Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 20 (2001), No. 1, 131–154

Boundary Integral Operators for Plate Bending in Domains with Corners

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Abstract. The paper studies boundary integral operators of the bi-Laplacian on piecewise smooth curves with corners and describes their mapping properties in the trace spaces of variational solutions of the biharmonic equation. We formulate a direct integral equation method for solving interior and exterior mixed boundary value problems on non-smooth plane domains, analyze the solvability of the corresponding systems of integral equations and prove their strong ellipticity.

Keywords: *Biharmonic equation, plate bending, non-smooth curves, boundary integral operators, boundary integral equations*

AMS subject classification: Primary 31A30, secondary 47G10, 65N38

1. Introduction

The aim of the present paper is to study boundary integral operators of the bi-Laplacian on piecewise smooth curves with corners and to analyze a direct integral equation method for solving the biharmonic equation with mixed boundary conditions on a non-smooth plane domain Ω with boundary Γ. Although boundary element methods offer important advantages over domain type methods and are frequently used for solving plate bending or related problems for fourth-order equations (cf. [2, 12] and also the references therein), their theoretical foundation is very limited compared with results for second-order equations.

For the case of a smooth curve quite satisfactory results are available by using nowadays standard tools from the theory of integral and pseudodifferential equations and of approximation methods. In connection with indirect boundary integral equation methods we mention Chapter 8 of the book [2], where a detailed analysis of mapping properties of biharmonic boundary integral operators and of indirect formulations for four types of boundary value problems can be found. As a rule, indirect methods are designed for specific classes of problems, but their application to other types of plate bending problems, for example to mixed boundary conditions, is complicated both in analytical and numerical respect. The study of direct methods can be based on the approach developed by Costabel and Wendland in [4, 9], which results in a complete description of mapping properties of boundary integral operators and strong ellipticity of systems of first kind integral equations corresponding to various types

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ISSN 0232-2064 / $ 2.50

of boundary conditions. This can be used to consider different numerical methods for solving corresponding integral equations, to prove stability and error estimates similarly to well-established techniques for second-order equations; for the case of a free plate see the interesting paper by Giroire and Nédélec [10].

If the boundary of the domain has corners, then the situation is quite different. The boundary integral operators are no longer classical pseudodifferential operators and biharmonic boundary value problems have in general only weak solutions. To extend the theory developed for second-order equations in non-smooth domains one has therefore to study the behaviour of biharmonic boundary integral operators applied to Cauchy data of H^2-functions. In [8], the first paper devoted to the study of boundary integral equations for the biharmonic equation in non-smooth domains, Costabel, Stephan and Wendland considered an indirect method for the solution of the boundary value problem $\operatorname{grad} u|_\Gamma = \mathbf{f}$. Using a layer potential ansatz with the gradient of the fundamental solution of the bi-Laplacian as integral kernel they obtained a system of two integral equations of the first kind with logarithmic principal part. Thus the above mentioned problem of dealing with biharmonic integral operators applied to Cauchy data of weak solutions could be avoided. This was first treated by Bourlard in [1], where the biharmonic Dirichlet problem on a polygonal domain was transformed into a variational formulation for the first kind boundary integral equation with biharmonic single layer potential. It was shown that the variational problem is coercive on the dual of the space of Dirichlet data of H^2-functions (the boundary values of the function and its normal derivative). That means, the single layer potential operator is a symmetric and strongly elliptic mapping from this dual into the trace space. Similar results were obtained in [16] by extending some methods for second-order equations from [5, 7] to define biharmonic boundary integral operators. These operators were associated with the bilinear form

$$\int_\Omega \Delta u \, \Delta v \, dx \tag{1.1}$$

which is positive definite on $H_0^2(\Omega)$ and corresponds to the biharmonic Dirichlet problem. The simple idea was to consider the two functions of the Dirichlet datum of a H^2-function, which obviously are subjected to some compatibility conditions at the corner points of Γ, as one element of a trace space and to define Neumann data of H^2-functions u with $\Delta^2 u \in L^2$ by using (1.1). Then the Neumann data belong to the dual of the trace space. The biharmonic layer potentials are simply the values of the duality functional applied to the Dirichlet datum (single layer) or the Neumann datum (double layer) of the biharmonic fundamental solution and to an element of the corresponding dual space, which becomes the density. Now the setting is the same as for potentials of second-order equations, and by using the approach of Costabel [5] we were able to prove the jump relations for the potentials, to define the boundary operators and analyze their mapping properties in the trace spaces of variational solutions. The obtained results were used to formulate boundary integral equations for interior and exterior biharmonic Dirichlet problems in non-smooth domains and to analyze their solvability.

In this paper we extend the approach of [16] to treat other types of boundary conditions, which appear in thin plate bending as free, simply supported or roller-supported plate. To this end form (1.1) has to be replaced by another form

$$a^\sigma(u,v) = \int_\Omega \Big(\sigma \Delta u \, \Delta v + (1-\sigma) \sum_{j,k=1}^{2} \partial_j \partial_k u \, \partial_j \partial_k v \Big) dx$$

connected with the bending strain energy of a Kirchhoff plate if $0 < \sigma < \frac{1}{2}$. In Section 2 we provide the analogous construction as in [16] to define the Neumann data of H^2-functions u with $\Delta^2 u \in L^2$, which now depend on σ and contain, even for smooth u, Dirac functionals supported at the corner points of the boundary. Further, we consider the existence of variational solutions of interior and exterior Dirichlet and Neumann problems. In Section 3 we introduce the biharmonic layer potentials associated with a^σ, characterize their behaviour at infinity and prove the jump relations and representation formulas for biharmonic functions. The corresponding boundary integral operators will be studied in Section 4. For $0 \le \sigma < 1$ these operators have similar properties as the boundary integral operators of the Laplacian. In Section 5 we transform biharmonic boundary value problems into equivalent systems of boundary integral equations. If the boundary value problem allows a coercive variational formulation, then the corresponding system of integral equation is strongly elliptic. We study the solvability of this system, which leads immediately to stability results for Galerkin boundary element methods.

2. Traces of H^2-functions on piecewise smooth boundaries

For the following let Γ be a simple closed curve in the (x_1, x_2)-plane composed of m smooth arcs Γ_i. Adjacent arcs Γ_{i-1} and Γ_i meet at corner points x^i $(i = 1, \dots, m)$ with interior angles α_i, $0 < \alpha_i < 2\pi$. The interior of Γ we denote by Ω_1, the exterior $\mathbb{R}^2 \setminus \overline{\Omega}_1$ by Ω_2, and direct the unit normal $n = (n_1, n_2)$ on Γ into Ω_2. In the following we denote by ∂_j $(j = 1, 2)$ the partial derivative with respect to x_j, by $\partial_n = n_1 \partial_1 + n_2 \partial_2$ the normal derivative and by $\partial_\tau = -n_2 \partial_1 + n_1 \partial_2$ the tangential derivative along Γ. The norm in the Sobolev space $H^2(\Omega_1)$ is defined by

$$\|u\|_{H^2(\Omega_1)} = \big(\|u\|^2_{L^2(\Omega_1)} + |u|^2_{H^2(\Omega_1)} \big)^{\frac{1}{2}}$$

where

$$|u|^2_{H^2(\Omega_1)} = \sum_{j,k=1}^{2} \|\partial_j \partial_k u\|^2_{L^2(\Omega_1)}.$$

The traces of functions from $H^2(\Omega_1)$ can be characterized by using the following general result.

Lemma 2.1 (see [13]). *There exists a constant $c > 0$ not depending on $u \in H^2(\Omega_1)$ such that*

$$\sum_{i=1}^{m} \big(\|u\|_{H^{3/2}(\Gamma_i)} + \|\partial_n u\|_{H^{1/2}(\Gamma_i)} \big) + \|\partial_1 u\|_{H^{1/2}(\Gamma)} + \|\partial_2 u\|_{H^{1/2}(\Gamma)} \le c \, \|u\|_{H^2(\Omega)}.$$

In the following we identify functions on Γ with periodic functions depending on the arc length s and denote the derivative with respect to s by $u' = \frac{du}{ds}$. Since with exception of the corner points x^i there holds

$$\left.\begin{aligned} \partial_1 u|_\Gamma &= n_1\,\partial_n u - n_2\,\partial_\tau u \\ \partial_2 u|_\Gamma &= n_2\,\partial_n u + n_1\,\partial_\tau u \\ \partial_\tau &= \tfrac{d}{ds} \end{aligned}\right\}$$

Lemma 2.1 suggests the definition of the trace space

$$V(\Gamma) = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} :\, u_1 \in H^1(\Gamma),\ n_1u_2 - n_2u_1',\ n_2u_2 + n_1u_1' \in H^{1/2}(\Gamma) \right\}$$

equipped with the canonical norm. We introduce the generalized trace mapping

$$\gamma u = \begin{pmatrix} u|_\Gamma \\ \partial_n u|_\Gamma \end{pmatrix} :\, H^2(\Omega_1) \to V(\Gamma).$$

Lemma 2.2 (see [13]). *The linear mapping* $\gamma :\, H^2_{loc}(\mathbb{R}^2) \to V(\Gamma)$ *is continuous and has a continuous right inverse* $\gamma^- :\, V(\Gamma) \to H^2_{loc}(\mathbb{R}^2)$. *In particular,* γ *maps* $C_0^\infty(\mathbb{R}^2)$ *onto a dense subspace of* $V(\Gamma)$.

If we define the duality form

$$\left[\begin{pmatrix} v_4 \\ v_3 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right] = -\langle v_4, u_1\rangle_\Gamma + \langle v_3, u_2\rangle_\Gamma \tag{2.1}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the extension of the L^2-scalar product on Γ, then the dual space of $V(\Gamma)$ can be described as follows.

Lemma 2.3. *The vector* $\binom{v_4}{v_3}$ *belongs to* $(V(\Gamma))'$ *if and only if there exist* $z_1, z_2 \in H^{-1/2}(\Gamma)$ *and a number* $a \in \mathbb{R}$ *such that for any* $\varphi \in C_0^\infty(\mathbb{R}^2)$ *the equations*

$$\begin{aligned} \langle \varphi|_\Gamma, v_4\rangle_\Gamma &= \big\langle (\varphi|_\Gamma)', n_2z_1 - n_1z_2 \big\rangle_\Gamma + a\textstyle\int_\Gamma \varphi\, ds \\ \langle \varphi|_\Gamma, v_3\rangle_\Gamma &= \big\langle \varphi|_\Gamma, n_1z_1 + n_2z_2 \big\rangle_\Gamma \end{aligned}$$

are satisfied.

To consider boundary integral equations connected with plate bending problems we introduce the bilinear form

$$a^\sigma(u,v) = a^\sigma_{\Omega_1}(u,v) := \int_{\Omega_1} \Big(\sigma \Delta u\, \Delta v + (1-\sigma) \sum_{j,k=1}^{2} \partial_j\partial_k u\, \partial_j\partial_k v \Big) dx \tag{2.2}$$

well-known in the variational formulation of bending problems for a thin plate with Poisson ratio $\sigma = \frac{\lambda}{2}(\lambda+\mu)$, λ and μ being the Lamé constants of the material. If u

represents the deflection function on Ω_1 corresponding to suitable loading and boundary conditions, then the value of

$$a^\sigma(u,u) = \sigma\,\|\Delta u\|^2_{L^2(\Omega_1)} + (1-\sigma)\,|u|^2_{H^2(\Omega_1)} \tag{2.3}$$

is exactly twice the bending strain energy of the plate.

The analysis of boundary value problems for the bi-Laplacian is based on the fact that the bilinear form a^σ is coercive on appropriate function spaces, for certain values of the parameter σ. By (2.3) the form a^σ is coercive on $H^2(\Omega_1)$ at least for $0 \le \sigma < 1$. We mention that in the case of a smooth boundary Γ the form a^σ is coercive on $H^2(\Omega_1)$ if and only if $-3 < \sigma < 1$, as stated in [11]. Furthermore, for $u, v \in C_0^\infty(\Omega_1)$

$$\int_{\Omega_1} \partial_j\partial_k u\,\partial_j\partial_k v\,dx = \int_{\Omega_1} \partial_j\partial_j u\,\partial_k\partial_k v\,dx$$

hence the value of $a^\sigma(u,v)$ does not depend on σ and $(a^\sigma(u,u))^{\frac{1}{2}} = |u|_{H^2(\Omega_1)}$ is a norm on $H_0^2(\Omega_1)$ equivalent to $\|\cdot\|_{H^2(\Omega_1)}$. Thus for given $f \in L^2(\Omega_1)$ and $\psi \in V(\Gamma)$ the problem

$$\left.\begin{aligned} a^\sigma(u,v) &= \langle f, v\rangle_{\Omega_1} \quad \forall\, v \in H_0^2(\Omega_1) \\ \gamma u &= \psi \end{aligned}\right\} \tag{2.4}$$

has a unique solution $u \in H^2(\Omega_1)$ being the weak solution of the Dirichlet problem

$$\left.\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega_1 \\ \gamma u &= \psi \end{aligned}\right\}. \tag{2.5}$$

It is obvious that the solution operator defined by $u = T(f,\psi)$ is a continuous mapping

$$T:\, L^2(\Omega_1)\times V(\Gamma) \to H^2(\Omega_1,\Delta^2) = \big\{u \in H^2(\Omega_1):\, \Delta^2 u \in L^2(\Omega_1)\big\}. \tag{2.6}$$

To consider other boundary value problems we define on Γ the differential operators

$$\begin{aligned} \partial_{nn} u &= n_1^2\,\partial_1^2 u + 2n_1n_2\,\partial_1\partial_2 u + n_2^2\,\partial_2^2 u \\ \partial_{\tau n} u &= (n_1^2 - n_2^2)\partial_1\partial_2 u - n_1n_2(\partial_1^2 u - \partial_2^2 u) \\ \partial_{\tau\tau} u &= n_2^2\,\partial_1^2 u - 2n_1n_2\,\partial_1\partial_2 u + n_1^2\,\partial_2^2 u. \end{aligned} \tag{2.7}$$

Lemma 2.4. *Let $u \in H^2(\Omega_1,\Delta^2)$ and $\sigma \in \mathbb{R}$. The mapping*

$$\delta_\sigma u:\, \psi \to [\delta_\sigma u, \psi] = a^\sigma(u, \gamma^-\psi) - \int_{\Omega_1} \gamma^-\psi\,\Delta^2 u\,dx \tag{2.8}$$

is a continuous linear functional on $V(\Gamma)$ that coincides for sufficiently smooth u with

$$\begin{aligned}[\delta_\sigma u, \psi] = &- \int_\Gamma \big(v_1\,\partial_n\Delta u - (1-\sigma)\,v_1'\,\partial_{\tau n} u\big)ds \\ &+ \int_\Gamma v_2\,\big(\sigma\Delta u + (1-\sigma)\,\partial_{nn} u\big)ds \end{aligned} \qquad \big(\psi = (v_1,v_2) \in V(\Gamma)\big). \tag{2.9}$$

Moreover, $\delta_\sigma : H^2(\Omega_1, \Delta^2) \to (V(\Gamma))'$ *is continuous.*

Proof. Since

$$\sigma\Delta u\,\Delta v + (1-\sigma)\sum_{j,k=1}^{2} \partial_j\partial_k u\,\partial_j\partial_k v$$
$$= \Delta u\Delta v + (1-\sigma)\big(2\,\partial_1\partial_2 u\,\partial_1\partial_2 v - \partial_1^2 u\,\partial_2^2 v - \partial_2^2 u\,\partial_1^2 v\big)\,,$$

after applying Green's formula with $u \in H^4(\Omega_1)$ and $v \in H^2(\Omega_1)$ one has

$$\int_{\Omega_1} \big(\Delta u\,\Delta v - v\,\Delta^2 u\big)dx = \int_\Gamma \big(\Delta u\,\partial_n v - v\,\partial_n \Delta u\big)ds$$
$$\int_{\Omega_1} \big(2\,\partial_1\partial_2 u\,\partial_1\partial_2 v - \partial_1^2 u\,\partial_2^2 v - \partial_2^2 u\,\partial_1^2 v\big)dx = \int_\Gamma \big(\partial_\tau v\,\partial_{\tau n} u - \partial_n v\,\partial_{\tau\tau} u\big)ds.$$

Thus the value of the domain integrals $a^\sigma_{\Omega_1}(u,v) - \int_{\Omega_1} v\,\Delta^2 u\,dx$ depends only on $\gamma v \in V(\Gamma)$ and we obtain the Rayleigh-Green formula (2.9). Since

$$|a^\sigma(u,v)| \le |\sigma|\|\Delta u\|_{L^2(\Omega_1)}\|\Delta v\|_{L^2(\Omega_1)} + |1-\sigma|\,|u|_{H^2(\Omega_1)}|v|_{H^2(\Omega_1)}$$

there exists a constant depending only on σ such that

$$\big|[\delta_\sigma u, \psi]\big| \le \|\Delta^2 u\|_{L^2(\Omega_1)}\|\gamma^- \psi\|_{L^2(\Omega_1)} + c_\sigma\,|u|_{H^2(\Omega_1)}|\gamma^-\psi|_{H^2(\Omega_1)}. \tag{2.10}$$

Hence the assertion follows by continuity from Lemma 2.2 and the fact that $C^\infty(\overline{\Omega}_1)$ is dense in $H^2(\Omega_1, \Delta^2)$ (see [16]) ∎

Corollary 2.1. *For* $u, v \in H^2(\Omega_1, \Delta^2)$ *Green's second formula*

$$\int_{\Omega_1} (v\,\Delta^2 u - u\,\Delta^2 v)dx = [\delta_\sigma v, \gamma u] - [\delta_\sigma u, \gamma v]$$

holds.

For $\psi = (v_1, v_2) \in V(\Gamma)$ we write formula (2.9) in the form

$$[\delta_\sigma u, \psi] = -\langle v_1, \widetilde{N}_\sigma u\rangle_\Gamma + \langle v_2, M_\sigma u\rangle_\Gamma \tag{2.11}$$

where for sufficiently smooth u, say $u \in H^4(\Omega_1)$,

$$\left.\begin{aligned} M_\sigma u &= \sigma\Delta u + (1-\sigma)\,\partial_{nn} u \\ \widetilde{N}_\sigma u &= \partial_n \Delta u + \frac{d}{ds}(T_\sigma u) \end{aligned}\right\} \tag{2.12}$$

with

$$T_\sigma u = (1-\sigma)\,\partial_{\tau n} u \tag{2.13}$$

and the derivative of $T_\sigma u$ is understood in distributional sense. In plate bending $M_\sigma u$ corresponds to the bending moment, $T_\sigma u$ to the twisting moment and $\widetilde{N}_\sigma u$ is known as transverse force. In general the twisting moment $T_\sigma u$ is discontinuous at the corner points of Γ. Therefore

$$\widetilde{N}_\sigma u = N_\sigma u + \sum_{i=1}^{m} \delta(\cdot - x^i)\big(T_\sigma u(x^i_+) - T_\sigma u(x^i_-)\big)$$

where $\delta(x)$ is the Dirac functional, $T_\sigma u(x^i_+) - T_\sigma u(x^i_-)$ is the corner force at x^i and the function $N_\sigma u$, known as Kirchhoff shear, is equal to

$$N_\sigma u = \partial_n \Delta u + \frac{d}{ds}(T_\sigma u) \qquad \text{on the arcs } \Gamma_i. \tag{2.14}$$

Since adjacent arcs meet at the corner point x^i with interior angle α_i, from (2.7) it follows easily that

$$T_\sigma u(x^i_+) - T_\sigma u(x^i_-) = (1-\sigma)\,\sin\alpha_i\,\big(\partial_{\tau^i\tau^i} u(x^i) - \partial_{n^i n^i} u(x^i)\big). \tag{2.15}$$

Here the unit vector

$$n^i = \Big(\cos\big(\varphi_i + \tfrac{\pi-\alpha_i}{2}\big), \sin\big(\varphi_i + \tfrac{\pi-\alpha_i}{2}\big)\Big) = \Big(-\sin\big(\varphi_i - \tfrac{\alpha_i}{2}\big), \cos\big(\varphi_i - \tfrac{\alpha_i}{2}\big)\Big)$$

is directed as the bisector of the angle between $n(x^i_-)$ and $n(x^i_+)$, φ_i denotes the angle between the x_1-axis and $n(x^i_-)$, and

$$\tau^i = -\Big(\cos\big(\varphi_i - \tfrac{\alpha_i}{2}\big), \sin\big(\varphi_i - \tfrac{\alpha_i}{2}\big)\Big).$$

Hence we get

$$\widetilde{N}_\sigma u = N_\sigma u + (1-\sigma)\sum_{i=1}^{m} \delta(\cdot - x^i)\sin\alpha_i\,\big(\partial_{\tau^i\tau^i} u(x^i) - \partial_{n^i n^i} u(x^i)\big). \tag{2.16}$$

The vector composed of the components of the Dirichlet and Neumann data

$$\begin{pmatrix} \gamma u \\ \delta_\sigma u \end{pmatrix} = \begin{pmatrix} u \\ \partial_n u \\ M_\sigma u \\ \widetilde{N}_\sigma u \end{pmatrix} \tag{2.17}$$

will be called *Cauchy datum* of $u \in H^2(\Omega_1, \Delta^2)$ *associated* with the bilinear form a^σ.

Let us now consider the problem to find $u \in H^2(\Omega_1)$ such that for given $\chi \in (V(\Gamma))'$

$$a^\sigma(u,v) = [\chi, \gamma v] \qquad \forall\, v \in H^2(\Omega_1). \tag{2.18}$$

By (2.8) this is equivalent to the Neumann problem for the biharmonic equation

$$\left.\begin{aligned} \Delta^2 u = 0 \quad &\text{in } \Omega_1 \\ \delta_\sigma u = \chi & \end{aligned}\right\}. \tag{2.19}$$

Let us denote by $\mathbb{P}_1$ the space of linear functions on $\mathbb{R}^2$ and introduce the factor space $\mathcal{H}^2(\Omega_1) = H^2(\Omega_1)/\mathbb{P}_1$. It is well known that

$$\|\dot{u}\|_{\mathcal{H}^2(\Omega_1)} = |u|_{H^2(\Omega_1)}$$

gives a norm on the Hilbert space $\mathcal{H}^2(\Omega_1)$ equivalent to the quotient norm

$$\inf_{p\in\mathbb{P}_1} \|u-p\|_{H^2(\Omega_1)}.$$

Further, we denote by $l\,(\Gamma)$ the traces of linear functions, $l\,(\Gamma) = \gamma(\mathbb{P}_1)$, and consider the space $W(\Gamma) = V(\Gamma)/l\,(\Gamma)$ equipped with the factor norm. The adjoint space $(W(\Gamma))'$ with respect to (2.1) can be identified with the polar set

$$l\,(\Gamma)^\perp = \Big\{\chi \in (V(\Gamma))' : \, [\chi,\psi] = 0 \;\; \forall\, \psi \in l\,(\Gamma)\Big\}.$$

Obviously, the assertions of Lemma 2.2 remain true for the mapping $\gamma : \mathcal{H}^2(\Omega_1) \to W(\Gamma)$.

Lemma 2.5. *Let $\dot{u} \in \mathcal{H}^2(\Omega_1)$ with $\Delta^2\dot{u} = 0$ and $0 \le \sigma < 1$. There exist constants c_1 and c_2 not depending on $\dot{u}$ such that*

$$c_1\|\dot{u}\|_{\mathcal{H}^2(\Omega_1)} \le \|\delta_\sigma \dot{u}\|_{(W(\Gamma))'} \le c_2\|\dot{u}\|_{\mathcal{H}^2(\Omega_1)}.$$

Proof. Since $\delta_\sigma p = 0$, $p \in \mathbb{P}_1$, the mapping δ_σ is defined on equivalence classes $\dot{u} \in \mathcal{H}^2(\Omega_1)$ with $\Delta^2 u \in L^2(\Omega_1)$. Further, for any $u \in H^2(\Omega_1)$ with $\Delta^2 u = 0$ there holds

$$[\delta_\sigma u, \gamma p] = 0 \quad (p \in \mathbb{P}_1), \qquad \text{i.e. } \delta_\sigma u \in l\,(\Gamma)^\perp. \tag{2.20}$$

From (2.10) we get

$$\big|[\delta_\sigma u,\psi]\big| \le c_\sigma |u|_{H^2(\Omega_1)} |\gamma^- \psi|_{H^2(\Omega_1)} \le c\,\|\dot{u}\|_{\mathcal{H}^2(\Omega_1)} \|\dot{\psi}\|_{W(\Gamma)}$$

hence δ_σ maps $\{\dot{u} \in \mathcal{H}^2(\Omega_1) : \, \Delta^2\dot{u} = 0\}$ into $(W(\Gamma))'$ and

$$\|\delta_\sigma \dot{u}\|_{(W(\Gamma))'} \le c_2\|\dot{u}\|_{\mathcal{H}^2(\Omega_1)}.$$

On the other hand, for $u \in H^2(\Omega_1)$ with $\Delta^2 u = 0$ we have

$$[\delta_\sigma u, \gamma u] = a^\sigma_{\Omega_1}(u,u) = \sigma\,\|\Delta u\|^2_{L^2(\Omega_1)} + (1-\sigma)\,|u|^2_{H^2(\Omega_1)}$$

so for $0 \le \sigma < 1$

$$[\delta_\sigma \dot{u}, \gamma\dot{u}] \ge (1-\sigma)\,\|\dot{u}\|^2_{\mathcal{H}^2(\Omega_1)}. \tag{2.21}$$

Hence we derive

$$\|\delta_\sigma \dot{u}\|_{(W(\Gamma))'}\,\|\gamma\dot{u}\|_{W(\Gamma)} \ge (1-\sigma)\,\|\dot{u}\|^2_{\mathcal{H}^2(\Omega_1)} \ge c_1\|\dot{u}\|_{\mathcal{H}^2(\Omega_1)}\|\gamma\dot{u}\|_{W(\Gamma)}$$

and the proof is finished ∎

Corollary 2.2. *Let $0 \le \sigma < 1$. The Neumann problem* (2.19) *has a solution $u \in H^2(\Omega_1)$ if and only if $\chi \in l\,(\Gamma)^{\perp}$. The corresponding equivalence class $\dot{u} \in \mathcal{H}^2(\Omega_1)$ is unique.*

Lemma 2.6. *The set $\{(\gamma\varphi, \delta_\sigma\varphi) : \varphi \in C_0^\infty(\mathbb{R}^2)\}$ is dense in $V(\Gamma) \times (V(\Gamma))'$.*

Proof. The assertion is proved if we show that for $(\psi, \chi) \in V(\Gamma) \times (V(\Gamma))'$ the relation

$$[\delta_\sigma\varphi, \psi] - [\chi, \gamma\varphi] = 0 \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^2) \tag{2.22}$$

implies $\psi = \chi = 0$. Choosing arbitrary $f \in L^2(\Omega_1)$ we obtain by applying Corollary 2.1 and (2.6)

$$\begin{aligned}\big[\delta_\sigma T(f,0), \psi\big] &= \big[\delta_\sigma T(f,0), \gamma T(0,\psi)\big] - \big[\delta_\sigma T(0,\psi), \gamma T(f,0)\big] \\ &= \int_{\Omega_1} \Big(T(f,0)\, \Delta^2 T(0,\psi) - T(0,\psi)\, \Delta^2 T(f,0)\Big) dx \\ &= -\int_{\Omega_1} f\, T(0,\psi)\, dx.\end{aligned}$$

Since $C^\infty(\overline{\Omega}_1)$ is dense in $H^2(\Omega_1, \Delta^2)$ relation (2.22) holds also for $\varphi = T(f,0)$, so $\int_{\Omega_1} f\, T(0,\psi)\, dx = 0$ for all $f \in L^2(\Omega_1)$. Thus $T(0,\psi) = 0$ yielding $\psi = \gamma T(0,\psi) = 0$. From (2.22) it follows now that $[\chi, \gamma\varphi] = 0$ for all $\varphi \in H^2(\Omega_1, \Delta^2)$ which together with Lemma 2.2 implies $\chi = 0$ ∎

Next we consider boundary value problems in the exterior domain Ω_2. The traces of functions given outside of Ω_1 are defined so that for any $\varphi \in C_0^\infty(\mathbb{R}^2)$

$$\left.\begin{aligned}\gamma(\varphi|_{\Omega_2}) &= \gamma(\varphi|_{\Omega_1}) \\ \delta_\sigma(\varphi|_{\Omega_2}) &= \delta_\sigma(\varphi|_{\Omega_1})\end{aligned}\right\}.$$

Hence, if $\widetilde{\Omega}$ denotes a domain containing $\overline{\Omega}_1$, $u \in H^2(\widetilde{\Omega}\backslash\Omega_1, \Delta^2)$ and $v \in H^2(\widetilde{\Omega}\backslash\Omega_1)$, then

$$[\delta_\sigma u, \gamma v] = \int\limits_{\widetilde{\Omega}\backslash\Omega_1} \Big((\varphi v)\, \Delta^2 u - \sigma\, \Delta(\varphi v)\, \Delta u - (1-\sigma) \sum_{j,k=1}^{2} \partial_j\partial_k u\, \partial_j\partial_k(\varphi v)\Big) dx$$

where $\varphi \in C_0^\infty(\widetilde{\Omega})$ with $\varphi \equiv 1$ on a neighbourhood of $\overline{\Omega}_1$.

Let us define the Hilbert space $W^2(\Omega_2)$ which is a special case in a family of weighted Sobolev spaces studied in [14] and allows variational formulations of exterior problems for the biharmonic equation. We denote $\rho(r) = \log(2 + r^2)$ and introduce

$$\begin{aligned} W^2(\Omega_2) &= \left\{ u : \frac{u}{(1+|x|^2)\rho(|x|)}, \frac{\partial_j u}{(1+|x|^2)^{\frac{1}{2}}\rho(|x|)}, \partial_j\partial_k u \in L^2(\Omega_2)\ (j,k = 1,2) \right\} \\ W_0^2(\Omega_2) &= \text{ closure of } C_0^\infty(\Omega_2) \text{ in } W^2(\Omega_2)\end{aligned}$$

equipped with the canonical norm. It is proved in [14] that the seminorm

$$|u|_{W^2(\Omega_2)} = \left(\sum_{j,k=1}^{2} \|\partial_j \partial_k u\|^2_{L^2(\Omega_2)} \right)^{\frac{1}{2}}$$

is a norm on $W_0^2(\Omega_2)$ and on the factor space $W^2(\Omega_2)/\mathbb{P}_1$ equivalent to the corresponding induced norms. Hence the bilinear form

$$a^\sigma_{\Omega_2}(u,v) = \int_{\Omega_2} \left(\sigma \Delta u \, \Delta v + (1-\sigma) \sum_{j,k=1}^{2} \partial_j \partial_k u \, \partial_j \partial_k v \right) dx \tag{2.23}$$

is positive definite on $W_0^2(\Omega_2)$ and, for $0 \le \sigma < 1$, on $\mathcal{H}^2(\Omega_2)$. Here we use the notations $\mathcal{H}^2(\Omega_2) = W^2(\Omega_2)/\mathbb{P}_1$ and $\|\dot{u}\|_{\mathcal{H}^2(\Omega_2)} := |u|_{W^2(\Omega_2)}$. Furthermore, for $u \in W^2(\Omega_2)$ with $\Delta^2 u = 0$ and $0 \le \sigma < 1$

$$[\delta_\sigma u, \gamma u] = -\, a^\sigma_{\Omega_2}(u,u) \le (\sigma - 1)\, |u|^2_{W^2(\Omega_2)}, \tag{2.24}$$

hence, analogously to Lemma 2.5 one obtains

Lemma 2.7. *Let $\dot{u} \in \mathcal{H}^2(\Omega_2)$ with $\Delta^2 \dot{u} = 0$ and $0 \le \sigma < 1$. There exist constants not depending on $\dot{u}$ such that $c_1 \|\dot{u}\|_{\mathcal{H}^2(\Omega_2)} \le \|\delta_\sigma \dot{u}\|_{(W(\Gamma))'} \le c_2 \|\dot{u}\|_{\mathcal{H}^2(\Omega_2)}$.*

Similarly to the interior problems the following assertions can be proved.

Lemma 2.8. *For any $\psi \in V(\Gamma)$ the weak formulation of the Dirichlet problem*

$$\left.\begin{aligned} \gamma u &= \psi \\ a^\sigma_{\Omega_2}(u,v) &= 0 \quad \forall\, v \in W_0^2(\Omega_2) \end{aligned}\right\}$$

has a unique solution $u \in W^2(\Omega_2)$. The exterior Neumann problem

$$a^\sigma_{\Omega_2}(u,v) = -[\chi, \gamma v] \qquad \forall\, v \in W^2(\Omega_2)$$

has a solution $u \in W^2(\Omega_2)$ if and only if $\chi \in l\,(\Gamma)^\perp \subset (V(\Gamma))'$. The corresponding equivalence class $\dot{u} \in \mathcal{H}^2(\Omega_2)$ is unique.

3. Layer potentials for the bi-Laplacian

Here we introduce the biharmonic layer potentials, which are based on the fundamental solution of the bi-Laplacian Δ^2

$$G(x) = \tfrac{1}{8\pi} |x|^2 \log |x| \qquad (x \in \mathbb{R}^2)$$

and are associated with the form a^σ. Note that the operator

$$\mathcal{G}u(x) = \langle G(x,\cdot), u \rangle_{\mathbb{R}^2} \qquad \text{with } G(x,y) = G(x-y)$$

is the inverse of Δ^2 on the space of compactly supported distributions on $\mathbb{R}^2$ and that

$$\mathcal{G} : H^s_{comp}(\mathbb{R}^2) \to H^{s+4}_{loc}(\mathbb{R}^2) \qquad (s \in \mathbb{R}) \tag{3.1}$$

is continuous. We have the following representation formula which follows immediately from the special case $\sigma = 1$ given in [16].

Lemma 3.1. *Let $u \in L^2(\mathbb{R}^2)$ be a function with compact support such that $u|_{\Omega_1} \in H^2(\Omega_1)$, $u|_{\Omega_2} \in H^2_{loc}(\Omega_2)$ and $f = \Delta^2 u|_{\mathbb{R}^2\setminus\Gamma} \in L^2(\mathbb{R}^2)$. Then for $x \in \mathbb{R}^2\setminus\Gamma$ the representation*

$$u(x) = \mathcal{G}f(x) - [\{\delta_\sigma u\}, \gamma G(x,\cdot)] + [\delta_\sigma G(x,\cdot), \{\gamma u\}]$$

holds where

$$\{\gamma u\} = \gamma(u|_{\Omega_2}) - \gamma(u|_{\Omega_1})$$
$$\{\delta_\sigma u\} = \delta_\sigma(u|_{\Omega_2}) - \delta_\sigma(u|_{\Omega_1})$$

denote the jumps of the Dirichlet and Neumann data, respectively, across Γ.

Lemma 3.1 leads to the definition of the layer potentials for $x \in \mathbb{R}^2\setminus\Gamma$

$$\begin{aligned} \mathcal{V}\chi(x) &= [\chi, \gamma G(x,\cdot)] \qquad &\big(\chi \in (V(\Gamma))'\big) \\ \mathcal{K}_\sigma\psi(x) &= [\delta_\sigma G(x,\cdot), \psi] \qquad &\big(\psi \in V(\Gamma)\big). \end{aligned} \tag{3.2}$$

Lemma 3.2. *The biharmonic layer potentials*

$$\mathcal{V} : (V(\Gamma))' \to H^2_{loc}(\mathbb{R}^2)$$
$$\mathcal{K}_\sigma : V(\Gamma) \to H^2(\Omega_1)$$

are continuous.

Proof. Because of $\mathcal{V}\chi(x) = \langle G(x,\cdot), \gamma'\chi\rangle_{\mathbb{R}^2}$ we can write

$$\mathcal{V}\chi = \mathcal{G}\gamma'\chi. \tag{3.3}$$

The adjoint of the trace map $\gamma' : (V(\Gamma))' \to H^{-2}_{comp}(\mathbb{R}^2)$ is continuous, therefore the assertion for $\mathcal{V}$ follows from (3.1). Due to Lemma 3.1 the solution $u = T(0,\psi)$ of the Dirichlet problem (2.5) can be represented by

$$T(0,\psi) = \mathcal{V}\delta_\sigma T(0,\psi) - \mathcal{K}_\sigma\psi.$$

So Lemma 2.4 and the continuity of T imply $\|\mathcal{K}_\sigma\psi\|_{H^2(\Omega_1)} \le c\|\psi\|_{V(\Gamma)}$ ∎

Note that definitions (2.1) and (3.2) lead to known representations of $\mathcal{V}$ and $\mathcal{K}_\sigma$ as integral operators [17, 12]. If the components of the vector $\chi = (v_1, v_2)$ are integrable functions, then we have

$$\begin{aligned} \mathcal{V}\chi(x) = &-\tfrac{1}{8\pi}\int_\Gamma v_1(y)\,|x-y|^2\,\log|x-y|\,ds_y \\ &+\tfrac{1}{8\pi}\int_\Gamma v_2(y)\,(n_y, y-x)(2\log|x-y|+1)\,ds_y. \end{aligned} \tag{3.4}$$

From (2.16) we derive that the potential $\mathcal{K}_\sigma\psi$, $\psi = (v_1, v_2) \in V(\Gamma)$, is the sum of two integrals and of a finite number of functions depending on $v_1(x^i)$:

$$\begin{aligned} \mathcal{K}_\sigma\psi(x) = &\int_\Gamma v_2(y)\,M_{\sigma,y}G(x,y)\,ds_y - \int_\Gamma v_1(y)\,N_{\sigma,y}G(x,y)\,ds_y \\ &- \tfrac{1-\sigma}{4\pi}\sum_{i=1}^m v_1(x^i)\sin\alpha_i\left(1 - \frac{2(n_i, x-x^i)^2}{|x-x^i|^2}\right) \end{aligned} \tag{3.5}$$

where

$$M_{\sigma,y}G(x,y) = \frac{1+\sigma}{4\pi}(\log|x-y|+1) + \frac{1-\sigma}{4\pi}\Big(\frac{(n_y,y-x)^2}{|x-y|^2} - \frac{1}{2}\Big)$$
$$N_{\sigma,y}G(x,y) = \frac{1+\sigma}{4\pi}\,\frac{(n_y,y-x)}{|x-y|^2} + \frac{1-\sigma}{2\pi}\Big(\frac{(n_y,y-x)^3}{|x-y|^4} - \kappa_y\Big(\frac{(n_y,y-x)^2}{|x-y|^2} - \frac{1}{2}\Big)\Big).$$

Here κ_y denotes the curvature of Γ at the boundary point y, $\kappa = \frac{d\varphi}{ds}$ where φ is the angle between the x_1-axis and n_y.

Let us define the linear spaces

$$L_j^\sigma = \Big\{u(x) = \mathcal{V}\chi(x) - \mathcal{K}_\sigma\psi(x) : (\psi,\chi) \in V(\Gamma)\times(V(\Gamma))',\ x\in\Omega_j\Big\}$$

of biharmonic functions representable via layer potentials. From Lemmas 3.1 and 3.2 we conclude that the space L_1^σ corresponding to the interior domain is independent of σ and coincides with the set of functions $u \in H^2(\Omega_1)$ satisfying $\Delta^2 u = 0$. Moreover, for $u \in L_1$ the representation formula

$$\mathcal{V}\delta_\sigma u(x) - \mathcal{K}_\sigma\gamma u(x) = \begin{cases} u(x) & \text{if } x\in\Omega_1 \\ 0 & \text{if } x\in\Omega_2 \end{cases} \tag{3.6}$$

holds. The space L_2^σ consists of functions $u \in H^2_{loc}(\Omega_2)$ characterized by $\Delta^2 u = 0$ and by a special asymptotics at infinity which will be described in the following lemma. To this end we introduce the functions of $(x,y)\in\mathbb{R}^2$

$$\left.\begin{aligned} g_1(x,y) &= 1 \\ g_2(x,y) &= (x,y) \\ g_3(x,y) &= |y|^2 \\ g_4(x,y) &= \tfrac{1}{2}|y|^2 + (x,y)^2 \end{aligned}\right\}$$

denote by $\hat{x} = \frac{x}{|x|}$ the direction of x and define

$$\begin{aligned} I_j\chi(x) &= [\chi, \gamma g_j(\hat{x},\cdot)] && \big(\chi\in(V(\Gamma))'\big) \\ J_j^\sigma\psi(x) &= [\delta_\sigma g_j(\hat{x},\cdot),\psi] && \big(\psi\in V(\Gamma)\big) \end{aligned} \qquad (j=1,\dots,4). \tag{3.7}$$

Note that J_1^σ and J_2^σ vanish, I_1, I_3 and J_3^σ are constants, while I_2, I_4 and J_4^σ depend on the direction of x. Since the asymptotics of the fundamental solution for $|x| = R \to \infty$ can be written in the form

$$\begin{aligned} G(x,y) = \tfrac{1}{8\pi}\Big(&R^2\log R - g_2(\hat{x},y)(2R\log R + R) + g_3(\hat{x},y)\log R + g_4(\hat{x},y)\Big) \\ &+ O(R^{-1}) \end{aligned} \tag{3.8}$$

(cf. [3]), definition (3.2) of the layer potentials implies

Lemma 3.3. *For given* $(\psi, \chi) \in V(\Gamma) \times (V(\Gamma))'$ *the function*

$$u(x) = \mathcal{K}_\sigma \psi(x) - \mathcal{V}\chi(x)$$

behaves for large $|x| = R$ *as*

$$\begin{aligned} u(x) = -\tfrac{1}{8\pi}\Big(& I_1\chi\, R^2 \log R - I_2\chi(x)(2R\log R + R) \\ &+ (I_3\chi - J_3^\sigma\psi)\log R + I_4\chi(x) - J_4^\sigma\psi(x)\Big) + O(R^{-1}). \end{aligned} \tag{3.9}$$

Corollary 3.1. *The operator* $\mathcal{K}_\sigma : V(\Gamma) \to W^2(\Omega_2)$ *is continuous.*

Now one can prove the representation formula for functions $u \in L_2^\sigma$.

Lemma 3.4. *For* $u \in L_2^\sigma$ *with Cauchy data* $(\gamma u, \delta_\sigma u)$ *there holds*

$$\mathcal{K}_\sigma\gamma u(x) - \mathcal{V}\delta_\sigma u(x) = \begin{cases} u(x) & \text{if } x \in \Omega_2 \\ 0 & \text{if } x \in \Omega_1. \end{cases} \tag{3.10}$$

Proof. We enclose Ω_1 by a ball B_R with radius $R > |x|$. Then representation formula (3.6) is valid for the bounded domain $\Omega_2 \cap B_R$ yielding

$$\begin{aligned} u(x) = {}& \mathcal{K}_\sigma\gamma u(x) - \mathcal{V}\delta_\sigma u(x) \\ &+ \int_{S_R}\Big(u\, N_{\sigma,z}G(x,z) - M_{\sigma,z}G(x,z)\,\partial_n u \\ &\qquad + M_\sigma u\,\partial_{n_z}G(x,z) - G(x,z)\,N_\sigma u\Big)ds_z. \end{aligned}$$

Using asymptotics (3.9) of $u(z)$ as $R = |z| \to \infty$ and asymptotics (3.8) of the fundamental solution it was shown in [16] that the integral

$$\int_{S_R}\Big(u\,\partial_{n_z}\Delta G(x,z) - \Delta G(x,z)\,\partial_n u + \Delta u\,\partial_{n_z}G(x,z) - G(x,z)\,\partial_n\Delta u\Big)ds_z$$

converges to 0 as $R \to \infty$. By the same technique one obtains after some lengthy computations that the remaining integral converges to 0, too ∎

Corollary 3.2. *A function* $u \in L_2^\sigma$ *belongs to the weighted Sobolev space* $W^2(\Omega_2)$ *if and only if* $\delta_\sigma u \in l\,(\Gamma)^\perp$.

Corollary 3.3. *Let* $0 \le \sigma < 1$. *If the exterior Neumann problem*

$$\left.\begin{aligned} \Delta^2 u = 0 \quad & \text{in } \Omega_2 \\ \delta_\sigma u = \chi \in (V(\Gamma))' & \end{aligned}\right\} \tag{3.11}$$

has a solution $u \in L_2^\sigma$, *then this solution is unique.*

Proof. Obviously, it suffices to show that $\delta_\sigma u = 0$ for $u \in L_2^\sigma$ implies $u = 0$. Due to Lemma 2.7 we have $\|\dot{u}\|^2_{\mathcal{H}^2(\Omega_2)} = 0$, hence $u \in \mathbb{P}_1$. But in view of asymptotics (3.9) this is only possible if $u = 0$ ∎

We note that the exterior Dirichlet problem

$$\left.\begin{aligned} \Delta^2 u = 0 \quad \text{in } \Omega_2 \\ \gamma u = \psi \in V(\Gamma) \end{aligned}\right\} \tag{3.12}$$

is not uniquely solvable in L_2^σ, in general. For example, the two biharmonic functions

$$u_j(x_1, x_2) = x_j \big(2 \log |x| + 1 + e^{-2} \, |x|^{-2} \big)$$

have vanishing trace $\gamma u_j = 0$ on the circle Γ with radius e^{-1}, whereas for any circle Γ with radius $r \neq e^{-1}$ the problem

$$\left.\begin{aligned} \Delta^2 u = 0 \quad \text{in } \Omega_2 \\ \gamma u = 0 \qquad \end{aligned}\right\} \tag{3.13}$$

has only the trivial solution.

In the following we say that the curve Γ satisfies the assumption (A_I) if the corresponding exterior homogeneous Dirichlet problem (3.13) has only the trivial solution or, equivalently,

(A_I) $\quad u \in L_2^\sigma$ with $\gamma u = 0$ implies $\delta_\sigma u = 0$.

Recently Costabel and Dauge proved in [6] that for any general curve Γ there exist between 1 and 4 values of the scaling factor $\rho > 0$ such that the scaled curve $\rho\Gamma = \{\rho x \in \mathbb{R}^2 : x \in \Gamma\}$ violates assumption (A_I).

Lemma 3.5. *The layer potentials provide the jump relations*

$$\begin{aligned} \{\gamma \mathcal{V} \chi\} = 0, \qquad \{\delta_\sigma \mathcal{V} \chi\} = -\chi \quad & \textit{for all } \chi \in (V(\Gamma))' \\ \{\gamma \mathcal{K}_\sigma \psi\} = \psi, \quad \{\delta_\sigma \mathcal{K}_\sigma \psi\} = 0 \qquad & \textit{for all } \psi \in V(\Gamma). \end{aligned}$$

Proof. Since $u = \mathcal{V}\chi \in H^2_{loc}(\mathbb{R}^2)$ we have $\gamma(u|_{\Omega_1}) = \gamma(u|_{\Omega_2})$. Further, from (3.3) we obtain that $\Delta^2 u = \gamma' \chi$ in distributional sense, i.e.

$$\int_{\mathbb{R}^2} u \, \Delta^2 \varphi \, dx = \langle \gamma' \chi, \varphi \rangle_{\mathbb{R}^2} = [\chi, \gamma\varphi] \qquad \forall \, \varphi \in C_0^\infty(\mathbb{R}^2).$$

On the other hand,

$$\begin{aligned} \int_{\Omega_1} u \, \Delta^2 \varphi \, dx &= a^\sigma_{\Omega_1}(u, \varphi) - [\delta_\sigma \varphi, \gamma u] = [\delta_\sigma (u|_{\Omega_1}), \gamma\varphi] - [\delta_\sigma \varphi, \gamma u] \\ \int_{\Omega_2} u \, \Delta^2 \varphi \, dx &= a^\sigma_{\Omega_2}(u, \varphi) + [\delta_\sigma \varphi, \gamma u] = -[\delta_\sigma (u|_{\Omega_2}), \gamma\varphi] + [\delta_\sigma \varphi, \gamma u]. \end{aligned}$$

Thus

$$[\chi, \gamma\varphi] = -\big[\delta_\sigma(\mathcal{V}\chi|_{\Omega_2}) - \delta_\sigma(\mathcal{V}\chi|_{\Omega_1}), \gamma\varphi\big] \qquad \forall \, \varphi \in C_0^\infty(\mathbb{R}^2).$$

Let now $u = \mathcal{K}_\sigma\psi$, $\psi \in V(\Gamma)$, and again $\varphi \in C_0^\infty(\mathbb{R}^2)$. Green's second formula yields

$$\int_{\mathbb{R}^2} u\,\Delta^2\varphi\,dx = -[\{\delta_\sigma u\}, \gamma\varphi] + [\delta_\sigma\varphi, \{\gamma u\}]. \tag{3.14}$$

The definition of $\mathcal{K}_\sigma$ provides

$$u = \mathcal{K}_\sigma\psi = \mathcal{G}\delta'_\sigma\psi \tag{3.15}$$

where $\delta'_\sigma\psi$ denotes the compactly supported distribution on $\mathbb{R}^2$ defined by

$$\langle\varphi, \delta'_\sigma\psi\rangle_{\mathbb{R}^2} = [\delta_\sigma\varphi, \psi] \qquad \forall\,\varphi \in C_0^\infty(\mathbb{R}^2).$$

So $\Delta^2 u = \delta'_\sigma\psi$ in distributional sense, therefore

$$\int_{\mathbb{R}^2} u\,\Delta^2\varphi\,dx = [\delta_\sigma\varphi, \psi]. \tag{3.16}$$

Comparing (3.14) and (3.16) we obtain

$$[\delta_\sigma\varphi, \psi - \{\gamma u\}] = -[\{\delta_\sigma u\}, \gamma\varphi] \qquad \forall\,\varphi \in C_0^\infty(\mathbb{R}^2).$$

Thus from (2.22) we conclude that $\{\gamma\mathcal{K}_\sigma\psi\} - \psi = 0 = \{\delta_\sigma\mathcal{K}_\sigma\psi\}$ ∎

4. Boundary integral operators for the bi-Laplacian

In this section we study some basic properties of boundary integral operators connected with the biharmonic layer potentials. These operators are defined as the traces

$$\begin{aligned}
\mathcal{A}\chi &= 2\,\gamma\mathcal{V}\chi \\
\mathcal{B}_\sigma\chi &= 2\,\delta_\sigma(\mathcal{V}\chi|_{\Omega_1}) \\
\mathcal{C}_\sigma\psi &= 2\,\gamma(\mathcal{K}_\sigma\psi|_{\Omega_1}) \\
\mathcal{D}_\sigma\psi &= -2\,\delta_\sigma(\mathcal{K}_\sigma\psi|_{\Omega_1}).
\end{aligned}$$

Formally this definition is the same as for the second order equations given in [5]. We will show that the biharmonic boundary integral operators have analogous properties as the corresponding operators of the Laplacian.

Lemma 4.1 [16]. *The operator $\mathcal{A} : (V(\Gamma))' \to V(\Gamma)$ is continuous, symmetric and strongly elliptic, and it is positive definite on $(W(\Gamma))'$, i.e. for any $\chi \in (W(\Gamma))' = l\,(\Gamma)^\perp$ there holds*

$$[\chi, \mathcal{A}\chi] \geq c\,\|\chi\|^2_{(V(\Gamma))'}$$

with a constant $c > 0$ not depending on χ. If additionally the curve Γ satisfies assumption (A_I), then $\mathcal{A}$ is bijective.

Here and in the following the adjoints of boundary integral operators are taken of course with respect to duality (2.1).

Lemma 4.2. *Let $0 \le \sigma < 1$. The operator $\mathcal{D}_\sigma : V(\Gamma) \to (V(\Gamma))'$ is symmetric and strongly elliptic with $\ker \mathcal{D}_\sigma = l\,(\Gamma)$ and $\operatorname{im} \mathcal{D}_\sigma = l\,(\Gamma)^\perp$. Moreover, the isomorphism $\mathcal{D}_\sigma : W(\Gamma) \to (W(\Gamma))'$ is positive definite.*

Proof. Note first that the boundedness and symmetry of $\mathcal{D}_\sigma$ follows immediately from Lemmas 2.2, 2.4, 3.2 and the symmetry of the kernel function G. To prove that $\mathcal{D}_\sigma$ is positive definite we take $\psi \in V(\Gamma)$ and set $u_1 = -\mathcal{K}_\sigma \psi|_{\Omega_1}$ and $u_2 = -\mathcal{K}_\sigma \psi|_{\Omega_2}$. The jump relations lead to $\delta_\sigma u_1 = \delta_\sigma u_2 = \frac{1}{2}\,\mathcal{D}_\sigma \psi$ and $\gamma u_2 - \gamma u_1 = -\psi$. Due to Corollary 3.1 we have $|u|^2_{W^2(\Omega_2)} < \infty$, so by (2.21) and (2.24)

$$\begin{aligned}
\tfrac{1}{2}\,[\mathcal{D}_\sigma \psi, \psi] &= [\delta_\sigma u_1, \gamma u_1] - [\delta_\sigma u_2, \gamma u_2] \\
&= a^\sigma_{\Omega_1}(u_1, u_1) + a^\sigma_{\Omega_2}(u_2, u_2) \\
&\ge (1-\sigma)\big(\|u_1\|^2_{\mathcal{H}^2(\Omega_1)} + \|u_2\|^2_{\mathcal{H}^2(\Omega_2)}\big).
\end{aligned}$$

Since

$$\|\psi\|_{W(\Gamma)} \le \|\gamma u_1\|_{W(\Gamma)} + \|\gamma u_2\|_{W(\Gamma)} \le c\left(\|u_1\|_{\mathcal{H}^2(\Omega_1)} + \|u_2\|_{\mathcal{H}^2(\Omega_2)}\right)$$

we obtain $[\mathcal{D}_\sigma \psi, \psi] \ge c_\sigma\, \|\psi\|^2_{W(\Gamma)}$, hence $\mathcal{D}_\sigma$ is strongly elliptic in $V(\Gamma)$. From (2.20) it is clear that $\ker \mathcal{D}_\sigma = l\,(\Gamma)$ ∎

Lemma 4.3. *The boundary operators $\mathcal{C}_\sigma : V(\Gamma) \to V(\Gamma)$ and $\mathcal{B}_\sigma : (V(\Gamma))' \to (V(\Gamma))'$ are continuous and connected by the relation $\mathcal{B}'_\sigma = \mathcal{C}_\sigma + 2\,I$.*

Proof. For any $(\psi, \chi) \in V(\Gamma) \times (V(\Gamma))'$ we obtain from (3.15) and Lemma 3.5

$$\begin{aligned}
[\mathcal{B}_\sigma \chi, \psi] &= [\delta_\sigma(\mathcal{V}\chi|_{\Omega_1}) + \delta_\sigma(\mathcal{V}\chi|_{\Omega_2}) + \chi, \psi] \\
&= \langle \mathcal{G}\gamma'\chi|_{\Omega_1} + \mathcal{G}\gamma'\chi|_{\Omega_2}, \delta'_\sigma \psi\rangle_{\mathbb{R}^2} + [\chi, \psi] \\
&= \langle \mathcal{G}\gamma'\chi, \delta'_\sigma \psi\rangle_{\mathbb{R}^2} + [\chi, \psi] \\
&= \langle \gamma'\chi, \mathcal{K}_\sigma \psi\rangle_{\mathbb{R}^2} + [\chi, \psi] \\
&= [\chi, \gamma(\mathcal{K}_\sigma \psi|_{\Omega_1}) + \gamma(\mathcal{K}_\sigma \psi|_{\Omega_2})] + [\chi, \psi] \\
&= [\chi, 2\gamma(\mathcal{K}_\sigma \psi|_{\Omega_1}) + \psi] + [\chi, \psi] \\
&= [\chi, \mathcal{C}_\sigma \psi] + 2[\chi, \psi]
\end{aligned}$$

and the proof is complete ∎

If we introduce the operator $\mathcal{W}_\sigma = I + \mathcal{C}_\sigma$, then $\mathcal{B}_\sigma = I + \mathcal{W}'_\sigma$, and Lemma 3.5 yields

$$\begin{aligned}
\gamma(\mathcal{K}_\sigma \psi|_{\Omega_j}) &= \tfrac{1}{2}\,(\mathcal{W}_\sigma + (-1)^j I)\,\psi \\
\delta_\sigma(\mathcal{V}\chi|_{\Omega_j}) &= \tfrac{1}{2}\,(\mathcal{W}'_\sigma - (-1)^j I)\,\chi
\end{aligned} \qquad (j = 1, 2). \tag{4.1}$$

Let us mention that in the special case $\sigma = 1$, where the form a^σ is not coercive, we obtained the following characterizations in [16]:

- The operator $\frac{1}{2}\,(I - \mathcal{W}_1) = -\frac{1}{2}\,\mathcal{C}_1$ is the Calderon projection onto the traces γu of harmonic functions $u \in H^2(\Omega_1)$.

- The operator $\frac{1}{2}(I + \mathcal{W}_1) = \frac{1}{2}\mathcal{B}_1'$ projects onto the traces γu of all harmonic functions $u \in H^2_{loc}(\Omega_2)$ with the asymptotics $u(x) = a(\log|x| + 1) + O(|x|^{-1})$, $|x| \to \infty$, for some real a.
- $\mathcal{D}_1\psi = 0$ for all $\psi \in V(\Gamma)$.

Now we introduce the bounded linear operator

$$\mathfrak{B}_\sigma = \begin{pmatrix} -\mathcal{W}_\sigma & \mathcal{A} \\ \mathcal{D}_\sigma & \mathcal{W}_\sigma' \end{pmatrix} : \begin{matrix} V(\Gamma) \\ \times \\ (V(\Gamma))' \end{matrix} \longrightarrow \begin{matrix} V(\Gamma) \\ \times \\ (V(\Gamma))' \end{matrix} \tag{4.2}$$

and define the mappings

$$\mathfrak{C}_{\sigma,j} = \tfrac{1}{2}\left(I - (-1)^j \mathfrak{B}_\sigma\right) \qquad (j = 1, 2). \tag{4.3}$$

Lemma 4.4. *The operators $C_{\sigma,j}$ $(j = 1, 2)$ are the Calderon projections which map $V(\Gamma) \times (V(\Gamma))'$ onto the set of Cauchy data $(\gamma u, \delta_\sigma u)$ of functions $u \in L_j^\sigma$.*

Proof. For arbitrary $(\psi, \chi) \in V(\Gamma) \times (V(\Gamma))'$ and $u = (-1)^j(\mathcal{K}_\sigma\psi - \mathcal{V}\chi) \in L_j^\sigma$ the jump relations of Lemmas 3.5 and (4.1) imply

$$\begin{aligned}
\begin{pmatrix} \gamma u \\ \delta_\sigma u \end{pmatrix} &= (-1)^j \begin{pmatrix} \gamma(\mathcal{K}_\sigma\psi|_{\Omega_j}) - \gamma(\mathcal{V}\chi|_{\Omega_j}) \\ \delta_\sigma(\mathcal{K}_\sigma\psi|_{\Omega_j}) - \delta_\sigma(\mathcal{V}\chi|_{\Omega_j}) \end{pmatrix} \\
&= (-1)^j \tfrac{1}{2} \begin{pmatrix} (\mathcal{W}_\sigma + (-1)^j I)\psi - \mathcal{A}\chi \\ -\mathcal{D}_\sigma\psi - (\mathcal{W}_\sigma' - (-1)^j I)\chi \end{pmatrix} \\
&= \tfrac{1}{2}\left(I - (-1)^j \mathfrak{B}_\sigma\right) \begin{pmatrix} \psi \\ \chi \end{pmatrix} \\
&= \mathfrak{C}_{\sigma,j} \begin{pmatrix} \psi \\ \chi \end{pmatrix}.
\end{aligned}$$

Let now $u \in L_j^\sigma$. Then representation formula (3.6) or (3.10) yields

$$u(x) = (-1)^j \left(\mathcal{K}_\sigma \gamma u(x) - \mathcal{V}\delta_\sigma u(x)\right) \qquad (x \in \Omega_j).$$

After applying the jump relations we obtain

$$\begin{pmatrix} \gamma u \\ \delta_\sigma u \end{pmatrix} = \mathfrak{C}_{\sigma,j} \begin{pmatrix} \gamma u \\ \delta_\sigma u \end{pmatrix}.$$

Hence the mappings $\mathfrak{C}_{\sigma,j}$ are bounded projections and the Cauchy data of all functions from L_j^σ belong to the image of $\mathfrak{C}_{\sigma,j}$ ∎

Since the Calderon projections for the interior and exterior problems are conjugate, $\mathfrak{C}_{\sigma,1} + \mathfrak{C}_{\sigma,2} = I$, the space $V(\Gamma) \times (V(\Gamma))'$ can be decomposed as the direct sum of closed subspaces

$$V(\Gamma) \times (V(\Gamma))' = \left\{(\gamma u, \delta_\sigma u) : u \in L_1\right\} \dot{+} \left\{(\gamma u, \delta_\sigma u) : u \in L_2^\sigma\right\}. \tag{4.4}$$

Corollary 4.1. *From $\mathfrak{C}^2_{\sigma,j} = \mathfrak{C}_{\sigma,j}$ we derive the relations*

$$\begin{aligned} \mathcal{W}_\sigma \, \mathcal{A} &= \mathcal{A} \, \mathcal{W}'_\sigma \\ \mathcal{W}'_\sigma \, \mathcal{D}_\sigma &= \mathcal{D}_\sigma \, \mathcal{W}_\sigma \\ \mathcal{A} \, \mathcal{D}_\sigma &= I - \mathcal{W}^2_\sigma . \end{aligned} \tag{4.5}$$

Lemma 4.5. *Let $0 \le \sigma < 1$. The operator $(I - \mathcal{W}_\sigma) : V(\Gamma) \to V(\Gamma)$ is bijective, whereas $(I + \mathcal{W}_\sigma) : V(\Gamma) \to V(\Gamma)$ is Fredholm with index zero,* $\ker (I + \mathcal{W}_\sigma) = l\,(\Gamma)$ *and* $\operatorname{im} (I + \mathcal{W}_\sigma) = \mathcal{A}(l\,(\Gamma)^\perp)$.

Proof. From (4.5) we have

$$\mathcal{A} \, \mathcal{D}_\sigma = (I + \mathcal{W}_\sigma)(I - \mathcal{W}_\sigma) = (I - \mathcal{W}_\sigma)(I + \mathcal{W}_\sigma). \tag{4.6}$$

Since $\mathcal{A}$ and $\mathcal{D}_\sigma$ $(0 \le \sigma < 1)$ are strongly elliptic, the operator $\mathcal{A} \, \mathcal{D}_\sigma$ is Fredholm with index zero, and by well-known arguments (cf. [15: Theorems 1.3.1 and 1.3.3]) the operators $(I \pm \mathcal{W}_\sigma)$ are Fredholm itself. Based on relations (4.1) one can use the uniqueness of the interior Dirichlet problem in L_1 and of the exterior Neumann problem in L_2^σ to derive that

$$\ker (I - \mathcal{W}_\sigma) = \ker (I - \mathcal{W}'_\sigma) = 0.$$

Therefore $(I + \mathcal{W}_\sigma)$ is a Fredholm operator with index 0, from (4.6) its kernel and image can be determined by using Lemmas 4.1 and 4.2 ∎

5. Boundary integral equations for plate bending problems

Using the layer potentials and boundary integral operators it is now quite easy to transform biharmonic boundary value problems into integral equations over the boundary. For example, the results of Sections 2 and 3 and certain layer potential representations lead immediately to equivalent integral equations for Dirichlet and Neumann problems. However, the analysis of indirect methods for other types of boundary conditions seems to be more involved. Here we concentrate on a direct method which produces strongly elliptic systems of boundary integral equations equivalent to mixed biharmonic boundary value problems. Having properties of boundary integral operators at hand the analysis of the proposed method simply extends the well-studied approach for second-order equations to our situation.

We introduce the bounded bilinear form on $V(\Gamma) \times (V(\Gamma))'$

$$\left\langle \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \begin{pmatrix} \rho \\ \tau \end{pmatrix} \right\rangle_{V(\Gamma) \times (V(\Gamma))'} := [\tau, \psi] + [\chi, \rho]. \tag{5.1}$$

From (4.2) we see that for any $(\psi, \chi) \in V(\Gamma) \times (V(\Gamma))'$ the equality

$$\begin{aligned} &\left\langle \mathfrak{B}_\sigma \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \begin{pmatrix} \psi \\ \chi \end{pmatrix} \right\rangle_{V(\Gamma) \times (V(\Gamma))'} \\ &\quad = -[\chi, \mathcal{W}_\sigma \psi] + [\chi, \mathcal{A} \chi] + [\mathcal{D}_\sigma \psi, \psi] + [\mathcal{W}'_\sigma \chi, \psi] \\ &\quad = [\chi, \mathcal{A} \chi] + [\mathcal{D}_\sigma \psi, \psi] \end{aligned} \tag{5.2}$$

holds. Let us denote by $\mathcal{P} : V(\Gamma) \to V(\Gamma)$ a bounded projection, set $\mathcal{Q} = I - \mathcal{P}$ and introduce the projection $\mathfrak{P}$ in $V(\Gamma) \times (V(\Gamma))'$ by

$$\mathfrak{P} = \begin{pmatrix} \mathcal{P} & 0 \\ 0 & \mathcal{Q}' \end{pmatrix} : \begin{matrix} V(\Gamma) \\ \times \\ (V(\Gamma))' \end{matrix} \longrightarrow \begin{matrix} \operatorname{im}\mathcal{P} \\ \times \\ \operatorname{im}\mathcal{Q}'. \end{matrix} \tag{5.3}$$

Note that

$$\begin{aligned} \operatorname{im}\mathcal{P} \times \operatorname{im}\mathcal{Q}' &= \operatorname{im}\mathfrak{P} \\ \operatorname{im}\mathcal{Q} \times \operatorname{im}\mathcal{P}' &= \operatorname{im}(I - \mathfrak{P}) \end{aligned}$$

are closed subspaces of $V(\Gamma) \times (V(\Gamma))'$ which are in duality with respect to (5.1). Since $(\operatorname{im}\mathcal{Q})^{\perp} = (\ker\mathcal{P})^{\perp} = \operatorname{im}\mathcal{P}'$, equality (5.2) leads to

$$\begin{aligned} &\Big\langle \mathfrak{C}_{\sigma,j} \binom{\mathcal{Q}\psi}{\mathcal{P}'\chi}, \binom{\mathcal{Q}\psi}{\mathcal{P}'\chi} \Big\rangle_{V(\Gamma)\times(V(\Gamma))'} \\ &\quad = \tfrac{1}{2} \Big\langle \big(I - (-1)^j \mathfrak{B}_\sigma\big) \binom{\mathcal{Q}\psi}{\mathcal{P}'\chi}, \binom{\mathcal{Q}\psi}{\mathcal{P}'\chi} \Big\rangle_{V(\Gamma)\times(V(\Gamma))'} \\ &\quad = (-1)^{j+1} \tfrac{1}{2} \Big([\mathcal{A}\mathcal{P}'\chi, \mathcal{P}'\chi] + [\mathcal{D}_\sigma \mathcal{Q}\psi, \mathcal{Q}\psi] \Big). \end{aligned}$$

Hence for any projection $\mathcal{P}$ the mappings

$$\mathfrak{A}_\sigma^{\mathcal{P}} = (-1)^{j+1} \mathfrak{P}\, \mathfrak{C}_{\sigma,j}\, (I - \mathfrak{P}) = \tfrac{1}{2}\, \mathfrak{P}\, \mathfrak{B}_\sigma (I - \mathfrak{P}) : \begin{matrix} \operatorname{im}\mathcal{Q} \\ \times \\ \operatorname{im}\mathcal{P}' \end{matrix} \longrightarrow \begin{matrix} \operatorname{im}\mathcal{P} \\ \times \\ \operatorname{im}\mathcal{Q}' \end{matrix} \tag{5.4}$$

do not depend on $j = 1, 2$. If $0 \le \sigma < 1$, then in view of Lemmas 4.1 and 4.2 the operator $\mathfrak{A}_\sigma^{\mathcal{P}}$ satisfies a Gårding inequality

$$\Big\langle (\mathfrak{A}_\sigma^{\mathcal{P}} + \mathfrak{T}) \binom{\psi}{\chi}, \binom{\psi}{\chi} \Big\rangle_{V(\Gamma)\times(V(\Gamma))'} \ge c \Big(\|\psi\|^2_{V(\Gamma)} + \|\chi\|^2_{(V(\Gamma))'} \Big)$$

for all $(\psi, \chi) \in \operatorname{im}\mathcal{Q} \times \operatorname{im}\mathcal{P}'$, with some constant $c > 0$ and a compact operator $\mathfrak{T}$. Since the adjoint of $\mathfrak{A}_\sigma^{\mathcal{P}}$ with respect to form (5.1) provides the same property we derive

Lemma 5.1. *Let* $0 \le \sigma < 1$ *and* $\mathcal{P}$ *be a bounded projection in* $V(\Gamma)$*. Then* $\mathfrak{A}_\sigma^{\mathcal{P}}$ *defined in* (5.4) *is a Fredholm operator with index zero from* $\operatorname{im}(I - \mathfrak{P})$ *into* $\operatorname{im}\mathfrak{P}$ *and strongly elliptic with respect to* (5.1).

Note that the two trivial cases $\mathcal{P} = I$ and $\mathcal{P} = 0$ are treated in Lemmas 4.1 and 4.2, respectively.

The mapping $\mathfrak{A}_\sigma^{\mathcal{P}}$ is closely connected with the following biharmonic boundary value problem:

$$\text{Find } \; u \in L_j^\sigma \; \text{ such that } \mathcal{P}\gamma u = \rho \quad \text{and} \quad \mathcal{Q}'\delta_\sigma u = \tau \tag{5.5}$$

where $(\rho, \tau) \in \operatorname{im}\mathcal{P} \times \operatorname{im}\mathcal{Q}'$ are given boundary values. Indeed, for $u \in L_j^\sigma$ we know from Lemma 4.4 that

$$\mathfrak{C}_{\sigma,j} \begin{pmatrix} \gamma u \\ \delta_\sigma u \end{pmatrix} = \begin{pmatrix} \gamma u \\ \delta_\sigma u \end{pmatrix}.$$

To solve problem (5.5) we decompose

$$\begin{pmatrix} \gamma u \\ \delta_\sigma u \end{pmatrix} = \begin{pmatrix} \mathcal{P}\gamma u \\ \mathcal{Q}'\delta_\sigma u \end{pmatrix} + \begin{pmatrix} \mathcal{Q}\gamma u \\ \mathcal{P}'\delta_\sigma u \end{pmatrix}$$

so that the unknowns $\psi = \mathcal{Q}\gamma u$ and $\chi = \mathcal{P}'\delta_\sigma u$ have to satisfy

$$(I - \mathfrak{C}_{\sigma,j}) \begin{pmatrix} \psi \\ \chi \end{pmatrix} = (\mathfrak{C}_{\sigma,j} - I) \begin{pmatrix} \rho \\ \tau \end{pmatrix}. \tag{5.6}$$

In particular, applying the projection $\mathfrak{P}$ to both sides we get the equation

$$\mathfrak{A}_\sigma^{\mathcal{P}} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = (-1)^j \, \mathfrak{P} \, (\mathfrak{C}_{\sigma,j} - I) \begin{pmatrix} \rho \\ \tau \end{pmatrix}. \tag{5.7}$$

Lemma 5.2. *Let* $(\rho, \tau) \in \operatorname{im} \mathcal{P} \times \operatorname{im} \mathcal{Q}'$.

(i) *If* $u \in L_j^\sigma$ *satisfies* (5.5), *then* $\psi = \mathcal{Q}\gamma u$ *and* $\chi = \mathcal{P}'\delta_\sigma u$ *solve equation* (5.7).

(ii) *If* $(\psi, \chi) \in \operatorname{im} \mathcal{Q} \times \operatorname{im} \mathcal{P}'$ *is a solution of* (5.7), *then the function* u *given in* Ω_j *by*

$$u = (-1)^j \big(\mathcal{K}_\sigma(\psi + \rho) - \mathcal{V}(\chi + \tau)\big) \tag{5.8}$$

solves boundary value problem (5.5).

Proof. It remains to show statement (ii). For u from (5.8) there holds in view of Lemma 4.4

$$\begin{pmatrix} \gamma u \\ \delta_\sigma u \end{pmatrix} = \mathfrak{C}_{\sigma,j} \begin{pmatrix} \psi + \rho \\ \chi + \tau \end{pmatrix}.$$

Since equation (5.7) is fulfilled we have

$$\mathfrak{P} \, (I - \mathfrak{C}_{\sigma,j}) \begin{pmatrix} \psi + \rho \\ \chi + \tau \end{pmatrix} = 0$$

implying $\mathcal{P}\gamma u = \mathcal{P}(\psi + \rho) = \rho$ and $\mathcal{Q}'\delta_\sigma u = \mathcal{Q}'(\chi + \tau) = \tau$ ∎

Thus any solution of the boundary value problem (5.5) can be obtained by solving the system of boundary integral equations (5.7). Note that in general this system has more linear independent solutions than (5.5).

Lemma 5.3. *Let* $0 \le \sigma < 1$ *and* $\beta_j \ \ (j = 1, 2)$ *be the dimension of the null space of the corresponding homogeneous problem* (5.5) *with* $\rho = \tau = 0$. *Then*

$$\dim \ker \mathfrak{A}_\sigma^{\mathcal{P}} = \beta_1 + \beta_2 \le 3 \qquad \text{and} \qquad \beta_1 = \dim \mathcal{Q}(l(\Gamma)).$$

Proof. Since $u \in L_j^\sigma$ with $(\gamma u, \delta_\sigma u) \in \operatorname{im} \mathcal{Q} \times \operatorname{im} \mathcal{P}'$, i.e. $\mathcal{P}\gamma u = \mathcal{Q}'\delta_\sigma u = 0$, determines an element $(\gamma u, \delta_\sigma u) \in \ker \mathfrak{A}_\sigma^{\mathcal{P}}$ and by (4.4)

$$\big\{(\gamma u, \delta_\sigma u) : \ u \in L_1\big\} \cap \big\{(\gamma u, \delta_\sigma u) : \ u \in L_2^\sigma\big\} = \emptyset$$

it is clear that $\dim \ker \mathfrak{A}_\sigma^{\mathcal{P}} \ge \beta_1 + \beta_2$. On the other hand, since $V(\Gamma) \times (V(\Gamma))'$ is the direct sum of these subspaces there exists a basis in $\ker \mathfrak{A}_\sigma^{\mathcal{P}}$ consisting of elements of the subspaces. Due to Lemma 5.2/(ii) and representation formulas (3.6) and (3.10) we get therefore $\dim\ker\mathfrak{A}_\sigma^{\mathcal{P}} \le \beta_1+\beta_2$.

Let now $u\in L_1$ with $\mathcal{P}\gamma u=\mathcal{Q}'\delta_\sigma u=0$. Then

$$a_{\Omega_1}^\sigma(u,u)=[\delta_\sigma u,\gamma u]=[\delta_\sigma u,\mathcal{P}\gamma u]+[\mathcal{Q}'\delta_\sigma u,\gamma u]=0$$

and Lemma 2.5 yields $u\in\mathbb{P}_1$, i.e. $\gamma u\in l\,(\Gamma)$ and $\delta_\sigma u=0$. Hence the homogeneous boundary conditions can be satisfied by $\beta_1=\dim\,\mathcal{Q}\,(l(\Gamma))$ linear independent elements of L_1. Using (2.24), Corollaries 3.2 and 3.3 it can be seen quite similarly that

$$u\in L_2^\sigma\cap W^2(\Omega_2)\ \text{ with }\ \mathcal{P}\gamma u=\mathcal{Q}'\delta_\sigma u=0 \qquad \text{implies}\ \ u=0.$$

Hence any non-trivial solution of the homogeneous boundary value problem in the outer domain Ω_2 satisfies $\delta_\sigma u\notin l\,(\Gamma)^\perp$ or, more precisely, the corresponding equivalence class $\delta_\sigma\dot u$ in the factor space $(V(\Gamma))'/l\,(\Gamma)^\perp$ is different from zero, $\delta_\sigma\dot u\neq 0$. Consequently, if $(\psi,\chi)\in(\operatorname{im}\mathcal{Q}\times\operatorname{im}\mathcal{P}')\cap\ker\mathfrak{A}_\sigma^{\mathcal{P}}$ and $\chi\neq 0$, then the equivalence class $\dot\chi\neq 0$ in $(V(\Gamma))'/l\,(\Gamma)^\perp$. This means that β_2 is not greater than the number of linear independent elements $\chi\in\operatorname{im}\mathcal{P}'$ with $\dot\chi\neq 0$ which equals to $\dim\mathcal{P}\,(l(\Gamma))=3-\dim\mathcal{Q}\,(l(\Gamma))=3-\beta_1$ ∎

Now we introduce the assumption

($\mathrm{A}_{\mathcal{P}}$) If $u\in L_2^\sigma$ satisfies $\mathcal{P}\gamma u=0$ and $(I-\mathcal{P}')\delta_\sigma u=0$, then $u=0$

and consider boundary value problem (5.5) for $j=2$.

Theorem 5.1. *Suppose that* Γ *satisfies assumption* ($\mathrm{A}_{\mathcal{P}}$), *let* $(\rho,\tau)\in\operatorname{im}\mathcal{P}\times\operatorname{im}\mathcal{Q}'$ *and* $0\le\sigma<1$. *Then the boundary value problem for the bi-Laplacian*

$$\left.\begin{aligned}&\Delta^2 u=0\quad \text{in } \Omega_2\\ &\mathcal{P}\gamma u=\rho,\ \ (I-\mathcal{P}')\,\delta_\sigma u=\tau\end{aligned}\right\}\tag{5.9}$$

has in the space L_2^σ *a unique solution given by*

$$u=\mathcal{K}_\sigma(\psi+\rho)-\mathcal{V}(\chi+\tau)$$

where $(\psi,\chi)\in\operatorname{im}\mathcal{Q}\times\operatorname{im}\mathcal{P}'$ *solves the system of boundary integral equations*

$$\mathfrak{A}_\sigma^{\mathcal{P}}\begin{pmatrix}\psi\\ \chi\end{pmatrix}=\tfrac12\,\mathfrak{P}\,(I-\mathfrak{B}_\sigma)\begin{pmatrix}\rho\\ \tau\end{pmatrix}.\tag{5.10}$$

If additionally the projection $\mathcal{P}$ *reproduces the traces of linear functions,* $\mathcal{P}\gamma p=\gamma p$ *for all* $p\in\mathbb{P}_1$, *then* (5.10) *is uniquely solvable.*

For $j=1$ boundary value problem (5.5) admits the following variational formulation:

$$\begin{aligned}&\text{Find}\ \ u\in H^2(\Omega_1)\ \text{ such that }\ \mathcal{P}\gamma u=\rho\ \text{ and}\\ &a_{\Omega_1}^\sigma(u,v)=[\tau,\mathcal{Q}\gamma v]\ \forall\, v\in H^2_{\mathcal{P}}(\Omega_1)=\{u\in H^2(\Omega_1):\,\mathcal{P}\gamma u=0\}.\end{aligned}\tag{5.11}$$

It is clear that problem (5.11) is uniquely solvable for $0\le\sigma<1$ if and only if the only linear function p satisfying $\mathcal{P}\gamma p=0$ is the trivial function $p=0$.

Theorem 5.2. *Suppose that* $\mathcal{P}(l(\Gamma)) = l(\Gamma)$, $0 \le \sigma < 1$ *and let* $(\rho, \tau) \in \operatorname{im}\mathcal{P} \times \operatorname{im}\mathcal{Q}'$. *The unique weak solution of the boundary value problem for the bi-Laplacian*

$$\left.\begin{aligned} &\Delta^2 u = 0 \quad \text{in } \Omega_1 \\ &\mathcal{P}\gamma u = \rho, \ (I - \mathcal{P}')\,\delta_\sigma u = \tau \end{aligned}\right\} \tag{5.12}$$

can be obtained by the formula

$$u = \mathcal{V}(\chi + \tau) - \mathcal{K}_\sigma(\psi + \rho)$$

where $(\psi, \chi) \in \operatorname{im}\mathcal{Q} \times \operatorname{im}\mathcal{P}'$ *solves the system of boundary integral equations*

$$\mathfrak{A}_\sigma^{\mathcal{P}} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = -\tfrac{1}{2}\,\mathfrak{P}\,(I + \mathfrak{B}_\sigma) \begin{pmatrix} \rho \\ \tau \end{pmatrix}. \tag{5.13}$$

If Γ *satisfies assumption* $(\mathrm{A}_{\mathcal{P}})$, *then system* (5.13) *is uniquely solvable.*

Roughly spoken, if the boundary conditions are such that the biharmonic boundary value problem can be transformed into a coercive variational problem, then it is equivalent to a strongly elliptic system of boundary integral equations.

As an example we now consider the choice of the projection $\mathcal{P}$ for mixed boundary conditions. We assume that the boundary Γ is composed of four disjoint parts Γ_c, Γ_h, Γ_r, and Γ_f such that

$$\Gamma = \overline{\Gamma_c \cup \Gamma_h \cup \Gamma_r \cup \Gamma_f}$$

and consider a bounded projection $\mathcal{P}$ in $V(\Gamma)$ providing

$$\mathcal{P}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in V(\Gamma) \qquad \text{with} \begin{cases} w_1 = v_1, w_2 = v_2 & \text{on } \Gamma_c \\ w_1 = v_1 & \text{on } \Gamma_h \\ w_2 = v_2 & \text{on } \Gamma_r \end{cases} \tag{5.14}$$

whereas the functions w_j are extended to the other parts of Γ in some specific way. Clearly, there exists of variety of projections giving (5.14), which differ only in the method of extending w_j. But the concrete form of the projection $\mathcal{P}$ is not important, we need only the existence of bounded projections, $\|\mathcal{P}\psi\|_{V(\Gamma)} \le c\,\|\psi\|_{V(\Gamma)}$, which is obvious. Since for the adjoint of $\mathcal{Q} = I - \mathcal{P}$ we have

$$\mathcal{Q}'\begin{pmatrix} v_4 \\ v_3 \end{pmatrix} = \begin{pmatrix} z_4 \\ z_3 \end{pmatrix} \in (V(\Gamma))' \qquad \text{with} \begin{cases} \langle v_4, w_1\rangle_\Gamma = \langle z_4, w_1\rangle_\Gamma \\ \langle v_3, w_2\rangle_\Gamma = \langle z_3, w_2\rangle_\Gamma \end{cases}$$

for all

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \ker\mathcal{P} = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V(\Gamma) : \ v_1|_{\Gamma_c \cup \Gamma_h} = 0 \ \text{ and } \ v_2|_{\Gamma_c \cup \Gamma_r} = 0 \right\}$$

we conclude that in weak sense

$$\mathcal{Q}'\begin{pmatrix} v_4 \\ v_3 \end{pmatrix} = \begin{pmatrix} z_4 \\ z_3 \end{pmatrix} \in (V(\Gamma))' \qquad \text{with} \begin{cases} z_3 = v_3, z_4 = v_4 & \text{on } \Gamma_f \\ z_3 = v_3 & \text{on } \Gamma_h \\ z_4 = v_4 & \text{on } \Gamma_r. \end{cases} \tag{5.15}$$

We note that the space $\ker \mathcal{P} \times \ker \mathcal{Q}'$ in which the unknowns (ψ, χ) of system (5.6) have to be sought is independent of the concrete choice of $\mathcal{P}$. Moreover, the definition of the trace spaces together with the description of $\ker \mathcal{P} \times \ker \mathcal{Q}'$ imposes certain compatibility conditions for the components of ψ and χ at singular boundary points, i.e. corners and points at which the type of boundary conditions changes. We will not go into detail, we mention only that it is important to take into account these compatibility conditions in choosing the approximation spaces for the numerical solution of (5.6).

If we formulate the boundary conditions in (5.5)

$$\mathcal{P}\gamma u = \rho = \binom{g_1}{g_2} \in \operatorname{im} \mathcal{P}$$
$$\mathcal{Q}'\delta_\sigma u = \tau = \binom{g_4}{g_3} \in \operatorname{im} \mathcal{Q}'$$

in terms of the Cauchy data of u which are defined in (2.17), then we obtain from (5.14) and (5.15) the following well-known mixed boundary conditions of plate bending:

(i) clamped: $u = g_1,\ \partial_n u = g_2$ on Γ_c,

(ii) hinged or simply supported: $u = g_1,\ M_\sigma u = g_3$ on Γ_h,

(iii) roller-supported: $\partial_n u = g_2,\ \widetilde{N}_\sigma u = g_4$ on Γ_r,

(iv) free: $M_\sigma u = g_3,\ \widetilde{N}_\sigma u = g_4$ on Γ_f.

Now the stability of the Galerkin method for solving the system of integral equations derived from mixed boundary conditions (i) - (iv) can be proved by standard arguments for sequences of finite-dimensional spaces of approximating functions $X_h \subset \ker \mathcal{P}$ and $Y_h \subset \ker \mathcal{Q}'$, $h \to 0$, so that

$$\bigcup_h X_h \times Y_h \qquad \text{is dense in } \ker \mathcal{P} \times \ker \mathcal{Q}'.$$

Theorem 5.3. *Suppose that $0 \le \sigma < 1$ and that the interior and the exterior boundary value problems for the biharmonic equation with homogeneous boundary conditions* (i) - (iv), *i.e. $g_i = 0$, have only trivial solutions. Then the Galerkin equations*

$$\Big\langle \mathfrak{B}_\sigma \binom{\psi_h}{\chi_h}, \binom{\varphi_h}{\phi_h} \Big\rangle_{V(\Gamma)\times(V(\Gamma))'} = 2(-1)^j \Big\langle (\mathfrak{C}_{\sigma,j} - I)\binom{\rho}{\tau}, \binom{\varphi_h}{\phi_h} \Big\rangle_{V(\Gamma)\times(V(\Gamma))'}$$

for all $\binom{\varphi_h}{\phi_h} \in X_h \times Y_h$ are uniquely solvable for all sufficiently small h and the approximate solutions

$$u_h = (-1)^j \big(\mathcal{K}_\sigma(\psi_h + \rho) - \mathcal{V}(\chi_h + \tau)\big)$$

converge quasioptimally to the biharmonic function u in Ω_j $(j = 1, 2)$ satisfying boundary conditions (i) - (iv). *For example, for any $x \in \Omega_j$ the estimate*

$$\big|u(x) - u_h(x)\big| \le c\Big(\inf_{\varphi_h \in X_h} \big\|\mathcal{Q}\gamma u - \varphi_h\big\|_{V(\Gamma)} + \inf_{\phi_h \in Y_h} \big\|\mathcal{P}'\delta_\sigma u - \phi_h\big\|_{(V(\Gamma))'}\Big)$$

holds with some constant $c > 0$ not depending on u and h.

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Received 24.05.2000