Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 20 (2001), No. 1, 115–130

Parabolic Equations with Functional Dependence

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Abstract. We consider the Cauchy problem for nonlinear parabolic equations with functional dependence and prove theorems on the existence of solutions to parabolic differential-functional equations.

Keywords: *Parabolic functional-differential equations, existence and uniqueness of solutions, iterative methods, Cauchy problems*

AMS subject classification: Primary 35K10, 35K15, secondary 35K55, 35R10

1. Introduction

Any relevant research of differential-functional equations starts with thorough studies of differential equations. The classical theory of linear parabolic equations was developed in [5, 6, 8]. The existence and behaviour of solutions to initial- or initial-boundary-value problems for equations of the form

$$\mathcal{P}u = \frac{\partial u}{\partial t} - \sum_{j,l=1}^{n} a_{jl}(t,x)\frac{\partial^2 u}{\partial x_j \partial x_l} = g(t,x) \qquad (*)$$

depend upon fundamental solutions. If the matrix $[a_{jl}(t,x)]_{j,l}$ is bounded, positive definite and Hölder continuous, then we can obtain useful estimates of the fundamental solutions and their derivatives (see [4, 5, 12]). These estimates apply to examinations of the inverse operator $\mathcal{P}^{-1}$:

$$u \mapsto g = \mathcal{P}u \quad \Longrightarrow \quad u = \mathcal{P}^{-1}g.$$

If g is continuous in t and locally Hölder continuous in x, then $u = \mathcal{P}^{-1}g$ has continuous derivatives $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x_j \partial x_l}$. In particular, there exists a classical solution of the Cauchy problem for equation $(*)$. Integro-differential equations

$$\mathcal{P}u(t,x) = \bar{f}\left(t, x, u(t,x), \int u(s,y)\, d\mu(s,y)\right)$$

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ISSN 0232-2064 / $ 2.50

were considered in [14]. Initial-boundary-value problems generate the fixed point equations $u \longrightarrow \mathcal{P}^{-1}\mathcal{F}[u]$ where

$$\mathcal{F}[u](t,x) = \bar{f}\left(t, x, u(t,x), \int u(s,y)\, d\mu(s,y)\right)$$

is the Nemytskii operator. Typical assumptions on the right-hand side $\bar{f}$ guarantee that the mapping $u \to \mathcal{P}^{-1}\mathcal{F}[u]$ is a contraction on a Banach space. Existence results for parabolic equations with delays and other Volterra functionals in [1, 10, 11, 13, 18] also undergo similar procedures. The Banach contraction principle can be viewed as a particular case of the direct iterative method (Picard iterations)

$$\mathcal{P}u^{(\nu+1)} = \mathcal{F}[u^{(\nu)}].$$

At each stage of this iterative method one solves the classical equation $\mathcal{P}u = g$ with $g = \mathcal{F}[u^{(\nu)}]$. Convergence criteria for general iterative methods were formulated in [17], and stated for parabolic differential-functional equations in [9]. It is demanded to impose certain comparison conditions on the right-hand side. These comparison conditions are weakened when monotone techniques are used (see [3, 7, 11]), but there is another strong assumption that provides a given pair of lower/upper solutions. We point out that Chaplygin's iterations play an important role among monotone techniques because Chaplygin's methods provide fast convergent sequences of approximate solutions (see [2]). One of Chaplygin's sequences obeys the recurrence relation

$$\mathcal{P}u^{(\nu+1)} = \mathcal{F}[u^{(\nu)}] + D\mathcal{F}[u^{(\nu)}]\big(u^{(\nu+1)} - u^{(\nu)}\big),$$

where $D\mathcal{F}$ is a partial differentiation with respect to the functional variable. The papers [10, 13] are devouted to parabolic problems with a general functional dependence which concerns also partial derivatives. Existence and uniqueness theorems are obtained by means of the Banach contraction principle. Due to a generalization of Bielecki's norms, the results in [10] cover the case of unbounded solutions with unbounded gradients, however, the leading term of the differential operator $\mathcal{P}$ contains only the diagonal (the Laplacian case). The present paper is aimed at existence results with general operator $\mathcal{P}$ whose coefficients are Hölder continuous. Since we do not assume any differentiability of the right-hand side, in particular $D\mathcal{F}$ may not exist, then, in general, Chaplygin's methods are not applicable. We introduce suitable Bielecki's norms. Original Bielecki's norms

$$\|v\|_\lambda = \sup_t |v(t)| / \exp(\lambda t) \qquad \text{for } v \in C([0,a]),$$

equivalent to the usual supremum norm, were used in order to establish the global Picard-Lindelöf theorem for ordinary differential equations: if the constant λ is sufficiently large, the integral operator becomes $\|\cdot\|_\lambda$-contraction with no restriction on the existence interval $[0,a]$. Based on the observation that the weight function $\psi : [0,a] \longrightarrow \mathbb{R}_+$ (alike $\psi(t) = \exp(\lambda t)$ in the above ordinary differential equation case) have to fulfill a comparison integral inequality such as

$$\theta\psi(t) \geq 1 + \int_0^t L\psi(s)\, ds \qquad \big(\theta \in (0,1)\big),$$

we construct Bielecki's norms in our partial differential equation case. Next we find closed, convex subspaces of functions which are mapped to itself by the operator $u \to \mathcal{P}^{-1}\mathcal{F}[u]$ where

$$\mathcal{F}[u](t,x) = f(t,x,u_{(t,x)}).$$

In Section 2 we study the Lipschitz-type case, but the Lipschitz constants are replaced by L^1-functions. The kind of the Volterra functional dependence and allowed derivatives are specified by the choice of suitable norms. We study there the following main cases:

(i) No dependene on ∂u in equation (1).

(ii) A pointwise dependence on ∂u in equation (1).

(iii) A full functional dependence on ∂u in equation (1).

In the case (ii), there are no additional assumptions on the initial function φ. The choice of Bielecki's norms is motivated by comparison integral equations (see Theorem 2.2). Section 3 is devouted to classical solutions. We give sufficient conditions that the mapping

$$E \ni (t,x) \longmapsto g(t,x) = f(t,x,u_{(t,x)}) \in \mathbb{R}$$

be locally Hölder continuous in x (cf. [15, 16]). The functional dependence makes serious difficulties especially at $t = 0$. The usual Lipschitz condition for f is insufficient. There are the following two ways of remedy:

1^0 Assuming some regularity of the initial function φ, for instance a local Hölder condition.

2^0 Modifying the Lipschitz or Hölder condition of the function f according to singularities of u at $t = 0$.

In the last section, Section 4, we give some existence results proved by means of the Schauder fixed point theorem. We distinguish two cases: with ∂u and without ∂u in f.

1.1 Formulation of the problem. Let

$$\begin{aligned} E &= (0,a] \times \mathbb{R}^n \\ E_0 &= [-\tau_0, 0] \times \mathbb{R}^n \\ \widetilde{E} &= E_0 \cup E \\ B &= [-\tau_0, 0] \times [-\tau, \tau] \end{aligned}$$

where $a > 0$, $\tau_0, \tau_1, \ldots, \tau_n \in \mathbb{R}_+ = [0, +\infty)$ and

$$\tau = (\tau_1, \ldots, \tau_n), \qquad [-\tau, \tau] = [-\tau_1, \tau_1] \times \ldots \times [-\tau_n, \tau_n].$$

The Hale-type functional $u_{(t,x)} : \ B \longrightarrow \mathbb{R} \ \ ((t,x) \in E)$ is defined by

$$u_{(t,x)}(s,y) = u(t+s, x+y) \qquad ((s,y) \in B).$$

Let $C(X)$ be the class of all continuous functions from a metric space X into $\mathbb{R}$, and let $CB(X)$ and $CB(X)^n$ be the classes of all continuous and bounded functions from

X into $\mathbb{R}$ and $\mathbb{R}^n$, respectively. Denote by ∂_0 and $\partial_1, \ldots, \partial_n$ the operators of partial derivatives with respect to t and $x_1, \ldots, x_n$, respectively. Let $\partial = (\partial_1, \ldots, \partial_n)$ and $\partial_{jl} = \partial_j \partial_l \ \ (j, l = 1, 2, \ldots, n)$. The operator $\mathcal{P}$ is defined by

$$\mathcal{P}u(t,x) = \partial_0 u(t,x) - \sum_{j,l=1}^{n} a_{jl}(t,x)\partial_{jl}u(t,x).$$

Suppose that $f : E \times C(B) \longrightarrow \mathbb{R}$ and $\varphi : E_0 \longrightarrow \mathbb{R}$ are given functions. Consider the Cauchy problem

$$\mathcal{P}u(t,x) = f(t,x,u_{(t,x)}) \tag{1}$$

$$u(t,x) = \varphi(t,x) \quad \text{on } E_0. \tag{2}$$

It is transformed into the integral equation

$$u(t,x) = \int_{\mathbb{R}^n} \Gamma(t,x,0,y)\,\varphi(0,y)\,dy + \int_0^t\!\!\int_{\mathbb{R}^n} \Gamma(t,x,s,y) f(s,y,u_{(s,y)})\,dyds \tag{3}$$

where $\Gamma(t,x,\sigma,\xi)$ is the fundamental solution of the above parabolic problem.

Definition 1.1. Let $u \in C(\widetilde{E})$.

(i) u is called a *classical solution* of problem (1)-(2) (in other words: a $C^{1,2}$ solution) if $\partial_0 u, \partial_j u, \partial_{jl} u \in C(E)$, u satisfies equation (1) on E and initial condition (2) on E_0.

(ii) u is called a C^0 *solution* of problem (1)-(2) if u coincides with φ on E_0 and it satisfies integral equation (3) on E.

(iii) u is called a $C^{0,1}$ solution of problem (1)-(2) if u is a C^0 solution whose derivatives $\partial_j u \ \ (j = 1, \ldots, n)$ are continuous on E.

The notion of $C^0, C^{0,1}, C^{1,2}$ *weak solutions* require only the existence of partial derivatives, being not necessarily continuous.

We are looking for $C^0, C^{0,1}, C^{1,2}$ weak and strong solutions to problem (1)-(2). The functional dependence has Volterra type, which can be easily recognized by location of the domain $B \subset E_0$, i.e., to the left from $t = 0$. Because the present paper mainly deals with bounded solutions, almost all results carry over to the limit case $B = E_0$. Nevertheless, an unbounded domain of the shift operator $u_{(t,x)}$ causes some problems with continuity, compactness, etc. For this reason we analyze the Cauchy problem with bounded domain B. In [16] some existence and comparison problems for some kinds of the data with unbounded growth as $\|x\| \to \infty$ are treated. These investigation show the complexity of the whole problem even for parabolic equations without functional dependence. Another topic concerning the limit behaviour of solutions, generated by nonlinearities with respect to u, is studied in [15].

1.2 Fundamental solution of problem (1)-(2). The supremum norm will be denoted by $\|\cdot\|_0$ while the symbol $\|\cdot\|$ stands for the Euclidean norm.

Assumption 1.1. Suppose $a_{jl} \in CB(E)$ $(j,l = 1,\dots,n)$, the operator $\mathcal{P}$ is parabolic, i.e.,

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \geq c'\|\xi\|^2 \qquad \big((t,x)\in E, \xi \in \mathbb{R}^n\big)$$

and the coefficients a_{ij} satisfy the Hölder condition

$$\big|a_{ij}(t,x) - a_{ij}(\tilde{t},\tilde{x})\big| \leq c''\big(|t-\tilde{t}|^{\frac{\alpha}{2}} + \|x-\tilde{x}\|^{\alpha}\big) \qquad (i,j = 1,\dots,n)$$

where $c', c'' > 0$.

Lemma 1.1. *If Assumption* 1.1 *holds, then there are constants* $k_0 > 0$ *and* $c_0, c_1, c_2 > 0$ *such that*

$$\begin{aligned}
|\Gamma(t,x,\sigma,\xi)| &\leq c_0(t-\sigma)^{-\frac{n}{2}} \exp\left(-\frac{k_0\|x-\xi\|^2}{4(t-\sigma)}\right)\\
|\partial_j\Gamma(t,x,\sigma,\xi)| &\leq c_1(t-\sigma)^{-\frac{n+1}{2}} \exp\left(-\frac{k_0\|x-\xi\|^2}{4(t-\sigma)}\right)\\
|\partial_0\Gamma(t,x,\sigma,\xi)|, |\partial_{jl}\Gamma(t,x,\sigma,\xi)| &\leq c_2(t-\sigma)^{-\frac{n+2}{2}} \exp\left(-\frac{k_0\|x-\xi\|^2}{4(t-\sigma)}\right)
\end{aligned}$$

for all $0 \leq \sigma < t \leq a$ *and* $x,\xi \in R^n,\ j,l = 1,\dots,n.$

Proof. A priori estimates for the fundamental solution and its partial derivatives can be found in [5, 8, 12].

Remark 1.1. Under Assumption 1.1 we obtain for the fundamental solution $\Gamma(t,x,\sigma,\xi)$ of problem (1)-(2) the more general Hölder-type estimates with any Hölder exponent $\delta \in (0,1]$

$$\begin{aligned}
&\big|\Gamma(t,x,\sigma,\xi) - \Gamma(\bar{t},\bar{x},\sigma,\xi)\big|\\
&\qquad \leq c_{0+\delta}(t-\sigma)^{-\frac{n+\delta}{2}} \exp\left(-\frac{k_0\|x-\xi\|^2}{4(t-\sigma)}\right)\big[|t-\bar{t}|^{\frac{\delta}{2}} + \|x-\bar{x}\|^{\delta}\big]\\
&\big|\partial_j\Gamma(t,x,\sigma,\xi) - \partial_j\Gamma(\bar{t},\bar{x},\sigma,\xi)\big|\\
&\qquad \leq c_{1+\delta}(t-\sigma)^{-\frac{n+1+\delta}{2}} \exp\left(-\frac{k_0\|x-\xi\|^2}{4(t-\sigma)}\right)\big[|t-\bar{t}|^{\frac{\delta}{2}} + \|x-\bar{x}\|^{\delta}\big]
\end{aligned}$$

for $0 \leq \sigma < t \leq \bar{t} \leq a$ and $x,\bar{x},\xi \in R^n,\ j,l = 1,\dots,n$ ∎

Lemma 1.2. *If* $\varphi \in CB(E_0)$, *then there exists a classical solution* $\widetilde{\varphi} \in CB(\widetilde{E})$ *of the problem*

$$\left.\begin{aligned}\mathcal{P}u &= 0\\ u &\succ \varphi\end{aligned}\right\}$$

where the symbol $u \succ \varphi$ *means the same as* $u(t,x) = \varphi(t,x)$ *for* $(t,x) \in E_0$.

Proof. This is a basic existence result for the Cauchy problem, see [5, 6, 8] ∎

We introduce also the following notation that will be used throughout the paper. Let $\widetilde{c_0} = c_0(\frac{4\pi}{k_0})^{\frac{n}{2}}$, $\widetilde{c_1} = c_1(\frac{4\pi}{k_0})^{\frac{n}{2}}$ and set

$$
\begin{aligned}
S_0(t) &= \frac{1}{\sqrt{t}} && (t>0)\\
f_\varphi(t,x) &= f(t,x,\widetilde{\varphi}_{(t,x)}) && ((t,x)\in E)\\
(g_1 * g_2)(t) &= \int_0^t g_1(t-s)\, g_2(s)\, ds && (t>0)
\end{aligned}
$$

where $g_1, g_2 \in L^1_{loc}(\mathbb{R}_+)$. In particular, $(1*g)(t) = \int_0^t g(s)\,ds$ for $g \in L^1_{loc}(\mathbb{R}_+)$ where the symbol $L^p_{loc}(\mathbb{R}_+)$ stands for the space of real locally integrable functions on $\mathbb{R}_+$ with exponent $p \geq 1$.

2. Existence and uniqueness

Let $L^1[0,a]$ denote the set of all real integrable functions on $[0,a]$. Define the operator $\mathcal{T}$ as follows: if $u \succ \varphi$, then $\mathcal{T}u \succ \varphi$ is determined on E by the right-hand side of integral equation (3).

Theorem 2.1. *Let $\varphi \in CB(E_0)$, $\lambda, m_f, f(\cdot,x,0) \in L^1[0,a]$ and $f(t,\cdot,0) \in C(R^n)$. Assume that $|f(t,x,0)| \leq m_f(t)$ and*

$$
\big|f(t,x,w) - f(t,x,\bar{w})\big| \leq \lambda(t)\, \|w-\bar{w}\|_0 \qquad \text{on } E\times C(B). \tag{4}
$$

Then there exists a unique bounded C^0 solution to problem (1) – (2).

Proof. We show that the operator $\mathcal{T}$ defined by the right-hand side of (3) is a contraction from $CB(\widetilde{E})$ into itself. Take $u, \bar{u} \in CB(\widetilde{E})$ and $(t,x) \in E$. From (4) and Lemma 1.1 we have

$$
\begin{aligned}
&\big|\mathcal{T}u(t,x) - \mathcal{T}\bar{u}(t,x)\big|\\
&\quad\leq \int_0^t\!\!\int_{\mathbb{R}^n} |\Gamma(t,x,s,y)|\, \big|f(s,y,u_{(s,y)}) - f(s,y,\bar{u}_{(s,y)})\big|\, dyds\\
&\quad\leq \int_0^t\!\!\int_{\mathbb{R}^n} c_0(t-s)^{-\frac{n}{2}} \exp\Big(-\frac{k_0\|x-y\|^2}{4(t-s)}\Big)\lambda(s)\Big\|\frac{u-\bar{u}}{\psi}\Big\|_0 \psi(s)\, dyds
\end{aligned}
$$

where the non-decreasing function $\psi \in C[0,a]$ satisfies the equation

$$
\psi(t) = 1 + \tfrac{1}{\theta}\int_0^t \widetilde{c_0}\, \lambda(s)\, \psi(s)\, ds
$$

with some $\theta \in (0,1)$. We change the variables $\eta = \frac{\sqrt{k_0}(x-y)}{2\sqrt{t-s}}$ to obtain

$$
\big|\mathcal{T}u(t,x) - \mathcal{T}\bar{u}(t,x)\big| \leq \int_0^t \widetilde{c_0}\lambda(s)\Big\|\frac{u-\bar{u}}{\psi}\Big\|_0 \psi(s)\, ds \leq \theta\, \psi(t)\Big\|\frac{u-\bar{u}}{\psi}\Big\|_0.
$$

Therefore, we have the contraction with respect to a Bielecki's type norm

$$\left\| \frac{\mathcal{T}u - \mathcal{T}\bar{u}}{\psi} \right\|_0 \leq \theta \left\| \frac{u - \bar{u}}{\psi} \right\|_0.$$

The boundedness and continuity of $\mathcal{T}u$ at $t = 0$ follows from the estimate

$$\left|\mathcal{T}u(t,x) - \widetilde{\varphi}(t,x)\right| \leq \int_0^t \widetilde{c_0}\, \psi(s)\, \lambda(s) \left\| \frac{u}{\psi} \right\|_0 ds + \int_0^t \widetilde{c_0}\, m_f(s)\, ds.$$

Thus the operator $\mathcal{T}$ maps $CB(\widetilde{E})$ into itself, and the proof is complete ∎

Proposition 2.1. *Let $\varphi \in CB(E_0)$, $\lambda, m_{f,\varphi}, f(\cdot, x, 0) \in L^1[0,a]$ and $f(t,\cdot,0) \in C(R^n)$ be such that*

$$|f_\varphi(t,x)| \leq m_{f,\varphi}(t).$$

Assume that condition (4) *holds true. Then the function u obtained in Theorem* 2.1 *satisfies the inequality*

$$|(u - \widetilde{\varphi})(t,x)| \leq \gamma_0(t) := \int_0^t \widetilde{c}_0\, m_{f,\varphi}(s)\, \exp\left(\frac{\widetilde{c_0}}{\theta} \int_s^t \lambda(\sigma)\, d\sigma \right) ds, \tag{5}$$

where $\theta \in (0,1)$. If $S_0 * m_{f,\varphi}$ and $S_0 * (\lambda \cdot \gamma_0)$ are continuous in $(0,a]$, then the derivative ∂u exists on E, and the function u is a $C^{0,1}$ weak solution. If additionally we assume

$$\lim_{\varepsilon \searrow 0} \varepsilon \int_0^{t(1-\varepsilon^2)} \left\{ m_{f,\varphi}(s) + \lambda(s)\gamma_0(s) \right\} (t-s)^{-1} ds = 0 \qquad (t > 0), \tag{6}$$

then u is a $C^{0,1}$ solution.

Proof. Estimate (5) for a C^0-solution u, whose existence follows from Theorem 2.1, is obtained similarly as in the proof of the previous theorem. Condition (6) implies continuous differentiability of u in x. Thus the C^0-solution u becomes a $C^{0,1}$-solution of problem (1)-(2) ∎

Example 2.1. We explain the sense of condition (6) which seems to be technical. Observe that this condition (as well as all assumptions of Proposition 2.1) is satisfied when $m_{f,\varphi}(t)$ and $\lambda(t)$ are constant, i.e., in the Lipschitz case. We generalize this simple example. Let $\varphi \in CB(E_0)$ and $|f_\varphi(t,x)| \leq m_{f,\varphi}(t) := Mt^{-\kappa'}$ where $\kappa' \in [0,1)$ and $M > 0$. Assume that the function f satisfies condition (4) where $\lambda(t) = Lt^{-\kappa}$ with $\kappa \in [0,1)$ and $L > 0$. Thus the function γ_0 from (5) is given by

$$\gamma_0(t) = c_0 \frac{M}{1-\kappa'}\, t^{1-\kappa'} \exp\left(\frac{\widetilde{c}_0 L}{\theta(1-\kappa)}\, t^{1-\kappa} \right).$$

In this case the functions $S_0 * m_{f,\varphi}$ and $S_0 * (\lambda \cdot \gamma_0)$ are continuous in $(0,a]$, and condition (6) holds if the integrals

$$I_1 = \varepsilon M \int_0^{t(1-\varepsilon^2)} s^{-\kappa'} (t-s)^{-1} ds$$

$$I_2 = \varepsilon \widetilde{c}_0 \frac{\kappa}{1-\kappa'} L \int_0^{t(1-\varepsilon^2)} s^{1-\kappa} s^{-\kappa'} (t-s)^{-1} ds$$

tend to 0 as $\varepsilon \to 0$. Indeed, considering I_1 and substituting $\eta = \frac{s}{t}$ we get

$$I_1 = \varepsilon M\, t^{-\kappa'} \left\{ \int_0^r \frac{1}{\eta^{\kappa'}(1-\eta)}\, d\eta + \int_r^{1-\varepsilon^2} \frac{1}{\eta^{\kappa'}(1-\eta)}\, d\eta \right\}$$

where $r = \frac{\kappa'}{\kappa'+1}$. Denote by $I_{1,1}$ and $I_{1,2}$ the integrals in the brackets. The integral $I_{1,1}$ is bounded. The integral $I_{1,2}$ has an estimate proportional to $|\ln \varepsilon|$. Thus $I_1 \to 0$ as $\varepsilon \to 0$. The convergence of the integral I_2 can be proved in a similar way.

Let C_0 denote the set of all continuous functions $\psi : [0,a] \to \mathbb{R}$ such that $\psi(0) = 0$, and let C_0^+ denote the set of all non-decreasing functions $\psi \in C_0$.

Theorem 2.2. *Let $\varphi \in CB(E_0)$ and $\lambda, \lambda_1, m_{f,\varphi} \in L^1[0,a]$, $f_\varphi(\cdot, x) \in L^1[0,a]$, $f_\varphi(t,\cdot) \in C(R^n)$. Assume that the functions $S_0 * \lambda_1$ and $S_0 * m_{f,\varphi}$ are bounded, $\lambda_1(t) = \lambda(t)\sqrt{t}$, $|f_\varphi(t,x)| \le m_{f,\varphi}(t)$ and*

$$\big|f(t,x,w) - f(t,x,\bar{w})\big| \le \lambda(t)\, \|w - \bar{w}\|_0 + \lambda_1(t)\, \|\partial(w-\bar{w})(0,0)\|_0 \tag{7}$$

$$(t-s)^{\frac{1}{2}} \int_s^t \frac{c_0}{c_1}\, \lambda_1(s)\, (t-\zeta)^{-\frac{1}{2}} (\zeta - s)^{-\frac{1}{2}} d\zeta \le \theta_1 < 1 \quad (t > s). \tag{8}$$

Then there is a unique $C^{0,1}$ weak solution to problem (1) – (2).

Proof. Let $\psi, \psi_1 \in C_0^+$ and $\psi_1(t) = \frac{\psi(t)}{k_1\sqrt{t}}$ where $k_1 = \frac{\widetilde{c_0} + \widetilde{c_1}}{\theta}$. Define

$$\mathcal{X}_{\varphi,\psi,\psi_1} = \left\{ u \in CB(\widetilde{E}) \;\middle|\; \begin{array}{l} u \succ \varphi \text{ and } |(u - \widetilde{\varphi})(t,x)| \le \psi(t) \\ |\partial_j (u - \widetilde{\varphi})(t,x)| \le \psi_1(t) \quad (j = 1,\dots,n) \end{array} \right\}. \tag{9}$$

Suppose that the functions ψ, ψ_1 are solutions of the system of inequalities

$$\int_0^t \tilde{c}_0 \Big\{ m_{f,\varphi}(s) + \lambda(s)\psi(s) + \lambda_1(s)\psi_1(s) \Big\} ds \le \theta\, \psi(t) \tag{10}$$

$$\int_0^t \tilde{c}_1 (t-s)^{-\frac{1}{2}} \Big\{ m_{f,\varphi}(s) + \lambda(s)\psi(s) + \lambda_1(s)\psi_1(s) \Big\} ds \le \theta\, \psi_1(t). \tag{11}$$

Take $u, \bar{u} \in \mathcal{X}_{\varphi,\psi,\psi_1}$ and $(t,x) \in E$. Applying (7), (8) and (10) we have

$$\begin{aligned} &\big|\mathcal{T}u(t,x) - \mathcal{T}\bar{u}(t,x)\big| \\ &\quad \le \int_0^t \!\! \int_{\mathbb{R}^n} |\Gamma(t,x,s,y)|\, \big|f(s,y,u_{(s,y)}) - f(s,y,\bar{u}_{(s,y)})\big|\, dy ds \\ &\quad \le \int_0^t \widetilde{c_0} \Big\{ \lambda(s) \Big\| \frac{u - \bar{u}}{\psi} \Big\|_0 \psi(s) + \lambda_1(s) \Big\| \frac{\partial(u - \bar{u})}{\psi_1} \Big\|_0 \psi_1(s) \Big\} ds \\ &\quad \le \theta\, \psi(t)\, \|u - \bar{u}\|_* \end{aligned}$$

where $\|u-\bar{u}\|_* = \max\left\{\left\|\frac{u-\bar{u}}{\psi}\right\|_0, \left\|\frac{\partial(u-\bar{u})}{\psi_1}\right\|_0\right\}$. Similarly, using (7), (8) and (11) we have

$$\begin{aligned}
&\left|\partial_j(\mathcal{T}u - \mathcal{T}\bar{u})(t,x)\right| \\
&\le \int_0^t\!\!\int_{\mathbb{R}^n} |\partial\Gamma(t,x,s,y)|\left|f(s,y,u_{(s,y)}) - f(s,y,\bar{u}_{(s,y)})\right| dyds \\
&\le \int_0^t \widetilde{c}_1(t-s)^{-\frac{1}{2}}\left\{\lambda(s)\left\|\frac{u-\bar{u}}{\psi}\right\|_0\psi(s) + \lambda_1(s)\left\|\frac{\partial(u-\bar{u})}{\psi_1}\right\|_0\psi_1(s)\right\}ds \\
&\le \theta\,\psi_1(t)\,\|u-\bar{u}\|_*.
\end{aligned}$$

If $u \in \mathcal{X}_{\varphi,\psi,\psi_1}$ and conditions (10) - (11) are satisfied, then

$$\begin{aligned}
\left|\mathcal{T}u(t,x) - \widetilde{\varphi}(t,x)\right| &\le \int_0^t\!\!\int_{\mathbb{R}^n} |\Gamma(t,x,s,y)|\,|f(s,y,u_{(s,y)})|\,dyds \\
&\le \int_0^t \widetilde{c}_0\Big\{m_{f,\varphi}(s) + \lambda(s)\psi(s) + \lambda_1(s)\psi_1(s)\Big\}ds \\
&\le \psi(t) \\
\left|\partial_j(\mathcal{T}u - \widetilde{\varphi})(t,x)\right| &\le \int_0^t\!\!\int_{\mathbb{R}^n} |\partial_j\Gamma(t,x,s,y)|\,|f(s,y,u_{(s,y)}|\,dyds \\
&\le \int_0^t \widetilde{c}_1\frac{1}{\sqrt{t-s}}\Big\{m_{f,\varphi}(s) + \lambda(s)\psi(s) + \lambda_1(s)\psi_1(s)\Big\}ds \\
&\le \psi_1(t).
\end{aligned}$$

Therefore the operator $\mathcal{T}$ maps $\mathcal{X}_{\varphi,\psi,\psi_1}$ into itself. By virtue of the Banach contraction principle the operator $\mathcal{T}$ has exactly one fixed point $u \in \mathcal{X}_{\varphi,\psi,\psi_1}$. The proof of Theorem 2.2 is complete ∎

Remark 2.1. Condition (8) plays a crucial role in the proof of Theorem 2.2. This condition implies the solvability of system (10) - (11), which can be reduced to the single equation

$$\int_0^t \tilde{c}_1(t-s)^{-\frac{1}{2}}\Big\{m_{f,\varphi(s)} + \lambda_1(s)\psi_1(s)[1+k_1]\Big\}ds = \psi_1(t). \tag{12}$$

The Lipschitz case, that is $\lambda(t) = \text{const}$ and $\lambda_1(t) = \text{const}$, implies condition (8) for sufficiently small values of $t-s$. The right-hand side of the Lipschitz-type condition (7) indicates the functional dependence in equation (1) on the past and spatial values of the unknown function u, whereas the derivative ∂u appears only at the point (t,x). In particular, there is no need to assume differentiability of the initial function φ. However, the derivative ∂u may have a singularity at $t=0$.

Example 2.2. Let $\varphi \in CB(E_0)$ and $|f_\varphi(t,x)| \le m_{f,\varphi}(t) := Mt^{-\kappa}$ where $\kappa \in [0,1)$ and $M > 0$. Assume that the function f satisfies condition (7) where $\lambda(t) = Lt^{-\kappa}$ with $\kappa \in [0,1)$ and $L > 0$. Equation (12) has the form

$$\int_0^t \tilde{c}_1(t-s)^{-\frac{1}{2}}\Big\{Ms^{-\kappa} + Ls^{-\kappa-\frac{1}{2}}\psi_1(s)[1+k_1]\Big\}ds = \psi_1(t).$$

This equation is solvable locally for small values of $t - s$. Any solution ψ_1 can be extended to the whole interval. Global solvability of this equation may be obtained directly via the von Neumann expansion.

Theorem 2.2 concerns equations with various types of Volterra functional dependence at the unknown function. The derivatives have the classical form (without functional dependence). Now we consider equation (1) with functionals at the derivatives. Therefore, we need differentiability of the initial function φ with respect to x.

Theorem 2.3. *Let $\varphi \in CB(E_0)$, $\partial\varphi \in CB(E_0)^n$, $\lambda, \lambda_1, m_{f,\varphi} \in L^1[0,a]$ and $f_\varphi(t,\cdot) \in C(R^n)$. Suppose that $|f_\varphi(t,x)| \le m_{f,\varphi}(t)$ and*

$$\big|f(t,x,w) - f(t,x,\bar{w})\big| \le \lambda(t)\|w-\bar{w}\|_0 + \lambda_1(t)\|\partial(w-\bar{w})\|_0.$$

Assume that the functions $S_0 * \lambda_1$ and $S_0 * m_{f,\varphi}$ are bounded, and condition (8) of Theorem 2.2 is satisfied. Then problem (1) – (2) has a unique $C^{0,1}$ solution.

Proof. Let ψ and ψ_1 satisfy inequalities (10) and (11), respectively. Consider the set $\mathcal{X}_{\varphi,\psi,\psi_1}$ given by (9). Similarly as in the proof of Theorem 2.2 we see that the operator $\mathcal{T}$ is a contraction. The continuity and boundedness of φ imply $\widetilde{\varphi} \in CB(\widetilde{E})$. By virtue of the assumption $\partial\varphi \in CB(E_0)^n$ we have also $\partial\widetilde{\varphi} \in CB(\widetilde{E})^n$. Hence the operator $\mathcal{T}$ maps $\mathcal{X}_{\varphi,\psi,\psi_1}$ into itself. Now we show the continuity of ∂u in x. We have

$$\begin{aligned}&\big|\partial_j u(t,x) - \partial_j u(t,\bar{x})\big| \\ &\quad \le \big|\widetilde{\varphi}(t,x) - \widetilde{\varphi}(t,\bar{x})\big| \\ &\qquad + \left|\int_0^t\!\!\int_{\mathbb{R}^n} \Big\{\partial_j\Gamma(t,x,s,y) - \partial_j\Gamma(t,\bar{x},s,y)\Big\} f(s,y,u_{(s,y)})\,dyds\right|.\end{aligned}$$

The last integral is estimated by

$$\int_0^t \widetilde{c}_{1+\delta}\|x-\bar{x}\|^\delta (t-s)^{-\frac{1+\delta}{2}} \Big\{m_{f,\varphi}(s) + \lambda(s)\|u-\widetilde{\varphi}\|_0 + \lambda_1(s)\|\partial(u-\widetilde{\varphi})\|_0\Big\} ds$$

The continuity of ∂u in t and of u at 0 can be proved in a similar way ∎

Remark 2.2. Suppose that $\frac{S_0^{1+\delta} * m_{f,\varphi}}{S_0^{\delta'}} \in L^\infty[0,a]$ for $0 \le \delta' < 1$ is satisfied in Theorem 2.2. In this case we obtain a $C^{0,1}$ solution to problem (1)-(2), but the derivative ∂u is not necessarily continuous at $t = 0$.

3. Classical solutions

In the present section, we give some sufficient conditions for the existence of classical ($C^{1,2}$) solutions to problem (1)-(2). We start with the simplest case when the right-hand side is independent of the unknowns function u.

Theorem 3.1. *Let $\varphi \in CB(E_0)$ and $g \in CB(E)$. Suppose that g is locally Hölder continuous in x with Hölder exponent $\delta \in (0,1]$. If $u \in CB(\widetilde{E})$ is a C^0 solution to the Cauchy problem*

$$\left.\begin{aligned} \mathcal{P}u(t,x) &= g(t,x) \\ u &\succ \varphi \end{aligned}\right\}, \tag{13}$$

then u is a classical solution ($C^{1,2}$).

We omit the proof of this theorem, for details we refer to [15, 16].

Remark 3.1. Instead of the boundedness of the functions g and φ it suffices to assume in Theorem 3.1 the growth condition $|g(t,x)|, |\varphi(t,x)| \le c \exp(\kappa\|x\|^2)$ where $\kappa \in [0,1)$.

If we put $g(t,x) = f(t,x,u_{(t,x)})$, then the differential-functional problem (1)-(2) coincides with problem (13). Hence the differentiability of weak solutions may be reduced to a local Hölder continuity condition of $g(t,\cdot)$. Unfortunately, the Hölder constant strongly depend on the unknown function which makes this problem difficult. We give sufficient conditions for the existence of classical solutions to problem (1)-(2).

Theorem 3.2. *Suppose that $\varphi \in CB(E_0)$ is locally Hölder continuous in x with exponent $\delta \in (0,1]$. Assume that the function f is locally δ-Hölder continuous in x and locally Lipschitz continuous in $w \in C(B)$. If $u \in CB(\widetilde{E})$ is a C^0 solution to problem $(1)-(2)$ such that*

$$|f(t,x,u_{(t,x)})| \le m(t) \qquad \big(m \in L^1[0,a], S_0^\delta * m \in L^\infty[0,a]\big), \tag{14}$$

then u is a classical solution ($C^{1,2}$) to problem $(1)-(2)$.

Proof. Observe that, if φ is δ-Hölder continuous in x, then $\widetilde{\varphi}$ is also δ-Hölder continuous in x. Hence the function u is also δ-Hölder in x. Since $S_0^\delta * m \in L^\infty[0,a]$, we have

$$\begin{aligned} &\big|u(t,x) - u(t,\bar{x})\big| \\ &\quad \le \big|\widetilde{\varphi}(t,x) - \widetilde{\varphi}(t,\bar{x})\big| \\ &\qquad + \|x-\bar{x}\|^\delta \int_0^t\!\!\int_{\mathbb{R}^n} c_{0+\delta}(t-s)^{-\frac{n+\delta}{2}} \exp\left(-\frac{k_0\|x-y\|^2}{4(t-s)}\right) m(s)\,dyds \\ &\quad \le L_2(x,\bar{x};u)\,\|x-\bar{x}\|^\delta \end{aligned}$$

where $L_2(x,\bar{x};u)$ is a Hölder coefficient. Now we explain why the function g is locally Hölder continuous in x:

$$\begin{aligned} &\big|g(t,x) - g(t,\bar{x})\big| \\ &\quad \le \big|f(t,x,u_{(t,x)}) - f(t,\bar{x},u_{(t,x)})\big| + \big|f(t,\bar{x},u_{(t,x)}) - f(t,\bar{x},u_{(t,\bar{x})})\big| \\ &\quad \le L_0(x,\bar{x};u)\|x-\bar{x}\|^\delta + L_1(x,\bar{x};u)\|u_{(t,x)} - u_{(t,\bar{x})}\| \\ &\quad \le L_0(x,\bar{x};u)\|x-\bar{x}\|^\delta + L_1(x,\bar{x};u)L_2(x,\bar{x};u)\|x-\bar{x}\|^\delta \end{aligned}$$

where $L_0(x,\bar{x};u)$ and $L_1(x,\bar{x};u)$ are some constants from the local Hölder and local Lipschitz continuity of the function f. By virtue of Theorem 3.1, the function u is a classical solution, which completes the proof ∎

Remark 3.2. The local Lipschitz and Hölder constants in Theorem 3.2 are independent of t. The result remains true if we multiply these constants by $m(t)$ (cf. (14)). Instead of the boundedness of φ and f one can assume their exponential growth like $\exp(C\|x\|^2)$ as $\|x\| \to \infty$. If we skip the local Hölder condition for the initial function φ, then the functions $\widetilde{\varphi}$ and u become less regular at $t=0$. Consequently, the functional dependence for the function f have to be restricted to the set E. The local Hölder and Lipschitz conditions will be modified as

$$\begin{aligned} &\big|f(t,x,w)-f(t,\bar{x},\bar{w})\big| \\ &\qquad \le m(t)\left\{\|x-\bar{x}\|^\delta + \|w-\bar{w}\|_{<t;\delta>}\right\}\omega_0\big(\|x\|+\|\bar{x}\|, \|w\|_0+\|\bar{w}\|_0\big) \end{aligned} \tag{15}$$

where $\omega_0 : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a non-decreasing function, and

$$\|w\|_{<t;\delta>} = \sup_{\sigma\in(-t,0]} \big\|w(\sigma,\cdot)(t+\sigma)^{\frac{\delta}{2}}\big\|_0.$$

The term $\|w-\bar{w}\|_{<t;\delta>}$ in the local Hölder-Lipschitz condition (15) can be also replaced by

$$\theta\|w-\bar{w}\|_{<t;\delta>} + (1+\theta)\|w-\bar{w}\|^\delta_{<t;\frac{1}{2}>} \qquad (\theta\in[0,1]).$$

We prove the local Hölder condition for $g(t,x) = f(t,x,u_{(t,x)})$:

$$\begin{aligned} &\big|g(t,x)-g(t,\bar{x})\big| \\ &\quad \le \big|f(t,x,u_{(t,x)}) - f(t,\bar{x},u_{(t,\bar{x})})\big| \\ &\quad \le m(t)\left\{\|x-\bar{x}\|^\delta + \|u_{(t,x)}-u_{(t,\bar{x})}\|_{<t;\delta>}\right\}\omega_0\big(\|x\|+\|\bar{x}\|, \|u_{(t,x)}\|_0+\|u_{(t,\bar{x})}\|_0\big) \\ &\quad \le m(t)\,\|x-\bar{x}\|^\delta\left\{1+\omega_1(\|x\|+\|\bar{x}\|+2\|\tau\|)\right\}\omega_0\big(\|x\|+\|\bar{x}\|, 2m(t)\big) \end{aligned}$$

since for all $(\sigma,\xi)\in B$ such that $t+\sigma>0$ we have

$$\big|(u_{(t,x)}-u_{(t,\bar{x})})(\sigma,\xi)\big| \le \omega_1\big(\|x\|+\|\bar{x}\|+2\|\xi\|\big)\frac{\|x-\bar{x}\|^\delta}{(t+\sigma)^{\frac{\delta}{2}}}$$

with some non-decreasing function $\omega_1 : \mathbb{R}_+ \to \mathbb{R}_+$.

We summarize these considerations in the following two theorems.

Theorem 3.3. *Suppose that $\varphi \in CB(E_0)$ and that f satisfies (14) and (15). Then any C^0 solution u of problem (1) – (2) is a classical ($C^{1,2}$) solution.*

Theorem 3.4. *Assume that $\varphi\in CB(E_0)$ and $\partial\varphi \in CB(E_0)^n$. Suppose that $m \in L^1[0,a]$, $S_0^{1+\delta} * m \in L^\infty[0,a]$, and $\omega_0 : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a non-decreasing function such that*

$$\begin{aligned} &\big|f(t,x,w)-f(t,\bar{x},\bar{w})\big| \\ &\qquad \le m(t)\left\{\|x-\bar{x}\|^\delta + \|w-\bar{w}\|^*_{<t;\delta>}\right\}\omega_0\big(\|x\|+\|\bar{x}\|, \|w\|_{0,1}+\|\bar{w}\|_{0,1}\big) \end{aligned}$$

where $\|w\|_{0,1} = \|w\|_{0,1} + \|\partial w\|_{0,1}$ *and*

$$\|w\|^*_{<t;\delta>} = \|w\|_{0,1} + \theta\|\partial w\|_{<t;\delta>} + (1-\theta)\|\partial w\|_{<t;\frac{1}{2}>} \qquad (\theta \in [0,1]).$$

Then any $C^{0,1}$ *solution* $u \in CB(\widetilde{E})$ *to problem* $(1)-(2)$ *is a classical solution.*

Remark 3.3. The condition $\partial\varphi \in CB(E_0)^n$ can be weakened. If we omit this condition, then we have to take into consideration singularities at $t = 0$. The norm $\|\cdot\|^*_{<t;\delta>}$ in the above local Hölder-Lipschitz condition should be replaced by

$$\|w\|^{**}_{<t;\delta>} = \|w\|_{<t;\delta>} + \theta\|\partial w\|_{<t;1+\delta>} + (1-\theta)\|\partial w\|^{\delta}_{<t;2>}.$$

4. Existence of solutions

Let $\mathcal{X}_\varphi = \{u \in CB(\widetilde{E}) : u(t,x) \succ \varphi\}$ and denote by $\|u\|_t$ the seminorms

$$\|u\|_t = \sup\{u(s,y) : (s,y) \in \widetilde{E},\, s \le t\} \qquad (t \in (0,a]).$$

Theorem 4.1. *Suppose that* $\varphi \in CB(E_0)$ *and that there are* $m, \lambda \in L^1[0,a]$ *such that*

$$|f(t,x,w)| \le m(t) + \lambda(t)\,\|w\|_0 \qquad \text{on } E \times C(B)$$

where $f(\cdot,x,w) \in L^1[0,a]$ *and* f *is continuous in* $(x,w) \in \mathbb{R}^n \times C(B)$*. Assume that* $\frac{S_0^\delta * m}{S^{\delta'}}, \frac{S_0^\delta * \lambda}{S^{\delta'}} \in L^\infty[0,a]$ *where* $\delta,\delta' \in [0,1)$*. Then there is a* $C^{0,1}$ *weak solution to problem* $(1)-(2)$*.*

Proof. We apply the Schauder-type approach. Let $\psi \in C_0^+$ be given by

$$\psi(t) = \int_0^t \{m(s) + \lambda(s)\|\widetilde{\varphi}\|_s\} \exp\left(\int_0^t \lambda(\sigma)\,d\sigma\right) ds.$$

Define

$$\mathcal{X}_{\varphi,\psi} = \Big\{u \in \mathcal{X}_\varphi : \big|u(t,x) - \widetilde{\varphi}(t,x)\big| \le \psi(t) \;\text{ on } \widetilde{E}\Big\}.$$

Observe that $\mathcal{X}_{\varphi,\psi}$ is a bounded, closed and convex subset of $\mathcal{X}_\varphi$. It is easy to see that the operator $\mathcal{T}$ maps the set $\mathcal{X}_{\varphi,\psi}$ into itself. We prove that $\mathcal{T}$ is compact, i.e., the closure of $\mathcal{T}(\mathcal{X}_{\varphi,\psi})$ is compact. It is sufficient to show that the set $\mathcal{T}(\mathcal{X}_{\varphi,\psi})$ is uniformly bounded and equicontinuous on all compact subsets of $CB(\widetilde{E})$. Take $t \in (0,a]$ and $x, \bar{x} \in \mathbb{R}^n$. If $u \in \mathcal{X}_{\varphi,\psi}$, then we have

$$|f(t,y,u_{(t,y)})| \le P(t) := m(t) + \lambda(t)\,\|u\|_t.$$

Therefore, we get

$$\begin{aligned}
&\big|\mathcal{T}u(t,\bar{x}) - \mathcal{T}u(t,x)\big| + \int_0^t\!\!\int_{\mathbb{R}^n} \big|\Gamma(t,\bar{x},s,y) - \Gamma(t,x,s,y)\big| P(s)\,dyds \\
&\quad\le \big|\widetilde{\varphi}(t,\bar{x}) - \widetilde{\varphi}(t,x)\big| + \int_0^t \widetilde{c}_\delta\,\|x-\bar{x}\|^\delta (t-s)^{-\frac{\delta}{2}} P(s)\,ds \\
&\quad\le \|x-\bar{x}\|^\delta \big\{t^{-\frac{\delta}{2}}\widetilde{c}_\delta\|\varphi\|_0 + \widetilde{c}_\delta (S_0^\delta * P)(t)\big\}.
\end{aligned}$$

Hence the set $\mathcal{T}(\mathcal{X}_{\varphi,\psi})$ is equicontinuous in x on all compact subsets of E. Take arbitrary t and $\bar{t}$ such that $a \geq \bar{t} > t > 0$. Then we have

$$\begin{aligned}
&\big|\mathcal{T}u(t,x) - \mathcal{T}u(\bar{t},x)\big| \\
&\quad\leq \big|\widetilde{\varphi}(t,x) - \widetilde{\varphi}(\bar{t},x)\big| \\
&\qquad + \int_0^t\!\!\int_{\mathbb{R}^n} \big|\Gamma(t,x,s,y) - \Gamma(\bar{t},x,s,y)\big| P(s)\,dyds \\
&\qquad + \int_t^{\bar{t}}\!\!\int_{\mathbb{R}^n} |\Gamma(\bar{t},x,s,y)|\, P(s)\,dyds \\
&\quad\leq \big|\widetilde{\varphi}(t,x) - \widetilde{\varphi}(\bar{t},x)\big| \\
&\qquad + \int_0^t \widetilde{c}_\delta |\bar{t}-t|^{\frac{\delta}{2}}\,(t-s)^{-\frac{\delta}{2}} P(s)\,ds + \int_t^{\bar{t}} \widetilde{c}_0 P(s)\,ds.
\end{aligned}$$

Hence the set $\mathcal{T}(\mathcal{X}_{\varphi,\psi})$ is equicontinuous in $t > 0$.

Now we show that $\mathcal{T}(\mathcal{X}_{\varphi,\psi})$ is almost uniformly bounded. We have the estimate

$$\begin{aligned}
&\big|(\mathcal{T}u - \widetilde{\varphi})(t,x)\big| \\
&\quad\leq \int_0^t\!\!\int_{\mathbb{R}^n} |\Gamma(t,x,s,y)| \Big\{m(s) + \lambda(s)\|u_{(s,y)} - \widetilde{\varphi}_{(s,y)}\|_0\Big\} dyds \\
&\quad\leq \widetilde{c}_0 \int_0^t \Big\{m(s) + \lambda(s)\|u-\widetilde{\varphi}\|_s + \lambda(s)\|\widetilde{\varphi}\|_s\Big\} ds \\
&\quad\leq \psi(t).
\end{aligned}$$

The continuity of the function f in x and w implies that the operator $\mathcal{T}$ is continuous. By the Schauder fixed point theorem the operator $\mathcal{T}$ has a fixed point in $\mathcal{X}_{\varphi,\psi}$. This completes the proof ∎

Theorem 4.2. *Let $\varphi \in CB(E_0)$ and there are $m, \lambda, \lambda_1 \in L^1[0,a]$ such that*

$$|f(t,x,w)| \leq m(t) + \lambda(t)\,\|w\|_0 + \lambda_1(t)\|\partial w(0,0)\|_0$$

where $\lambda_1(t) = \lambda(t)\sqrt{t}$. Assume that the functions $\frac{S_0^\delta * m}{\delta'}, \frac{S_0^\delta * \lambda}{\delta'} \in L^\infty[0,a]$ and

$$(t-s)^{\frac{1}{2}} \int_s^t \frac{c_0}{c_1}\,\lambda_1(s)\,(t-\zeta)^{-\frac{1}{2}}(\zeta-s)^{-\frac{1}{2}}d\zeta \leq \theta_1 < 1 \qquad (t > s).$$

Then there is a $C^{0,1}$ weak solution to problem (1) – (2).

Proof. We recall that

$$\mathcal{X}_{\varphi,\psi,\psi_1} = \left\{ u \in CB(\widetilde{E}) \;\middle|\; \begin{array}{l} u \succ \varphi \text{ and } |(u-\widetilde{\varphi})(t,x)| \leq \psi(t) \\ |\partial_j(u-\widetilde{\varphi})(t,x)| \leq \psi_1(t) \quad (j = 1,\dots,n) \end{array} \right\}$$

where $\psi, \psi_1 \in C_0^+$ and $\psi(t) = k\,\psi_1(t)\sqrt{t}$ with $k = \frac{c_1}{c_0}$. Suppose that ψ, ψ_1 satisfy the integral inequalities

$$\int_0^t \tilde{c}_0 \Big\{ \widetilde{m}(s) + \lambda(s)\psi(s) + \lambda_1(s)\psi_1(s) \Big\}\, ds \le \psi(t)$$

$$\int_0^t \tilde{c}_1 (t-s)^{-\frac{1}{2}} \Big\{ \widetilde{m}(s) + \lambda(s)\psi(s) + \lambda_1(s)\psi_1(s) \Big\}\, ds \le \psi_1(t)$$

where $\widetilde{m}(s) = \lambda(s)\|\widetilde{\varphi}\|_s + \lambda_1(s)\|\partial\widetilde{\varphi}(s,\cdot)\|_0$. Then similarly as in Theorem 2.2 we can show that

$$\left.\begin{aligned} |(\mathcal{T}u - \widetilde{\varphi})(t,x)| &\le \psi(t) \\ |\partial_j(\mathcal{T}u - \widetilde{\varphi})(t,x)| &\le \psi_1(t) \end{aligned}\right\}$$

for $u \in \mathcal{X}_{\varphi,\psi,\psi_1}$. We prove that the set $(\partial_j \mathcal{T})(\mathcal{X}_{\varphi,\psi,\psi_1})$ is equicontinuous in t on all compact subsets of E. If $u \in (\partial_j \mathcal{T})(\mathcal{X}_{\varphi,\psi,\psi_1})$, then we have

$$\begin{aligned} &\big|\partial_j \mathcal{T}u(\bar{t},x) - \partial_j \mathcal{T}u(t,x)\big| \\ &\quad \le \big|\partial_j \widetilde{\varphi}(\bar{t},x) - \partial_j \widetilde{\varphi}(\bar{t},x)\big| \\ &\qquad + \int_0^t \widetilde{c}_{1+\delta}(t-s)^{-\frac{1+\delta}{2}} |\bar{t}-t|^{\frac{\delta}{2}} P_1(s)\, ds + \int_t^{\bar{t}} \widetilde{c}_{1+\delta}(t-s)^{-\frac{1+\delta}{2}} P_1(s)\, ds \end{aligned}$$

for all $0 < t < \bar{t} \le a$ where $P_1(s) = \widetilde{m}(s) + \lambda(s)\psi(s) + \lambda_1(s)\psi_1(s)$. Observe that the right-hand side does not depend on u. The equicontinuity of $(\partial_j \mathcal{T})(\mathcal{X}_{\varphi,\psi,\psi_1})$ in x can be proved in a similar way. Since f is continuous, we easily check that the operator $\mathcal{T}$ is continuous, too. We conclude from the Schauder fixed point theorem that the operator $\mathcal{T}$ has a fixed point in $\mathcal{X}_{\varphi,\psi,\psi_1}$ which completes the proof ∎

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Received 24.05.2000; in revised form 01.12.2000