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A Sequence of Integro-Differential Equations Approximating a Viscous Porous Medium Equation

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Abstract. We consider a sequence of particular integro-differential equations, whose solutions ρ_N converge as $N \to \infty$ to the solution ρ of a viscous porous medium equation. First, it is demonstrated that under suitable regularity conditions the functions ρ_N are smooth uniformly in $N \in \mathbb{N}$. Furthermore, an asymptotic expansion for ρ_N as $N \to \infty$ is provided, which precisely describes the convergence to ρ. The results of this paper are needed in particular for the numerical simulation of a viscous porous medium equation by a particle method.

Keywords: *Integro-differential equations, porous medium equations, asymptotic expansions*

AMS subject classification: Primary 45K05, secondary 35B65, 35C20, 35K55

1. Introduction

In this paper we study the solutions ρ_N $(N \in \mathbb{N})$ of some sequence of particular integro-differential equations, namely

$$\left.\begin{aligned} \partial_t \rho_N(x,t) &= \tfrac{1}{2}\Delta\rho_N(x,t) + \nabla \cdot \big(\rho_N(x,t)\nabla(\rho_N(\cdot,t) * \phi_N)(x)\big) \\ \rho_N(x,0) &= \rho_0(x) \end{aligned}\right\} \qquad \left(\begin{smallmatrix} t\geq 0 \\ x\in\mathbb{R}^d \end{smallmatrix}\right) \tag{1.1}$$

where $*$ denotes convolution. For different N the equations in (1.1) differ in the *interaction kernel* ϕ_N. We suppose

$$\phi_N(x) = \theta_N^d \phi_1(\theta_N x) \qquad (x \in \mathbb{R}^d,\ N \in \mathbb{N}) \tag{1.2}$$

where ϕ_1 is some smooth, symmetric probability density, i.e.,

$$\phi_1 \in C_b^\infty(\mathbb{R}^d) \quad \text{with } \phi_1 \geq 0,\ \int_{\mathbb{R}^d} dx\, \phi_1(x) = 1,\ \phi_1(x) = \phi_1(-x) \ \ (x \in \mathbb{R}^d) \tag{1.3}$$

and the scaling coefficients θ_N satisfy

$$\lim_{N\to\infty} \theta_N = \infty. \tag{1.4}$$

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Additionally, we assume that the initial state ρ_0 of ρ_N is non-negative and smooth.

We are interested in the regularity of the functions ρ_N and in their asymptotics as $N \to \infty$. In particular, we shall demonstrate that regularity holds uniformly in $N \in \mathbb{N}$. Moreover, we shall provide an expansion representing ρ_N as $N \to \infty$. As far as the asymptotics is concerned, we note that (1.2) - (1.4) yield $\lim_{N\to\infty} \phi_N = \delta_0$ in $\mathcal{S}(\mathbb{R}^d)$, where δ_a is Dirac's *delta function* concentrated at $a \in \mathbb{R}^d$. Therefore, from (1.1) we formally obtain the convergence of ρ_N to the solution ρ of

$$\left.\begin{aligned} \partial_t \rho(x,t) &= \tfrac{1}{2}\Delta\rho(x,t) + \nabla\cdot\big(\rho(x,t)\nabla\rho(x,t)\big) \\ &= \tfrac{1}{2}\Delta\rho(x,t) + \tfrac{1}{2}\Delta\rho^2(x,t) \\ \rho(x,0) &= \rho_0(x) \end{aligned}\right\} \quad (t \geq 0,\, x \in \mathbb{R}^d). \tag{1.5}$$

The partial differential equation (1.5) is a simple nonlinear reaction-diffusion equation, namely a *porous medium equation* with an additional linear viscous term. Results of classical analysis imply that the regularity properties of ρ_0 are preserved by (1.5), i.e., its solution $\rho(\cdot, t)$ is non-negative and smooth for any $t \geq 0$ (cf. Remark (i) in Section 2). Our considerations will show that uniformly in $N \in \mathbb{N}$ the functions ρ_N are also non-negative and smooth. These regularity properties will turn out to be essential for the derivation of the asymptotic expansion

$$\rho_N \simeq \rho + \theta_N^{-2}\rho_{\{1\}} + \theta_N^{-4}\rho_{\{2\}} + \dots \qquad (N \text{ large}) \tag{1.6}$$

where $\rho_{\{1\}}, \rho_{\{2\}}, \dots$ are smooth, too. In particular, (1.6) specifies the rate of convergence of ρ_N to ρ.

The present paper is part of a more extensive study of general systems of reaction-diffusion equations extending the simple example (1.5). Such systems are often used as mathematical models for the time evolution of large collections of many components or particles, which belong to a few different species, such that within each species the components are of the same kind. Typical examples arise in population biology, fluid dynamics or in spatially inhomogeneous, reacting chemical systems. In these cases systems of reaction-diffusion equations serve as models for particular *many-particle systems* with interaction, and they typically describe the dynamics of population-, mass-, velocity- or energy densities.

To study the correctness of these models we derived convergence results for the empirical processes of several types of mathematically idealized many-particle systems. In our studies particular emphasis was given to so-called *moderately interacting* many-particle systems, which are characterized by the property that the range of the interaction between the particles is both large in comparison to the typical distance between neighbouring particles and small with respect to the size of the whole system (cf. [9 - 13]). In these papers the respective dynamics of the particle positions is given in terms of coupled stochastic differential equations (cf. [9 - 11]) or ordinary differential equations (cf. [12, 13]). As central objects of our investigations we chose the *empirical processes* of the various species, which for any time give equal positive mass to the positions of the particles of the respective species. Aiming at a characterization of the empirical processes for large particle numbers we introduced for any $N \in \mathbb{N}$ one particular many-particle system consisting of approximately N particles. To retain finiteness of the empirical processes as $N \to \infty$ we also defined the mass of individual particles as $\frac{1}{N}$. As result we demonstrate in [9 - 13] the convergence of the measure-valued empirical processes to the components of the solution of some particular system of reaction-diffusion equations.

We note that equation (1.5) appears in [9, 10] as limit dynamics. The time evolution of the positions $X_N^k(\cdot)$ $(k = 1, \dots, N)$ of the particles in the associated N-particle system is defined by

$$dX_N^k(t) = -\tfrac{1}{N} \sum_{l=1}^{N} \nabla \kappa_N \big(X_N^k(t) - X_N^l(t) \big) dt + dW^k(t) \quad (k = 1, \dots, N). \tag{1.7}$$

Here W^k denote independent, standard $\mathbb{R}^d$-valued Brownian motions. Moreover, the interaction potential κ_N is obtained by the scaling

$$\kappa_N(x) = N^{\beta} \kappa_1(N^{\beta/d} x) \qquad (x \in \mathbb{R}^d,\ N \in \mathbb{N}) \tag{1.8}$$

from a fixed function κ_1, which has the same properties as ϕ_1 (cf. (1.3)). In the system with N particles the typical distance between neighbouring particles is $O(N^{-\frac{1}{d}})$, whereas the size of the whole system is $O(1)$ as $N \to \infty$. Therefore, to match the above mentioned characterization of moderate interaction we suppose $\beta \in (0, 1)$ for the scaling parameter β. We note that the empirical processes for the N-particle systems $\big((X_N^1(\cdot), \dots, X_N^N(\cdot) \big)$, which as indicated above converge to the solution ρ of (1.5) as $N \to \infty$, are given by

$$t \to \mathbb{X}_N(t) = \tfrac{1}{N} \sum_{k=1}^{N} \delta_{X_N^k(t)} \qquad (t \geq 0,\ N \in \mathbb{N}),$$

i.e., they take values in the space of probability measures on $\mathbb{R}^d$.

The convergence results in [9 - 13] suggest to utilize *particle methods* based on the many-particle systems in those papers to solve or simulate the corresponding systems of reaction-diffusion equations numerically. The analysis of such simulations, in particular, of their convergence properties, leads to additional problems, which are related to discretizations of space and/or time. Those problems did not arise in [9 - 13], where by working with finitely many particles only the mass of the respective populations is discretized.

To handle such problems related to the numerical simulation of moderately interacting many-particle systems in a less complicated example we investigate in [15] a particle-method based on (1.7) in order to solve (1.5). In the analysis of the simulation procedure the solutions ρ_N of (1.1) appear as auxiliary functions, which are used as intermediate objects between the empirical processes of the many-particle systems and their limit ρ. In particular, both main results of the present paper, i.e., the regularity of ρ_N uniformly in N and expansion (1.6), are needed in [15].

It is expected that analogues of our results here also hold for extensions of (1.1), which are related to systems $(\mathcal{S})$ of reaction-diffusion equations like those appearing in [11]. Hence, for any such system $(\mathcal{S})$ we may find a sequence of systems $(\mathcal{S}_N)$ $(N \in \mathbb{N})$

of integro-differential equations, whose solutions ρ_N are smooth uniformly in $N \in \mathbb{N}$, and admit an expansion like (1.6), where now ρ solves $(\mathcal{S})$.

In the literature many more contributions on the derivation of (1.5) or closely related partial differential equations as limit dynamics of interacting many-particle systems are available. For example, in [16] or in an extension [2] a modification of (1.7) with *short-range interaction* corresponding to the case $\beta = 1$ is studied in the limit $N \to \infty$, i.e., in the *hydrodynamic limit*. In these papers the limit dynamics is given by

$$\partial_t \rho = \nabla \cdot \big(D(\rho)\nabla\rho\big), \tag{1.9}$$

which is a *generalized porous medium equation*. The functional D is determined in the framework of statistical physics and, in particular, depends on details of the interaction potential corresponding to the function κ_1 in (1.8). Other interacting many-particle systems, especially systems on a discrete lattice like $\mathbb{Z}^d$, which also have partial differential equations like (1.9) as limit dynamics, are discussed in [7]. In particular, that book contains an exhaustive bibliography on the subject of interacting many-particle systems.

We conclude this introduction by presenting some notation, which will be utilized later on. In particular, we shall employ the summation convention, i.e., indices appearing twice in a product are summed from 1 to d.

We denote by $C, C', \ldots$ positive, finite constants, which may vary from place to place. In general, these constants are independent of N or other variables being involved in the respective calculations. If however the dependence on particular parameters $\alpha_1, \ldots, \alpha_M$ is to be emphasized, the notation $C(\alpha_1, \ldots, \alpha_M)$ is employed. To be able to refer to particular constants later in calculations we also use $C_1(\ldots), C_2(\ldots), \ldots$. Without explicit hints we shall apply the notations $|\cdot|$ and $\|\cdot\|_{\ldots}$ for norms of both $\mathbb{R}$- and $\mathbb{R}^n$-valued objects, where $n > 1$. For example,

$$\begin{aligned}
\|\nabla^{\otimes m} f\|_2 &= \left(\int_{\mathbb{R}^d} dx\, |\nabla^{\otimes m} f(x)|^2\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}^d} dx \sum_{i_1,\ldots,i_m=1}^{d} \Big|\frac{\partial^m}{\partial_{i_1}\cdots\partial_{i_m}} f(x)\Big|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i_1,\ldots,i_m=1}^{d} \Big\|\frac{\partial^m}{\partial_{i_1}\cdots\partial_{i_m}} f\Big\|_2^2\right)^{\frac{1}{2}}
\end{aligned}$$

denotes the L^2-norm of the tensor of all partial derivatives of order m of some sufficiently smooth and integrable real-valued function f on $\mathbb{R}^d$.

To quantify regularity properties of real-valued functions we primarily shall utilize *Sobolev norms* given by

$$\|f\|_{(k)} = \left(\sum_{m=0}^{k} \|\nabla^{\otimes m} f\|_2^2\right)^{\frac{1}{2}} \qquad (k \in \mathbb{N}_0) \tag{1.10}$$

for any f, where the right side is well defined. The corresponding *Sobolev spaces* are

$$H_k^2(\mathbb{R}^d) = \big\{ f : \mathbb{R}^d \to \mathbb{R} \,|\, \|f\|_{(k)} < \infty \big\}.$$

Additionally, certain weighted Sobolev norms, namely

$$\|f\|_{(k,\alpha)} = \Bigg(\sum_{m=0}^{k} \int_{\mathbb{R}^d} dy \; (1+|y|^\alpha)|\nabla^{\otimes m} f(y)|^2 \Bigg)^{\frac{1}{2}} \qquad (k \in \mathbb{N}_0, \alpha > 0) \tag{1.11}$$

are used. We note that L^∞-norms may be estimated by employing *Sobolev's inequality*

$$\sum_{r=0}^{m} \|\nabla^{\otimes r} f\|_\infty \leq C(d,m,p)\|f\|_{(p+m)} \quad \big(f \in H_{p+m}^2(\mathbb{R}^d),\ m \in \mathbb{N}_0,\ p > \tfrac{d}{2}\big). \tag{1.12}$$

Furthermore, by [1: Theorem 2.21] we conclude that

$$\left.\begin{array}{l} \text{a sequence } f_n : \mathbb{R}^d \to \mathbb{R} \;\; (n \in \mathbb{N}) \text{ satisfying } \sup_{n\in\mathbb{N}}\|f_n\|_{(k+1,\alpha)} < \infty \\ \text{for some } k \in \mathbb{N}_0 \text{ and some } \alpha > 0 \text{ is relatively compact in } H_k^2(\mathbb{R}^d) \end{array}\right\}. \tag{1.13}$$

To simplify our calculations involving norms $\|\cdot\|_{(k)}$ we often shall utilize the relations

$$\|f\|_{(k)} \leq \begin{cases} C_1(k)\big(\|f\|_2^2 + \|(-\Delta)^{\frac{k}{2}} f\|_2^2\big)^{\frac{1}{2}} & \text{if } k \text{ is even} \\ C_1(k)\big(\|f\|_2^2 + \|\nabla(-\Delta)^{\frac{k-1}{2}} f\|_2^2\big)^{\frac{1}{2}} & \text{if } k \text{ is odd.} \end{cases} \tag{1.14}$$

As abbreviation for integrals we occasionally shall apply the notation

$$\langle f, g\rangle = \int_{\mathbb{R}^d} dx \; f(x)g(x),$$

whenever the right side is well-defined for functions f and g.

In the next section we shall present our results. Their proofs can be found in Section 3. Finally, two appendices in Sections 4 and 5 contain a formal derivation of expansion (1.6) and the proof of (3.63), respectively.

2. Results

Before presenting the precise formulation of our results, we mention some assumptions about ϕ_1, which according to (1.2) is the basis for the interaction kernels ϕ_N, and the initial state ρ_0 of the solutions ρ_N $(N \in \mathbb{N})$ and ρ of (1.1) and (1.5), respectively. ϕ_1 is supposed to be a convolution product

$$\phi_1 = \phi_1^r * \phi_1^r \tag{2.1}$$

where ϕ_1^r is a smooth, symmetric probability density, i.e.,

$$\phi_1^r \in C_b^\infty(\mathbb{R}^d) \quad \text{with } \phi_1^r \geq 0, \; \int_{\mathbb{R}^d} dx\, \phi_1^r(x) = 1, \; \phi_1^r(x) = \phi_1^r(-x) \;\; (x \in \mathbb{R}^d). \tag{2.2}$$

Furthermore, we assume that ϕ_1 has bounded moments of all orders, i.e.,

$$\int_{\mathbb{R}^d} dx\ |x|^m \phi_1(x) < \infty \qquad (m \in \mathbb{N}). \tag{2.3}$$

Obviously, (1.3) is an immediate consequence of (2.1) and (2.2). Analogously to (1.2) we also introduce rescaled versions of ϕ_1^r, namely

$$\phi_N^r(x) = \theta_N^d \phi_1^r(\theta_N x) \qquad (x \in \mathbb{R}^d,\ N \in \mathbb{N}). \tag{2.4}$$

The functions $\phi_N^r\ \ (N \in \mathbb{N})$ satisfy similar relations as ϕ_1^r, namely

$$\phi_N^r \in C_b^\infty(\mathbb{R}^d), \quad \phi_N^r \geq 0,\ \int_{\mathbb{R}^d} dx\ \phi_N^r(x) = 1,\ \phi_N^r(x) = \phi_N^r(-x)\ \ (x \in \mathbb{R}^d) \tag{2.5}$$

and

$$\phi_N = \phi_N^r * \phi_N^r \qquad (N \in \mathbb{N}) \tag{2.6}$$

which immediately follow from (1.2), (2.1), (2.2) and (2.4).

As far as the initial state of ρ and $\rho_N\ \ (N \in \mathbb{N})$ is concerned, we suppose that ρ_0 is a smooth probability density, i.e.,

$$\rho_0 \geq 0, \quad \int_{\mathbb{R}^d} dx\ \rho_0(x) = 1 \tag{2.7}$$

$$\|\rho_0\|_{(m,1)} < \infty\ \ (m \in \mathbb{N}_0). \tag{2.8}$$

We remark that (2.7) is assumed in view of the application of the results of the present paper in [15], where this assumption also appears. Whereas the positivity of ρ_0 is essential for our calculations, e.g., for the derivation of the positivity of $\rho_N(\cdot,t)\ \ (t \geq 0, N \in \mathbb{N};$ cf. (2.11)), the second part of (2.7) is only a normalization condition, which might be replaced by $\int_{\mathbb{R}^d} dx\ \rho_0(x) < \infty$. The regularity hypothesis (2.8), which by (1.10) - (1.12) yields

$$\rho_0 \in C_b^\infty(\mathbb{R}^d), \tag{2.9}$$

would not be needed in this strength. However, it allows to work in a C^∞-environment, where essentially all functions related to ρ_N or ρ are arbitrarily smooth.

Now, we may formulate our result about uniform regularity of the functions $\rho_N\ \ (N \in \mathbb{N})$.

Theorem 1. *Suppose that the interaction kernels $\phi_N\ \ (N \in \mathbb{N})$ are determined by $(1.2), (1.4)$ and $(2.1) - (2.3)$. Moreover, assume that the initial state ρ_0 of (1.1) satisfies $(2.7) - (2.8)$. Then, for any $N \in \mathbb{N}$ there exists a unique solution ρ_N of (1.1) satisfying*

$$\rho_N \in C_b^\infty(\mathbb{R}^d \times [0,T]) \qquad (T > 0,\ N \in \mathbb{N}). \tag{2.10}$$

For any fixed time these functions ρ_N are probability densities, i.e.,

$$\rho_N(\cdot,t) \geq 0, \quad \int_{\mathbb{R}^d} dx\ \rho_N(x,t) = 1 \qquad (t \geq 0,\ N \in \mathbb{N}). \tag{2.11}$$

Furthermore, they are regular uniformly in $N \in \mathbb{N}$, i.e.,

$$\sup_{t\le T, N\in\mathbb{N}} \|\rho_N(\cdot,t)\|_{(m,1)} < \infty \qquad (m \in \mathbb{N}_0,\, T>0) \tag{2.12}$$

and therefore

$$\sup_{x\in\mathbb{R}^d, t\le T, N\in\mathbb{N}} \left|\nabla^{\otimes m}\partial_t^k \rho_N(x,t)\right| < \infty \qquad (m,k \in \mathbb{N}_0,\, T>0). \tag{2.13}$$

To demonstrate a simple convergence of ρ_N to the solution ρ of (1.5) as $N \to \infty$, we can apply (1.1) - (1.4), (2.10) and (2.13). However, for our considerations in [15] we need more information about that convergence. This additional knowledge is provided by

Theorem 2. *For $N \in \mathbb{N}$ let ρ_N be the solution of (1.1), where the initial state ρ_0 and the interaction kernel ϕ_N satisfy the same conditions as in Theorem 1. Then, there exist functions $\rho_{\{r\}} \in C^\infty(\mathbb{R}^d \times [0,\infty))$ $(r \in \mathbb{N})$ such that*

$$\rho_{\{r\}} \in C_b^\infty(\mathbb{R}^d \times [0,T]) \qquad (T>0,\, r\in\mathbb{N}) \tag{2.14}$$

which solve

$$\left.\begin{aligned} \partial_t\rho_{\{r\}}(x,t) &= \big(\mathcal{L}_{\rho,t}\rho_{\{r\}}(\cdot,t)\big)(x) + \mathcal{G}_r(\rho,\rho_{\{1\}},\dots,\rho_{\{r-1\}},x,t) \\ \rho_{\{r\}}(x,0) &= 0 \quad \big(x\in\mathbb{R}^d, t\ge 0, r\in\mathbb{N}\big) \end{aligned}\right\} \tag{2.15}$$

where

$$\begin{gathered}(\mathcal{L}_{\rho,t}f)(x) = \tfrac{1}{2}\Delta f(x) + \nabla\cdot\big(f(x)\nabla\rho(x,t)\big) + \nabla\cdot\big(\rho(x,t)\nabla f\big) \\ \big(x\in\mathbb{R}^d, t\ge 0, f\in C_b^2(\mathbb{R}^d)\big)\end{gathered} \tag{2.16}$$

and

$$\begin{aligned}&\mathcal{G}_r\big(\rho,\rho_{\{1\}},\dots,\rho_{\{r-1\}},x,t\big) = \\ &\sum_{\substack{p,q=0,1,\dots,r-1\\ p+q\le r}} \; \sum_{\substack{0\le l_1,\dots,l_d\le 2(r-p-q)\\ l_1+\dots+l_d=2(r-p-q)}} \sigma^*(l_1,\dots,l_d;\phi_1)\nabla\cdot\left(\rho_{\{p\}}(x,t)\nabla\frac{\partial^{2(r-p-q)}}{\partial_1^{l_1}\cdots\partial_d^{l_d}}\rho_{\{q\}}(x,t)\right) \\ &\big(x\in\mathbb{R}^d, t\ge 0, r\in\mathbb{N}\big).\end{aligned} \tag{2.17}$$

In (2.15) – (2.17) the function $\rho = \rho_{\{0\}}$ is the solution of (1.5). Moreover, we employed

$$\sigma^*(l_1,...,l_d;\phi_1) = \tfrac{1}{l_1!\cdots l_d!}\int_{\mathbb{R}^d} dy\; y_1^{l_1}\cdots y_d^{l_d}\phi_1(y) \qquad (l_1,...,l_d\in\mathbb{N}_0). \tag{2.18}$$

Using ρ, $\rho_{\{1\}}$, $\rho_{\{2\}}$, … we can expand ρ_N as in (1.6). More precisely, for any $L \in \mathbb{N}_0$ we get

$$\sup_{x\in\mathbb{R}^d, t\le T, N\in\mathbb{N}} \theta_N^{2L+2}\left|\nabla^{\otimes m}\partial_t^k\left(\rho_N(x,t) - \rho(x,t) - \sum_{r=1}^{L}\theta_N^{-2r}\rho_{\{r\}}(x,t)\right)\right| < \infty \tag{2.19}$$

for $m, k \in \mathbb{N}_0$ and $T > 0$.

Remarks.

(i) The limit dynamics (1.5) with the initial state ρ_0 satisfying (2.7) - (2.8) represents a simple nonlinear parabolic problem and may be handled quite well by classical methods, which can be found, e.g., in [5, 8]. In particular, by using existence and uniqueness theorems for classical solutions of parabolic equations and the maximum principle it can be verified that (1.5) has a unique smooth solution ρ with

$$\rho \in C_b^\infty(\mathbb{R}^d \times [0,T]) \qquad (T > 0) \tag{2.20}$$

such that for any t the function $\rho(\cdot, t)$ is a probability density, i.e.,

$$\rho(\cdot, t) \geq 0, \quad \int_{\mathbb{R}^d} dx\ \rho(x,t) = 1 \qquad (t \geq 0). \tag{2.21}$$

Furthermore, we observe

$$\begin{aligned} \|(-\Delta)^m \rho(\cdot,t)\|_2^2 &= \langle (-\Delta)^{2m} \rho(\cdot,t), \rho(\cdot,t)\rangle \\ &\leq \|(-\Delta)^{2m}\rho(\cdot,t)\|_\infty \langle \rho(\cdot,t), 1\rangle \end{aligned} \qquad (m \in \mathbb{N}_0,\ t \geq 0). \tag{2.22}$$

To justify the neglect of the boundary terms in integration by parts in (2.22) we may replace for fixed $t \geq 0$ the function $\rho(\cdot,t)$ by $\rho_{[\eta]}(\cdot,t) = \rho(\cdot,t)\psi_\eta(\cdot)$, where $\psi_\eta(x) = \psi_0(|x| - \eta)$ $(x \in \mathbb{R}^d, \eta \geq 1)$ with $\psi_0 \in C_b^\infty(\mathbb{R})$, $\psi_0' \leq 0$, $\psi_0(u) = 1$ if $u \leq 0$ and $\psi_0(u) = 0$ if $u \geq 1$. Obviously, since $\rho_{[\eta]}(x,t) = 0$, if $|x| \geq \eta + 1$, the arguments in (2.22) hold for $\rho_{[\eta]}(\cdot,t)$, and also with (2.20) and (2.21) imply

$$\sup_{\eta \geq 1} \|(-\Delta)^m \rho_{[\eta]}(\cdot,t)\|_2 < \infty \qquad (m \in \mathbb{N}_0).$$

Since

$$\lim_{\eta \to \infty} \nabla^{\otimes m} \rho_{[\eta]}(x,t) = \nabla^{\otimes m}\rho(x,t) \qquad (x \in \mathbb{R}^d, m \in \mathbb{N}_0)$$

the validity of (2.22) for $\rho(\cdot,t)$ follows in the limit $\eta \to \infty$. Now, (1.14) and (2.20) - (2.22) imply

$$\sup_{t \leq T} \|\rho(\cdot,t)\|_{(m)} < \infty \qquad (m \in \mathbb{N}_0, T > 0). \tag{2.23}$$

(ii) The N-dependent nonlinear integro-differential equations in (1.1) are *McKean-Vlasov equations* as considered, e.g., in [6]. The results in that paper ensure for fixed $N \in \mathbb{N}$ the unique existence of a weak, i.e., measure-valued solution $\rho_N = \rho_N(t)$ $(t \geq 0)$ of (1.1). In particular, since ρ_0 is a probability density (cf. (2.7)), ρ_N is associated to the solution $X_N = X_N(t)$ $(t \geq 0)$ of a nonlinear martingale problem, i.e., $\rho_N(t)$ coincides with the law of $X_N(t)$ for any $t \geq 0$. Consequently, ρ_N takes values in the space of probability measures, i.e.,

$$\langle \rho_N(t), 1\rangle = 1, \quad \langle \rho_N(t), f\rangle \geq 0 \quad \big(f \in L^\infty(\mathbb{R}^d),\ f \geq 0,\ t \geq 0\big). \tag{2.24}$$

By these relations and the smoothness of ϕ_N (cf. (1.2) - (1.3)) the drift vector $\nabla(\rho_N(t)*\phi_N)(x)\quad(x\in\mathbb{R}^d, t\geq 0)$ in (1.1) and its partial derivatives of all orders are bounded uniformly in $x\in\mathbb{R}^d$ and $t\geq 0$ independently of further regularity properties of ρ_N. Therefore, (2.9) and in particular the fact that $\frac{1}{2}\Delta-\partial_t$, which determines the principal part of (1.1), has a Gaussian fundamental solution (cf. (3.35)) imply that the measure $\rho_N(t)$ is absolutely continuous with respect to Lebesgue measure for any $t\geq 0$ and also satisfies the smoothness property (2.10). Of course, (2.11) then immediately follows from (2.24).

(iii) For the function ϕ_1, which determines the interaction kernel ϕ_N, we use only a few assumptions, namely (2.1) - (2.3). However, in our considerations in [15] apart from the results of the present paper further ingredients from [9 - 13] are employed. As a consequence, additional regularity properties of ϕ_1 have to be supposed in that paper. As far as the calculations here are concerned, the presence of the diffusive term $\Delta\rho_N(x,t)$ in (1.1) will turn out to be very advantageous. In particular, to estimate the right sides of (3.2) - (3.3), which is a crucial part of the proofs of our results, we obtain contributions (3.4) and (3.14), which ultimately allow to use for fixed m the L^2-norms of $\nabla(-\Delta)^m\rho_N(\cdot,t)$ and $\nabla(-\Delta)^m(\rho_N(\cdot,t)*\phi_N^r)$, respectively, in the remaining estimates of the terms on the right sides of (3.2) and (3.3). As a consequence, calculations like those in [12: Section 4/(ii)], which would necessitate further regularity properties of ϕ_1, are not required in the present paper.

(iv) Also, as a consequence of the diffusive part $\Delta\rho_N(x,t)$ in (1.1) it is expected that ρ_N is smooth for $t>0$, even if the initial state ρ_0 is not regular at all. However, then results holding uniformly in $t\in[0,T]\quad(T>0)$, like (2.12), (2.13) and (2.19) cannot be valid any more.

(v) Of course, it would be interesting to consider also the non-diffusive case, where $\frac{1}{2}\Delta\rho_N(x,t)$ is missing in (1.1), and to look for appropriate extensions of Theorems 1 - 3. An exact application of the analysis of the present paper obviously is not feasible in that situation, cf. e.g. the derivation of (3.23). Nevertheless, it should be possible to find a suitable modification of our method, at least if the considerations are restricted to a sufficiently small finite time interval, and if additional assumptions on ρ_0 and ϕ_1 are used. This presumption is supported in particular by the results in [14], where the convergence of another sequence of integro-differential equations without diffusion to a nonlinear wave equation is proved. In the considerations in [14] it is essential that the limit dynamics is hyperbolic, i.e., like the parabolic partial differential equation (1.5) possesses certain regularity properties.

(vi) The smoothness of ϕ_1^r (cf. (2.2)) ensures unique existence of a smooth solution ρ_N of (1.1) for any fixed $N\in\mathbb{N}$ (cf. Remark (ii)). On the other hand, the calculations in the proofs of Theorems 1 and 2 turn out to be completely independent of smoothness properties of ϕ_1^r or ϕ_1. This observation can be employed to deduce the subsequent extension of our previous results to more general interaction kernels.

Theorem 3. *Suppose that the interaction kernels $\phi_N\quad(N\in\mathbb{N})$ are determined by* (1.2), (1.4), (2.1), (2.3) *and*

$$\phi_1^r\geq 0,\qquad \int_{\mathbb{R}^d}dx\;\phi_1^r(x)=1,\qquad \phi_1^r(x)=\phi_1^r(-x)\quad(x\in\mathbb{R}^d)\tag{2.2*}$$

instead of (2.2). *Furthermore, assume that* ρ_0 *satisfies* (2.7) − (2.8). *Then, for any* $T' > 0$ *some* $N(T') \in \mathbb{N}$ *may be chosen, such that for* $N \geq N(T')$ *there exists a unique solution* ρ_N *of* (1.1) *in* $[0, T']$ *satisfying*

$$\rho_N \in C_b^\infty(\mathbb{R}^d \times [0, T']) \qquad \big(N \geq N(T')\big) \tag{2.25}$$

and

$$\rho_N(\cdot, t) \geq 0, \quad \int_{\mathbb{R}^d} dx\ \rho_N(x, t) = 1 \qquad \big(t \in [0, T'],\ N \geq N(T')\big). \tag{2.26}$$

These functions are regular uniformly in $N \geq N(T')$, *i.e.,*

$$\sup_{t \leq T', N \geq N(T')} \|\rho_N(\cdot, t)\|_{(m,1)} < \infty \qquad (m \in \mathbb{N}_0) \tag{2.27}$$

and therefore

$$\sup_{x \in \mathbb{R}^d, t \leq T', N \geq N(T')} |\nabla^{\otimes m} \partial_t^k \rho_N(x, t)| < \infty \qquad (m, k \in \mathbb{N}_0). \tag{2.28}$$

Moreover, for $L \in \mathbb{N}_0$ *they may be expanded as*

$$\sup_{\substack{x \in \mathbb{R}^d \\ t \leq T', N \geq N(T')}} \theta_N^{2L+2} \left| \nabla^{\otimes m} \partial_t^k \left(\rho_N(x, t) - \rho(x, t) - \sum_{r=1}^L \theta_N^{-2r} \rho_{\{r\}}(x, t) \right) \right| < \infty \tag{2.29}$$

for $m, k \in \mathbb{N}_0$, *where* ρ, $\rho_{\{1\}}$, $\rho_{\{2\}}$, ... *are described in Theorem* 2.

As final part of this section we state an auxiliary result needed for the proof of Theorem 2. It will be used, in particular, to clarify the relation between the term $\nabla(\rho_N(\cdot, t) * \phi_N)$ in (1.1) and $\nabla \rho_N(\cdot, t)$.

Lemma 1. *Let* $f \in C_b^\infty(\mathbb{R}^d)$ *and let* ϕ_N *be some kernel described by* (1.2), (1.4), (2.3) *and*

$$\phi_1 \geq 0, \qquad \int_{\mathbb{R}^d} dx\ \phi_1(x) = 1, \qquad \phi_1(x) = \phi_1(-x) \quad (x \in \mathbb{R}^d).$$

Then, for any $L \in \mathbb{N}_0$ *the function* $f * \phi_N$ *may be expanded as*

$$\begin{aligned}(f * \phi_N)(x) &= f(x) \\ &\quad + \sum_{l=1}^L \theta_N^{-2l} \sum_{\substack{0 \leq l_1, \ldots, l_d \leq 2l \\ l_1 + \ldots + l_d = 2l}} \sigma^*(l_1, \ldots, l_d; \phi_1) \frac{\partial^{2l}}{\partial_1^{l_1} \cdots \partial_d^{l_d}} f(x) \\ &\quad + \theta_N^{-2L-2} R_{N,L}(f; \phi_1)(x)\end{aligned} \tag{2.30}$$

where $\sigma^*(\ldots)$ *is defined by* (2.18) *and*

$$\sup_{N \in \mathbb{N}} \|\nabla^{\otimes m} R_{N,L}(f; \phi_1)\|_\infty \leq C(f, m, L) < \infty \qquad (m, L \in \mathbb{N}_0).$$

As function of f *the constant* $C(f, m, L)$ *depends only on the* L^∞*-norms of* f *and its partial derivatives of order* $\leq m + 2L + 3$.

In a slightly modified version Lemma 1 can be found in [10], where it is also proved for $d = 1$. The extension to $d > 1$ is obvious. We note that for $d > 1$ the right side of the version of (2.30) in [10] is not complete. Indeed, in [10] we have forgotten all those contributions, where l_k is odd for some $k = 1, \ldots, d$.

3. Proofs

A first part of this section contains the derivation of our basic result Theorem 1. In the second part Theorem 2 will be deduced. We note that the proofs of Theorems 1 and 2 are not independent. As mentioned at the end of the proof of Theorem 1 there exist $0 < T_*^0 < T_*^1 < \ldots$ such that successively both theorems are deduced in $[0, T_*^0], [T_*^0, T_*^1], \ldots$ Finally, in this section we shall present the proof of Theorem 3.

3.1 Proof of Theorem 1. For verification of (2.10) and (2.11) we refer to Remark (ii) in Section 2. As in the derivation of (2.23) these relations and (1.14) may be applied to get additionally

$$\sup_{t \le T} \|\rho_N(\cdot, t)\|_{(m)} < \infty \qquad (m \in \mathbb{N}_0, T > 0, N \in \mathbb{N}).$$

Furthermore, (2.8) and the arguments from Remark (ii) in Section 2 immediately yield

$$\sup_{t \le T} \|\rho_N(\cdot, t)\|_{(m,1)} < \infty \qquad (m \in \mathbb{N}_0, T > 0, N \in \mathbb{N}).$$

In particular, these regularity properties, (2.10) and (2.11) ensure that for any fixed $N \in \mathbb{N}$ the subsequent calculations in Subsections 3.1 and 3.2 for the proofs of Theorems 1 and 2 are justified. More precisely, we do not have to care about the existence of any of the partial derivatives or integrals involving ρ_N, which will appear.

For convenience, our considerations will be restricted to a finite time interval $[0, T]$, where $T \in (0, \infty)$. To simplify notation we also shall perform the subsequent calculations for d even. Apart from some minor notational changes the calculations would be the same for d odd.

First, we employ (1.1) to determine a description of the evolution of quantities like $\|(-\Delta)^m \rho_N(\cdot, t)\|_2$ $(m \in \mathbb{N}_0)$, which by (1.14) appear in upper bounds to the Sobolev norms $\|\rho_N(\cdot, t)\|_{(k)}$ $(k \in \mathbb{N}_0, t \ge 0)$ of ρ_N. It will turn out to be convenient to study also the expressions $\|(-\Delta)^m (\rho_N(\cdot, t) * \phi_N^r)\|_2$, where ϕ_N^r is defined in (2.4). Hence, we shall compute

$$\frac{d}{dt} \big\|(-\Delta)^m \rho_N(\cdot, t)\big\|_2^2 \qquad \text{and} \qquad \frac{d}{dt} \big\|(-\Delta)^m (\rho_N(\cdot, t) * \phi_N^r)\big\|_2^2$$

for $m \in \mathbb{N}_0$.

By (1.1) we first obtain the relation

$$\begin{aligned}
&\partial_t (-\Delta)^m \rho_N(x, t) \\
&\quad = \tfrac{1}{2} (-\Delta)^m \Delta \rho_N(x, t) + (-\Delta)^m \nabla \cdot \Big(\rho_N(x, t) \nabla (\rho_N(\cdot, t) * \phi_N)(x) \Big) \\
&\quad = -\tfrac{1}{2} (-\Delta)^{m+1} \rho_N(x, t) + \nabla_i \Big(\rho_N(x, t) (-\Delta)^m \nabla_i (\rho_N(\cdot, t) * \phi_N)(x) \Big) \\
&\qquad - 2m \nabla_i \Big(\nabla_{i_1} \rho_N(x, t) (-\Delta)^{m-1} \nabla_{i_1} \nabla_i (\rho_N(\cdot, t) * \phi_N)(x) \Big)
\end{aligned}$$

$$
\begin{aligned}
&- m\nabla_i\Big(\Delta\rho_N(x,t)(-\Delta)^{m-1}\nabla_i(\rho_N(\cdot,t)*\phi_N)(x)\Big)\\
&+ 4\binom{m}{2}\nabla_i\Big(\nabla_{i_1}\nabla_{i_2}\rho_N(x,t)(-\Delta)^{m-2}\nabla_{i_1}\nabla_{i_2}\nabla_i(\rho_N(\cdot,t)*\phi_N)(x)\Big)\\
&+ \nabla_i\Big((-\Delta)^m\rho_N(x,t)\nabla_i(\rho_N(\cdot,t)*\phi_N)(x)\Big)\\
&- 2m\nabla_i\Big((-\Delta)^{m-1}\nabla_{i_1}\rho_N(x,t)\nabla_{i_1}\nabla_i(\rho_N(\cdot,t)*\phi_N)(x)\Big)\\
&+ (-1)^m\sum_{i_1,\ldots,i_m=1}^{d}\ \sum_{\substack{\alpha_{i_1},\ldots,\alpha_{i_m}=0,1,2\\ 3\le\alpha_{i_1}+\ldots+\alpha_{i_m}\le 2m-2}}\ \prod_{r=1}^{m}\binom{2}{\alpha_{i_r}}\\
&\qquad \nabla_i\Big(\nabla_{i_1}^{\alpha_{i_1}}\cdots\nabla_{i_m}^{\alpha_{i_m}}\rho_N(x,t)\nabla_{i_1}^{2-\alpha_{i_1}}\cdots\nabla_{i_m}^{2-\alpha_{i_m}}\nabla_i(\rho_N(\cdot,t)*\phi_N)(x)\Big).
\end{aligned}
\tag{3.1}
$$

In (3.1) we have specified separately those contributions, which contain partial derivatives of ρ_N with highest order. Of course, for $m \le 2$ some terms in (3.1) have to be omitted.

An immediate consequence of (3.1) is

$$
\begin{aligned}
&\frac{d}{dt}\|(-\Delta)^m\rho_N(\cdot,t)\|_2^2\\
&\quad = 2\Big\langle(-\Delta)^m\rho_N(\cdot,t),\partial_t(-\Delta)^m\rho_N(\cdot,t)\Big\rangle\\
&\quad = -\Big\langle(-\Delta)^m\rho_N(\cdot,t),(-\Delta)^{m+1}\rho_N(\cdot,t)\Big\rangle\\
&\qquad + 2\Big\langle(-\Delta)^m\rho_N(\cdot,t),\nabla_i\Big(\rho_N(\cdot,t)(-\Delta)^m\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\\
&\qquad - 4m\Big\langle(-\Delta)^m\rho_N(\cdot,t),\nabla_i\Big(\nabla_{i_1}\rho_N(\cdot,t)(-\Delta)^{m-1}\nabla_{i_1}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\\
&\qquad - 2m\Big\langle(-\Delta)^m\rho_N(\cdot,t),\nabla_i\Big(\Delta\rho_N(\cdot,t)(-\Delta)^{m-1}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\\
&\qquad + 8\binom{m}{2}\Big\langle(-\Delta)^m\rho_N(\cdot,t),\\
&\qquad\qquad \nabla_i\Big(\nabla_{i_1}\nabla_{i_2}\rho_N(\cdot,t)(-\Delta)^{m-2}\nabla_{i_1}\nabla_{i_2}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\\
&\qquad + 2\Big\langle(-\Delta)^m\rho_N(\cdot,t),\nabla_i\Big((-\Delta)^m\rho_N(\cdot,t)\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\\
&\qquad - 4m\Big\langle(-\Delta)^m\rho_N(\cdot,t),\nabla_i\Big((-\Delta)^{m-1}\nabla_{i_1}\rho_N(\cdot,t)\nabla_{i_1}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\\
&\qquad + 2(-1)^m\sum_{i_1,\ldots,i_m=1}^{d}\ \sum_{\substack{\alpha_{i_1},\ldots,\alpha_{i_m}=0,1,2\\ 3\le\alpha_{i_1}+\ldots+\alpha_{i_m}\le 2m-2}}\ \prod_{r=1}^{m}\binom{2}{\alpha_{i_r}}\Big\langle(-\Delta)^m\rho_N(\cdot,t),\\
&\qquad\qquad \nabla_i\Big(\nabla_{i_1}^{\alpha_{i_1}}\cdots\nabla_{i_m}^{\alpha_{i_m}}\rho_N(\cdot,t)\nabla_{i_1}^{2-\alpha_{i_1}}\cdots\nabla_{i_m}^{2-\alpha_{i_m}}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle.
\end{aligned}
\tag{3.2}
$$

Quite similarly, (2.5), (2.6) and (3.1) yield

$$
\frac{d}{dt}\big\|(-\Delta)^m(\rho_N(\cdot,t)*\phi_N^r)\big\|_2^2
$$

$$
\begin{aligned}
&= 2\Big\langle(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), \partial_t(-\Delta)^m\rho_N(\cdot,t)\Big\rangle\\
&= -\Big\langle(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), (-\Delta)^{m+1}\rho_N(\cdot,t)\Big\rangle\\
&\quad + 2\Big\langle(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), \nabla_i\Big(\rho_N(\cdot,t)(-\Delta)^m\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\\
&\quad - 4m\Big\langle(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), \nabla_i\big(\nabla_{i_1}\rho_N(\cdot,t)(-\Delta)^{m-1}\nabla_{i_1}\nabla_i(\rho_N(\cdot,t)*\phi_N)\big)\Big\rangle\\
&\quad - 2m\Big\langle(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), \nabla_i\Big(\Delta\rho_N(\cdot,t)(-\Delta)^{m-1}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\\
&\quad + 8\binom{m}{2}\Big\langle(-\Delta)^m(\rho_N(\cdot,t)*\phi_N),\\
&\qquad\qquad \nabla_i\Big(\nabla_{i_1}\nabla_{i_2}\rho_N(\cdot,t)(-\Delta)^{m-2}\nabla_{i_1}\nabla_{i_2}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\\
&\quad + 2\Big\langle(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), \nabla_i\Big((-\Delta)^m\rho_N(\cdot,t)\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\\
&\quad - 4m\Big\langle(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), \nabla_i\Big((-\Delta)^{m-1}\nabla_{i_1}\rho_N(\cdot,t)\nabla_{i_1}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\\
&\quad + 2(-1)^m\sum_{i_1,\dots,i_m=1}^{d}\ \sum_{\substack{\alpha_{i_1},\dots,\alpha_{i_m}=0,1,2\\ 3\le\alpha_{i_1}+\dots+\alpha_{i_m}\le 2m-2}}\ \prod_{r=1}^{m}\binom{2}{\alpha_{i_r}}\Big\langle(-\Delta)^m(\rho_N(\cdot,t)*\phi_N),\\
&\qquad\qquad \nabla_i\Big(\nabla_{i_1}^{\alpha_{i_1}}\cdots\nabla_{i_m}^{\alpha_{i_m}}\rho_N(\cdot,t)\nabla_{i_1}^{2-\alpha_{i_1}}\cdots\nabla_{i_m}^{2-\alpha_{i_m}}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle.
\end{aligned}
\tag{3.3}
$$

For the various contributions to the right sides of (3.2) and (3.3) we now have to determine suitable upper bounds. In the corresponding calculations (cf. (3.4) - (3.16)) we assume $m \geq 3$. Otherwise, some of these estimates could be omitted.

For the first and the second term on the right side of (3.2) we obtain

$$
\begin{aligned}
&-\big\langle(-\Delta)^m\rho_N(\cdot,t), (-\Delta)^{m+1}\rho_N(\cdot,t)\big\rangle\\
&\qquad = \big\langle(-\Delta)^m\rho_N(\cdot,t), \Delta(-\Delta)^m\rho_N(\cdot,t)\big\rangle\\
&\qquad = -\big\|\nabla(-\Delta)^m\rho_N(\cdot,t)\big\|_2^2
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
&\Big|\Big\langle(-\Delta)^m\rho_N(\cdot,t), \nabla_i\Big(\rho_N(\cdot,t)(-\Delta)^m\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\Big|\\
&\quad = \Big|\Big\langle\nabla_i(-\Delta)^m\rho_N(\cdot,t), \rho_N(\cdot,t)(-\Delta)^m\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big\rangle\Big|\\
&\quad \le \big\|\nabla(-\Delta)^m\rho_N(\cdot,t)\big\|_2\big\|\rho_N(\cdot,t)\nabla(-\Delta)^m(\rho_N(\cdot,t)*\phi_N)\big\|_2\\
&\quad \le C_2\big\|\nabla(-\Delta)^m\rho_N(\cdot,t)\big\|_2^2\\
&\qquad + \tfrac{1}{C_2}\Big\langle\nabla(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), \rho_N(\cdot,t)\nabla(-\Delta)^m(\rho_N(\cdot,t)*\phi_N)\Big\rangle\|\rho_N(\cdot,t)\|_\infty.
\end{aligned}
\tag{3.5}
$$

In (3.5) we also utilize the positivity of ρ_N (cf. (2.11)). Moreover, here and in subsequent estimates we employ some positive constant C_2, which can be chosen arbitrarily. A precise value will be fixed later in (3.18).

By using in particular Sobolev's inequality (1.12) we get for the next terms on the right side of (3.2) in a similar way as in (3.5) the estimates

$$\begin{aligned}&\Big|\Big\langle(-\Delta)^m\rho_N(\cdot,t),\nabla_i\Big(\nabla_{i_1}\rho_N(\cdot,t)(-\Delta)^{m-1}\nabla_{i_1}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\Big|\\&\quad\le C_2\big\|\nabla(-\Delta)^m\rho_N(\cdot,t)\big\|_2^2+\tfrac{C}{C_2}\big\|\nabla^{\otimes 2}(-\Delta)^{m-1}(\rho_N(\cdot,t)*\phi_N)\big\|_2^2\|\nabla\rho_N(\cdot,t)\|_\infty^2\\&\quad\le C_2\big\|\nabla(-\Delta)^m\rho_N(\cdot,t)\big\|_2^2+\tfrac{C}{C_2}\big\|\nabla^{\otimes(2m)}(\rho_N(\cdot,t)*\phi_N)\big\|_2^2\|\rho_N(\cdot,t)\|_{(2+d/2)}^2\end{aligned}\tag{3.6}$$

$$\begin{aligned}&\Big|\Big\langle(-\Delta)^m\rho_N(\cdot,t),\nabla_i\Big(\Delta\rho_N(\cdot,t)(-\Delta)^{m-1}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\Big|\\&\quad\le C_2\big\|\nabla(-\Delta)^m\rho_N(\cdot,t)\big\|_2^2+\tfrac{C}{C_2}\big\|\nabla^{\otimes(2m-1)}(\rho_N(\cdot,t)*\phi_N)\big\|_2^2\|\rho_N(\cdot,t)\|_{(3+d/2)}^2\end{aligned}\tag{3.7}$$

$$\begin{aligned}&\Big|\Big\langle(-\Delta)^m\rho_N(\cdot,t),\nabla_i\Big(\nabla_{i_1}\nabla_{i_2}\rho_N(\cdot,t)(-\Delta)^{m-2}\nabla_{i_1}\nabla_{i_2}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\Big|\\&\quad\le C_2\big\|\nabla(-\Delta)^m\rho_N(\cdot,t)\big\|_2^2+\tfrac{C}{C_2}\big\|\nabla^{\otimes(2m-1)}(\rho_N(\cdot,t)*\phi_N)\|_2^2\|\rho_N(\cdot,t)\|_{(3+d/2)}^2\end{aligned}\tag{3.8}$$

$$\begin{aligned}&\Big|\Big\langle(-\Delta)^m\rho_N(\cdot,t),\nabla_i\Big((-\Delta)^m\rho_N(\cdot,t)\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\Big|\\&\quad\le C_2\big\|\nabla(-\Delta)^m\rho_N(\cdot,t)\big\|_2^2+\tfrac{C}{C_2}\big\|\nabla^{\otimes(2m)}\rho_N(\cdot,t)\big\|_2^2\|\rho_N(\cdot,t)*\phi_N\|_{(2+d/2)}^2\end{aligned}\tag{3.9}$$

$$\begin{aligned}&\Big|\Big\langle(-\Delta)^m\rho_N(\cdot,t),\nabla_i\Big((-\Delta)^{m-1}\nabla_{i_1}\rho_N(\cdot,t)\nabla_{i_1}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\Big|\\&\quad\le C_2\big\|\nabla(-\Delta)^m\rho_N(\cdot,t)\big\|_2^2+\tfrac{C}{C_2}\big\|\nabla^{\otimes(2m-1)}\rho_N(\cdot,t)\|_2^2\big\|\rho_N(\cdot,t)*\phi_N\big\|_{(3+d/2)}^2.\end{aligned}\tag{3.10}$$

To determine an estimate for the last term on the right side of (3.2) we have to find for any summand an upper bound for a product of a derivative of $\rho_N(\cdot,t)$ and a derivative of $\rho_N(\cdot,t)*\phi_N$. As in (3.5) - (3.10) we choose the L^2- and L^∞-norm, respectively, such that after application of Sobolev's inequality (1.12) to the L^∞-expression the orders of the derivatives in the resulting term are as small as possible. Hence, we deduce

$$\begin{aligned}&\sum_{i_1,\dots,i_m=1}^{d}\ \sum_{\substack{\alpha_{i_1},\dots,\alpha_{i_m}=0,1,2\\3\le\alpha_{i_1}+\dots+\alpha_{i_m}\le 2m-2}}\ \prod_{r=1}^{m}\binom{2}{\alpha_{i_r}}\\&\qquad\Big|\Big\langle(-\Delta)^m\rho_N(\cdot,t),\nabla_i\Big(\nabla_{i_1}^{\alpha_{i_1}}\dots\nabla_{i_m}^{\alpha_{i_m}}\rho_N(\cdot,t)\nabla_{i_1}^{2-\alpha_{i_1}}\dots\nabla_{i_m}^{2-\alpha_{i_m}}\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle\Big|\\&\quad\le C_2\big\|\nabla(-\Delta)^m\rho_N(\cdot,t)\big\|_2^2+\tfrac{C}{C_2}\sum_{q=3}^{m}\big\|\nabla^{\otimes(2m+1-q)}(\rho_N(\cdot,t)*\phi_N)\big\|_2^2\|\rho_N(\cdot,t)\|_{(q+1+d/2)}^2\\&\qquad+\tfrac{C}{C_2}\sum_{q=3}^{m}\big\|\rho_N(\cdot,t)*\phi_N\big\|_{(q+1+d/2)}^2\big\|\nabla^{\otimes(2m+1-q)}\rho_N(\cdot,t)\big\|_2^2.\end{aligned}\tag{3.11}$$

Estimates (3.4) - (3.11) obtained so far for the terms on the right side of (3.2) may be collected in a single relation, namely

$$
\begin{aligned}
&\frac{d}{dt}\big\|(-\Delta)^m \rho_N(\cdot,t)\big\|_2^2 \\
&\qquad \le (C_2C_3(m)-1)\big\|\nabla(-\Delta)^m \rho_N(\cdot,t)\big\|_2^2 \\
&\qquad\quad + \tfrac{2}{C_2}\Big\langle \nabla(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), \rho_N(\cdot,t)\nabla(-\Delta)^m(\rho_N(\cdot,t)*\phi_N)\Big\rangle \\
&\qquad\quad \times \|\rho_N(\cdot,t)\|_\infty \\
&\qquad\quad + \tfrac{C_4(m)}{C_2}\sum_{q=1}^{m}\Big(\|\rho_N(\cdot,t)*\phi_N\|^2_{(2m+1-q)}\|\rho_N(\cdot,t)\|^2_{(q+1+d/2)} \\
&\qquad\quad + \|\rho_N(\cdot,t)*\phi_N\|^2_{(q+1+d/2)}\|\rho_N(\cdot,t)\|^2_{(2m+1-q)}\Big),
\end{aligned}
\tag{3.12}
$$

where $C_3(m)$ and $C_4(m)$ in particular depend on m, e.g.,

$$
C_3(m) = 6 + 10m + 8\binom{m}{2}. \tag{3.13}
$$

The precise value of $C_4(m)$ is not important.

Now, we turn to the estimation of the summands on the right side of (3.3). By (2.5) and (2.6) we deduce for the first and the second term

$$
\begin{aligned}
-\Big\langle(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), (-\Delta)^{m+1}\rho_N(\cdot,t)\Big\rangle& \\
= -\big\|\nabla(-\Delta)^m(\rho_N(\cdot,t)*\phi_N^r)\big\|_2^2&
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
&\Big\langle(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), \nabla_i\Big(\rho_N(\cdot,t)(-\Delta)^m\nabla_i(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle \\
&\quad = -\Big\langle\nabla_i(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), \rho_N(\cdot,t)\nabla_i(-\Delta)^m(\rho_N(\cdot,t)*\phi_N)\Big\rangle.
\end{aligned}
\tag{3.15}
$$

Obviously, the structure of the right side of (3.3) is closely related to that of (3.2). More precisely, any term $\langle(-\Delta)^m\rho_N(\cdot,t), \nabla_i(\ldots)\rangle$ in (3.2) corresponds to some term $\langle(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), \nabla_i(\ldots)\rangle$ in (3.3), where in both expressions "..." coincide. Hence, we only have to modify (3.6) - (3.11) slightly to deduce upper bounds for the remaining contributions to the right side of (3.3). With (3.14) and (3.15) we obtain as summary an analogue of (3.12), namely

$$
\begin{aligned}
&\frac{d}{dt}\big\|(-\Delta)^m(\rho_N(\cdot,t)*\phi_N^r)\big\|_2^2 \\
&\qquad \le (C_2C_3(m)-1)\big\|\nabla(-\Delta)^m(\rho_N(\cdot,t)*\phi_N^r)\big\|_2^2 \\
&\qquad\quad - 2\Big\langle \nabla(-\Delta)^m(\rho_N(\cdot,t)*\phi_N), \rho_N(\cdot,t)\nabla(-\Delta)^m(\rho_N(\cdot,t)*\phi_N)\Big\rangle \\
&\qquad\quad + \tfrac{C_4(m)}{C_2}\sum_{q=1}^{m}\Big(\|\rho_N(\cdot,t)*\phi_N\|^2_{(2m+1-q)}\|\rho_N(\cdot,t)\|^2_{(q+1+d/2)} \\
&\qquad\quad + \|\rho_N(\cdot,t)*\phi_N\|^2_{(q+1+d/2)}\|\rho_N(\cdot,t)\|^2_{(2m+1-q)}\Big).
\end{aligned}
\tag{3.16}
$$

Next, we shall determine an upper bound for a suitable linear combination of $\|(-\Delta)^m \rho_N(\cdot,t)\|_2^2$ and $\|(-\Delta)^m(\rho_N(\cdot,t) * \phi_N^r)\|_2^2$. For that purpose we have to adjust some parameters appearing on the right sides of (3.12) and (3.16).

First, we have got sums of products of squared Sobolev norms. As a consequence of applications of Sobolev's inequality (1.12) in these contributions derivatives of order $q + 1 + \frac{1}{2}d$ $(q = 1, \dots, m)$ appeared. These orders should be $\leq 2m$, which is the order of the derivatives on the left sides of (3.12) and (3.16). Hence, we should choose some $m \geq 1 + \frac{1}{2}d$, and so we temporarily suppose

$$m = m_0 = 1 + d/2. \tag{3.17}$$

We also have to care for terms on the right sides of (3.12) and (3.16), which explicitly contain partial derivatives of order $2m_0 + 1$. The terms with $\|\nabla(-\Delta)^{m_0}\rho_N(\cdot,t)\|_2^2$ or $\|\nabla(-\Delta)^{m_0}(\rho_N(\cdot,t) * \phi_N^r)\|_2^2$ are negative if we assume

$$C_2 = C_2(m_0) \leq \frac{1}{2C_3(m_0)} \tag{3.18}$$

where $C_3(m_0)$ is defined by (3.13). Next, to handle expressions containing

$$\Big\langle \nabla(-\Delta)^{m_0}(\rho_N(\cdot,t) * \phi_N), \rho_N(\cdot,t)\nabla(-\Delta)^{m_0}(\rho_N(\cdot,t) * \phi_N)\Big\rangle$$

we temporarily restrict the time interval for our considerations to a subinterval of $[0, T]$. For that purpose we choose some constant C_5 satisfying

$$\begin{gathered} C_1(2m_0)C_5 > 2\sup_{t\leq T}\|\rho(\cdot,t)\|_{(2m_0)} \\ \sup_{t\leq T}\|\rho(\cdot,t)\|_{(2m_0)}^2\Big(1 + \frac{C(d,m_0,1+d/2)C_1(2m_0)C_5}{C_2(m_0)}\Big) \leq \tfrac{1}{2}C_5^2, \end{gathered} \tag{3.19}$$

where ρ is the solution of the limit dynamics (1.5), and $C(d, m_0, 1 + d/2)$ and $C_1(2m_0)$ are introduced in (1.12) and (1.14), respectively. We note that by (2.23) both relations in (3.19) hold for C_5 sufficiently large. Now, we define

$$T_N = \inf\Big\{t \in [0,T] : \|\rho_N(\cdot,t)\|_{(2m_0)} > C_1(2m_0)C_5\Big\} \wedge T \qquad (N \in \mathbb{N}) \tag{3.20}$$

and then consider the function

$$[0, T_N) \ni t \to \Big\|(-\Delta)^{m_0}\rho_N(\cdot,t)\Big\|_2^2 + C_6\Big\|(-\Delta)^{m_0}(\rho_N(\cdot,t) * \phi_N^r)\Big\|_2^2$$

with

$$C_6 = \frac{C(d,m_0,1+d/2)C_1(2m_0)C_5}{C_2(m_0)}. \tag{3.21}$$

Since $\rho_N(\cdot,0) = \rho(\cdot,0) = \rho_0$ $(N \in \mathbb{N})$, and by continuity (cf. (2.10)), the first relation in (3.19) yields $T_N > 0$ $(N \in \mathbb{N})$. Moreover, (1.12), (3.17) and (3.20) imply

$$\|\rho_N(\cdot,t)\|_\infty \leq C(d,m_0,1+d/2)C_1(2m_0)C_5 \qquad (0 \leq t \leq T_N, N \in \mathbb{N}). \tag{3.22}$$

By (2.5) and (2.6), which imply $\|f * \phi_N\|_2 \leq \|f * \phi_N^r\|_2$ for $f \in L^2(\mathbb{R}^d)$, and by (3.12), (3.16) - (3.18), (3.21) and (3.22) we get

$$\begin{aligned}
&\frac{d}{dt}\Big(\|(-\Delta)^{m_0}\rho_N(\cdot,t)\|_2^2 + C_6\|(-\Delta)^{m_0}(\rho_N(\cdot,t) * \phi_N^r)\|_2^2\Big) \\
&\quad\leq -\tfrac{1}{2}\Big(\|\nabla(-\Delta)^{m_0}\rho_N(\cdot,t)\|_2^2 + C_6\|\nabla(-\Delta)^{m_0}(\rho_N(\cdot,t) * \phi_N^r)\|_2^2\Big) \\
&\qquad + C\sum_{q=1}^{m_0}\Big(\|\rho_N(\cdot,t) * \phi_N\|_{(2m_0+1-q)}^2\|\rho_N(\cdot,t)\|_{(q+1+d/2)}^2 \\
&\qquad + \|\rho_N(\cdot,t) * \phi_N\|_{(q+1+d/2)}^2\|\rho_N(\cdot,t)\|_{(2m_0+1-q)}^2\Big) \\
&\quad\leq C\|\rho_N(\cdot,t)\|_{(2m_0)}^2\|\rho_N(\cdot,t) * \phi_N\|_{(2m_0)}^2 \\
&\quad\leq C\Big(\|\rho_N(\cdot,t)\|_{(2m_0)}^2 + C_6\|\rho_N(\cdot,t) * \phi_N^r\|_{(2m_0)}^2\Big)^2
\end{aligned} \tag{3.23}$$

for $t \leq T_N$ and $N \in \mathbb{N}$. According to (1.4), to obtain an estimate for

$$\|\rho_N(\cdot,t)\|_{(2m_0)}^2 + C_6\|\rho_N(\cdot,t) * \phi_N^r\|_{(2m_0)}^2$$

we need in addition to (3.23) an upper bound for

$$\frac{d}{dt}\Big(\|\rho_N(\cdot,t)\|_2^2 + C_6\|\rho_N(\cdot,t) * \phi_N^r\|_2^2\Big).$$

Therefore, we have to repeat the calculations in (3.1) - (3.16) for $m = 0$. In this particular case on the right sides of (3.1) - (3.3) only the first and the second term have to be retained, and consequently, for the desired upper bound we only have to consider estimates (3.4), (3.5), (3.14) and (3.15) with $m = 0$. With analogues of (3.12) and (3.16) containing also only the first and the second term on their right sides we finally deduce

$$\frac{d}{dt}\Big(\|\rho_N(\cdot,t)\|_2^2 + C_6\|\rho_N(\cdot,t) * \phi_N^r\|_2^2\Big) \leq 0 \qquad (t \leq T_N,\ N \in \mathbb{N}). \tag{3.24}$$

Now, combined with (1.10) and (1.14) estimates (3.23) and (3.24) lead to

$$\begin{aligned}
&\|\rho_N(\cdot,t)\|_{(2m_0)}^2 + C_6\|\rho_N(\cdot,t) * \phi_N^r\|_{(2m_0)}^2 \\
&\quad\leq C_1(2m_0)^2\Big(\|\rho_N(\cdot,t)\|_2^2 + \|(-\Delta)^{m_0}\rho_N(\cdot,t)\|_2^2 \\
&\qquad + C_6\Big(\|\rho_N(\cdot,t) * \phi_N^r\|_2^2 + \|(-\Delta)^{m_0}(\rho_N(\cdot,t) * \phi_N^r)\|_2^2\Big)\Big) \\
&\quad\leq C_1(2m_0)^2\Big(\|\rho_N(\cdot,0)\|_{(2m_0)}^2 + C_6\|\rho_N(\cdot,0) * \phi_N^r\|_{(2m_0)}^2\Big) \\
&\qquad + C_7\int_0^t ds\ \Big(\|\rho_N(\cdot,s)\|_{(2m_0)}^2 + C_6\|\rho_N(\cdot,s) * \phi_N^r\|_{(2m_0)}^2\Big)^2
\end{aligned} \tag{3.25}$$

for $t \leq T_N$ and $N \in \mathbb{N}$. We emphasize that the constant C_7 here, which depends on the previously introduced constants $C_1, \ldots, C_6$, is independent of N. Moreover, (1.1), (1.5), (2.5), (3.19) and (3.21) yield

$$\|\rho_N(\cdot,0)\|_{(2m_0)}^2 + C_6\|\rho_N(\cdot,0) * \phi_N^r\|_{(2m_0)}^2 \leq \|\rho_N(\cdot,0)\|_{(2m_0)}^2(1 + C_6) \leq \tfrac{1}{2}C_5^2 \tag{3.26}$$

for $N \in \mathbb{N}$. Hence, with (3.25) we deduce the existence of some $T_*^0 \in (0,T]$ such that

$$\sup_{t \le T_*^0 \wedge T_N, N \in \mathbb{N}} \Big(\|\rho_N(\cdot,t)\|^2_{(2m_0)} + C_6 \|\rho_N(\cdot,t) * \phi_N^r\|^2_{(2m_0)} \Big) \le \tfrac{3}{4} C_1 (2m_0)^2 C_5^2.$$

Consequently, (3.20) and the continuity of the functions

$$t \to \big\|\rho_N(\cdot,t)\big\|^2_{(2m_0)} + C_6 \big\|\rho_N(\cdot,t) * \phi_N^r\big\|^2_{(2m_0)} \qquad (N \in \mathbb{N})$$

yield

$$T_*^0 \le T_N \qquad (N \in \mathbb{N}) \tag{3.27}$$

and we get

$$\sup_{t \le T_*^0, N \in \mathbb{N}} \Big(\|\rho_N(\cdot,t)\|^2_{(2m_0)} + C_6 \|\rho_N(\cdot,t) * \phi_N^r\|^2_{(2m_0)} \Big) \le \tfrac{3}{4} C_1 (2m_0)^2 C_5^2. \tag{3.28}$$

Next, we have to extend (3.28) to Sobolev norms of order $> 2m_0$. For that purpose we need as supplement to (3.12) and (3.16) upper bounds for

$$\frac{d}{dt} \big\|\nabla(-\Delta)^m \rho_N(\cdot,t)\big\|_2^2 \qquad \text{and} \qquad \frac{d}{dt} \big\|\nabla(-\Delta)^m (\rho_N(\cdot,t) * \phi_N^r)\big\|_2^2$$

for $m \in \mathbb{N}_0$. After modifying relations (3.1) - (3.11), (3.14) and (3.15) slightly in some rather obvious way we can deduce

$$\begin{aligned}
&\frac{d}{dt} \big\|\nabla(-\Delta)^m \rho_N(\cdot,t)\big\|_2^2 \\
&\qquad \le \big(C_8 C_9(m) - 1\big) \big\|(-\Delta)^{m+1} \rho_N(\cdot,t)\big\|_2^2 \\
&\qquad\quad + \tfrac{2}{C_8} \Big\langle (-\Delta)^{m+1} (\rho_N(\cdot,t) * \phi_N), \rho_N(\cdot,t) (-\Delta)^{m+1} (\rho_N(\cdot,t) * \phi_N) \Big\rangle \\
&\qquad\quad \times \|\rho_N(\cdot,t)\|_\infty \\
&\qquad\quad + \tfrac{C_{10}(m)}{C_8} \sum_{q=1}^{m+1} \Big(\|\rho_N(\cdot,t) * \phi_N\|^2_{(2m+2-q)} \|\rho_N(\cdot,t)\|^2_{(q+1+d/2)} \\
&\qquad\quad + \|\rho_N(\cdot,t) * \phi_N\|^2_{(q+1+d/2)} \|\rho_N(\cdot,t)\|^2_{(2m+2-q)} \Big)
\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}
&\frac{d}{dt} \big\|\nabla(-\Delta)^m (\rho_N(\cdot,t) * \phi_N^r)\big\|_2^2 \\
&\qquad \le \big(C_8 C_9(m) - 1\big) \big\|(-\Delta)^{m+1} (\rho_N(\cdot,t) * \phi_N^r)\big\|_2^2 \\
&\qquad\quad - 2 \Big\langle (-\Delta)^{m+1} (\rho_N(\cdot,t) * \phi_N), \rho_N(\cdot,t) (-\Delta)^{m+1} (\rho_N(\cdot,t) * \phi_N) \Big\rangle \\
&\qquad\quad + \tfrac{C_{10}(m)}{C_8} \sum_{q=1}^{m+1} \Big(\|\rho_N(\cdot,t) * \phi_N\|^2_{(2m+2-q)} \|\rho_N(\cdot,t)\|^2_{(q+1+d/2)} \\
&\qquad\quad + \|\rho_N(\cdot,t) * \phi_N\|^2_{(q+1+d/2)} \|\rho_N(\cdot,t)\|^2_{(2m+2-q)} \Big)
\end{aligned} \tag{3.30}$$

as analogues of (3.12) and (3.16), where C_8, C_9 and C_{10} are suitable constants corresponding to C_2, C_3 and C_4, respectively.

For estimation of the remaining Sobolev norms $\|\rho_N(\cdot,t)\|_{(k)}$ and $\|\rho_N(\cdot,t)*\phi_N^r\|_{(k)}$ $(k>2m_0)$ we remark that for $m>m_0$ the sums on the right sides of the basic inequalities (3.12) and (3.16) are linear in $\|\rho_N(\cdot,t)\|^2_{(2m)}$ and $\|\rho_N(\cdot,t)*\phi_N\|^2_{(2m)}$, and that for $m\ge m_0$ the sums on the right sides of (3.29) and (3.30) are linear in $\|\rho_N(\cdot,t)\|^2_{(2m+1)}$ and $\|\rho_N(\cdot,t)*\phi_N\|^2_{(2m+1)}$. Furthermore, Sobolev norms with higher order do not appear in the respective sums. Consequently, similar to (3.25) we now obtain

$$\begin{aligned}
&\|\rho_N(\cdot,t)\|^2_{(k)}+C_{11}(k)\|\rho_N(\cdot,t)*\phi_N^r\|^2_{(k)}\\
&\qquad\le C_{12}(k)\big(\|\rho_0\|^2_{(k)}+C_{11}(k)\|\rho_0*\phi_N^r\|^2_{(k)}\big)\\
&\qquad\quad+C\int_0^t ds\ \|\rho_N(\cdot,s)\|^2_{(k-1)}\Big(\|\rho_N(\cdot,s)\|^2_{(k)}+C_{11}(k)\|\rho_N(\cdot,s)*\phi_N^r\|^2_{(k)}\Big)\\
&\qquad\big(t\le T_*^0,\ k=2m_0+1,2m_0+2,\dots;\ N\in\mathbb{N}\big)
\end{aligned}\tag{3.31}$$

where $C_{11}(k)$ and $C_{12}(k)$ are suitable constants depending on k. Obviously, (2.8), (3.28) and (3.31) can be used in an iteration scheme to show

$$\sup_{t\le T_*^0,N\in\mathbb{N}}\Big(\|\rho_N(\cdot,t)\|^2_{(k)}+C_{11}(k)\|\rho_N(\cdot,t)*\phi_N^r\|^2_{(k)}\Big)<\infty\qquad(k>2m_0).$$

By (2.5) this relation is equivalent to

$$\sup_{t\le T_*^0,N\in\mathbb{N}}\|\rho_N(\cdot,t)\|_{(k)}<\infty\qquad(k\in\mathbb{N}_0).\tag{3.32}$$

As a consequence of (1.1), which gives a relation between partial derivatives with respect to time and spatial derivatives, Sobolev's inequality (1.12), (2.5) and (2.6) we additionally obtain

$$\sup_{x\in\mathbb{R}^d,t\le T_*^0,N\in\mathbb{N}}|\nabla^{\otimes m}\partial_t^k\rho_N(x,t)|<\infty\qquad(m,k\in\mathbb{N}_0).\tag{3.33}$$

Next, we turn to the derivation of upper bounds for

$$\|\rho_N(\cdot,t)\|_{(m,1)}\qquad(m\in\mathbb{N}_0,t\le T_*^0,N\in\mathbb{N}).$$

Therefore, we write the dynamics (1.1) of ρ_N as integral equation, namely

$$\begin{aligned}
\rho_N(x,t)&=\int_{\mathbb{R}^d}dz\ \sigma(x-z;t)\rho_0(z)\\
&\quad+\int_0^t ds\int_{\mathbb{R}^d}dz\ \sigma(x-z;t-s)\nabla\cdot\big(\rho_N(z,s)\nabla(\rho_N(\cdot,s)*\phi_N)(z)\big)\\
&\quad\big(x\in\mathbb{R}^d,t\ge 0,N\in\mathbb{N}\big)
\end{aligned}\tag{3.34}$$

where

$$\sigma(z;u) = \tfrac{1}{(2\pi u)^{d/2}} \exp\Big(-\frac{z^2}{2u}\Big) \qquad (z \in \mathbb{R}^d, u > 0) \tag{3.35}$$

is the fundamental solution of $\frac{1}{2}\Delta - \partial_t$. Relations (2.5), (2.6), (2.11), (3.33) - (3.35) and integration by parts yield

$$\begin{aligned}\rho_N(x,t) \leq & \int_{\mathbb{R}^d} dz\ \sigma(x-z;t)\rho_0(z) \\ & + C\int_0^t ds \int_{\mathbb{R}^d} dz\ \frac{1}{(t-s)^{(d+1)/2}} \exp\Big(-\frac{(x-z)^2}{4(t-s)}\Big)\rho_N(z,s) \\ & \big(x \in \mathbb{R}^d, 0 \leq t \leq T_*^0, N \in \mathbb{N}\big)\end{aligned}$$

and consequently

$$\begin{aligned}\rho_N(x,t)^2 \leq & \ 2\int_{\mathbb{R}^d} dz\ \sigma(x-z;t)\rho_0(z)^2 \\ & + C\sqrt{t}\int_0^t ds\ \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^d} dz\ \sigma(x-z;2(t-s))\rho_N(z,s)^2 \\ & \big(x \in \mathbb{R}^d, 0 \leq t \leq T_\star^0, N \in \mathbb{N}\big).\end{aligned} \tag{3.36}$$

Both sides of this relation may be multiplied by $(1+|x|)$ and integrated with respect to x. By (2.8), (3.32), (3.35) and (3.36) we then get

$$\begin{aligned}& \int_{\mathbb{R}^d} dx\ (1+|x|)\rho_N(x,t)^2 \\ & \leq C + C\sqrt{t}\int_0^t ds\ \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^d} dz\ (1+|z|)\rho_N(z,s)^2 \qquad \big(0 \leq t \leq T_*^0, N \in \mathbb{N}\big). \\ & \leq C + C_{13} t \sup_{0\leq s\leq t} \int_{\mathbb{R}^d} dx\ (1+|x|)\rho_N(x,s)^2\end{aligned}$$

Hence,

$$\sup_{\substack{0\leq s\leq T_*^0\wedge(1/(2C_{13})) \\ N\in\mathbb{N}}} \int_{\mathbb{R}^d} dx\ (1+|x|)\rho_N(x,s)^2 < \infty. \tag{3.37}$$

The constants appearing in computations between (3.34) and (3.37) may be chosen in such a way that they depend on ρ_N and its partial derivatives only on terms of upper bounds in the time interval $[0,T_*^0]$. Consequently, by (3.32) and (3.33) the arguments between (3.34) and (3.37) may be repeated word for word in $[T_*^0 \wedge (1/(2C_{13})), T_*^0 \wedge (1/C_{13})]$. As initial estimate we only have to use (3.37) instead of (2.8). By iterating this procedure in further intervals with length $1/(2C_{13})$ we finally deduce

$$\sup_{0\leq s\leq T_*^0, N\in\mathbb{N}} \|\rho_N(\cdot,s)\|_{(0,1)} < \infty. \tag{3.38}$$

To continue with the proof of (2.12) for $T = T_*^0$ we now consider

$$\begin{aligned}
&\frac{\partial^{p_1+\ldots+p_d}}{\partial_1^{p_1}\cdots\partial_d^{p_d}}\rho_N(x,t) \\
&\quad = \int_{\mathbb{R}^d} dz\ \sigma(x-z;t)\frac{\partial^{p_1+\ldots+p_d}}{\partial_1^{p_1}\cdots\partial_d^{p_d}}\rho_0(z) \\
&\qquad + \int_0^t ds \int_{\mathbb{R}^d} dz\ \sigma(x-z;t-s) \\
&\qquad\qquad \nabla\cdot\Bigg(\frac{\partial^{p_1+\ldots+p_d}}{\partial_1^{p_1}\cdots\partial_d^{p_d}}\rho_N(z,s)\nabla(\rho_N(\cdot,s)*\phi_N)(z) \\
&\qquad + \sum_{\substack{0\le r_1\le p_1,\ldots,0\le r_d\le p_d\\ r_1+\ldots+r_d<p_1+\ldots+p_d}} C(r_1,\ldots,r_d)\frac{\partial^{r_1+\ldots+r_d}}{\partial_1^{r_1}\cdots\partial_d^{r_d}}\rho_N(z,s) \\
&\qquad\qquad \nabla\frac{\partial^{p_1-r_1+\ldots+p_d-r_d}}{\partial_1^{p_1-r_1}\cdots\partial_d^{p_d-r_d}}(\rho_N(\cdot,s)*\phi_N)(z)\Bigg) \\
&\qquad \big(x\in\mathbb{R}^d, t\ge 0,\ p_1,\ldots,p_d\in\mathbb{N}_0,\ N\in\mathbb{N}\big)
\end{aligned} \tag{3.39}$$

which is obtained from (3.34) by differentiating both sides with respect to $\frac{\partial^{p_1+\ldots+p_d}}{\partial_1^{p_1}\cdots\partial_d^{p_d}}$. Integral equation (3.39) is the basic ingredient to derive

$$\sup_{0\le s\le T_*^0, N\in\mathbb{N}} \|\rho_N(\cdot,s)\|_{(p,1)} < \infty \qquad (p\in\mathbb{N}_0) \tag{3.40}$$

by induction on p. Obviously, the case $p=0$ is treated in (3.38). Suppose now that (3.40) has been verified for $p = 0,1,\ldots,q-1$. After squaring both sides of (3.39) we then obtain in analogy to (3.36) the relation

$$\begin{aligned}
&\Big|\frac{\partial^{q_1+\ldots+q_d}}{\partial_1^{q_1}\cdots\partial_d^{q_d}}\rho_N(x,t)\Big|^2 \\
&\quad \le 2\int_{\mathbb{R}^d} dz\ \sigma(x-z;t)\Big|\frac{\partial^{q_1+\ldots+q_d}}{\partial_1^{q_1}\cdots\partial_d^{q_d}}\rho_0(z)\Big|^2 \\
&\qquad + C\sqrt{t}\int_0^t ds\ \frac{1}{\sqrt{t-s}}\int_{\mathbb{R}^d} dz\ \sigma\big(x-z;2(t-s)\big) \\
&\qquad \times\Bigg(\Big|\frac{\partial^{q_1+\ldots+q_d}}{\partial_1^{q_1}\cdots\partial_d^{q_d}}\rho_N(z,s)\Big|^2 + \sum_{p=0}^{q-1}|\nabla^{\otimes p}\rho_N(z,s)|^2\Bigg) \\
&\qquad\quad \Big(x\in\mathbb{R}^d, 0\le t\le T_*^0,\ 0\le q_1,\ldots,q_d\le q,\ q_1+\ldots+q_d=q,\ N\in\mathbb{N}\Big).
\end{aligned} \tag{3.41}$$

In the derivation of (3.41) we have applied in particular integration by parts with respect to the operator "$\nabla\cdot$". Additionally, (2.5), (2.6) and (3.33) have been used to obtain upper bounds for derivatives of $\rho_N(\cdot,s)*\phi_N$.

Next, both sides of (3.41) are multiplied by $(1+|x|)$ and integrated with respect to x. By (2.8), (3.32), (3.41) and the induction hypothesis we conclude

$$\begin{aligned}
&\int_{\mathbb{R}^d} dx\ (1+|x|)\Big|\frac{\partial^{q_1+\ldots+q_d}}{\partial_1^{q_1}\cdots\partial_d^{q_d}}\rho_N(x,t)\Big|^2 \\
&\quad\le C + C\sqrt{t}\int_0^t ds\ \frac{1}{\sqrt{t-s}} \\
&\qquad\times\Big(\int_{\mathbb{R}^d} dz\ (1+|z|)\Big|\frac{\partial^{q_1+\ldots+q_d}}{\partial_1^{q_1}\cdots\partial_d^{q_d}}\rho_N(z,s)\Big|^2 + \sum_{p=0}^{q-1}\|\rho_N(\cdot,s)\|^2_{(p,1)}\Big) \\
&\quad\le C + C_{14}t\sup_{0\le s\le t}\int_{\mathbb{R}^d} dx\ (1+|x|)\Big|\frac{\partial^{q_1+\ldots+q_d}}{\partial_1^{q_1}\cdots\partial_d^{q_d}}\rho_N(x,s)\Big|^2 \\
&\qquad\big(0\le t\le T_*^0,\ 0\le q_1,\ldots,q_d\le q,\ q_1+\ldots+q_d=q,\ N\in\mathbb{N}\big).
\end{aligned}$$

As in (3.37) we therefore get

$$\begin{aligned}
&\sup_{\substack{0\le s\le T_*^0\wedge(1/(2C_{14}))\\ N\in\mathbb{N}}}\int_{\mathbb{R}^d} dx\ (1+|x|)\Big|\frac{\partial^{q_1+\ldots+q_d}}{\partial_1^{q_1}\cdots\partial_d^{q_d}}\rho_N(x,s)\Big|^2<\infty \\
&\qquad\big(0\le q_1,\ldots,q_d\le q,\ q_1+\ldots+q_d=q\big).
\end{aligned}\tag{3.42}$$

The arguments leading to (3.42) may be repeated in

$$\begin{gathered}
\big[T_*^0\wedge(1/(2C_{14})),T_*^0\wedge(1/C_{14})\big]\\
\big[T_*^0\wedge(1/C_{14}),T_*^0\wedge(3/(2C_{14}))\big]\\
\vdots
\end{gathered}$$

such that finally (3.40) for $p=q$ follows. Hence, by induction on p relation (3.40) holds for any $p\in\mathbb{N}_0$. By (3.33) and (3.40) the restriction of Theorem 1 to $\mathbb{R}^d\times[0,T_*^0]$ is proved now.

If $T>T_*^0$, we have to extend the proof of Theorem 1. For that purpose we first may verify Theorem 2 in $\mathbb{R}^d\times[0,T_*^0]$, and then we can repeat our considerations done so far in the time interval $[T_*^0,T]$. We observe

$$\begin{aligned}
&\big\|(-\Delta)^m\rho_N(\cdot,t)-(-\Delta)^m\rho(\cdot,t)\big\|_2^2 \\
&\quad=\Big\langle(-\Delta)^{2m}\rho_N(\cdot,t)-(-\Delta)^{2m}\rho(\cdot,t),\rho_N(\cdot,t)-\rho(\cdot,t)\Big\rangle \\
&\quad\le\big\|(-\Delta)^{2m}\rho_N(\cdot,t)-(-\Delta)^{2m}\rho(\cdot,t)\big\|_\infty\langle\rho_N(\cdot,t)+\rho(\cdot,t),1\rangle \\
&\qquad\big(m\in\mathbb{N}_0,t\in[0,T_*^0],N\in\mathbb{N}\big),
\end{aligned}$$

i.e., as consequence of (1.14), (2.11), the restriction of (2.19) to $\mathbb{R}^d\times[0,T_*^0]$ and (2.21) we obtain

$$\lim_{N\to\infty}\big\|\rho_N(\cdot,T_*^0)-\rho(\cdot,T_*^0)\big\|_{(m)}=0\qquad(m\in\mathbb{N}_0).$$

Therefore, the existence of some $N_1 \in \mathbb{N}$ follows such that C_5 chosen in (3.19) also satisfies

$$\begin{aligned} &C_1(2m_0)C_5 > \tfrac{3}{2}\|\rho_N(\cdot,T_*^0)\|_{(2m_0)} \\ \|\rho_N(\cdot,T_*^0)\|_{(2m_0)}^2 &\Big(1+\frac{C(d,m_0,1+d/2)C_1(2m_0)C_5}{C_2(m_0)}\Big) \le \tfrac{3}{5}C_5^2 \end{aligned} \qquad (N \ge N_1) \qquad (3.43)$$

where m_0, $C(d,m_0,1+d/2)$, $C_1(2m_0)$ and $C_2(m_0)$ are defined by (3.17), (1.12), (1.14) and (3.18), respectively. Analogously to (3.20), we then can define

$$T_N^1 = \inf\Big\{t \in [T_*^0,T] :\ \|\rho_N(\cdot,t)\|_{(2m_0)} > C_1(2m_0)C_5\Big\} \wedge T \qquad (N \ge N_1).$$

Now, we may repeat the considerations between (3.20) and (3.27) word for word, where however the time interval $[0,T_N]$ has to be replaced by $[T_*^0,T_N^1]$. In particular, similarly as in (3.25) we observe

$$\begin{aligned} &\|\rho_N(\cdot,t)\|_{(2m_0)}^2 + C_6\|\rho_N(\cdot,t)*\phi_N^r\|_{(2m_0)}^2 \\ &\quad\le C_1(2m_0)^2\Big(\|\rho_N(\cdot,T_*^0)\|_{(2m_0)}^2 + C_6\|\rho_N(\cdot,T_*^0)*\phi_N^r\|_{(2m_0)}^2\Big) \\ &\qquad + C_7\int_{T_*^0}^t ds\ \big(\|\rho_N(\cdot,s)\|_{(2m_0)}^2 + C_6\|\rho_N(\cdot,s)*\phi_N^r\|_{(2m_0)}^2\big)^2 \\ &\quad\le \tfrac{3}{5}C_1(2m_0)^2C_5^2 + C_7(t-T_*^0)(1+C_6)^2C_1(2m_0)^4C_5^4 \\ &\quad\ \big(t\in[T_*^0,T_N^1], N \ge N_1\big) \end{aligned} \qquad (3.44)$$

where in the last line we use (2.5), (3.21) and (3.43). Hence, for

$$T_*^1 = \Big(T_*^0 + \tfrac{3}{20}\big(C_7(1+C_6)^2C_1(2m_0)^2C_5^2\big)^{-1}\Big) \wedge T \qquad (3.45)$$

we deduce $T_*^1 \le T_N^1$ $(N \ge N_1)$ since (3.44) yields

$$\sup_{\substack{T_*^0\le t\le T_*^1\wedge T_N^1\\ N\ge N_1}} \Big(\|\rho_N(\cdot,t)\|_{(2m_0)}^2 + C_6\|\rho_N(\cdot,t)*\phi_N^r\|_{(2m_0)}^2\Big) \le \tfrac{3}{4}C_1(2m_0)^2C_5^2. \qquad (3.46)$$

Consequently, when applied in $[T_*^0,T_*^1]$ the calculations between (3.29) and (3.42) may be employed to complete the proof of the restriction of Theorem 1 to $\mathbb{R}^d \times [0,T_*^1]$.

If necessary, i.e., if $T_*^1 < T$, the arguments after (3.42) until (3.46) may be utilized in an iteration procedure to introduce successively

$$\begin{aligned} &N_2 \ge N_1, \quad T_N^2 > T_*^1, \quad T_*^2 > T_*^1 \\ &N_3 \ge N_2, \quad T_N^3 > T_*^2, \quad T_*^3 > T_*^2 \\ &\qquad\qquad\vdots \end{aligned}$$

until finally $T_*^q \ge T$ for some $q \in \mathbb{N}$ holds. Relation (3.45) and its respective modifications for T_*^2, T_*^3,... show that

$$\inf\Big\{T_*^{l+1} - T_*^l :\ l \in \mathbb{N}_0,\ T_*^{l+1} < T\Big\} > 0.$$

Hence, the iteration procedure indeed will terminate. We note that as consequence of (2.10) and (2.11) we finally can get rid of restrictions like $N \geq N_1$ used in calculations after (3.43). Consequently, the proof of the restriction of Theorem 1 to $\mathbb{R}^d \times [0,T]$, where T has been chosen arbitrarily at the beginning of this subsection, is finished.

3.2 Proof of Theorem 2. As indicated at the end of the preceding subsection, the proofs of Theorems 1 and 2 are not independent. However, for convenience we may suppose in this subsection that Theorem 1 has been verified completely for any $T > 0$.

By positivity of ρ (cf. (2.21)) the partial differential equations described by (2.15) - (2.18) are uniformly parabolic for any $r \in \mathbb{N}$. Therefore, the unique existence of $\rho_{\{r\}}$ $(r \in \mathbb{N})$ and the smoothness (2.14) can be proved by induction on r and standard techniques for linear parabolic equations, which can be found, e.g., in [5] or [8]. To start the induction the regularity (2.20) of $\rho = \rho_{\{0\}}$ has to be employed.

To demonstrate (2.19) we shall derive for any fixed $L \in \mathbb{N}_0$ a suitable description of the dynamics of $\rho_N - \rho - \sum_{r=1}^{L} \theta_N^{-2r} \rho_{\{r\}}$. In order that our calculations get as clear as possible we shall omit the arguments x and t of the respective functions. Additionally, we define $\rho_{\{0\}} = \rho$. As consequence of (1.1), (1.5), (2.15) - (2.17) and Lemma 1 we then obtain

$$
\begin{aligned}
&\partial_t \Bigg(\rho_N - \sum_{k=0}^{L} \theta_N^{-2k} \rho_{\{k\}} \Bigg) \\
&\quad = \tfrac{1}{2} \Delta \Bigg(\rho_N - \sum_{k=0}^{L} \theta_N^{-2k} \rho_{\{k\}} \Bigg) + \nabla \cdot (\rho_N \nabla (\rho_N * \phi_N)) - \nabla \cdot (\rho_{\{0\}} \nabla \rho_{\{0\}}) \\
&\qquad - \sum_{k=1}^{L} \theta_N^{-2k} \Big(\nabla \cdot \big(\rho_{\{k\}} \nabla \rho_{\{0\}} + \rho_{\{0\}} \nabla \rho_{\{k\}} \big) + \mathcal{G}_k \big(\rho_{\{0\}}, \ldots, \rho_{\{k-1\}} \big) \Big) \\
&\quad = \tfrac{1}{2} \Delta \Bigg(\rho_N - \sum_{k=0}^{L} \theta_N^{-2k} \rho_{\{k\}} \Bigg) + \nabla \cdot \Bigg(\Bigg(\rho_N - \sum_{k=0}^{L} \theta_N^{-2k} \rho_{\{k\}} \Bigg) \nabla (\rho_N * \phi_N) \Bigg) \\
&\qquad + \nabla \cdot \Bigg(\Bigg(\sum_{k=0}^{L} \theta_N^{-2k} \rho_{\{k\}} \Bigg) \\
&\qquad\qquad \nabla \Bigg(\rho_N * \phi_N - \sum_{l=0}^{L} \theta_N^{-2l} \sum_{\substack{0 \leq l_1, \ldots, l_d \leq 2l \\ l_1 + \ldots + l_d = 2l}} \sigma^*(l_1, \ldots, l_d; \phi_1) \frac{\partial^{2l}}{\partial_1^{l_1} \cdots \partial_d^{l_d}} \rho_N \Bigg) \Bigg) \\
&\qquad + \nabla \cdot \Bigg(\Bigg(\sum_{k=0}^{L} \theta_N^{-2k} \rho_{\{k\}} \Bigg) \\
&\qquad\qquad \nabla \Bigg(\sum_{l=0}^{L} \theta_N^{-2l} \sum_{\substack{0 \leq l_1, \ldots, l_d \leq 2l \\ l_1 + \ldots + l_d = 2l}} \sigma^*(l_1, \ldots) \frac{\partial^{2l}}{\partial_1^{l_1} \cdots \partial_d^{l_d}} \Bigg(\rho_N - \sum_{r=0}^{L-l} \theta_N^{-2r} \rho_{\{r\}} \Bigg) \Bigg) \Bigg) \\
&\qquad - \nabla \cdot \Bigg(\Bigg(\sum_{k=0}^{L} \theta_N^{-2k} \rho_{\{k\}} \Bigg)
\end{aligned}
\tag{3.47}
$$

$$
\begin{aligned}
&\nabla\Bigg(\sum_{l=0}^{L}\theta_N^{-2l}\sum_{\substack{0\le l_1,\dots,l_d\le 2l\\ l_1+\dots+l_d=2l}}\sigma^*(l_1,\dots)\frac{\partial^{2l}}{\partial_1^{l_1}\cdots\partial_d^{l_d}}\Bigg(\sum_{r=L-l+1}^{L}\theta_N^{-2r}\rho_{\{r\}}\Bigg)\Bigg)\Bigg)\\
&+\Bigg(\sum_{k=0}^{L}\theta_N^{-2k}\sum_{p=0}^{k}\sum_{q=0}^{k-p}\sum_{\substack{0\le l_1,\dots,l_d\le 2(k-p-q)\\ l_1+\dots+l_d=2(k-p-q)}}\sigma^*(\dots)\nabla\cdot\Bigg(\rho_{\{p\}}\nabla\frac{\partial^{2(k-p-q)}}{\partial_1^{l_1}\cdots\partial_d^{l_d}}\rho_{\{q\}}\Bigg)\\
&-\nabla\cdot\big(\rho_{\{0\}}\nabla\rho_{\{0\}}\big)\\
&-\sum_{k=1}^{L}\theta_N^{-2k}\Big(\nabla\cdot\big(\rho_{\{k\}}\nabla\rho_{\{0\}}+\rho_{\{0\}}\nabla\rho_{\{k\}}\big)+\mathcal{G}_k\big(\rho_{\{0\}},\dots,\rho_{\{k-1\}}\big)\Big)\Bigg)\\
&+\sum_{\substack{k,l,r=0,\dots,L\\ k+l+r>L}}\theta_N^{-2(k+l+r)}\sum_{\substack{0\le l_1,\dots,l_d\le 2l\\ l_1+\dots+l_d=2l}}\sigma^*(l_1,\dots;\phi_1)\nabla\cdot\Bigg(\rho_{\{k\}}\nabla\frac{\partial^{2l}}{\partial_1^{l_1}\cdots\partial_d^{l_d}}\rho_{\{r\}}\Bigg)
\end{aligned}
$$

for $N\in\mathbb{N}$. To obtain a condensed version of (3.47) we define

$$
f_{N,L}=\sum_{k=0}^{L}\theta_N^{-2k}\rho_{\{k\}} \tag{3.48}
$$

$$
g_{N,L}=\rho_N*\phi_N-\sum_{l=0}^{L}\theta_N^{-2l}\sum_{\substack{0\le l_1,\dots,l_d\le 2l\\ l_1+\dots+l_d=2l}}\sigma^*(l_1,\dots,l_d;\phi_1)\frac{\partial^{2l}}{\partial_1^{l_1}\cdots\partial_d^{l_d}}\rho_N \tag{3.49}
$$

$$
\begin{aligned}
h_{N,L}=-\nabla\cdot\Bigg(f_{N,L}\nabla\Bigg(\sum_{l=0}^{L}\theta_N^{-2l}&\\
\sum_{\substack{0\le l_1,\dots,l_d\le 2l\\ l_1+\dots+l_d=2l}}\sigma^*(l_1,\dots)&\frac{\partial^{2l}}{\partial_1^{l_1}\cdots\partial_d^{l_d}}\Bigg(\sum_{r=L-l+1}^{L}\theta_N^{-2r}\rho_{\{r\}}\Bigg)\Bigg)\Bigg)\\
+\sum_{\substack{k,l,r=0,\dots,L\\ k+l+r>L}}\theta_N^{-2(k+l+r)}\sum_{\substack{0\le l_1,\dots,l_d\le 2l\\ l_1+\dots+l_d=2l}}&\sigma^*(l_1,\dots)\nabla\cdot\Bigg(\rho_{\{k\}}\nabla\frac{\partial^{2l}}{\partial_1^{l_1}\cdots\partial_d^{l_d}}\rho_{\{r\}}\Bigg)
\end{aligned} \tag{3.50}
$$

$$
F_{N,L}=\theta_N^{2L+2}\big(\nabla\cdot(f_{N,L}\nabla g_{N,L})+h_{N,L}\big). \tag{3.51}
$$

Next, we introduce with

$$
\mathcal{L}_{N,L}f=\nabla\cdot\big((\tfrac{1}{2}+f_{N,L})\nabla f\big)\qquad\big(f\in C_b^2(\mathbb{R}^d);\ N,L\in\mathbb{N}_0\big) \tag{3.52}
$$

a family of second order differential operators depending on time. We also note that by (2.17) the sixth contribution to the right side of (3.47) vanishes. Now, with

$$
\rho_{N,r}^{\delta}=\theta_N^{2r+2}\Bigg(\rho_N-\sum_{k=0}^{r}\theta_N^{-2k}\rho_{\{k\}}\Bigg)\qquad(r,N\in\mathbb{N}_0) \tag{3.53}
$$

relation (3.47) may be written as

$$\begin{aligned}\partial_t \rho^\delta_{N,L} &= \mathcal{L}_{N,L}\rho^\delta_{N,L} + \nabla\cdot\big(\rho^\delta_{N,L}\nabla(\rho_N * \phi_N)\big)\\ &\quad + \sum_{l=1}^{L} \sum_{\substack{0\le l_1,\dots,l_d\le 2l\\ l_1+\dots+l_d=2l}} \sigma^*(l_1,\dots,l_d;\phi_1)\nabla\cdot\left(f_{N,L}\nabla\frac{\partial^{2l}}{\partial_1^{l_1}\cdots\partial_d^{l_d}}\rho^\delta_{N,L-l}\right)\\ &\quad + F_{N,L}\end{aligned} \tag{3.54}$$

for all $N, L \in \mathbb{N}_0$. By differentiating both sides of (3.54) we immediately obtain

$$\begin{aligned}&\partial_t \frac{\partial^{p_1+\dots+p_d}}{\partial_1^{p_1}\cdots\partial_d^{p_d}}\rho^\delta_{N,L}\\ &\quad = \mathcal{L}_{N,L}\Big(\frac{\partial^{p_1+\dots+p_d}}{\partial_1^{p_1}\cdots\partial_d^{p_d}}\rho^\delta_{N,L}\Big)\\ &\qquad + \sum_{\substack{0\le q_1,\dots,q_d\le 1+p_1+\dots+p_d\\ q_1+\dots+q_d\le 1+p_1+\dots+p_d}} G^{p_1,\dots,p_d}_{N,L,q_1,\dots,q_d}\frac{\partial^{q_1+\dots+q_d}}{\partial_1^{q_1}\cdots\partial_d^{q_d}}\rho^\delta_{N,L}\\ &\qquad + \sum_{l=1}^{L}\sum_{\substack{0\le q_1,\dots,q_d\le 2+2l+p_1+\dots+p_d\\ q_1+\dots+q_d\le 2+2l+p_1+\dots+p_d}} H^{p_1,\dots,p_d}_{N,L,l,q_1,\dots,q_d}\frac{\partial^{q_1+\dots+q_d}}{\partial_1^{q_1}\cdots\partial_d^{q_d}}\rho^\delta_{N,L-l}\\ &\qquad + \frac{\partial^{p_1+\dots+p_d}}{\partial_1^{p_1}\cdots\partial_d^{p_d}}F_{N,L} \qquad \big(N, L, p_1, \dots, p_d \in \mathbb{N}_0\big)\end{aligned} \tag{3.55}$$

where $G^{p_1,\dots,p_d}_{N,L,q_1,\dots,q_d}$ and $H^{p_1,\dots,p_d}_{N,L,l,q_1,\dots,q_d}$ linearly depend on $\rho, \rho_{\{1\}}, \dots, \rho_{\{L\}}, \rho_N * \phi_N$ and their partial derivatives of order $\le 2 + p_1 + \dots + p_d$, such that the coefficients are proportional to θ_N^{-2k} $(k = 0, 1, \dots, L)$. By (1.1), (1.5) and (2.15) the functions $\rho^\delta_{N,L}$ and their partial derivatives vanish at $t = 0$, i.e.,

$$\frac{\partial^{p_1+\dots+p_d}}{\partial_1^{p_1}\cdots\partial_d^{p_d}}\rho^\delta_{N,L}(\cdot,0) = 0 \qquad (N, L, p_1, \dots, p_d \in \mathbb{N}_0). \tag{3.56}$$

For the further study of the solution of (3.55), (3.56) we need some estimates. First, (1.2) - (1.4), (2.13), (2.14) and (2.20) yield

$$\begin{aligned}&\sup_{x\in\mathbb{R}^d, t\in[0,T], N\in\mathbb{N}} \left|\nabla^{\otimes m} G^{p_1,\dots,p_d}_{N,L,q_1,\dots,q_d}(x,t)\right| < \infty\\ &\Big(L, m, p_1, \dots, p_d, q_1, \dots, q_d \in \mathbb{N}_0,\ q_1 + \dots + q_d \le 1 + p_1 + \dots + p_d;\ T > 0\Big)\end{aligned} \tag{3.57}$$

and

$$\begin{aligned}&\sup_{x\in\mathbb{R}^d, t\in[0,T], N\in\mathbb{N}} \left|H^{p_1,\dots,p_d}_{N,L,l,q_1,\dots,q_d}(x,t)\right| < \infty\\ &\big(L, p_1, \dots, p_d, q_1, \dots, q_d \in \mathbb{N}_0, l = 1, \dots, L,\\ &q_1 + \dots + q_d \le 2 + 2l + p_1 + \dots + p_d,\ T > 0\big)\end{aligned} \tag{3.58}$$

Next, (2.13), (2.14), (2.20), (3.48) - (3.51) and Lemma 1 imply

$$\sup_{x\in\mathbb{R}^d,t\in[0,T],N\in\mathbb{N}} \Big|\frac{\partial^{p_1+\ldots+p_d}}{\partial_1^{p_1}\cdots\partial_d^{p_d}} F_{N,L}(x,t)\Big| < \infty \quad \big(L,p_1,...,p_d\in\mathbb{N}_0, T>0\big). \tag{3.59}$$

Both in (3.54) and (3.55) the principal part is determined by the time dependent partial differential operator $\mathcal{L}_{N,L}$, which for any fixed L has a bounded, smooth coefficient $\frac{1}{2}+f_{N,L}$ and is uniformly positive definite uniformly for N sufficiently large. More precisely, (1.4), (2.14), (2.20) and (3.48) yield

$$\sup_{x\in\mathbb{R}^d,t\in[0,T],N\in\mathbb{N}} \big|\nabla^{\otimes m}\partial_t^r f_{N,L}(x,t)\big| < \infty \qquad (L,m,r\in\mathbb{N}_0, T>0). \tag{3.60}$$

Using additionally (2.21) we obtain for any L and T the existence of some $N_0(L,T)\in\mathbb{N}$ such that

$$\inf\Big\{\tfrac{1}{2}+f_{N,L}(x,t) : x\in\mathbb{R}^d, t\in[0,T], N>N_0(L,T)\Big\} \geq \tfrac{1}{4} \quad (L\in\mathbb{N}_0, T>0). \tag{3.61}$$

For our further calculations in the proof of Theorem 2 we now choose some fixed but arbitrary $T_1>0$, and then restrict our considerations to the time interval $[0,T_1]$. By (2.13), (2.14), and (2.20) we then may suppose that $N\geq N_0(L,T_1)$ $(L\in\mathbb{N}_0)$.

As consequence of (3.60) and (3.61) the operator $\mathcal{L}_{N,L}-\partial_t$ has a fundamental solution

$$\Gamma_{N,L} : \mathbb{R}^{2d}\times\big\{(s,t)\in[0,T_1]^2 : 0\leq s<t\leq T_1\big\}\to\mathbb{R}$$

with a Gaussian upper bound, i.e.,

$$\begin{gathered} 0\leq \Gamma_{N,L}(x,y;s,t)\leq \frac{C_{15}}{(t-s)^{d/2}}\exp\Big(-C_{16}\frac{(x-y)^2}{t-s}\Big) \\ \Big(x,y\in\mathbb{R}^d, 0\leq s<t\leq T_1, L\in\mathbb{N}_0, N\geq N_0(L,T_1)\Big) \end{gathered} \tag{3.62}$$

where the constants $C_{15}=C_{15}(L,T_1)$ and $C_{16}=C_{16}(L,T_1)$ may depend on L and T_1, however, are independent of $N=N_0(L,T_1), N_0(L,T_1)+1,\ldots$ (cf. [4]). As supplement of (3.62) also an estimate

$$\begin{gathered} \big|\nabla_x\Gamma_{N,L}(x,y;s,t)\big| + \big|\nabla_y\Gamma_{N,L}(\ldots)\big| \leq \frac{C_{17}}{(t-s)^{(d+1)/2}}\exp\Big(-C_{18}\frac{(x-y)^2}{t-s}\Big) \\ \Big(x,y\in\mathbb{R}^d, 0\leq s<t\leq T_1, L\in\mathbb{N}_0, N\geq N_0(L,T_1)\Big) \end{gathered} \tag{3.63}$$

for the gradients of $\Gamma_{N,L}$ with respect to the spatial variables holds. In the literature we only could find the derivation of (3.63) for fixed $N\geq N_0(L,T_1)$ (cf. [5: Chapter 1]). Hence, a proof that it indeed holds uniformly in $N\geq N_0(L,T_1)$ is given in Appendix B. As crucial prerequisite for that result the fact that (3.60) and (3.61) hold uniformly in $N\geq N_0(L,T_1)$ is needed.

Using the fundamental solution $\Gamma_{N,L}$ of $\mathcal{L}_{N,L} - \partial_t$ equations (3.55) may be written as

$$
\begin{aligned}
&\frac{\partial^{p_1+\ldots+p_d}}{\partial_1^{p_1}\cdots\partial_d^{p_d}}\,\rho^\delta_{N,L}(x,t) \\
&\quad = \int_{\mathbb{R}^d} dz\ \Gamma_{N,L}(z,x;s,t)\frac{\partial^{p_1+\ldots+p_d}}{\partial_1^{p_1}\cdots\partial_d^{p_d}}\rho^\delta_{N,L}(z,s) \\
&\qquad + \int_s^t du \int_{\mathbb{R}^d} dz\ \Gamma_{N,L}(z,x;u,t) \\
&\qquad \times \Bigg(\sum_{\substack{0\le q_1,\ldots,q_d\le 1+p_1+\ldots+p_d \\ q_1+\ldots+q_d\le 1+p_1+\ldots+p_d}} G^{p_1,\ldots,p_d}_{N,L,q_1,\ldots,q_d}(z,u)\frac{\partial^{q_1+\ldots+q_d}}{\partial_1^{q_1}\cdots\partial_d^{q_d}}\rho^\delta_{N,L}(z,u) \\
&\qquad + \sum_{l=1}^{L}\sum_{\substack{0\le q_1,\ldots,q_d\le 2+2l+p_1+\ldots+p_d \\ q_1+\ldots+q_d\le 2+2l+p_1+\ldots+p_d}} H^{p_1,\ldots,p_d}_{N,L,l,q_1,\ldots,q_d}(z,u)\frac{\partial^{q_1+\ldots+q_d}}{\partial_1^{q_1}\cdots\partial_d^{q_d}}\rho^\delta_{N,L-l}(z,u) \\
&\qquad + \frac{\partial^{p_1+\ldots+p_d}}{\partial_1^{p_1}\cdots\partial_d^{p_d}}F_{N,L}(z,u)\Bigg) \\
&\qquad \Big(x\in\mathbb{R}^d, 0\le s<t\le T_1;\ L,p_1,\ldots,p_d\in\mathbb{N}_0, N\ge N_0(L,T_1)\Big).
\end{aligned}
\tag{3.64}
$$

To obtain upper bounds for $\rho^\delta_{N,L}$ and its partial derivatives we shall employ (3.64) as basic relation of some induction procedure with respect to $p = p_1+\ldots+p_d$ and L. First, the order of the spatial derivatives of $\rho^\delta_{N,L}$ in the integrand on the right side of (3.64) may be reduced by 1 by performing integration by parts. As consequence of estimates (3.57) - (3.59), (3.62) and (3.63) we then get

$$
\begin{aligned}
&\|\nabla^{\otimes p}\rho^\delta_{N,L}(\cdot,t)\|_\infty \\
&\quad \le C_{19}(L,p)\|\nabla^{\otimes p}\rho^\delta_{N,L}(\cdot,s)\|_\infty \\
&\qquad + C_{20}(L,p)\int_s^t du\ \frac{1}{\sqrt{t-u}}\int_{\mathbb{R}^d} d\zeta\ \frac{1}{(t-u)^{d/2}}\exp\Big(-C_{21}(L)\frac{\zeta^2}{t-u}\Big) \\
&\qquad \times \Bigg(\sum_{q=0}^{p}\|\nabla^{\otimes q}\rho^\delta_{N,L}(\cdot,u)\|_\infty + \sum_{l=1}^{L}\sum_{q=0}^{2+2l+p}\|\nabla^{\otimes q}\rho^\delta_{N,L-l}(\cdot,u)\|_\infty + 1\Bigg) \\
&\quad \le C_{19}(L,p)\|\nabla^{\otimes p}\rho^\delta_{N,L}(\cdot,s)\|_\infty + C_{22}(L,p)\int_s^t du\ \frac{1}{\sqrt{t-u}} \\
&\qquad \times \Bigg(\sum_{q=0}^{p}\|\nabla^{\otimes q}\rho^\delta_{N,L}(\cdot,u)\|_\infty + \sum_{l=1}^{L}\sum_{q=0}^{2+2l+p}\|\nabla^{\otimes q}\rho^\delta_{N,L-l}(\cdot,u)\|_\infty + 1\Bigg) \\
&\qquad \Big(0\le s<t\le T_1,\ L,p\in\mathbb{N}_0, N\ge N_0(L,T_1)\Big).
\end{aligned}
\tag{3.65}
$$

To start the induction we choose $L=p=s=0$. Then, by (3.56) relation (3.65) turns into

$$
\|\rho^\delta_{N,0}(\cdot,t)\|_\infty \le C + C_{22}(0,0)\int_0^t du\ \frac{1}{\sqrt{t-u}}\|\rho^\delta_{N,0}(\cdot,u)\|_\infty \tag{3.66}
$$

for $0 \le t \le T_1$ and $N \ge N_0(0,T_1)$, i.e., we may conclude

$$\sup_{\substack{u \le T_1 \wedge (4C_{22}(0,0))^{-2} \\ N \ge N_0(0,T_1)}} \|\rho^\delta_{N,0}(\cdot,u)\|_\infty < \infty. \tag{3.67}$$

Hence, (2.19) for $L = m = k = 0$ and $T \le T_1 \wedge (4C_{22}(0,0))^{-2}$ is proved. To deduce (2.19) for $L = m = k = 0$ and any $T \le T_1$ we then obviously have to repeat the arguments leading to (3.66) and (3.67) successively in the intervals

$$\begin{aligned}&\big[T_1 \wedge (4C_{22}(0,0))^{-2}, T_1 \wedge 2(4C_{22}(0,0))^{-2}\big]\\ &\big[T_1 \wedge 2(4C_{22}(0,0))^{-2}, T_1 \wedge 3(4C_{22}(0,0))^{-2}\big]\\ &\quad\vdots\end{aligned}$$

Suppose now that (2.19) for $L = k = 0$, $m = p$ and $T = T_1$ is shown. Then, by a further application of (3.65) an analogue of (3.66) for

$$\begin{aligned}&\Big\| \frac{\partial^{p_1+\ldots+p_d}}{\partial_1^{p_1}\cdots\partial_d^{p_d}} \rho^\delta_{N,0}(\cdot,t) \Big\|_\infty \\ &\big(p_1,\ldots,p_d = 0,\ldots,p+1;\ p_1+\ldots+p_d = p+1, t \in [0,T_1]\big)\end{aligned}$$

may be derived. By arguments like those in the preceding paragraph we next get (2.19) for $L = k = 0$, $m = p+1$ and $T = T_1$. Consequently, by induction on p we obtain (2.19) for $L = k = 0$, $T = T_1$ and any $m \in \mathbb{N}_0$.

Now, assume that (2.19) for $L = L_1$, $k = 0$, $T = T_1$ and any $m \in \mathbb{N}_0$ is verified. Then, (3.65) may be applied once more to yield successively relations similar to (3.66) for the L^∞-norms of ρ^δ_{N,L_1+1} and its partial derivatives in certain time intervals exhausting $[0,T_1]$. These relations finally imply (2.19) for $L = L_1 + 1$, $k = 0$, $T = T_1$ and any $m \in \mathbb{N}_0$, i.e., by another induction procedure any $L \in \mathbb{N}_0$ is handled.

We still have to study the cases $k > 0$. However, since by (3.54) and (3.55) time derivatives of $\rho^\delta_{N,L}$ may be written in terms of spatial derivatives of $\rho^\delta_{N,L-l}$ $(l = 0,\ldots,L)$, ρ_N, ρ and $\rho_{\{r\}}$ $(r = 1,\ldots,L)$, our previous regularity results (2.13), (2.14), (2.20) and (2.19) for $k = 0$ suffice to complete the proof of Theorem 2.

3.3 Proof of Theorem 3. The calculations in Subsections 3.1 and 3.2 are independent of regularity properties of ϕ_1^r or ϕ_1. However, if (2.2)* holds instead of (2.2), we still have to justify these calculations. In particular, we have to check, if for any $T' > 0$, which from now on is fixed for the remainder of this subsection, some $N(T') \in \mathbb{N}$ may be chosen, such that for $N \ge N(T')$ there exists a unique solution ρ_N of (1.1) in $[0,T']$ satisfying (2.25) and (2.26). To construct these solutions we shall apply an approximation procedure, where a sequence of smooth kernels satisfying (2.2) converging to an arbitrary ϕ_1^r, which only satisfies (2.2)*, is involved.

Therefore, for any fixed pair (ϕ_1^r, ϕ_1) of functions satisfying (2.1), (2.3) and (2.2)* we introduce some sequence $(\phi_{1,k}^r, \phi_{1,k})$ $(k \in \mathbb{N})$ whose elements satisfy (2.1) - (2.3) and

$$\lim_{k\to\infty} \langle \phi_{1,k}, f\rangle = \langle \phi_1, f\rangle \qquad (f \in C_b(\mathbb{R}^d)). \tag{3.68}$$

This condition states weak convergence of $\phi_{1,k}$ to ϕ_1 as $k \to \infty$ in the space of probability densities on $\mathbb{R}^d$. By applying now for any fixed $k \in \mathbb{N}$ scalings (1.2) and (2.4) to $\phi_{1,k}$ and $\phi^r_{1,k}$, respectively, we obtain the kernels

$$\left.\begin{aligned} \phi^r_{N,k}(x) &= \theta^d_N \phi^r_{1,k}(\theta_N x) \\ \phi_{N,k}(x) &= \theta^d_N \phi_{1,k}(\theta_N x) \end{aligned}\right\} \qquad (x \in \mathbb{R}^d;\ k, N \in \mathbb{N}). \tag{3.69}$$

Next, a family $\rho_{N,k}$ $(k, N \in \mathbb{N})$ of solutions of (1.1) with interaction kernels $\phi_{N,k}$, which satisfy (2.10) and (2.11), i.e.,

$$\rho_{N,k} \in C^\infty_b(\mathbb{R}^d \times [0,T]) \qquad (T > 0;\ k, N \in \mathbb{N})$$

and

$$\rho_{N,k}(\cdot,t) \geq 0, \quad \int_{\mathbb{R}^d} dx\ \rho_{N,k}(x,t) = 1 \qquad (t \geq 0;\ k, N \in \mathbb{N}) \tag{3.70}$$

can be introduced. By our considerations in Remark (ii) in Section 2 these functions exist uniquely. Furthermore, the above-mentioned independence of our estimates in Subsection 3.1 leads to

$$\sup_{t \leq T^0_*,\ k,N\in\mathbb{N}} \|\rho_{N,k}(\cdot,t)\|_{(m,1)} < \infty \qquad (m \in \mathbb{N}_0) \tag{3.71}$$

$$\sup_{x\in\mathbb{R}^d,\ t \leq T^0_*,\ k,N\in\mathbb{N}} |\nabla^{\otimes m} \partial^r_t \rho_{N,k}(x,t)| < \infty \qquad (m, r \in \mathbb{N}_0) \tag{3.72}$$

where T^0_* is introduced between (3.26) and (3.27). These relations may be proved in quite the same way as the restrictions of (2.12) and (2.13) to $\mathbb{R}^d \times [0, T^0_*]$ in Subsection 3.1.

For any fixed $N \in \mathbb{N}$ the uniform regularity (3.72) and the Ascoli-Arzelà Theorem imply the existence of some subsequence $k_1 = k_1(N) < k_2 = k_2(N) < \ldots$ in $\mathbb{N}$, such that

$$\begin{gathered} \lim_{n\to\infty} \sup_{(x,t)\in K} \left|\nabla^{\otimes m}\partial^r_t \rho_{N,k_n}(x,t) - \nabla^{\otimes m}\partial^r_t \rho_N(x,t)\right| = 0 \\ \big(m, r \in \mathbb{N}_0, K \subseteq \mathbb{R}^d \times [0, T^0_*] \text{ compact }, N \in \mathbb{N}\big) \end{gathered} \tag{3.73}$$

for some $\rho_N \in C^\infty_b(\mathbb{R}^d \times [0, T^0_*])$.

By utilizing the boundedness of $\rho_{N,k}$ with respect to the norms $\|\cdot\|_{(m,1)}$ $(m \in \mathbb{N}_0)$ uniformly in $k, N \in \mathbb{N}$ (cf. (3.71)) we can extend (3.73), which states convergence locally in space, to a global convergence result. To prove this improvement we first note that by (1.1) and (3.72) any partial derivative

$$\frac{\partial^{m_1+\ldots+m_d}}{\partial^{m_1}_1 \cdots \partial^{m_d}_d} \partial^r_t \rho_{N,k} \qquad \big(m_1, \ldots, m_d, r \in \mathbb{N}_0\big)$$

may be written as sum of products of spatial derivatives of $\rho_{N,k}$ and $\rho_{N,k} * \phi_{N,k}$. Consequently, by (2.2)*, (3.69), (3.71) and (3.72) we get

$$\sup_{t\leq T^0_*,\ k,N\in\mathbb{N}} \int_{\mathbb{R}^d} dx\ (1 + |x|)\big|\nabla^{\otimes m}\partial^r_t \rho_{N,k}(x,t)\big|^2 < \infty \qquad (m, r \in \mathbb{N}_0) \tag{3.74}$$

as extension of (3.71). In a next step, (1.12), (3.73) and (3.74) imply

$$\begin{aligned}
&\lim_{n,n'\to\infty} \sup_{x\in\mathbb{R}^d, t\le T_*^0} \Big|\nabla^{\otimes m}\partial_t^r \rho_{N,k_n}(x,t) - \nabla^{\otimes m}\partial_t^r \rho_{N,k_{n'}}(x,t)\Big|^2 \\
&\quad \le C(d,m,1+[d/2])^2 \lim_{n,n'\to\infty} \sup_{t\le T_*^0} \Big\|\partial_t^r \rho_{N,k_n}(\cdot,t) - \partial_t^r \rho_{N,k_{n'}}(\cdot,t)\Big\|^2_{(m+1+[d/2])} \\
&\quad \le C\gamma^d \lim_{n,n'\to\infty} \sup_{|x|\le\gamma, t\le T_*^0} \sum_{p=0}^{m+1+[d/2]} \Big|\nabla^{\otimes p}\partial_t^r \rho_{N,k_n}(x,t) - \nabla^{\otimes p}\partial_t^r \rho_{N,k_{n'}}(x,t)\Big|^2 \\
&\qquad + \frac{C}{1+\gamma} \sup_{t\le T_*^0,\ k,M\in\mathbb{N}} \sum_{p=0}^{m+1+[d/2]} \int_{\mathbb{R}^d} dx\ (1+|x|)\big|\nabla^{\otimes p}\partial_t^r \rho_{M,k}(x,t)\big|^2 \\
&\quad \le \frac{C}{1+\gamma} \qquad \big(m,r\in\mathbb{N}_0;\ \gamma>0, N\in\mathbb{N}\big),
\end{aligned}$$

i.e., for $m,r\in\mathbb{N}_0$ and $N\in\mathbb{N}$, $\nabla^{\otimes m}\partial_t^r \rho_{N,k_q}$ $(q\in\mathbb{N})$ is a Cauchy sequence in $C_b(\mathbb{R}^d\times[0,T_*^0])$. Therefore, with (3.73) we obtain the desired global convergence, i.e.,

$$\lim_{n\to\infty} \sup_{(x,t)\in\mathbb{R}^d\times[0,T_*^0]} \Big|\nabla^{\otimes m}\partial_t^r \rho_{N,k_n}(x,t) - \nabla^{\otimes m}\partial_t^r \rho_N(x,t)\Big| = 0 \tag{3.75}$$

for $m,r\in\mathbb{N}_0$ and $N\in\mathbb{N}$. By (3.68), (3.69), (3.72) and (3.75) we immediately observe that in $[0,T_*^0]$ for any $N\in\mathbb{N}$ the function ρ_N solves (1.1) with interaction kernel ϕ_N and satisfies (2.25) with $N(T_*^0)=1$. As far as (2.26) is concerned, $\rho_N(\cdot,t)\ge 0$ $(t\in[0,T_*^0], N\in\mathbb{N})$ follows from the corresponding positivity of $\rho_{N,k}$ (cf. (3.70)) and (3.75). Next, considering

$$\mathbb{R}^d\times[0,T_*^0] \ni (x,t) \to \nabla(\rho_N(\cdot,t)*\phi_N)(x)$$

as given drift vector field, (1.1) may be interpreted as a *linear Fokker-Planck equation* with smooth coefficients. The probabilistic interpretation of such equations and the uniqueness of their solutions (cf. [6]), and the validity of (2.7) for the initial state ρ_0 of ρ_N imply that $\rho_N(\cdot,t)$ is the density of the law of the state $Y_N(t)$ of some diffusion process Y_N at time t. Consequently, ρ_N satisfies (2.26).

To verify for some fixed $N\in\mathbb{N}$ the uniqueness of ρ_N we suppose that another function ρ_N^* has the same properties. With (2.5) and (2.6) we then get, for $t\in[0,T_*^0]$,

$$\begin{aligned}
&\frac{d}{dt}\big\|\rho_N(\cdot,t)-\rho_N^*(\cdot,t)\big\|_2^2 \\
&\quad = \frac{d}{dt}\Big(\|\rho_N(\cdot,t)\|_2^2 - 2\big\langle\rho_N(\cdot,t),\rho_N^*(\cdot,t)\big\rangle + \|\rho_N^*(\cdot,t)\|_2^2\Big) \\
&\quad = \big\langle\rho_N(\cdot,t),\Delta\rho_N(\cdot,t)\big\rangle - \big\langle\Delta\rho_N(\cdot,t),\rho_N^*(\cdot,t)\big\rangle - \big\langle\rho_N(\cdot,t),\Delta\rho_N^*(\cdot,t)\big\rangle \\
&\qquad + \big\langle\rho_N^*(\cdot,t),\Delta\rho_N^*(\cdot,t)\big\rangle - 2\big\langle\nabla\rho_N(\cdot,t),\rho_N(\cdot,t)\nabla(\rho_N(\cdot,t)*\phi_N)\big\rangle \\
&\qquad + 2\big\langle\rho_N(\cdot,t)\nabla(\rho_N(\cdot,t)*\phi_N),\nabla\rho_N^*(\cdot,t)\big\rangle \\
&\qquad + 2\big\langle\nabla\rho_N(\cdot,t),\rho_N^*(\cdot,t)\nabla(\rho_N^*(\cdot,t)*\phi_N)\big\rangle
\end{aligned}$$

$$
\begin{aligned}
&\quad - 2\langle \nabla\rho^*_N(\cdot,t), \rho^*_N(\cdot,t)\nabla(\rho^*_N(\cdot,t)*\phi_N)\rangle \\
&= -\big\|\nabla(\rho_N(\cdot,t)-\rho^*_N(\cdot,t))\big\|_2^2 \\
&\quad + 2\Big\langle \nabla\rho^*_N(\cdot,t)-\nabla\rho_N(\cdot,t), \rho_N(\cdot,t)\nabla(\rho_N(\cdot,t)*\phi_N)\Big\rangle \\
&\quad + 2\Big\langle \nabla\rho_N(\cdot,t)-\nabla\rho^*_N(\cdot,t), \rho^*_N(\cdot,t)\nabla(\rho^*_N(\cdot,t)*\phi_N)\Big\rangle \\
&= -\big\|\nabla(\rho_N(\cdot,t)-\rho^*_N(\cdot,t))\big\|_2^2 \\
&\quad + 2\Big\langle \nabla\rho^*_N(\cdot,t)-\nabla\rho_N(\cdot,t), \rho_N(\cdot,t)\Big(\nabla(\rho_N(\cdot,t)*\phi_N)-\nabla(\rho^*_N(\cdot,t)*\phi_N)\Big)\Big\rangle \\
&\quad + 2\Big\langle \nabla\rho^*_N(\cdot,t)-\nabla\rho_N(\cdot,t), \big(\rho_N(\cdot,t)-\rho^*_N(\cdot,t)\big)\nabla(\rho^*_N(\cdot,t)*\phi_N)\Big\rangle \\
&\le C\big\|\rho_N(\cdot,t)-\rho^*_N(\cdot,t)\big\|_2^2 + C_{23}\big\|\nabla(\rho_N(\cdot,t)*\phi_N)-\nabla(\rho^*_N(\cdot,t)*\phi_N)\big\|_2^2 \\
&\le C\big\|\rho_N(\cdot,t)-\rho^*_N(\cdot,t)\big\|_2^2 + C_{23}\big\|\nabla(\rho_N(\cdot,t)*\phi^r_N)-\nabla(\rho^*_N(\cdot,t)*\phi^r_N)\big\|_2^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\big\|\rho_N(\cdot,t)*\phi^r_N-\rho^*_N(\cdot,t)*\phi^r_N\big\|_2^2 \\
&= -\big\|\nabla(\rho_N(\cdot,t)*\phi^r_N)-\nabla(\rho^*_N(\cdot,t)*\phi^r_N)\big\|_2^2 - 2\langle\rho_N(\cdot,t), |\nabla(\rho_N(\cdot,t)*\phi_N)|^2\rangle \\
&\quad + 2\Big\langle \rho_N(\cdot,t)\nabla(\rho_N(\cdot,t)*\phi_N), \nabla(\rho^*_N(\cdot,t)*\phi_N)\Big\rangle \\
&\quad + 2\Big\langle \nabla(\rho_N(\cdot,t)*\phi_N), \rho^*_N(\cdot,t)\nabla(\rho^*_N(\cdot,t)*\phi_N)\Big\rangle \\
&\quad - 2\langle\rho^*_N(\cdot,t), |\nabla(\rho^*_N(\cdot,t)*\phi_N)|^2\rangle \\
&= -\big\|\nabla(\rho_N(\cdot,t)*\phi^r_N)-\nabla(\rho^*_N(\cdot,t)*\phi^r_N)\big\|_2^2 \\
&\quad - 2\Big\langle \rho_N(\cdot,t), |\nabla(\rho_N(\cdot,t)*\phi_N)-\nabla(\rho^*_N(\cdot,t)*\phi_N)|^2\Big\rangle \\
&\quad + 2\Big\langle \rho_N(\cdot,t)-\rho^*_N(\cdot,t), \nabla(\rho^*_N(\cdot,t)*\phi_N)\Big(\nabla(\rho^*_N(\cdot,t)*\phi_N)-\nabla(\rho_N(\cdot,t)*\phi_N)\Big)\Big\rangle \\
&\le -\tfrac{1}{2}\big\|\nabla(\rho_N(\cdot,t)*\phi^r_N)-\nabla(\rho^*_N(\cdot,t)*\phi^r_N)\big\|_2^2 + C\big\|\rho_N(\cdot,t)-\rho^*_N(\cdot,t)\big\|_2^2.
\end{aligned}
$$

Obviously, these relations yield

$$
\begin{aligned}
\frac{d}{dt}\Big(\big\|\rho_N(\cdot,t)-\rho^*_N(\cdot,t)\big\|_2^2 + 2C_{23}\big\|\rho_N(\cdot,t)*\phi^r_N-\rho^*_N(\cdot,t)*\phi^r_N\big\|_2^2\Big)& \\
\le C\big\|\rho_N(\cdot,t)-\rho^*_N(\cdot,t)\big\|_2^2 \qquad (t\in[0,T^0_*]),&
\end{aligned}
$$

and therefore Gronwall's Lemma implies $\|\rho_N(\cdot,t)-\rho^*_N(\cdot,t)\|_2 = 0 \quad (t\in[0,T^0_*])$ which proves the desired uniqueness of ρ_N.

By (3.71), (3.72) and (3.75) relations (2.27), (2.28) hold for the sequence $\rho_N \quad (N\in\mathbb{N})$ if $T'\le T^0_*$. In particular, in that time interval the prerequisites for performing the calculations of Subsection 3.2 are given now, i.e., we also may deduce (2.29) in the case $T'\le T^0_*$. Note that in this situation we may choose $N(T')=1$.

If $T' > T^0_*$, we have to utilize the arguments from the end of the proof of Theorem 1. More precisely, first some $N_1\in\mathbb{N}$ may be determined, such that (3.43) holds. Next, using the kernels $\phi_{1,k}^r, \phi_{1,k}$ $(k \in \mathbb{N})$ introduced above we may continue to work with solutions $\rho_{N,k}^1$ $(k \in \mathbb{N}, N \geq N_1)$ of (1.1) in $[T_*^0, \infty)$ with initial conditions $\rho_{N,k}^1(\cdot, T_*^0) = \rho_N(\cdot, T_*^0)$ $(k \in \mathbb{N}, N \geq N_1)$. Then, just as between (3.43) and (3.46) some T_*^1 exists with

$$\sup_{\substack{T_*^0 \leq t \leq T_*^1 \\ k\in\mathbb{N}, N \geq N_1}} \Big(\|\rho_{N,k}^1(\cdot,t)\|_{(2m_0)}^2 + C_6(T')\|\rho_{N,k}^1(\cdot,t) * \phi_{N,k}^r\|_{(2m_0)}^2 \Big) \leq \tfrac{3}{4} C_1(2m_0)^2 C_5(T')^2$$

in analogy to (3.46). Note that we have emphasized here that the constants C_5 and C_6 may depend on T'. Now, additional estimates for Sobolev norms with higher orders for $\rho_{N,k}^1(\cdot,t)$ $(t \in [T_*^0, T_*^1], k \in \mathbb{N}, N \geq N_1)$ may be obtained as in (3.29) - (3.42). Then, for any $N \geq N_1$ the limit $k \to \infty$ may be performed to obtain the extension to $[T_*^0, T_*^1]$ of the solution ρ_N of (1.1) pertaining to (ϕ_1^r, ϕ_1). In particular, we now get Theorem 3 for $T' \leq T_*^1$.

Similarly as in the proof of Theorem 1 these arguments may be iterated as long as necessary, i.e., if for some q we get $T_*^q \geq T'$. The associated N_q is the desired $N(T')$.

4. Appendix A: Formal derivation of expansion (1.6)

In this appendix we shall demonstrate how expansion (1.6) of ρ_N, where ρ solves (1.5) and the functions $\rho_{\{k\}}$ $(k \in \mathbb{N})$ satisfy (2.15) - (2.18), can be determined by some quite straightforward formal calculations.

First, we insert (1.6) with $\rho = \rho_{\{0\}}$ as ansatz into (1.1), and then expand the convolution with ϕ_N according to Lemma 1. In particular, if for some fixed $L \in \mathbb{N}_0$ we omit any term of order $o(\theta_N^{-2L})$ as $N \to \infty$, we obtain

$$\begin{aligned}
&\partial_t \rho_N \quad \Big(\approx \sum_{k=0}^{L} \theta_N^{-2k} \partial_t \rho_{\{k\}} \Big) \\
&\quad = \tfrac{1}{2}\Delta\rho_N + \nabla\cdot\big(\rho_N \nabla(\rho_N * \phi_N)\big) \\
&\quad \approx \tfrac{1}{2}\sum_{k=0}^{L} \theta_N^{-2k}\Delta\rho_{\{k\}} + \nabla\cdot\Bigg(\Big(\sum_{k=0}^{L}\theta_N^{-2k}\rho_{\{k\}}\Big)\nabla\Big(\sum_{k=0}^{L}\theta_N^{-2k}\rho_{\{k\}} * \phi_N\Big)\Bigg) \\
&\quad \approx \tfrac{1}{2}\sum_{k=0}^{L} \theta_N^{-2k}\Delta\rho_{\{k\}} + \nabla\cdot\Bigg(\Big(\sum_{k=0}^{L}\theta_N^{-2k}\rho_{\{k\}}\Big) \\
&\qquad\qquad \nabla\Bigg(\sum_{k=0}^{L}\theta_N^{-2k}\Bigg(\sum_{l=0}^{L}\theta_N^{-2l}\sum_{\substack{0\leq l_1,\ldots,l_d\leq 2l\\ l_1+\ldots+l_d=2l}}\sigma^*(l_1,\ldots,l_d;\phi_1)\frac{\partial^{2l}}{\partial_1^{l_1}\cdots\partial_d^{l_d}}\rho_{\{k\}}\Bigg)\Bigg)\Bigg) \\
&\quad \approx \tfrac{1}{2}\sum_{k=0}^{L}\theta_N^{-2k}\Delta\rho_{\{k\}} \\
&\qquad + \sum_{k=0}^{L}\theta_N^{-2k}\sum_{p=0}^{k}\sum_{q=0}^{k-p}\sum_{\substack{0\leq l_1,\ldots,l_d\leq 2(k-p-q)\\ l_1+\ldots+l_d=2(k-p-q)}}\sigma^*(l_1,\ldots,l_d;\phi_1)\nabla\cdot\Big(\rho_{\{p\}}\nabla\frac{\partial^{2(k-p-q)}}{\partial_1^{l_1}\cdots\partial_d^{l_d}}\rho_{\{q\}}\Big).
\end{aligned}$$

Now, by comparing the coefficients of different powers of θ_N^{-2} we observe that $\rho_{\{0\}} = \rho$ solves (1.5) and that $\rho_{\{k\}}$ $(k \in \mathbb{N})$ satisfy (2.15) - (2.18).

5. Appendix B: Proof of (3.63)

Since $\mathcal{L}_{N,L}$ is in divergence form (cf. (3.52)), we get

$$\Gamma_{N,L}(x,y;s,t) = \Gamma_{N,L}(y,x;s,t)$$
$$\big(x,y \in \mathbb{R}^d, 0 \le s < t \le T_1, L \in \mathbb{N}_0, N \ge N_0(L,T_1)\big).$$

Hence, it suffices to estimate

$$\big|\nabla_y \Gamma_{N,L}(x,y;s,t)\big| \qquad \big(y \in \mathbb{R}^d, t \in (s,T_1]\big)$$

with $x \in \mathbb{R}^d, s \in [0,T_1), L \in \mathbb{N}_0$ and $N \ge N_0(L,T_1)$ being fixed in the remaining parts of this appendix.

To obtain the desired estimate (3.63) we shall apply [3: Corollary 1.2.22] where scaled modifications of $\Gamma_{N,L}$ have to be considered in order to cover all $y \in \mathbb{R}^d$ and $t \in (s,T_1]$. In particular, we define

$$\gamma^\lambda(\eta,\tau) = \Gamma_{N,L}(x, y+\lambda\eta; s, t+\lambda^2\tau) \qquad \big(\eta \in \mathbb{R}^d, \tau \in \big(\tfrac{s-t}{\lambda^2}, \tfrac{T_1-t}{\lambda^2}\big]\big) \tag{B.1}$$

where $y \in \mathbb{R}^d$ and $t \in (s,T_1]$ are also fixed from now on. Since $\Gamma_{N,L}$ is a fundamental solution of $\mathcal{L}_{N,L} - \partial_t$, we deduce

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\gamma^\lambda(\eta,\tau)\\
&\quad = \lambda^2 \frac{\partial}{\partial u}\Gamma_{N,L}(x,y+\lambda\eta;s,u)\big|_{u=t+\lambda^2\tau}\\
&\quad = \lambda^2 \nabla_z \cdot \Big(\big(\tfrac12 + f_{N,L}(z,t+\lambda^2\tau)\big)\nabla_z \Gamma_{N,L}(x,z;s,t+\lambda^2\tau)\Big)\Big|_{z=y+\lambda\eta}\\
&\quad = \nabla_\eta \cdot \Big(\big(\tfrac12 + f^\lambda_{N,L;y,t}(\eta,\tau)\big)\nabla_\eta \gamma^\lambda(\eta,\tau)\Big) \quad \big(\eta \in \mathbb{R},\ \tau \in \big(\tfrac{s-t}{\lambda^2}, \tfrac{T_1-t}{\lambda^2}\big]\big)
\end{aligned} \tag{B.2}$$

where

$$f^\lambda_{N,L;y,t}(\eta,\tau) = f_{N,L}(y+\lambda\eta, t+\lambda^2\tau). \tag{B.3}$$

In our further calculations we shall use the scaling parameter $\lambda = \sqrt{(t-s)/8} = \lambda(s,t)$. As consequence of (3.60) and (3.61) the associated functions $f^{\lambda(s,t)}_{N,L;y,t}$ satisfy

$$\sup\left\{ \big|\nabla_\eta^{\otimes m}\partial_\tau^r f^{\lambda(s,t)}_{N,L;y,t}(\eta,\tau)\big| \;\middle|\; \begin{array}{l} y,\eta \in \mathbb{R}^d, N \ge N_0(L,T_1) \\ 0 \le s < t \le T_1,\, \frac{s-t}{\lambda(s,t)^2} < \tau \le \frac{T_1-t}{\lambda(s,t)^2} \end{array} \right\} < \infty \tag{B.4}$$
$$(L,m,r \in \mathbb{N}_0, T_1 > 0)$$

$$\inf\left\{ \tfrac12 + f^{\lambda(s,t)}_{N,L;y,t}(\eta,\tau) \;\middle|\; \begin{array}{l} y,\eta \in \mathbb{R}^d, N \ge N_0(L,T_1) \\ 0 \le s < t \le T_1,\, \frac{s-t}{\lambda(s,t)^2} < \tau \le \frac{T_1-t}{\lambda(s,t)^2} \end{array} \right\} \ge \frac14 \tag{B.5}$$
$$(L \in \mathbb{N}_0, T_1 > 0).$$

Applying now [3: Corollary 1.2.22] to the solution $(\eta,\tau)\to\gamma^{\lambda(s,t)}(\eta,\tau)$ of (B.2) we get

$$\|\nabla\gamma^{\lambda(s,t)}\|_{L^\infty(Q_{1/2})}\le C_{CK}\|\gamma^{\lambda(s,t)}\|_{L^2(Q_2)} \tag{$B.6$}$$

where

$$Q_r=\big\{\eta\in\mathbb{R}^d:\ |\eta|<r\big\}\times\big\{\tau\in\mathbb{R}:\ -r^2<\tau\le 0\big\}.$$

As consequence of (B.4) and (B.5) for any $L\in\mathbb{N}_0$ and $T_1>0$ the constant C_{CK} is independent of $x,y\in\mathbb{R}^d, 0\le s<t\le T_1$ and $N\ge N_0(L,T_1)$, which are fixed during the present calculations.

To deduce (3.63) we now consider both sides of (B.6) in more detail. First, we observe

$$\begin{aligned}
&\|\nabla\gamma^{\lambda(s,t)}\|_{L^\infty(Q_{1/2})}\\
&\quad=\sup\Big\{\big|\nabla_\eta\Gamma_{N,L}(x,y+\lambda(s,t)\eta;s,t+\lambda(s,t)^2\tau)\big|\ |\eta|<\tfrac12,\tau\in\big(-\tfrac14,0\big]\Big\}\\
&\quad\ge\lambda(s,t)\big|\nabla_y\Gamma_{N,L}(x,y;s,t)\big|\\
&\quad=\sqrt{\tfrac{t-s}{8}}\big|\nabla_y\Gamma_{N,L}(x,y;s,t)\big|.
\end{aligned}\tag{$B.7$}$$

Next, by (3.62) we obtain

$$\begin{aligned}
&\|\gamma^{\lambda(s,t)}\|^2_{L^2(Q_2)}\\
&\quad=\int_{\{|\eta|<2\}}d\eta\int_{-4}^0 d\tau\ \Big|\Gamma_{N,L}\big(x,y+\lambda(s,t)\eta;s,t+\lambda(s,t)^2\tau\big)\Big|^2\\
&\quad\le C\int_{\{\eta\in\mathbb{R}^d:|\eta|<2\}}d\eta\int_{-4}^0 d\tau\ \frac{\exp\Big(-C'\frac{|y+\lambda(s,t)\eta-x|^2}{|t+\lambda(s,t)^2\tau-s|}\Big)}{|t+\lambda(s,t)^2\tau-s|^d}\\
&\quad=\frac{C}{\lambda(s,t)^{d+2}}\int_{\{\eta'\in\mathbb{R}^d:|\eta'|<2\lambda(s,t)\}}d\eta'\int_{-4\lambda(s,t)^2}^0 d\tau'\frac{\exp\Big(-C'\frac{|y+\eta'-x|^2}{|t+\tau'-s|}\Big)}{|t+\tau'-s|^d}\\
&\quad=\frac{C}{(t-s)^{(d+2)/2}}\int_{\{\eta'\in\mathbb{R}^d:|\eta'|<\sqrt{(t-s)/2}\}}d\eta'\int_{-(t-s)/2}^0 d\tau'\ \frac{\exp\Big(-C'\frac{|y+\eta'-x|^2}{|t+\tau'-s|}\Big)}{|t+\tau'-s|^d}\\
&\quad=I(x,y;s,t).
\end{aligned}\tag{$B.8$}$$

To estimate $I(x,y;s,t)$ we consider two cases. First, if $\sqrt{\frac{t-s}{2}}<\frac{|x-y|}{2}$, then

$$\begin{aligned}
I(x,y;s,t)&\le\frac{C}{|t-s|^{(d+2)/2}}\frac{\exp\Big(-C'\frac{|x-y|^2}{|t-s|}\Big)}{|t-s|^d}|t-s|^{d/2}|t-s|\\
&\le C\frac{\exp\Big(-C'\frac{|x-y|^2}{|t-s|}\Big)}{|t-s|^d}.
\end{aligned}\tag{$B.9$}$$

On the other hand, if $\sqrt{\frac{t-s}{2}} \geq \frac{|x-y|}{2}$, then we observe also

$$
\begin{aligned}
I(x,y;s,t) &\leq \frac{C}{|t-s|^{(d+2)/2}} \int_{\mathbb{R}^d} d\eta' \int_{-(t-s)/2}^{0} d\tau' \, \frac{\exp\Big(-C' \frac{|y+\eta'-x|^2}{|t+\tau'-s|}\Big)}{|t+\tau'-s|^d} \\
&\leq \frac{C}{|t-s|^d} \\
&\leq C \frac{\exp\Big(-C' \frac{|x-y|^2}{|t-s|}\Big)}{|t-s|^d}.
\end{aligned} \tag{B.10}
$$

Relation (3.63) now follows from (B.6) - (B.10). In particular, we need the fact that all constants herein are independent of $x, y \in \mathbb{R}^d, 0 \leq s < t \leq T_1$ and $N \geq N_0(L, T_1)$ for any $L \in \mathbb{N}_0$ and $T_1 > 0$.

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