alpha+1)]u'(x) + \lambda u(x) = 0 \qquad (1')$$

with $u(x) = 0$ at $x = (a+h)^\alpha$. Observe that the coefficient of $u'(x)$ in $(1')$ is non-negative. From [3: Theorem 2.1] it reveals that $u'(x) < 0$ for $x \in \left(\frac{a^\alpha+(a+h)^\alpha}{2}, (a+h)^\alpha\right)$. It follows that $u(z)$ is decreasing from $z = \left(\frac{a^\alpha+(a+h)^\alpha}{2}\right)^{1/\alpha}$ straightforward to $z = a+h$.

Secondly, consider the case $a < 0$. We only transform (1) as in the previous case for $z \in (0, a+h)$ to $(1')$ for $x \in (0, (a+h)^\alpha)$. Then we could know that $u'(x) < 0$ in $\left(\frac{(a+h)^\alpha}{2}, (a+h)^\alpha\right)$ and to get that $u(z)$ is decreasing from $z = \frac{a+h}{\sqrt[\alpha]{2}}$ straightforward to $z = a+h$ ∎

Lemma 4. *For each $a \in [-\frac{1}{2}h, 1-h)$ we have:*

(i) *$u'(a) \leq -u'(a+h)$.*

(ii) *$u(z) \leq u(z^*)$ for $z \in (a, a+\frac{1}{2}h)$ where z^* is the reflection point of z with respect to $z = a + \frac{1}{2}h$.*

Proof. Let $t(z)$ be defined as in the proof of Lemma 2. We also define

$$\xi(z) = u(z) - t(z) = u(z) - u(z^*) \qquad \text{for } z \in [a, a+h].$$

Then

$$\left.\begin{aligned} \xi(z) &\equiv -\xi(z^*) \\ \xi'(z) &\equiv -\xi'(z^*) \end{aligned}\right\}$$

and for $z \in (a, a+h)$ it satisfies

$$(1-z^2)\xi''(z) - 2z\xi'(z) + \lambda\xi(z) = 4(a+\tfrac{1}{2}h)\{[z-(a+\tfrac{1}{2}h)]t''(z) + t'(z)\}$$

with $\xi(a) = \xi(a+h) = 0$. Since $z_0 > a + \frac{1}{2}h$, so $z_0^* < z_0$ and $u(z)$ is increasing for $z \in [z_0^*, z_0]$. Hence $\xi(z) = u(z) - u(z^*) < 0$ and $\xi'(z) = u'(z) + u'(z^*) > 0$ for $z \in [z_0^*, a+\frac{1}{2}h]$. We claim that $\xi(z) \leq 0$ for $z \in (a, z_0^*)$. Then we would have $\xi'(a) \leq 0$ by the fact that $\xi(a) = 0$. In fact, suppose there were a point $z_1 \in (a, z_0^*)$ and a constant $\delta > 0$ such that $\xi(z_1) = 0$ as well as $\xi(z) > 0$ for $z \in (z_1 - \delta, z_1) \subseteq (a, z_1)$ and $\xi(z) < 0$ for $z \in (z_1, z_0^*)$. Then $\xi'(z_1) \leq 0$. These yield

$$\begin{aligned} u(z_1^*) - t(z_1^*) &= \xi(z_1^*) = -\xi(z_1) = 0 \\ u'(z_1^*) - t'(z_1^*) &= \xi'(z_1^*) = \xi'(z_1) \leq 0. \end{aligned}$$

Observe that

$$\begin{aligned} &\int_{z_1^*}^{a+h} (1-z^2)[t'(z)]^2 dz - \int_{z_1^*}^{a+h} [1-(z^*)^2][t'(z)]^2 dz \\ &= \int_{z_1^*}^{a+h} [(z^*)^2 - z^2][t'(z)]^2 dz \\ &= -\int_{z_1^*}^{a+h} t(z)\frac{d}{dz}\{[z^2-(z^*)^2][-t'(z)]\}dz - \{[z_1^2 - (z_1^*)^2]t'(z_1^*)\}t(z_1^*). \end{aligned}$$

Also, that $u(z)$ satisfies (1) would yield

$$\int_a^{z_1^*}(1-z^2)[u'(z)]^2dz = \lambda\int_a^{z_1^*}u^2(z)\,dz + [1-(z_1^*)^2]u'(z_1^*)u(z_1^*).$$

Moreover, from (3′) and $t(a+h)=0$ we have

$$\int_{z_1^*}^{a+h}[1-(z^*)^2]\,[t'(z)]^2dz = \lambda\int_{z_1^*}^{a+h}t^2(z)dz - (1-z_1^2)\,t'(z_1^*)\,t(z_1^*).$$

Adding the three identities together and taking into account $u(z_1^*)=t(z_1^*)$ we get

$$\begin{aligned}
&\int_a^{z_1^*}(1-z^2)[u'(z)]^2dz + \int_{z_1^*}^{a+h}(1-z^2)[t'(z)]^2dz\\
&= \lambda\left[\int_a^{z_1^*}u^2(z)\,dz + \int_{z_1^*}^{a+h}t^2(z)\,dz\right]\\
&\quad - \int_{z_1^*}^{a+h}t(z)\frac{d}{dz}\{[z^2-(z^*)^2]\,[-t'(z)]\}dz \qquad (6)\\
&\quad + [1-(z_1^*)^2]\,[u'(z_1^*)-t'(z_1^*)]u(z_1^*)\\
&< \lambda\left[\int_a^{z_1^*}u^2(z)\,dz + \int_{z_1^*}^{a+h}t^2(z)\,dz\right]
\end{aligned}$$

where the last inequality is due to $u'(z_1^*)\le t'(z_1^*)$ as well as that $[z^2-(z^*)^2]$ and $[-t'(z)]$ are both strictly increasing for $z\in(z_1^*,a+h)$ (since $z_1^*>z_0>a+\frac{1}{2}h$) and that $t(z)$ is positive there. So the integrand of

$$\int_{z_1^*}^{a+h}t(z)\frac{d}{dz}\{[z^2-(z^*)^2]\,[-t'(z)]\}dz$$

is positive everywhere. Now we define a function η in $[a,a+h]$ by $\eta(z)=u(z)$ for $z\in[a,z_1^*]$ and $\eta(z)=t(z)$ for $z\in[z_1^*,a+h]$ (recall that by assumption $u(z_1^*)=t(z_1^*)$). It follows from (6) that

$$\frac{\int_a^{a+h}(1-z^2)[\eta'(z)]^2dz}{\int_a^{a+h}\eta^2(z)\,dz} < \lambda$$

which would raise a contradiction as in the proof of Lemma 2. This proves our claim to be true, that is,

$$\xi(z) = u(z)-u(z^*) \le 0 \quad \text{for } z\in(a,a+\tfrac{1}{2}h)$$
$$\xi'(a) = u'(a)+u'(a^*) = u'(a)+u'(a+h) \le 0$$

and the proof is completed ∎

Lemma 5. *If* $a \in (-\frac{1}{2}h, 0)$, *then* $u'(z) > |u'(-z)|$ *for* $z \in (a, 0)$.

Proof. For $z \in (a, -a)$, define $v(z) = u(z) - u(-z)$. Then v is an odd function and

$$[(1 - z^2)v']'(z) + \lambda v(z) = 0 \qquad \text{in } (a, -a).$$

Knowing that $z_0 > a + \frac{1}{2}h > 0$, we consider first the case $0 < z_0 < -a$. Since $u'(z) > 0$ for $z \in (-z_0, z_0)$, so $v(z) = u(z) - u(-z) < 0$ and $v'(z) = u'(z) + u'(-z) > 0$ for $z \in (-z_0, 0)$. Also, a direct computation yields $v'(-z_0) = u'(-z_0) > 0$.

We claim that $v(z) < 0$ in $(a, -z_0)$. In fact, if there were a point $z = -z_1 \in (a, -z_0)$ such that $v(-z_1) = 0$, then we would have $u(-z_1) = u(z_1)$. Since the first eigenvalue of problem (1) is non-degenerate, it follows from a standard symmetrization argument that $u(z)$ must be symmetric in $(-z_1, z_1)$ and attain the maximum at $z = 0$ which induces a contradiction, hence the claim is true. Accordingly, $[(1 - z^2)v']'(z) = -\lambda v(z) > 0$ in $(a, -z_0)$ which implies that $(1 - z^2)v'(z)$ is increasing in $(a, -z_0)$.

Next we claim that $v'(z) > 0$ in $(a, -z_0)$. Indeed, if there were a point $z = -z_2 \in (a, -z_0)$ such that $v'(-z_2) = 0$ and $v'(z) > 0$ in $(-z_2, -z_0)$, let $m(z) = v'(z)$ in $[-z_2, z_2]$. Then m is even and positive in $(-z_2, z_2)$ and it satisfies

$$[(1 - z^2)m]''(z) + \lambda m(z) = 0 \quad \text{for } z \in (-z_2, z_2) \text{ with } \; m(-z_2) = m(z_2) = 0.$$

Let $w(z) = (1 - z^2)m(z)$ for $z \in (-z_2, z_2)$. Then $w(z) > 0$ for $z \in (0, z_2)$ and satisfies

$$w''(z) + \frac{\lambda w(z)}{1 - z^2} = 0 \quad \text{for } z \in (0, z_2) \text{ with } \begin{cases} w(z_2) = 0 \\ w'(0) = m'(0) = 0. \end{cases} \tag{7}$$

We see that $(\lambda, w(z))$ is the first eigenpair for the mixed boundary eigenvalue problem (7). On the other hand, let $n(z) = u'(z)$. We define $q(z) = (1 - z^2)n(z)$ for $z \in (0, z_0]$ and $q(z) = 0$ for $z \in (z_0, z_2)$. Then $q(z_0) = 0$, q will be a Lipschitz continuous function in $(0, z_2)$ and it would satisfy the differential equation in (7) for $z \in (0, z_0)$. Since the infimum of the corresponding Rayleigh quotient for (7) in $(0, z_2)$ is attained by $w(z)$ with value being λ and the Rayleigh quotient of $q(z)$ over $(0, z_2)$ equals λ, too (notice that $q(z) \equiv 0$ in $[z_0, z_2)$), the variational principle tells that owing to that λ is non-degenerate (see [2: p.164]), we would have $q(z) \equiv cw(z)$ in $(0, z_2)$ for some non-zero constant c. But $q'(0) = u'(0) \neq 0 = cw'(0)$ raises a contradiction. Hence our claim is true, and we have $v'(z) > 0$ in $(a, -z_0]$. Thus $u'(z) > -u'(-z) = |u'(-z)|$ for $z \in (a, -z_0]$. Moreover, we know that $u'(z) > 0$ and is strictly decreasing in $(-z_0, z_0)$, hence we also have $u'(z) > u'(-z) = |u'(-z)|$ for $z \in (-z_0, 0)$.

As for the case $z_0 > -a$ notice that, now in $(a, -a)$, $u'(z) > 0$ and is decreasing, it is easy to see the statement to be true ∎

Through Lemmas 1 - 5 we have completed the proof of Theorem 1. Now we are going to apply Theorem 1 to prove the monotonicity of λ_a:

Theorem 2. *If* $a < b$ *are both in* $[-\frac{1}{2}h, 1 - h)$, *then* $\lambda_a > \lambda_b$.

Proof. We write u for u_a in this proof for convenience. Set $d = b - a > 0$ and define $j(z) = u(z - d)$ for $z \in (b, b + h)$. Then $j \in H_b$ (recall that, for $c \in [-\frac{1}{2}h, 1 - h)$,

$H_c = W_0^{1,2}(c, c+h)$). We see that H_a and H_b form a one-one correspondence via this translation. It is trivial that $\int_a^{a+h} u^2(z)\,dz = \int_b^{b+h} j^2(z)\,dz$. We will show that

$$\int_a^{a+h} (1-z^2)[u'(z)]^2 dz > \int_b^{b+h} (1-z^2)[j'(z)]^2 dz$$

if $a < b$. Then the theorem will be true from the variational principle.

First consider the case $a < 0$. Since $z_0 > a + \frac{1}{2}h \geq 0$, we have $u'(z) > 0$ for $z \in (a, 0)$. Let

$$A = \left\{ z \in (0, a+h) : |u'(z)| = u'(\sigma) \text{ for some } \sigma \in (a, 0) \right\}.$$

Then A is an open interval contained in $(0, a+h)$. Write $B = A + d$ to be the set translated from A by the distance d. Then we have

$$\begin{aligned}
&\int_a^{a+h} (1-z^2)[u'(z)]^2 dz - \int_b^{b+h} (1-z^2)[j'(z)]^2 dz \\
&= \left(\int_a^0 + \int_A + \int_{(0,a+h)-A} \right) (1-z^2)[u'(z)]^2 dz \\
&\quad - \left(\int_{a+d}^d + \int_B + \int_{(d,a+h+d)-B} \right) (1-z^2)[j'(z)]^2 dz \\
&= I + II \\
&> 0
\end{aligned}$$

where

$$I = \left(\int_a^0 + \int_A \right) (1-z^2)[u'(z)]^2 dz - \left(\int_{a+d}^d + \int_B \right) (1-z^2)[j'(z)]^2 dz$$

$$II = \int_{(0,a+h)-A} (1-z^2)[u'(z)]^2 dz - \int_{(d,d+h)-B} (1-z^2)[j'(z)]^2 dz.$$

Now we prove that $I > 0$. In fact, for each $\sigma \in (a, 0)$ we know that there exists a unique $\tau \in A$ such that $u'(\sigma) = |u'(\tau)| = j'(\sigma + d) = |j'(\tau + d)|$. Then

$$\left(\int_a^0 + \int_A \right) (1-z^2)[u'(z)]^2 dz = \int_a^0 [(1-\sigma^2) + (1-\tau^2)]\,[u'(\sigma)]^2 d\sigma$$

and

$$\left(\int_{a+d}^d + \int_B \right) (1-z^2)[j'(z)]^2 dz = \int_a^0 \{[1-(\sigma+d)^2] + [1-(\tau+d)^2]\}[j'(\sigma+d)]^2 d\sigma.$$

By Theorem 1 we know that $\sigma + \tau > 0$ which implies

$$\begin{aligned}
&[(1-\sigma^2) + (1-\tau^2)]\,[u'(\sigma)]^2 - [1-(\sigma+\tau)^2 + 1 - (\tau+d)^2]\,[j'(\sigma+d)]^2 \\
&\quad = [u'(\sigma)]^2 [2\sigma d + 2\tau d + 2d^2] \\
&\quad = 2d[u'(\sigma)]^2 [\sigma + \tau + d] \\
&\quad > 0
\end{aligned}$$

for each $d > 0$. Hence $I > 0$.

Next we prove that $II > 0$. In fact, for $z \in (0, a+h) - A$, $(1 - z^2)$ is decreasing, so

$$\begin{aligned} II &= \int_{(0,a+h)-A} (1 - z^2)[u'(z)]^2 dz - \int_{(0,a+h)-A} [1 - (z+d)^2]\,[j'(z+d)]^2 dz \\ &= \int_{(0,a+h)-A} \{(1 - z^2) - [1 - (z+d)^2]\}[u'(z)]^2 dz \\ &> 0. \end{aligned}$$

Secondly, consider the case $a > 0$. Since $(1 - z^2)$ is decreasing in $(a, a+h)$, the result is easily seen to be true as the previous computation in the proof of $II > 0$ ∎

The technique of proving $II > 0$ in Theorem 2 could be used to get some generalization of Theorem 2. Also the monotonicity of λ_a reveals that the distance between two consecutive zeros of any Legendre function would be shorter as the zeros get larger. We have the following

Corollary 1. *Let μ_a be the first eigenvalue of the Dirichlet- Sturm-Liouville operator $\frac{d}{dz}[p(z)\frac{d}{dz}]$ over $(a, a+h)$ with p being positive and continuous in $[a, a+h]$. If p is decreasing in $(a, a+h)$, then μ_a will decrease with respect to a.*

Corollary 2. *Suppose p_v is a solution of the differential equation in (1) over $(-1, 1)$ with $v(v+1) = \lambda$, and $y_1 < y_2 < y_3$ are consecutive zeros of p_v with $y_1 \geq y_0$ where y_0 is the largest non-positive zero of p_v. Then $y_3 - y_2 < y_2 - y_1$.*

Proof. If $y_3 - y_2 \geq y_2 - y_1$, then from Theorem 2 and "domain monotonicity " of the first Dirichlet eigenvalues (see [1: p.100]), p_v would not satisfy the differential equation in (1) over both (y_1, y_2) and (y_2, y_3) with the same λ, which is a contradiction ∎

3. Applications

Denote by S^2 the unit sphere in $\mathbb{R}^3$. For $a \in [\frac{1}{2}h, 1-h)$, let D_a be the spherical band on S^2 parametrized according to cylindrical coordinates:

$$b(z, \theta) = ((1 - z^2)^{1/2} \cos\theta, (1 - z^2)^{1/2} \sin\theta, z)$$

for $a < z < a + h$ and $0 \leq \theta < 2\pi$. We see that each D_a has the same area $2\pi h$, that is the equal-area of D_a implies the irrelevance of h to a. The Beltrami-Laplace operator on S_2 in our coordinates is written as

$$\Delta = \frac{\partial}{\partial z}\left[(1 - z^2)\frac{\partial}{\partial z}\right] + \frac{\partial}{\partial \theta}\left[\frac{1}{1 - z^2}\frac{\partial}{\partial \theta}\right].$$

Due to the non-degeneracy of the first Dirichlet eigenvalue, the corresponding eigenfunction is independent of θ, so $u_a(z)$ (in Section 1) is the eigenfunction corresponding to the first eigenvalue λ_a for the Laplacian in D_a with Dirichlet boundary condition. Hence we know:

Corollary 3. *The first Dirichlet eigenvalue is decreasing as the spherical band moved toward the north pole.*

Now we will give some estimates of λ_a. First we shall give a lower bound of the first Dirichlet eigenvalue of the spherical cap D_0 (defined later), which would turn out to be a universal lower bound of all λ_a for $a \in [-\frac{1}{2}h, 1-h)$.

Let D_0 be the spherical cap parametrized by $b(z,\theta)$ for $z \in (1-h, 1]$ and $\theta \in [0, 2\pi)$. As before the first Dirichlet eigenfunction will be independent of θ. Let (λ_0, u_0) be the first eigenpair for the Laplacian in D_0 with Dirichlet boundary condition. Then u_0 would satisfy the differential equation in (1) with the boundary conditions $u_0(1-h) = 0$ and $u_0'(1)$ being finite (since $\nabla u(z,\theta) = (1-z^2)^{1/2}u'(z)(-z\cos\theta, -z\sin\theta, (1-z^2)^{1/2})$). If we write $\lambda_0 = v_0(v_0+1)$, then $u_0 = c\,p_{v_0}$ where c is a non-zero constant and p_{v_0} is the Legendre function of degree v_0 of the first kind [5: p.165] with v_0 being chosen so that p_{v_0} has $z = 1-h$ to be its largest zero in $(-1,1)$. The other boundary condition is naturally held by p_{v_0} since $p'_{v_0}(z) = \frac{1}{2}(-v)(v_0+1)F(1-v_0, v_0+2, 2, \frac{1-z}{2})$ as well as $F(1-v_0, v_0+2, 2, 0) = 1$ where F is the herpergeometric series [5: p.193, 197, 238]. Now let P_n be the Legendre polynomial of degree n, z_n denote the largest (the n-th) zero of P_n in $(-1,1)$. Then we have

Theorem 3.

(i) $\lim_{a\to 1-h}\lambda_a = \lambda_0$ *and* $\lambda_0 \le \lambda_a$ *for all* a *in* $[-\frac{1}{2}h, 1-h)$.

(ii) *If* $1-h = z_n$ *for some integer* $n > 0$, *then* $\lambda_0 = n(n+1)$.

(iii) *If* $z_n < 1-h < z_{n+1}$, *then* $n(n+1) < \lambda_0 < (n+1)(n+2)$.

Proof. By a straight modification of [7, p. 551/Lemma] and the decreasing property of λ_a in Theorem 2, the result of statement (i) is established (also, see [8]). We need only to treat the latter half of the theorem.

It is well known that P_n satisfies the differential equation in (1) over $(-1,1)$ with $\lambda = \lambda_n = n(n+1)$ and it has exactly n distinct zeros in $(-1,1)$ for $n \in \mathbb{N}$. Also, we could view (λ_n, P_n) as the first eigenpair of the differential equation in (1) over $(z_n, 1)$ with $P_n(z_n) = 0$, $P_n'(1)$ being finite. If $1-h = z_n$ for some n, then $u_0 = cP_n$ in $[1-h, 1]$ and $\lambda_0 = n(n+1)$. On the other hand, from Sturm's Fundamental Theorem we know that z_n is increasing as n increases [4: p. 225]. If $1-h \in (z_n, z_{n+1})$ for some positive integer n, then P_{n+1} would be a trial function for the Rayleigh quotient associated with λ_0 over $(1-h, 1]$. Similarly, u_0 would be a trial function for the Rayleigh quotient associated with $\lambda_n = n(n+1)$ over $(z_n, 1]$. Hence we have $n(n+1) < \lambda_0 < (n+1)(n+2)$. This completes the proof ∎

In [6], z_n is tabulated from $n = 2$ to $n = 16$:

$z_2 \cong 0.57735$	$z_7 \cong 0.94911$	$z_{12} \cong 0.98156$
$z_3 \cong 0.77460$	$z_8 \cong 0.96029$	$z_{13} \cong 0.98418$
$z_4 \cong 0.86114$	$z_9 \cong 0.96816$	$z_{14} \cong 0.98628$
$z_5 \cong 0.90618$	$z_{10} \cong 0.97390$	$z_{15} \cong 0.98799$
$z_6 \cong 0.93247$	$z_{11} \cong 0.97823$	$z_{16} \cong 0.98940.$

We observe that when $1-h$ near 1 (that is, a small cap), λ_0 is very sensitive to h (or, to the area of the cap). Also, when $1-h < z_2$, the lower bound will not be available. Hence in Theorem 4 we give another form of the lower bound of λ_a by observing that

$$\inf_{f\in H_a} \frac{\int_a^{a+h}[f'(z)]^2\,dz}{\int_a^{a+h} f^2(z)\,dz} = \Lambda = \left(\frac{\pi}{h}\right)^2$$

(since it is the first Dirichlet eigenvalue of $f''(z) + \Lambda f(z) = 0$ in $(a, a+h)$). Also, we use the corresponding first eigenfunction $\sin\frac{\pi(z-a)}{h}$ as a trial function of the Rayleigh quotient associated with λ_a to get an upper bound:

Theorem 4. *For $a < 0$, we have*

$$1-(a+h)^2 < \left(\frac{h}{\pi}\right)^2\lambda_a < 1 - a(a+h) - \frac{(2\pi^2+3)h^2}{6\pi^2}$$

and for $a \geq 0$ we have

$$1-(a+h)^2 < \left(\frac{h}{\pi}\right)^2\lambda_a < 1 - a(a+h) - \frac{(2\pi^2+3)h^2}{6\pi^2} < 1-a^2.$$

However, from Theorem 1 we could also give a somewhat improved lower bound. Letting α and M be as in Theorem 1, we have

Corollary 4. $\lambda_a > \min\{1-M^2, 1-a^2\}[\frac{\pi}{2(M-a)}]^2$.

Proof. Since (λ, u) (this is the abbreviation of (λ_a, u_a) is also the first eigenpair of the differential equation in (1) over (a, z_0) with mixed boundary conditions $u(a) = u'(z_0) = 0$, we have

$$(\min\{1-z_0^2, 1-a^2\})\int_a^{z_0}[u'(z)]^2\,dz < \int_a^{z_0}(1-z^2)[u'(z)]^2\,dz = \lambda\int_a^{z_0} u^2(z)\,dz.$$

Notice that

$$\Omega = \left[\frac{\pi}{2(z_0-a)}\right]^2 = \inf_H \frac{\int_a^{z_0}[f'(z)]^2\,dz}{\int_a^{z_0} f^2(z)\,dz}$$

(since it is the first eigenvalue of $f''(z) + \Omega f(z) = 0$ over (a, z_0) with mixed boundary condition $f(a) = f'(z_0) = 0$), where H is the Sobolev space consisting of functions in $W^{1,2}(a, z_0)$ that vanish at $z = a$. From Lemma 3 we know that $\Omega > [\frac{\pi}{2(M-a)}]^2$. Hence

$$\lambda_a > (\min\{1-z_0^2, 1-a^2\})\Omega > (\min\{1-M^2, 1-a^2\})\left[\frac{\pi}{2(M-a)}\right]^2$$

and the corollary is proved ∎

4. On general surfaces of revolution

As in Section 3, a general surface of revolution S will be parametrized as $\Phi(z,\theta) = (p(z)\cos\theta, p(z)\sin\theta, z)$ for $0 \le \theta < 2\pi$ and $z \in I$, with Riemannian measure being $dA = \sqrt{g}\,dzd\theta$, $g(z,\theta) = g(z) = p^2(z)[1+(p'(z))^2]$ where $p(z) > 0$ is a smooth function in an open interval I on the z-axis. We consider the Dirichlet eigenvalue problems for the "equal-area bands" on S:

Let D_μ be a band on S parametrized by $\Phi(z,\theta)$ for $0 \le \theta < 2\pi$ and $z \in (\mu,\mu^*) \subseteq I$. Then as before, on D_μ, the first Dirichlet eigenpair (λ_μ, u_μ) would satisfy

$$\left(\sqrt{g}\frac{1}{1+[(p'(z)]^2}u'_\mu\right)'(z) = -\lambda_\mu\sqrt{g}u_\mu(z) \qquad \text{with } u_\mu(\mu) = u_\mu(\mu^*) = 0.$$

After changing the variable by $y(z) = \int_\chi^z \sqrt{g(z)}\,dz$ with χ being the left end point of I, the area of

$$D_\mu = 2\pi\int_\mu^{\mu^*}\sqrt{g(z)}\,dz = 2\pi\int_c^{c+k} dy = 2\pi k$$

with $c = z^{-1}(\mu)$, $c+k = z^{-1}(\mu^*)$ and $u_\mu(z(y))$ would satisfy

$$\frac{d}{dy}\left[p^2(y)\frac{du_\mu}{dy}\right] = -\lambda_\mu u_\mu(y) \qquad \text{in } (c, c+k)$$

with Dirichlet boundary condition. So

$$\lambda_\mu = \inf_{w\in H_c}\frac{\int_c^{c+k} p^2(y)[w'(y)]^2dy}{\int_c^{c+k} w^2(y)dy}$$

and we could get similar results as in Section 3:

Theorem 5. *Let $k = \frac{\text{area}\, D_\mu}{2\pi}$ as above. Then*

$$\inf_{z\in(\mu,\mu^*)}[p^2(z)]\left(\tfrac{\pi}{k}\right)^2 \le \lambda_\mu \le \sup_{z\in(\mu,\mu^*)}[p^2(z)]\left(\tfrac{\pi}{k}\right)^2$$

for each μ.

Also, since $y'(z) > 0$ for z in I, we get

Theorem 6. *Suppose p is decreasing (increasing respectively) in I. Then λ_μ is a decreasing (increasing respectively) function of μ. Moreover, the distance of two consecutive zeros in I would be decreasing (increasing respectively) as the zeros go larger.*

Acknowledgment. *The auther would like to thank the referee and Prof. R. Finn for some constructive suggestions.*

References

[1] Courant, R. and D. Hilbert: *Methods of Mathematical Physics*, Vol. 1. New York: Intersci. 1953.

[2] Churchill, R. V. and J. W. Brown: *Fourier Series and Boundary Value Problems*, 4th ed. Singapore: McGraw-Hill 1987.

[3] Gidas, B., NI, W. M. and L. Nirenberg: *Symmetry and related properties via the maximum principle*. Comm. Math. Phys 68 (1979), 209 - 243.

[4] Ince, E. L.: *Ordinary Differential Equations*. New York: Dover Publ. 1926.

[5] Lebedev, N. N.: *Special Functions and Their Applications*. New Jersey: Prentice-Hall 1965.

[6] Lowan, A. N., Davids, N. and A. Levenson: *Table of the Zeros of the Legendre Polynomials of Order 1-16 and the Weight Coefficients for Gauss' Mechanical Quadrature Formula*. Bull. Amer. Math. Soc. 48, (1942), 739 - 743.

[7] Osserman, R.: *A note on Hayman's theorem on the bass note of a drum*. Comment. Math. Helvetici 52 (1977), 545 - 555.

[8] Shen, C. L. and C. T. Shieh: *Some properties of the first eigenvalue of the Laplace operator on the spherical bands in S^2*. SIAM J. Math. Anal. 23 (1992), 1305 - 1308.

Received 01.12.1999