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Blow-Up and Convergence Results for a One-Dimensional Non-Local Parabolic Problem

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Abstract. Considering a one-dimensional non-local semilinear parabolic problem, it is shown that blow-up in finite time occurs for suitable large initial conditions. The asymptotic behavior of global solutions corresponding to small initial conditions is also investigated. Their convergence in H^1-norm to a well determinated stationary solution is proved.

Keywords: *Parabolic problems, blow-up, convergence to steady states*

AMS subject classification: 35K, 35K55, 35B40

1. Introduction

We would like to consider the problem

$$\left.\begin{aligned} u_t - u'' &= -a(l(u(t))) && \text{in} \quad (0,T)\times\Omega \\ u(t,0) &= 0 && \text{on} \quad (0,T) \\ u'(t,L) &= b(l(u(t))) && \text{on} \quad (0,T) \\ u(0,\cdot) &= u_0(\cdot) && \text{in} \quad \Omega \\ l(u(t)) &\in D && \text{on} \quad (0,T) \end{aligned}\right\}. \qquad (\mathbf{P}\, l, u_0)$$

Hereafter, T is a positive real number, $\Omega = (0, L)$ is a bounded open interval of $\mathbb{R}$, a, b are numerical functions defined on some interval D of $\mathbb{R}$ and l is a continuous linear form on $L^2(\Omega)$ or on

$$V = \big\{v \in H^1(\Omega) |\ v(0) = 0\big\}.$$

We will deal first with *variational solutions*, i.e. with solutions to the problem

$$\left.\begin{aligned} u &\in L^2(0,T;V) \\ \tfrac{du}{dt} &\in L^2(0,T;V') \\ \langle u_t, \varphi\rangle + \textstyle\int_\Omega u'\varphi' dx &= -a(l(u(t)))\textstyle\int_\Omega \varphi\, dx + b(l(u(t)))\varphi(L) && \text{in}\ \ D'(0,T),\ \forall \varphi \in V \\ u(0,\cdot) &= u_0(\cdot) && \text{in}\ \ L^2(\Omega) \\ l(u(t)) &\in D && \text{on}\ (0,T) \end{aligned}\right\}.$$

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The reader is referred to [4] or [9] for all the questions regarding Sobolev spaces and variational solutions. For applications, we have for instance in mind the **M**odel **P**roblem

$$\left.\begin{array}{lll} u_t - u'' = -(\int_\Omega u(t,x)\,dx)^p & \text{in} & (0,T)\times\Omega \\ u(t,0) = 0 & \text{on} & (0,T) \\ u'(t,L) = (\int_\Omega u(t,x)\,dx)^q & \text{on} & (0,T) \\ u(0,\cdot) = u_0(\cdot) & \text{in} & \Omega \\ \int_\Omega u(t,x)\,dx \geq 0 & \text{on} & (0,T) \end{array}\right\} \qquad (\mathbf{MP}\,p,q)$$

where p and q are real numbers greater or equal to 1.

This paper is a continuation of the paper [8]. We refer to it for physical motivations which have lead us to study this class of problems and for the proof of the following known result.

Theorem 1.1. *Assume the following:*

(i) *The functions a and b are locally Lipschitz continuous on $D = R$.*

(ii) *There exist $\frac{1}{2} \leq p \in \mathbb{R}$ and $0 < c_0 \in \mathbb{R}$ with $|a(s)|+|b(s)| \leq c_0(1+|s|^p)$ $(s \in \mathbb{R})$.*

(iii) *The initial condition u_0 belongs to $L^2(\Omega)$.*

(iv) *l is a continuous linear form on $L^2(\Omega)$.*

Then problem $(\mathrm{P}\,l,u_0)$ has a unique maximal variational solution u. Moreover, if $T_{max}(u_0)$ denotes its maximal time of existence, u belongs to $C\big([0,T_{max}(u_0)),L^2(\Omega)\big)$, and if $T_{max}(u_0)$ is finite, then $\lim_{t\to T_{max}(u_0)} |u(t)|_{L^2(\Omega)} = +\infty$.

The later assertion follows in a classical way (see, for instance, [5: Theorem 4.3.4] or [13: Theorem 5.1.1]) from [8: Proposition 3.1]. Note that the above result remains valid if, instead of assumption (iv), we suppose only l defined and Lipschitz continuous on $L^2(\Omega)$.

The paper is devoted to the study of the asymptotic behavior of the solutions to problem $(\mathrm{P}\,l,u_0)$. We will first address the issue of blow-up in finite time. Considering problem $(\mathrm{MP}\,p,q)$ we note that three phenomena work against blow-up:

- the different signs of the non-linearities
- the value of $u(t)$ at the point $x=0$
- the "weak growth" of the non-local functional $l(u) = \int_\Omega u\,dx$.

Let us explain briefly this last point. Going back to problem $(\mathrm{P}\,l,u_0)$, let us assume for simplicity that $a=b$. Then the key point allowing to prove that blow-up occurs is the inequality

$$l(u(t)) \geq c\,u_1(t) \tag{1.1}$$

where u_1 denotes the first coordinate of the solution u in some spectral basis $(\varphi_k)_{k\geq 1}$ (see Section 2) and $c>0$ is some constant. If, for instance, $l(u) = \big(\int_\Omega u^2 dx\big)^{\frac{1}{2}}$, then the above inequality holds for all u in $L^2(\Omega)$ since

$$u_1 = \int_\Omega u\varphi_1 dx \leq \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \varphi_1^2 dx\right)^{\frac{1}{2}} = c^{-1} l(u)$$

by the Cauchy-Schwarz inequality. But in the case where $l(u) = \int_\Omega u\,dx$, inequality (1.1) does not hold in V as we see by taking $u = -\varphi_2$.

We point out also difficulties coming from non-local terms. There is no global Liapunov's function and the maximum principle does not hold. More precisely, we see by numerical simulations that the sign of $u(t)$ can change in Ω even if u_0 is positive. Furthermore, $u(t)$ is not necessarily bounded from below in Ω when $\int_\Omega u(t)\,dx$ tends to $+\infty$ — see Figure 1 depicting the shape of the solution u to problem $(\mathrm{MP}\,2,2)$ for t close to the blow-up time.

Figure 1

In Section 3 we introduce another kind of solutions which allows us in Section 4 to study the convergence of variational solutions toward some steady state. Finally, in Section 5 the results are discussed and some open problems are formulated.

2. Blow-up in finite time

In this section we will assume that $l \equiv 1$. More precisely, we will consider the particular problem

$$\left.\begin{aligned} u_t - u'' &= -a(\textstyle\int_\Omega u(t,x)\,dx) && \text{in} \quad (0,T)\times\Omega \\ u(t,0) &= 0 && \text{on} \quad (0,T) \\ u'(t,L) &= b(\textstyle\int_\Omega u(t,x)\,dx) && \text{on} \quad (0,T) \\ u(0,\cdot) &= u_0(\cdot) && \text{in} \quad \Omega \end{aligned}\right\}. \qquad (\mathbf{P}\,1, u_0)$$

2.1 The main result. We have the following

Theorem 2.1. *Additionally to the assumptions of Theorem* 1.1 *let us assume the following:*

(i) $\Omega = (0, L)$ *with* $L \in (0, \frac{3\pi}{10})$.

(ii) b *is non-negative, non-decreasing on* $(0, +\infty)$ *and* $\int^{+\infty} \frac{ds}{b(s)} < +\infty$.

(iii) *The initial condition u_0 is equal to $\beta\phi$ where $\beta \in \mathbb{R}$ and ϕ is a function of $L^2(\Omega)$ satisfying one of the two conditions*

(A1) *$\phi \geq \alpha$ a.e. in Ω, for some $0 < \alpha \in \mathbb{R}$*

or

(A2) *ϕ is continuous on $\overline{\Omega}$, positive on $(0, L]$, differentiable from the right at 0, $\phi(0) = 0$ and $\phi'(0^+) > 0$.*

(iv) *$a(s) \leq b(s)$ for all $s > 0$.*

Then there exists a real number β_c such that for all $\beta \geq \beta_c$ the variational solution to problem $(\mathrm{P}\,1, \beta\phi)$ blows up in finite time in L^2-norm.

We would like first to introduce some notation. Let $(\varphi_k)_{k\geq 1}$ be the Hilbertian basis of $L^2(\Omega)$ defined by

$$\left.\begin{aligned} -\varphi_k'' &= \lambda_k \varphi_k \quad \text{in } \Omega \\ \varphi_k(0) &= \varphi_k'(L) = 0 \\ \textstyle\int_\Omega \varphi_k dx &> 0, \quad |\varphi_k|_{L^2(\Omega)} = 1 \end{aligned}\right\}.$$

An easy computation shows that

$$\begin{aligned} \lambda_k &= \tfrac{\pi^2}{4L^2}(2k-1)^2 \\ \varphi_k(x) &= \sqrt{\tfrac{2}{L}}\sin(\sqrt{\lambda_k}x) = \sqrt{\tfrac{2}{L}}\sin(\tfrac{\pi}{2L}(2k-1)x). \end{aligned} \tag{2.1}$$

We put

$$D(\varphi_k) = \varphi_k(L) - \int_\Omega \varphi_k dx = \sqrt{\tfrac{2}{L}}\Big((-1)^{k+1} - \frac{1}{\sqrt{\lambda_k}}\Big) \qquad (k \in \mathbb{N}). \tag{2.2}$$

Assuming that problem $(\mathrm{P}\,1, u_0)$ has a maximal variational solution u, we introduce the linear problem

$$v_k'(t) + \lambda_k v_k(t) = b(\textstyle\int_\Omega u(t)\,dx) D(\varphi_k) \quad \text{in } [0, T_{max}(u_0)) \tag{2.3}$$

$$v_k(0) = u_{0_k} \tag{2.4}$$

where u_{0_k} denotes the k^{th} coordinate of the initial condition, i.e. $u_{0_k} = \int_\Omega u_0(x)\varphi_k(x)dx$. In order to prove Theorem 2.1 one will show that for $k = 1$ problem (2.3) - (2.4) has no global solution. Thus $T_{max}(u_0)$ must be finite and by Theorem 1.1 we will obtain $\lim_{t\to T_{max}(u_0)} |u(t)|_{L^2(\Omega)} = +\infty$. To this aim we first need

Lemma 2.1. *Under the assumptions of Theorem 2.1 there exists a real number $c > 0$ depending only on L and ϕ such that, for all $\beta > 0$, the variational solution to problem $(\mathrm{P}\,1, u_0)$ satisfies $\int_\Omega u(t,x)\,dx \geq c\,v_1(t)$ for all $t \in [0, T_{max}(u_0))$.*

Proof. Let $t \in [0, T_{max}(u_0))$. From (2.3) - (2.4) we deduce the representation

$$v_k(t) = e^{-\lambda_k t} u_{0_k} + \int_0^t e^{-\lambda_k(t-s)} b\left(\int_\Omega u(s)\,dx\right) ds D(\varphi_k) \tag{2.5}$$

for v_k. For all $n \in \mathbb{N}$ let us consider the series

$$S_n(t) = \sum_{k=1}^{n} v_k(t) \int_\Omega \varphi_k dx.$$

With (2.5) and the notation

$$\begin{aligned} E_k &= D(\varphi_k) \int_\Omega \varphi_k dx = \tfrac{2}{L} \left(\frac{(-1)^{k+1}}{\sqrt{\lambda_k}} - \frac{1}{\lambda_k} \right) \\ I_k &= \int_0^t e^{-\lambda_k (t-s)} b \left(\int_\Omega u(s,x)\, dx \right) ds \end{aligned} \tag{2.6}$$

we write $S_n(t)$ in the form

$$S_n(t) = \sum_{k=1}^{n} e^{-\lambda_k t} u_{0_k} \int_\Omega \varphi_k dx + \sum_{k=1}^{n} I_k E_k =: S_n^1(t) + S_n^2(t) \tag{2.7}$$

where $S_n^1(t)$ and $S_n^2(t)$ are defined in an obvious way. Let us begin to minimize $S_n^1(t)$ by a quantity of the form $\varepsilon e^{-\lambda_1 t} u_{0_1} \int_\Omega \varphi_1 dx$. Under assumption (A1) of Theorem 2.1 choose

$$\varepsilon = \sqrt{\tfrac{L}{2}} \frac{\alpha}{\int_\Omega \phi \varphi_1 dx}.$$

Since $\varphi_1(x) = \sqrt{\frac{2}{L}} \sin(\frac{\pi}{2L} x) > 0$ on $(0,L)$, $\phi \in L^2(\Omega)$ and $\phi \geq \alpha > 0$ on Ω, it is clear that ε belongs to $(0, +\infty)$. Let $\beta > 0$. Then, for a.e. $x \in \Omega$ one has

$$\begin{aligned} u_0(x) - \varepsilon u_{0_1} \varphi_1(x) &= \beta \left(\phi(x) - \varepsilon \int_\Omega \phi \varphi_1 dx \varphi_1(x) \right) \\ &\geq \beta(\alpha - \sqrt{\tfrac{L}{2}} \alpha \varphi_1(x)) \\ &\geq \beta\alpha(1 - \sqrt{\tfrac{L}{2}} \varphi_1(x)). \end{aligned}$$

Since $\varphi_1 < \sqrt{\frac{2}{L}}$ in Ω one deduces that

$$u_0 - \varepsilon u_{0_1} \varphi_1 > 0 \qquad \text{a.e. in } \Omega. \tag{2.8}$$

If assumption (A2) holds, it is clear that the function

$$w(x) = \frac{u_0(x)}{u_{0_1} \varphi_1(x)} = \frac{\phi(x)}{\int_\Omega \phi \varphi_1 dx \; \varphi_1(x)} \qquad (x \in (0, L])$$

is positive on $(0, L]$ and

$$w(x) = \frac{1}{\int_\Omega \phi \varphi_1 dx} \frac{\phi(x)}{x} \frac{x}{\varphi_1(x)} \quad \to \quad \frac{1}{\int_\Omega \phi \varphi_1 dx} \frac{\phi'(0^+)}{\varphi_1'(0)} > 0 \qquad \text{as } x \to 0^+.$$

Thus extending w by

$$w(0) = \frac{\phi(0^+)}{\int_\Omega \phi\varphi_1 dx\, \varphi_1'(0)}$$

we see that w is continuous and positive on the compact set $\overline{\Omega}$, thus there exists some positive real number (depending only on L and ϕ) that we may assume to be equal to ε such that $w(x) > \varepsilon$ for a.e. $x \in \Omega$, i.e. for all $\beta > 0$

$$u_0 - \varepsilon u_{0_1}\varphi_1 > 0 \qquad \text{a.e. in } \Omega. \tag{2.9}$$

Moreover, the variational solution w_n to the linear problem

$$\left.\begin{aligned} w_{n_t} - w_n'' &= 0 && \text{in} \quad (0,+\infty)\times\Omega \\ w_n(\cdot,0) = w_n'(\cdot,L) &= 0 && \text{on} \quad (0,+\infty) \\ w_n(0,x) = u_{0_1}(1-\varepsilon)\varphi_1(x) &+ \textstyle\sum_{k=2}^n u_{0_k}\varphi_k(x) && \text{in} \quad \Omega \end{aligned}\right\}$$

converges in $C\big([0,t],L^2(\Omega)\big)$ towards the variational solution w to the problem

$$\left.\begin{aligned} w_t - w'' &= 0 && \text{in} \quad (0,+\infty)\times\Omega \\ w(\cdot,0) = w'(\cdot,L) &= 0 && \text{on} \quad (0,+\infty) \\ w(0,x) = u_0(x) &- \varepsilon u_{0_1}\varphi_1(x) && \text{in} \quad \Omega \end{aligned}\right\}.$$

Now, clearly,

$$w_n(t) = e^{-\lambda_1 t}u_{0_1}(1-\varepsilon)\varphi_1 + \sum_{k=2}^n e^{-\lambda_k t}u_{0_k}\varphi_k.$$

Thus, by continuity of the integral,

$$\sum_{k=1}^n e^{-\lambda_k t}u_{0_k}\int_\Omega \varphi_k - \varepsilon e^{-\lambda_1 t}u_{0_1}\int_\Omega \varphi_1 \quad\to\quad \int_\Omega w(t)\,dx \qquad \text{as } n\to+\infty.$$

It follows from (2.8) and (2.9) that $u_0 - \varepsilon\varphi_1 > 0$ a.e. in Ω. Thus with the strong maximum principle $\int_\Omega w(t)dx > 0$. Therefore, there exists some integer $n(t)$ such that, for all $n \geq n(t)$,

$$S_n^1(t) = \sum_{k=1}^n e^{-\lambda_k t}u_{0_k}\int_\Omega \varphi_k \geq \varepsilon u_{0_1}e^{-\lambda_1 t}\int_\Omega \varphi_1 dx. \tag{2.10}$$

Considering the sum $S_n^2(t)$, a direct computation leads to (see (2.6) and (2.1))

$$E_k + E_{k+1} = \frac{\pi^2}{2L^4\lambda_k\lambda_{k+1}}\big(\pi(4k^2-1) - 2(4k^2+1)L\big).$$

Thus

$$L < \tfrac{3\pi}{10} \implies E_k + E_{k+1} > 0 \qquad (k\in\mathbb{N} \text{ odd}). \tag{2.11}$$

In particular, for $k = 1$ there exists a positive real number that we may again denote by ε such that

$$E_1 + E_2 \geq \varepsilon E_1. \tag{2.12}$$

Furthermore, since $u_0 = \beta\phi > 0$ a.e. in Ω, [8: Theorem 4.1] leads to

$$\int_\Omega u(\cdot, x)\, dx > 0 \qquad \text{on } [0, T_{max}(u_0)). \tag{2.13}$$

Thus, since b is non-negative on $(0, +\infty)$, $b(\int_\Omega u(\cdot, x)\, dx) \geq 0$ on $[0, T_{max}(u_0))$ and $k \mapsto I_k$ is non-increasing. It follows that for all $k \in \mathbb{N}$

$$I_{k+1}E_{k+1} \geq I_k E_{k+1} \tag{2.14}$$

since $E_{k+1} \leq 0$. For all even integers $n \geq 4$, writing $S_n^2(t)$ in the form

$$S_n^2(t) = I_1E_1 + I_2E_2 + \sum_{k=3,\, odd}^{n-1} I_kE_k + I_{k+1}E_{k+1}$$

and using (2.14) it comes

$$S_n^2(t) \geq I_1(E_1 + E_2) + \sum_{k=3,\, odd}^{n-1} I_k(E_k + E_{k+1}).$$

Now $I_k \geq 0$, thus with (2.11) and (2.12) we obtain

$$S_n^2(t) \geq \varepsilon I_1 E_1. \tag{2.15}$$

Going back to (2.7) and using (2.10), (2.15) and (2.5) with $k = 1$, we may write for all even integers n greater than

$$\begin{aligned} S_n(t) &\geq \varepsilon\left(e^{-\lambda_1 t}u_{0_1}\int_\Omega \varphi_1 dx + I_1E_1\right) \\ &\geq \varepsilon\left(e^{-\lambda_1 t}u_{0_1} + \int_0^t e^{-\lambda_1(t-s)}b\left(\int_\Omega u(s)dx\right)dsD(\varphi_1)\right)\int_\Omega \varphi_1 dx \\ &\geq \varepsilon v_1(t)\int_\Omega \varphi_1 dx. \end{aligned} \tag{2.16}$$

Moreover, if $(u_k(t))_{k\geq 1}$ denotes the coordinates of $u(t)$ in the basis $(\varphi_k)_{k\geq 1}$, then

$$u_k(t) \geq v_k(t). \tag{2.17}$$

Indeed, taking $\varphi = \varphi_k$ in the variational form of problem $(\mathrm{P}\,1, u_0)$ we get in $C([0,t])$

$$\frac{d}{dt}u_k(s) + \lambda_k u_k(s) = -a\left(\int_\Omega u\right)\int_\Omega \varphi_k dx + b\left(\int_\Omega u\right)\varphi_k(L). \tag{2.18}$$

Now (2.13) and assumption (iv) of Theorem 2.1 imply $-a\left(\int_\Omega u\right) \geq -b\left(\int_\Omega u\right)$. Thus with (2.18) it comes since $\int_\Omega \varphi_k dx \geq 0$

$$u_k'(s) + \lambda_k u_k(s) \geq b\left(\int_\Omega u(s)\,dx\right) D(\varphi_k) \qquad (k \in \mathbb{N}, s \in [0,t])$$

which together with (2.3) - (2.4) proves (2.17). Furthermore, (2.16) - (2.17) lead to

$$\sum_{k=1}^n u_k(t) \int_\Omega \varphi_k dx \geq \sum_{k=1}^n v_k(t) \int_\Omega \varphi_k dx = S_n(t) \geq \varepsilon v_1(t) \int_\Omega \varphi_1 dx.$$

Letting $n \to +\infty$ it comes $\int_\Omega u(t)\,dx \geq \varepsilon v_1(t) \int_\Omega \varphi_1 dx$ for all $t \in [0, T_{max}(u_0))$. Since ε depends only on L and ϕ, we conclude setting $c = \varepsilon \int_\Omega \varphi_1 dx$ ∎

We can now give the

Proof of Theorem 2.1. Let u be the maximal solution to problem $(\mathrm{P}\,1, u_0)$. Lemma 2.1 says there exists a constant $c = c(L,\phi) > 0$ such that $\int_\Omega u(t)\,dx \geq cv_1(t)$. Now $u_{0_1} = \int_\Omega u_0 \varphi_1 dx > 0$ and $b(\int_\Omega u\,dx)D(\varphi_1) \geq 0$. Thus from (2.5) $v_1(t)$ is non-negative. Since b is non-decreasing on $(0,+\infty)$, we deduce that $b(\int_\Omega u(t)\,dx) \geq b(cv_1(t))$ for all $t \in [0, T_{max}(u_0))$. Thus with (2.3) we get since $D(\varphi_1) > 0$

$$v_1'(t) + \lambda_1 v_1(t) \geq b(cv_1(t))D(\varphi_1). \tag{2.19}$$

Denote again by $b(\cdot)$ the function $b(\cdot)cD(\varphi_1)$, and set $s = cv_1$ and $\lambda = \lambda_1$. The previous inequality becomes

$$s'(t) \geq b(s(t)) - \lambda s(t). \tag{2.20}$$

Thanks to the following lemma (see [13] for its proof) we control the sign of the right-hand side.

Lemma 2.2. *If a non-decreasing function $b : \mathbb{R} \to [0,+\infty)$ verifies $\int^{+\infty} \frac{ds}{b(s)} < +\infty$, then the set of zeros of the function $s \mapsto b(s) - \lambda s$ is bounded from above.*

According to hypothesis (iii) $\int_\Omega \phi\varphi_1 dx > 0$. Thus by Lemma 2.2 we can choose β large enough such that $s(0) = \beta c \int_\Omega \phi\varphi_1 dx$ is greater than every zero of $s \mapsto b(s) - \lambda s$ (recall that $c > 0$). Therefore one deduces easily with (2.20) that $b(v(t)) - \lambda v(t) > 0$ for all $t \in [0, T_{max}(u_0))$. Using again (2.20) and the fact that $\int^{+\infty} \frac{ds}{b(s)-\lambda s} < +\infty$ we infer that $T_{max}(u_0) < +\infty$ and conclude the proof of Theorem 2.1 with the blow-up result of Theorem 1.1 ∎

Remark 2.1. It can be proved in a similar way that if $l > \frac{\pi}{2}$, then blow-up driven by the source term can occur (see [14] for details).

Let us apply now Theorem 2.1 to problem $(\mathrm{MP}\,p,p)$.

Corollary 2.1. *Let $p > 1$ and $\Omega = (0,L)$ with $0 < L < \frac{3\pi}{10}$. If the initial condition $u_0 = \beta\phi$ satisfies hypothesis* (iii) *of Theorem* 2.1*, then there exists $\beta_c > 0$ such that for all $\beta \geq \beta_c$ the solution to problem* $(\mathrm{MP}\,p,p)$ *corresponding to the initial condition $u_0 = \beta\phi$ blows up in finite time in L^2-norm.*

Proof. We apply Theorem 2.1 to the problem

$$\left.\begin{array}{lll} u_t - u'' = -((\int_\Omega u(t,x)\,dx)^+)^p & \text{in} & (0,T)\times\Omega \\ u(t,0) = 0 & \text{on} & (0,T) \\ u'(t,L) = ((\int_\Omega u(t,x)\,dx)^+)^p & \text{on} & (0,T) \\ u(0,\cdot) = u_0(\cdot) & \text{in} & \Omega \end{array}\right\}$$

where $(\cdot)^+ : \mathbb{R} \to [0,+\infty)$ denotes the Lipschitz continuous function defined by

$$(s)^+ = \begin{cases} s & \text{if } s > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $u_0 \geq 0$, we know from [8: Corollary 4.1] that this problem is equivalent to problem $(\mathrm{MP}\, p, p)$.

2.2 Extension by rescaling. We would like to extend now Theorem 2.1 for larger domains. To this aim let us introduce the set

$$I_{bup} = \left\{ L > 0 \;\middle|\; \begin{array}{l} \text{Theorem 2.1 holds under assumptions (ii) - (iv)} \\ \text{without constraint on } L \text{ for } \Omega = (0,L) \end{array} \right\}.$$

Then we have

Corollary 2.2. *I_{bup} is a subinterval of $(0,+\infty)$ which contains $(0, \frac{3\pi}{10})$.*

Proof. The second part of the assertion follows from Theorem 2.1. The proof of the first part is very similar to that of [8: Corollary 4.2], therefore we omit it here ∎

Set $L_1 = \sup I_{bup}$ (recall that I_{bup} is not empty) and for every numerical functions a, b defined on $[0,+\infty)$ put

$$\lambda(a,b) = \inf \left\{ \frac{b(s)}{a(s)} \;\middle|\; s > 0 \text{ and } a(s) > 0 \right\}.$$

If the above set is empty, we put $\lambda(a,b) = +\infty$. Then we have the following result (see [8] for a similar proof).

Theorem 2.2. *Let a and b be two functions satisfying assumptions* (i) - (ii) *of Theorem* 1.1. *Assume, in addition, that*

(i) $\Omega = (0,L)$ *with* $0 < L < \lambda(a,b)L_1$

(ii) *b is non-negative, non-decreasing on* $(0,+\infty)$ *and* $\int^{+\infty} \frac{ds}{b(s)} < +\infty$.

Then, if the initial condition u_0 is large enough in the sense of Theorem 2.1, *the variational solution to problem* $(\mathrm{P}\,1, u_0)$ *blows up in finite time in L^2-norm.*

Remark 2.2.

(i) If $a \leq 0$ on $[0,+\infty)$, then $\lambda(a,b) = +\infty$ and Theorem 2.2 holds for all bounded domains $\Omega = (0,L)$.

(ii) The assumption $a \le b$ on $[0,+\infty)$ of Theorem 2.1 is here no more needed since the condition $\lambda(a,b) < 1$ is allowed. For instance, if

$$\left.\begin{aligned} a(s) &= s^p \\ b(s) &= s^{p+1} + \alpha s^p \end{aligned}\right\} \qquad (s > 0)$$

where $p \ge 1$ and $\alpha \in (0,1)$ are given, then $\lambda(a,b) = \alpha$ and $b < a$ on $(0, 1-\alpha)$. The assumptions of Theorem 2.2 can also be satisfied if a dominates b. Indeed, if

$$\left.\begin{aligned} a(s) &= s^p \\ b(s) &= \alpha s^p \end{aligned}\right\} \qquad (s > 0)$$

where $p > 1$ and $\alpha \in (0,1)$ are given, then $\lambda(a,b) = \alpha$ and $b < a$ on $(0,+\infty)$.

3. The semigroup approach

We refer to [2] for a heuristic introduction to semigroup theory. In this section, we follow [1] and will assume the following:

$$\Omega = (0,L) \text{ is a bounded interval of } \mathbb{R} \tag{3.1}$$

$$\text{the functions } a, b \text{ are locally Lipschitz continuous on } D = \mathbb{R} \tag{3.2}$$

$$\text{the initial condition } u_0 \text{ belongs to } V \tag{3.3}$$

$$\text{the functional } l \text{ belongs to } V'. \tag{3.4}$$

Let us introduce the following notation: if E and F are Banach spaces, we denotes by $\mathcal{L}(E,F)$ the vector space of all continuous linear operators from E to F. Moreover, $\mathcal{L}(E) := \mathcal{L}(E,E)$ and $\mathcal{L}(E,F)$ is equipped with the usual norm. For all $s \in \mathbb{R}$, $H^s(\Omega)$ denotes the standard Sobolev-Slobodeckii space.

Let us now define the operators

$$\begin{aligned} &A:\ D(A) \subset L^2(\Omega) \to L^2(\Omega), && u \mapsto -u'' \\ &\mathcal{B}:\ H^2(\Omega) \to L^2(\partial\Omega) \simeq \mathbb{R}^2, && u \mapsto \mathcal{B}u = (u(0), u'(L)) \end{aligned} \tag{3.5}$$

where

$$D(A) := H^2_{\mathcal{B}}(\Omega) := \Big\{ u \in H^2(\Omega) \Big|\ u(0) = 0 \text{ and } u'(L) = 0 \Big\}.$$

We put

$$b_1 : \mathbb{R} \to L^2(\partial\Omega) \simeq \mathbb{R}^2, \qquad s \mapsto (0, b(s))$$

which means that $b_1(s)(0) = 0$ and $b_1(s)(L) = b(s)$. Then following [1] we can define a new class of solutions to problem (P l, u_0).

Definition 3.1. Let $u_0 \in V$. We say that u is a V-*weak solution* to problem $(\mathrm{P}\, l, u_0)$ on $[0,T]$ if $u \in C([0,T],V)$, $u(0) = u_0$ and

$$\int_0^T \left(-\langle \xi_t, u\rangle + \int_\Omega u'\xi' dx \right) dt$$
$$= \int_0^T \left(-a(l(u)) \int_\Omega \xi \, dx + b(l(u))\xi(t,L) \right) dt + \int_\Omega \xi(0,x)u_0 dx$$

for all $\xi \in C([0,T],V) \cap C^1([0,T],V')$ verifying $\xi(T,\cdot) = 0$ in V.

We have then the following existence result.

Theorem 3.1. *Under assumptions* (3.1) – (3.4) *problem* $(\mathrm{P}\, l, u_0)$ *admits a unique maximal* V-*weak solution* $u \in C([0, T_{max}(u_0)), V)$.

Proof. It is well known that the operator A generates a strongly continuous analytic semigroup on $L^2(\Omega)$. Moreover, from (3.2) and (3.4), $a \circ l$ and $b_1 \circ l$ are locally Lipschitz continuous from V into $L^2(\Omega)$ and $\{v : \{0, L\} \to \mathbb{R} |\ v(0) = 0\} \simeq \mathbb{R}$, respectively. Thus according to [1: Theorem 12.3] problem $(\mathrm{P}\, l, u_0)$ has a unique maximal V-weak solution ∎

The next result motivates the introduction of such solutions since it provides compactness for trajectories.

Theorem 3.2. *Under the hypothesis of Theorem* 3.1*, let us denote by* u *the maximal* V-*weak solution to problem* $(\mathrm{P}\, l, u_0)$. *If there exists some constant* $M > 0$ *such that*

$$|a \circ l(u(t))| + |b \circ l(u(t))| \leq M \qquad \forall t \in [0, T_{max}(u_0)),$$

then $T_{max}(u_0) = +\infty$ *and the trajectory* $\{u(t)|\ t \geq 0\}$ *is relatively compact in* $L^2(\Omega)$.

Remark 3.1. It can be also proved that the trajectory is relatively compact in V (see [1: Theorem 12.3]).

Proof of Theorem 3.2. It is well known that the *type* of the semigroup generated by A is strictly negative, i.e. there exists $M \geq 1$ and $\omega \in (-\infty, 0)$ such that

$$\|e^{-tA}\|_{\mathcal{L}(V)} \leq M e^{\omega t} \qquad (t \geq 0). \tag{3.6}$$

Using the fact that the functions $a \circ l(u)$ and $b_1 \circ l(u)$ are independent of $x \in \Omega$, we show easily with [1: Theorem 12.8] that $T_{max}(u_0) = +\infty$ and $\{u(t) :\ t \geq 0\}$ is bounded in V. Thus by the Rellich Theorem, this trajectory is relatively compact in $L^2(\Omega)$ ∎

Before to give a convergence result in V we would like to recall some results about semigroup theory. Set $E_0 = L^2(\Omega), E_1 = D(A)$ and $A_1 = A$. Then following [1: Section 8] we are able to construct a scale $\{(E_\alpha, A_\alpha) :\ \alpha \in \mathbb{R}\}$ whose principal properties are presented in the following

Theorem 3.3. *Let $-\infty < \beta < \alpha < +\infty$ and A be defined by* (3.5). *Then the following statements hold:*

(1) *Each E_α is a Banach space and $A_\alpha \in \mathcal{L}(E_{\alpha+1}, E_\alpha)$.*

(2) *$-A_\alpha$ generates a strongly continuous analytic semigroup $\{e^{-tA_\alpha}\}_{t\geq 0}$ on E_α of type ω. Moreover, for all $t \geq 0$, $e^{-tA_\alpha} = e^{-tA_\beta}|_{E_\alpha}$.*

(3) *For all $\sigma > \omega$ there exists a constant $c = c(\alpha, \beta)$ such that*

$$\|e^{-tA_\beta}\|_{\mathcal{L}(E_\beta, E_\alpha)} \leq c\, t^{\beta-\alpha} e^{\sigma t}.$$

(4) *The injection of E_α into E_β is continuous and compact.*

When $\alpha \in (\frac{1}{4}, \frac{3}{4})$, we know that

$$E_\alpha = H^{2\alpha}_{\mathcal{B}}(\Omega) := \left\{u \in H^{2\alpha}(\Omega) |\; u(0) = 0\right\}.$$

To have a consistant notation, we put therefore

$$H^{2\alpha}_{\mathcal{B}}(\Omega) = E_\alpha \qquad (\alpha \in \mathbb{R}).$$

Considering now the map $\mathcal{R}_1 :\ R^2 \to H^2(\Omega)$, $g \mapsto \mathcal{R}g = u$ with u being solution to the problem

$$\left.\begin{aligned} -u'' &= 0 \qquad \text{in } \Omega \\ \mathcal{B}u &= g \qquad \text{on } \partial\Omega \end{aligned}\right\}$$

it is clear that $\mathcal{R}_1 \in \mathcal{L}(\mathbb{R}^2, H^2(\Omega))$. By interpolation we define (see [1: Section 11]) $\mathcal{R}_\alpha \in \mathcal{L}(R^2, H^{2\alpha}(\Omega))$. Moreover, if $\alpha \in (\frac{1}{4}, \frac{3}{4})$, one has

$$\mathcal{R}_\alpha \in \mathcal{L}(\{0\} \times \mathbb{R}, H^{2\alpha}_{\mathcal{B}}(\Omega)). \tag{3.7}$$

A "variation-of-constant" formula. Let $\alpha \in (\frac{1}{2}, \frac{3}{4})$. We put for all $u \in H^{2\alpha}_{\mathcal{B}}(\Omega)$

$$F_{\alpha-1}(u) = a(l(u)) + A_{\alpha-1}\mathcal{R}_\alpha(b_1(l(u))).$$

Then $F_{\alpha-1}(u) \in H^{2\alpha-2}_{\mathcal{B}}(\Omega)$ since

$$\begin{aligned} &|F_{\alpha-1}(u)|_{H^{2\alpha-2}_{\mathcal{B}}(\Omega)} \\ &\quad\leq |a(l(u))|_{H^{2\alpha-2}_{\mathcal{B}}(\Omega)} + \|A_{\alpha-1}\|_{\mathcal{L}(H^{2\alpha}_{\mathcal{B}}(\Omega), H^{2\alpha-2}_{\mathcal{B}}(\Omega))} \|\mathcal{R}_\alpha\|_{\mathcal{L}(\{0\}\times\mathbb{R}, H^{2\alpha}_{\mathcal{B}}(\Omega))} |b_1(l(u))|. \end{aligned}$$

Thus, using Theorem 3.3/(1), (3.7) and the fact that, by Theorem 3.3/(4), $L^2(\Omega)$ is continuously embedded into $H^{2\alpha-2}_{\mathcal{B}}(\Omega)$ (since $2\alpha - 2 < 0$), one gets for some constant $C > 0$

$$|F_{\alpha-1}(u)|_{H^{2\alpha-2}_{\mathcal{B}}(\Omega)} \leq C\big(|a(l(u))| + |b(l(u))|\big). \tag{3.8}$$

Then from [1: Theorem 10.2 and Lemma 11.1] u is a weak solution to problem $(\mathrm{P}\, l, u_0)$ on $[0, T]$ if and only if $u \in C([0, T], V)$ and

$$u(t) = e^{-tA_{\alpha-1}} u_0 + \int_0^t e^{-(t-s)A_{\alpha-1}} F_{\alpha-1}(u(s))\, ds \ \text{ in } V \qquad (t \in [0, T]). \tag{3.9}$$

After these preliminaries we can show the following convergence result.

Theorem 3.4. *Under the hypothesis of Theorem* 3.1, *assume that u and v are global V-weak solutions to problems* $(\mathrm{P}\, l, u_0)$ *and* $(\mathrm{P}\, l, v_0)$, *respectively. If, in addition, v is bounded in V-norm and $l(u(t)) - l(v(t)) \to 0$ as $t \to +\infty$, then*

$$(u-v)(t) \to 0 \qquad \text{in } V \text{ as } t \to +\infty.$$

Proof. Fix $\alpha \in (\frac{1}{2}, \frac{3}{4})$. Using (3.9) one gets by difference

$$(u-v)(t) = e^{-tA_{\alpha-1}}(u_0 - v_0) + \int_0^t e^{-(t-s)A_{\alpha-1}}\big(F_{\alpha-1}(u(s)) - F_{\alpha-1}(v(s))\big)ds \quad (3.10)$$

in V, for all $t \geq 0$. Set

$$\varepsilon(s) = \big|F_{\alpha-1}(u(s)) - F_{\alpha-1}(v(s))\big|_{H_{\mathcal{B}}^{2\alpha-2}(\Omega)}.$$

Arguing as in (3.8) leads to

$$\varepsilon(s) \leq C\Big(\big|a(l(u)) - a(l(v))\big| + \big|b(l(u)) - b(l(v))\big|\Big)$$

for some finite constant C. Next, since v is bounded in V-norm, l belongs to V' and $l(u(\cdot)) - l(v(\cdot)) \to 0$, it follows that $l(u(\cdot))$ and $l(v(\cdot))$ are bounded. Then the Lipschitz continuity of a and b implies

$$\varepsilon(s) \to 0 \qquad \text{when } s \to +\infty. \quad (3.11)$$

Going back to (3.10) one gets by (3.6) and Theorem 3.3/(2)-(3)

$$\begin{aligned} &|(u-v)(t)|_V \\ &\leq \|e^{-tA_{1/2}}\|_{\mathcal{L}(V)}|u_0 - v_0|_V + \int_0^t \|e^{-(t-s)A_{\alpha-1}}\|_{\mathcal{L}(H_{\mathcal{B}}^{2\alpha-2}(\Omega), V)}\varepsilon(s)\, ds \\ &\leq Me^{\omega t}|u_0 - v_0|_V + c\int_0^t (t-s)^{\alpha-\frac{3}{2}}e^{\sigma(t-s)}\varepsilon(s)\, ds \end{aligned} \quad (3.12)$$

where σ is taken in $(\omega, 0)$. Moreover, according to (3.11), for all $\delta > 0$ there exists a time t_δ such that

$$\varepsilon(t) \leq \tfrac{\delta}{c}\left(\int_0^{+\infty} y^{\alpha-3/2}e^{\sigma y}dy\right)^{-1} \qquad (t \geq t_\delta).$$

Note that the above integral converges since $\alpha - \frac{3}{2} > -1$ and $\sigma < 0$. Moreover, by a change of variable

$$\int_{t_\delta}^t (t-s)^{\alpha-\frac{3}{2}}e^{\sigma(t-s)}\varepsilon(s)\, ds \leq \sup_{[t_\delta, +\infty)} \varepsilon(\cdot) \int_0^{t-t_\delta} y^{\alpha-\frac{3}{2}}e^{\sigma y}dy \leq \frac{\delta}{c}.$$

Thus the integral in (3.12) that we write as

$$\int_0^{t_\delta} (t-s)^{\alpha-\frac{3}{2}}e^{\sigma(t-s)}\varepsilon(s)\, ds + \int_{t_\delta}^t (t-s)^{\alpha-\frac{3}{2}}e^{\sigma(t-s)}\varepsilon(s)\, ds$$

is bounded by

$$\sup_{[0,t_\delta]} \varepsilon(\cdot) \int_{t-t_\delta}^t y^{\alpha-\frac{3}{2}}e^{\sigma y}dy + \frac{\delta}{c}.$$

Going back to (3.12) one deduces since ω, σ are negative that $\limsup_{t\to+\infty} |(u-v)(t)|_V \leq \delta$ for all $\delta > 0$. Hence the result follows ∎

Now we would like to connect this theory together with the variational one by mean of the following

Theorem 3.5. *Under assumptions* (3.1) – (3.3) *and* (ii), (iv) *of Theorem* 1.1 *problem* $(\mathrm{P}\, l, u_0)$ *admits a unique maximal V-weak solution* $u \in C([0,T_1),V)$ *and a unique maximal variational solution* $v \in C\big([0,T_2),L^2(\Omega)\big)$. *Moreover,* $T_1 = T_2$ *and* $u \equiv v$ *on* $[0,T_1)$.

Proof. Existence and uniqueness of u follow from Theorem 3.1 since $l \in (L^2(\Omega))' \subset V'$. Existence and uniqueness of v follow from Theorem 1.1. Moreover, using the identification $(L^2(\Omega))' = L^2(\Omega)$ we deduce from [1: Corollary 12.2 and Theorem 8.1] that, for all $T < T_1$,

$$\frac{d}{dt} u \in C([0,T],V').$$

Thus, going back to Definition 3.1 one deduces easily that u is a variational solution to problem $(\mathrm{P}\, l, u_0)$ on $[0,T]$. Hence $T_1 \le T_2$ and $u \equiv v$ on $[0,T_1)$ by uniqueness of the variational solution. If $T_1 < T_2$, then $l(u(\cdot)) = l(v(\cdot))$ is bounded in absolute value on $[0,T_1)$ since by Theorem 1.1 $v \in C\big([0,T_1],L^2(\Omega)\big)$ and $l \in (L^2(\Omega))'$. From continuity of a, b and Theorem 3.2 it follows that $T_1 = +\infty$. Hence $T_1 = T_2$ which completes the proof of the theorem ∎

Remark 3.2. According to [3] we know that if the data are smooth enough, then V-weak solutions are classical solutions.

4. Long time behavior of global bounded solutions

In this section we suppose that the functions a and b are equal. More precisely, we will deal with the one-dimensional problem

$$\left.\begin{array}{lll} u_t - u'' = -a(\int_\Omega u(t,x)\,dx) & \text{in} & (0,T)\times\Omega \\ u(t,0) = 0 & \text{on} & (0,T) \\ u'(t,L) = a(\int_\Omega u(t,x)\,dx) & \text{on} & (0,T) \\ u(0,\cdot) = u_0(\cdot) & \text{in} & \Omega \end{array}\right\}. \qquad (\mathbf{P}\, u_0)$$

Let us introduce the variational solution ϕ to the problem

$$\left.\begin{array}{l} -\phi'' = 1 \;\; \text{in } \Omega = (0,L) \\ \phi(0) = \phi'(L) = 0 \end{array}\right\}$$

and let us set

$$D(\phi) = \phi(L) - \int_\Omega \phi\,dx = \tfrac{1}{2}L^2 - \tfrac{1}{3}L^3. \qquad (4.1)$$

We will assume the following:

$$L \in (0, \tfrac{3\pi}{10}] \qquad (4.2)$$

$$a \;\text{ is locally Lipschitz continuous on } \mathbb{R} \qquad (4.3)$$

$$|a(s)| \le c_0(1+|s|^p) \;\; (s \in \mathbb{R}) \qquad (4.4)$$

for some $p \geq \frac{1}{2}$ and $c_0 > 0$, and there exist two real numbers $s_1 < s_2$ such that (see Figure 2)

$$D(\phi)a(s) - s = 0 \qquad \forall s \in \{s_1, s_2\} \tag{4.5}$$

$$D(\phi)a(s) - s < 0 \qquad \forall s \in (s_1, s_2) \tag{4.6}$$

$$a(s_1) \leq a(s) \qquad \forall s \in [s_1, s_2]. \tag{4.7}$$

Figure 2

4.1 A comparison principle for the integral. For $i = 1, 2$ we deduce from (4.5) that the function $u_i : \Omega \to \mathbb{R}$ defined by

$$u_i(x) = \frac{a(s_i)}{2}x^2 + a(s_i)(1 - L)x \qquad (x \in \Omega) \tag{4.8}$$

fulfils $\int_\Omega u_i dx = s_i$. Hence we prove easily that it is a stationary solution to problem $(\mathrm{P}\, u_0)$. Moreover, since by (4.5) $a(s_1) < a(s_2)$ and $1 - L > 0$, we deduce from (4.8) that $u_1 < u_2$ in Ω. Then we have

Theorem 4.1. *Under assumptions* $(4.2) - (4.7)$, *if in addition the initial condition* u_0 *belongs to* $L^2(\Omega)$ *and verifies* $u_1 \leq u_0 \leq u_2$ *a.e. in* Ω, *then the variational solution* u *to problem* $(\mathrm{P}\, u_0)$ *is global, bounded in* L^2*-norm and satisfies*

$$\int_\Omega u_1 dx \leq \int_\Omega u(t,x)\, dx \leq \int_\Omega u_2 dx \qquad (t \geq 0). \tag{4.9}$$

Proof. Arguing as in [8: Proof of Theorem 4.3] we can show that

$$\int_\Omega u_1 dx \leq \int_\Omega u(t,x)\, dx \leq \int_\Omega u_2 dx \qquad \forall t \in [0, T_{max}(u_0)) \tag{4.10}$$

since by (4.5) - (4.7) $a(s_1) \leq a(s) \leq a(s_2)$ for all $s \in [s_1, s_2]$. Note that (4.9) will be proved if we show that $T_{max}(u_0) = +\infty$. From (4.3) and (4.10) there exists a constant $M > 0$ such that

$$\left| a\left(\int_\Omega u(\cdot, x)\, dx \right) \right| \leq M \qquad \text{in } [0, T_{max}(u_0)). \tag{4.11}$$

Moreover, since $u \in C\big([0, T_{max}(u_0)), L^2(\Omega)\big) \cap L^2(0,T;V)$, there exists a time $t_0 \in [0, T_{max}(u_0))$ such that $u(t_0) \in V$. Thus since $u(\cdot + t_0)$ is the variational solution to problem $(\mathrm{P}\, u(t_0))$, we deduce from Theorem 3.5 that it is also a V-weak solution. We conclude this proof using (4.11) and Theorem 3.2 ∎

4.2 A dynamical system. We have the following

Theorem 4.2. *Under the assumptions of Theorem* 4.1*, let us denote by* u *the global variational solution to problem* $(\mathrm{P}\, u_0)$*. Setting*

$$Z = \overline{u([0,+\infty))} \qquad (\textit{the closure in } L^2(\Omega)) \tag{4.12}$$

and for all $z \in Z$ *denoting by*

$$S(\cdot)z \qquad \textit{the maximal solution to problem } (\mathrm{P}\, z) \tag{4.13}$$

the following statements hold:

(i) $T_{max}(z) = +\infty$ *for all* $z \in Z$.

(ii) $\{S(t) |\, t \geq 0\}$ *is a dynamical system on* Z.

Proof. Since the solution u to problem $(\mathrm{P}\, u_0)$ is global from Theorem 4.1, (4.12) makes sense. Let $z = \lim u(t_n) \in Z$ and t be any real number in $(0, T_{max}(z))$. According to classical continuity properties of solutions with respect to initial conditions (see [13: Proposition 5.2.1] or [5: Proposition 4.3.7]), for n sufficiently large $S(t)u(t_n)$ is well defined and

$$\big|S(t)u(t_n) - S(t)z\big|_{L^2(\Omega)} \leq 1.$$

Moreover, since we easily show that $S(t)u(t_n) = u(t + t_n)$, Theorem 4.1 ensures the existence of a finite constant M such that

$$|S(t)u(t_n)|_{L^2(\Omega)} \leq M \qquad (t, n \geq 0).$$

Thus for n sufficiently large

$$|S(t)z|_{L^2(\Omega)} \leq |S(t)u(t_n) - S(t)z|_{L^2(\Omega)} + |S(t)u(t_n)|_{L^2(\Omega)} \leq 1 + M$$

for all $t < T_{max}(z)$ which proves assertion (i). Using the fact that by Theorem 4.1 Z is bounded in $L^2(\Omega)$ assertion (ii) can be proved in a standard way (see, for instance, [10: Theorem 1.2.2]) that we omit here ∎

4.3 A convergence result. Following [7: Theorem 5.6] we have the following

Theorem 4.3. *Under the assumptions and notation of Theorem* 4.2*, if* $u_0 \not\equiv u_2$ *a.e. in* Ω*, then*

$$u(t) = S(t)u_0 \to u_1 \qquad \textit{in } V \textit{ when } t \to +\infty.$$

Remark 4.1. In [7] the authors define a dynamical system using the weak topology of $L^2(\Omega)$. This method seems here not to apply since problem $(\mathrm{P}\, u_0)$ admits unbounded solutions (see Theorem 2.1). Note also that thanks to the semigroup theory we obtain convergence in H^1-norm while in [7] the convergence takes place in $L^2(\Omega)$.

Proof of Theorem 4.3. We begin to find out a Liapunov function for the dynamical system $\{S(t) : t \geq 0\}$. First, note that the test function ϕ introduced at the beginning of this section verifies

$$\int_\Omega \phi' v' dx = \int_\Omega v\, dx \qquad \forall v \in V. \tag{4.14}$$

For $z \in Z$ let us choose $\varphi = \phi$ in the variational form of problem (P z). Using notation (4.1) it comes

$$\frac{d}{dt}\left(\int_\Omega S(t)z\phi\, dx\right) + \int_\Omega \frac{d}{dx}(S(t)z)\phi' dx = D(\phi)\, a\left(\int_\Omega S(t)z\, dx\right).$$

Next, applying (4.14) with $v = S(t)z$ leads to

$$\frac{d}{dt}\int_\Omega S(t)z\phi\, dx = D(\phi)\, a\left(\int_\Omega S(t)z\, dx\right) - \int_\Omega S(t)z\, dx. \tag{4.15}$$

Since $z \in Z$, $z = \lim_{n\to\infty} u(t_n) = \lim_{n\to\infty} S(t_n)u_0$ for some sequence (t_n). Moreover, according to Theorem 4.1 one has

$$\int_\Omega u_1 dx \leq \int_\Omega S(t_n + t)u_0 dx \leq \int_\Omega u_2 dx.$$

Since $S(t_n + t)u_0 = S(t)S(t_n)u_0$, one gets passing to the limit in the above inequalities

$$\int_\Omega u_1 dx \leq \int_\Omega S(t)z\, dx \leq \int_\Omega u_2 dx \qquad (t \geq 0). \tag{4.16}$$

Going back to (4.15) we deduce with (4.16) and (4.5) - (4.6) that

$$\frac{d}{dt}\int_\Omega S(t)z\phi\, dx \leq 0 \qquad (t \geq 0). \tag{4.17}$$

This means that $u \mapsto \int_\Omega u\phi\, dx$ is a Liapunov function for our dynamical system.

Arguing as in the proof of Theorem 4.1 we obtain that, for some $t_0 \in [0, T_{max}(u_0))$, $u(t_0) \in V$ and $u(\cdot + t_0)$ is the V-weak solution to problem (P $u(t_0)$). Hence, according to (4.9) and Theorem 3.2 the trajectory $\{u(t) : t \geq 0\}$ is relatively compact in Z. Applying the La Salle invariance principle (see, for instance, [10: Theorem 2.1.3]), one gets for all z belonging to $\omega(u_0)$ (the *ω-limit set* of u_0),

$$\int_\Omega S(t)z\phi\, dx = \int_\Omega z\phi\, dx \qquad \text{for all } t \geq 0.$$

Thus combining the above identity with (4.15) and using (4.16) and (4.5) - (4.6) we deduce that

$$\int_\Omega S(t)z\, dx \in \{s_1, s_2\} \qquad \text{for all } t \geq 0.$$

In particular, $\int_\Omega z\,dx = s_1$ or $\int_\Omega z\,dx = s_2$ for all $z \in \omega(u_0)$. Next set

$$\omega_i = \left\{ z \in \omega(u_0) \middle| \int_\Omega z\,dx = s_i \right\} \qquad (i \in \{1,2\}).$$

These are disjoint closed subsets of $L^2(\Omega)$ such that $\omega(u_0) = \omega_1 \cup \omega_2$. Since $\omega(u_0)$ is connected in $L^2(\Omega)$ (see [10]), it follows that there exists $i \in \{1,2\}$ such that

$$\omega(u_0) = \omega_i. \tag{4.18}$$

Then we claim

$$\int_\Omega S(t)u_0 dx \to s_i \qquad \text{when } t \to \infty. \tag{4.19}$$

Otherwise, there exists a sequence of real numbers $t_n \to \infty$ such that $\int_\Omega S(t_n)u_0 dx$ does not converge toward s_i. By compactness of the trajectory we may assume that, up to a subsequence, $(S(t_n)u_0)_{n\geq 0}$ converges in $L^2(\Omega)$ toward some z which by definition of $\omega(u_0)$ belongs to $\omega(u_0)$. Thus (4.18) implies $\int_\Omega z\,dx = s_i$. We get a contradiction, hence (4.19) holds.

Next we want to show that $i = 1$ in (4.19). Indeed, if we assume $\int_\Omega S(t)u_0 dx \to s_2$, then (4.19) implies by Theorem 3.4 that $S(t)u_0 = S(t-t_0)u(t_0) \to u_2$ as $t \to +\infty$ (recall that $u(t_0) \in V$ and $\int_\Omega u_2 dx = s_2$). Moreover, since (see (4.17)) $t \mapsto \int_\Omega S(t)u_0\phi\,dx$ is non-increasing, one has

$$\int_\Omega S(t)u_0\phi\,dx \leq \int_\Omega u_0\phi\,dx.$$

Thus, letting $n \to +\infty$ we obtain $\int_\Omega u_2\phi\,dx \leq \int_\Omega u_0\phi\,dx$ which is impossible since $u_0 \leq u_2, u_0 \neq u_2$ and $\phi > 0$ in Ω. Hence $\int_\Omega S(t)u_0 dx \to s_1$ and applying Theorem 3.4 in the same way as above we get $S(t)u_0 \to u_1$ in V when $t \to \infty$. This is just what we had to prove ∎

We would like to apply now this result to problem (MP p,p) with $p > 1$ and $0 < L \leq \frac{3\pi}{10}$. For this it is enough (see the proof of Corollary 2.1) to consider the problem

$$\left.\begin{aligned} u_t - u'' &= -((\textstyle\int_\Omega u(t,x)\,dx)^+) && \text{in } (0,T)\times\Omega \\ u(t,0) &= 0 && \text{on } (0,T) \\ u'(t,L) &= ((\textstyle\int_\Omega u(t,x)\,dx)^+) && \text{on } (0,T) \\ u(0,\cdot) &= u_0(\cdot) && \text{in } \Omega. \end{aligned}\right\}$$

Since the function $a(s) = (s)^{+p}$ satisfies assumptions (4.3) - (4.7) with $s_1 = 0$ and $s_2 = (\frac{1}{D(\phi)})^{\frac{1}{p-1}} = (\frac{6}{L^2(3-2L)})^{\frac{1}{p-1}}$, Theorem 4.3 applies with (see (4.8))

$$\left.\begin{aligned} u_1(x) &= 0 \\ u_2(x) &= \frac{s_2^p}{2}x^2 + s_2^p(1-L)x \end{aligned}\right\} \qquad \forall x \in (0,L). \tag{4.20}$$

At this point, a natural question is that on the “size” of the set

$$\left\{u_0 \in L^2(\Omega) \,\middle|\, u_1 \leq u_0 \leq u_2 \ \text{ a.e. in } \Omega\right\}.$$

Since $u_1(0) = u_2(0) = 0$ and u_2 is increasing on Ω, we may answer studying $u_2(L)$. One has

$$u_2(L) = \frac{6^{\frac{p}{p-1}}}{2} \frac{L(2-L)}{(L^2(3-2L))^{\frac{p}{p-1}}}.$$

An easy computation shows that

$$u_2'(L) \qquad \text{and} \qquad \frac{2(1-L)L^4}{p-1}(p+2)\left(L - \frac{3(p+1)}{p+2}\right)$$

have the same sign. Since $p > 1$, it holds $\frac{3(p+1)}{p+2} = 3 - \frac{3}{p+2} \geq 2$. Thus

$$L - \frac{3(p+1)}{p+2} \leq L - 2 < 0$$

since $L \leq \frac{3\pi}{10}$. Therefore $u_2(L)$ decreases with L on $(0, \frac{3\pi}{10})$ from $+\infty$ to $u_2(\frac{3\pi}{10})$.

In order to study the variations of $u_2(L)$ with respect to p, let us set

$$u_2(L,p) = u_2(L).$$

Rewriting

$$u_2(L,p) = \frac{L(2-L)}{2}\left(\frac{6}{L^2(3-2L)}\right)^{\frac{p}{p-1}}$$

we deduce that, for fixed L, $u_2(L,p)$ is decreasing with $p \in (1,+\infty)$ from $+\infty$ to $\frac{3(2-L)}{L(3-2L)}$. In order to estimate this quantity from below we put for all $L \in (0, \frac{3}{2})$

$$u_2(L,+\infty) = \lim_{p\to+\infty} u_2(L,p) = 3\frac{2-L}{L(3-2L)}.$$

Since $L \mapsto u_2(L,+\infty)$ decreases on $(0,1)$ and $\frac{3\pi}{10} < 1$ we get $u_2(\frac{3\pi}{10},+\infty) > u_2(1,+\infty) = 3$. Note that a numerical computation leads to $u_2(\frac{3\pi}{10},+\infty) \simeq 3.019$.

Finally, for all $(L,p) \in (0, \frac{3\pi}{10}] \times (1,+\infty)$ it holds

$$\lim_{L\to 0^+} u_2(L,p) = \lim_{p\to 1^+} u_2(L,p) = +\infty$$

and

$$u_2(L,p) \geq u_2(\frac{3\pi}{10}, p) \geq \lim_{p\to+\infty} u_2(\frac{3\pi}{10}, p) \geq 3.$$

5. Conclusion and open problems

Considering the theoretical and numerical results, it appears that the asymptotic behavior of the solutions is essentially governed by the semilinear structure of our problem. For references on semilinear parabolic equations see, for instance, [6, 11, 12, 15, 16].

In Theorem 2.1 we prove that the L^2-norm of the solution blows up and we could also show that

$$\limsup_{t \to T_{max}(u_0)} \int_\Omega u(t)\, dx = +\infty.$$

But it remains to prove as suggested by numerical simulation that the integral of $u(t)$ blows up also. Moreover, from Figure 1 we conjecture that the blow-up set is equal to $(0, L]$ and

$$\lim_{t \to T_{max}(u_0)} u(t,x) = -\infty \quad (x \in (0,L)) \qquad \text{and} \qquad \lim_{t \to T_{max}(u_0)} u(t,L) = +\infty.$$

Finally, note that we just have to obtain estimates related on linear problems in order to be able to extend our results to higher dimensions.

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