Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 20 (2001), No. 1, 155–182

On the Fredholm Property of the Stokes Operator in a Layer-Like Domain

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Abstract. The Stokes problem is studied in the domain $\Omega \subset \mathbb{R}^3$ coinciding with the layer $\Pi = \{x = (y, z) : y = (y_1, y_2) \in \mathbb{R}^2, z \in (0, 1)\}$ outside some ball. It is shown that the operator of such problem is of Fredholm type; this operator is defined on a certain weighted function space $\mathcal{D}^l_\beta(\Omega)$ with norm determined by a stepwise anisotropic distribution of weight factors (the direction of z is distinguished). The smoothness exponent l is allowed to be a positive integer, and the weight exponent β is an arbitrary real number except for the integer set $\mathbb{Z}$ where the Fredholm property is lost. Dimensions of the kernel and cokernel of the operator are calculated in dependence of β. It turns out that, at any admissible β, the operator index does not vanish. Based on the generalized Green formula, asymptotic conditions at infinity are imposed to provide the problem with index zero.

Keywords: *Stokes equations, layer-like domains, Fredholm property, weighted spaces*

AMS subject classification: 35Q30, 76D07

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a domain coinciding outside the ball $B_{R_0} = \{x \in \mathbb{R}^3 : |x| < R_0\}$ with the infinite layer

$$\Pi = \Big\{ x = (y, z) : y = (y_1, y_2) \in \mathbb{R}^2, \, z \in (0, 1) \Big\}. \tag{1.1}$$

For simplicity we assume the boundary $\partial\Omega$ to be smooth. Without loss of generality we also fix $R_0 = 1$. The set $\partial\Omega \setminus B_1$ contains infinite parts of two planes

$$S^{(0)} = \Big\{ x : y \in \mathbb{R}^2, \, z = 0 \Big\}$$
$$S^{(1)} = \Big\{ x : y \in \mathbb{R}^2, \, z = 1 \Big\}$$

which form the boundary $\partial\Pi$ of the layer Π. We consider the Stokes system

$$\left.\begin{aligned} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \\ -\operatorname{div} \mathbf{u} &= g \end{aligned}\right\} \qquad (\text{in } \Omega) \tag{1.2}$$

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ISSN 0232-2064 / $ 2.50

with the boundary conditions

$$\mathbf{u} = \mathbf{h} \qquad (\text{on } \partial\Omega) \tag{1.3}$$

where

$\mathbf{u} = (u_1, u_2.u_3)$ is the velocity field
p is the pressure in the fluid
$\mathbf{f} = (f_1, f_2, f_3)$ is an external force
g is a given scalar-valued function in Ω
$\mathbf{h}$ is a given vector-valued function on $\partial\Omega$
ν is the constant coefficient of viscosity
$\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, $\Delta = \nabla \cdot \nabla$, $\operatorname{div} \mathbf{u} = \nabla \cdot \mathbf{u}$
" $\cdot$ " means the scalar product in $\mathbb{R}^3$.

In the previous paper [15] we have studied the properties of solutions $(\mathbf{u}, p)$ to problem (1.2) - (1.3) in a two-parametric scale of weighted function spaces $\mathcal{D}^l_\beta(\Omega)$ and $\mathcal{R}^l_\beta(\Omega; \partial\Omega)$ such that the mapping

$$\mathcal{D}^l_\beta(\Omega) \ni (\mathbf{u}, p) \longmapsto \mathcal{S}^l_\beta(\mathbf{u}, p) = (\mathbf{f}, g, \mathbf{h}) \in \mathcal{R}^l_\beta(\Omega; \partial\Omega), \tag{1.4}$$

where $\mathcal{S}^l_\beta$ is the operator of the Stokes probem (1.2) - (1.3), becomes continuous. In (1.4) l is a regularity index and β a weight index. The exact definitions of these spaces and their properties are presented in Section 2. In terms of these spaces we have proved (see [15]) regularity results and a coercive estimate for the solution $(\mathbf{u}, p) \in L^2_\beta(\Omega) \times L^2_\beta(\Omega)$ where the latter space consists of functions with finite norm

$$\big\|(\mathbf{u}, p); L^2_\beta(\Omega) \times L^2_\beta(\Omega)\big\| = \bigg(\int_\Omega (1 + |y|^2)^\beta (|\mathbf{u}|^2 + |p|^2)\, dx \bigg)^{\frac{1}{2}}.$$

Moreover, in [15] the asymptotic representation of the solution $(\mathbf{u}, p) \in L^2_\beta(\Omega) \times L^2_\beta(\Omega)$ is constructed.

In this paper we prove the Fredholm property of mapping (1.4), calculate the dimensions of the kernel and cokernel and therefore the index of the operator $\mathcal{S}^l_\beta$ in (1.4). Moreover, we derive integral formulae for the coefficients in the asymptotic representation of the solution, which lead to a generalized Green formula. This formula, in particular, furnishes asymptotic conditions at infinity (in the same way as in the paper [16] where the Stokes operator was studied in domains with cylindrical outlets to infinity). Note also that the Fredholm property of the Neumann problem operator for a second order elliptic equation in a layer-like domain was proved in [13].

Acknowledgement. This work was concluded during the authors' stay in Portugal in April - June, 1999, funded by grants from CMA/IST (Lisbon) and from CIM/Fundação Calouste Gulbenkian (Coimbra) - Thematic Term on Theoretical and Computational Fluid Dynamics. The Portuguese financial support is deeply acknowledged.

2. Weighted function spaces and preliminary results

2.1 Function spaces. Let G be an arbitrary domain in $\mathbb{R}^n$ $(n \geq 2)$. As usual, denote by $C^\infty(G)$ the set of all indefinitely differentiable functions in $\overline{G}$ and let $C_0^\infty(G)$ be a subset of functions from $C^\infty(G)$ with compact supports in G. Further, $W^{l,2}(G)$ $(l \geq 0)$ indicates the Sobolev space and $W^{l-\frac{1}{2},2}(\partial G)$ $(l \geq 1)$ the space of traces on the boundary ∂G of functions from $W^{l,2}(G)$. Besides, $W^{0,2}(G) = L^2(G)$ and $W^{l,2}_{loc}(G)$ consists of functions which belong to $W^{l,2}(K)$ for every compact $K \subset \overline{G}$. The spaces of scalar- and vector-valued functions are not distinguished in notations. The norm of an element u in the function space X is denoted by $\|u; X\|$.

Let $\Omega \subset \mathbb{R}^3$ be a layer-like domain. Denote by $C_0^\infty(\overline{\Omega})$ the subset of functions from $C^\infty(\Omega)$ with compact supports in $\overline{\Omega}$ (functions from $C_0^\infty(\overline{\Omega})$ are equal to zero for large $|x|$, but not necessarily on $\partial\Omega$). We define the norm

$$\|u; V^l_\beta(\Omega)\| = \left(\int_\Omega \sum_{|\mu|=0}^{l} (1+r^2)^{\beta-l+|\mu|} |\nabla^\mu_x u(x)|^2 dx \right)^{\frac{1}{2}} \tag{2.1}$$

with homogeneous isotropic weight distribution. In (2.1) $r = |y|$ $(y \in \mathbb{R}^2)$, $x = (y,z) \in \mathbb{R}^3$, $\mu = (\mu_1, \mu_2, \mu_3)$ with $\mu_1, \mu_2, \mu_3 \geq 0$ is a multi-index, and

$$\nabla^\mu_x u = \frac{\partial^{|\mu|} u}{\partial x_1^{\mu_1} \partial x_2^{\mu_2} \partial x_3^{\mu_3}} \qquad (|\mu| = \mu_1 + \mu_2 + \mu_3).$$

Analogously,

$$\|u; V^l_\beta(\mathbb{R}^2)\| = \left(\int_{\mathbb{R}^2} \sum_{|\gamma|=0}^{l} (1+r^2)^{\beta-l+|\gamma|} |\nabla^\gamma_y u(y)|^2 dy \right)^{\frac{1}{2}} \tag{2.2}$$

for functions u depending on $y \in \mathbb{R}^2$ only where $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1, \gamma_2 \geq 0$. The spaces $V^l_\beta(\Omega)$ and $V^l_\beta(\mathbb{R}^2)$ are the closures of $C_0^\infty(\overline{\Omega})$ and $C_0^\infty(\mathbb{R}^2)$ in norms (2.1) and (2.2), respectively. The spaces $V^l_\beta(G)$ with norm (2.1) or (2.2) were first employed by V. A. Kondratiev [1] (Kondratiev spaces) while treating solutions of elliptic boundary value problems in domains $G \subset \mathbb{R}^n$ $(n \geq 2)$ with conical outlets to infinity (in this case the weight in (2.1) should be changed to $(1+|x|^2)$).

Let $\beta \in \mathbb{R}$ and let l, κ denote integers such that $l \geq 0$ and $0 \leq \kappa \leq l$. We introduce the space $\mathcal{V}^l_{\beta,\kappa}(\Omega)$ as the closure of $C_0^\infty(\overline{\Omega})$ in the norm

$$\|v; \mathcal{V}^l_{\beta,\kappa}(\Omega)\| = \left(\sum_{\alpha+|\gamma|\leq l} \int_\Omega (1+r^2)^{\beta+|\gamma|-(|\gamma|-\kappa)_+} |\partial^\alpha_z \partial^\gamma_y v(y,z)|^2 dydz \right)^{\frac{1}{2}} \tag{2.3}$$

where $\alpha \geq 0$, $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1, \gamma_2 \geq 0$, $|\gamma| = \gamma_1 + \gamma_2$, $\partial^\alpha_z = \frac{\partial^\alpha}{\partial z^\alpha}$, $\partial^\gamma_y = \frac{\partial^{|\gamma|}}{\partial y_1^{\gamma_1} \partial y_2^{\gamma_2}}$ and $(t)_+ = \frac{t+|t|}{2}$ is the positive part of $t \in \mathbb{R}$.

As it can be observed in (2.3), differentiation in z does not change the weight multiplier. Differentiation in y of order $|\gamma| \le \kappa$ increases the weight exponent by $|\gamma|$ (i.e. reflects the Kondratiev distribution of weights [1]). At $|\gamma| = \kappa$ the weight distribution function has a step. Namely, the subtrahend $(|\gamma| - \kappa)_+$ compensates the growth of the weight exponent provided $|\gamma| > \kappa$. In the case of a cone where all directions are equivalent such step-weighted spaces were introduced and investigated in [4, 5].

It is easy to see that

$$V^0_\beta(\Omega) = \mathcal{V}^0_{\beta,0}(\Omega) = L^2_\beta(\Omega)$$

while

$$\|v; L^2_\beta(\Omega)\| = \Big(\int_\Omega (1 + r^2)^\beta |v(x)|^2 dx \Big)^{\frac{1}{2}}.$$

Finally, for $l \ge 1$ we introduce the trace space $\mathcal{V}^{l-\frac{1}{2}}_{\beta,\kappa}(\partial\Omega)$ of functions $v \in \mathcal{V}^l_{\beta,\kappa}(\Omega)$ supplied with the norm

$$\|w; \mathcal{V}^{l-\frac{1}{2}}_{\beta,\kappa}(\partial\Omega)\| = \inf \big\{ \|v; \mathcal{V}^l_{\beta,\kappa}(\Omega)\| : v = w \text{ on } \partial\Omega \big\}. \tag{2.4}$$

The trace w on $\partial\Omega$ of $v \in \mathcal{V}^l_{\beta,\kappa}(\Omega)$ is forgetting the normal direction z and the weight distribution in the norm of $\mathcal{V}^{l-\frac{1}{2}}_{\beta,\kappa}(\partial\Omega)$ turns into an isotropic one while preserving the step property. This becomes evident after using an equivalent norm in $\mathcal{V}^{l-\frac{1}{2}}_{\beta,\kappa}(\partial\Omega)$.

Lemma 2.1 (see [15]). *The norm $\|\zeta; \mathcal{V}^{l-\frac{1}{2}}_{\beta,\kappa}(\partial\Omega)\|$ $(\kappa \le l)$ is equivalent to*

$$\begin{aligned} |||\, \zeta \,||| = \Bigg\{ & \big\|\zeta; W^{l-\frac{1}{2},2}(\partial\Omega \cap B_2)\big\|^2 \\ & + \sum_{j=0}^{1} \Bigg(\sum_{0 \le |\gamma| \le l-1} \int_{S^{(j)} \setminus B_1} (1 + r^2)^{\beta + |\gamma| - (|\gamma| - \kappa)_+} |\partial_y^\gamma \zeta(y)|^2 dy \\ & + \sum_{|\gamma| = l-1} \int_{S^{(j)} \setminus B_1} \int_{S^{(j)} \setminus B_1} \Big| \partial_y^\gamma \big((1 + |y|^2)^{\beta+\kappa} \zeta(y) \big) \\ & - \partial_{\tilde{y}}^\gamma \big((1 + |\tilde{y}|^2)^{\beta+\kappa} \zeta(\tilde{y}) \big) \Big|^2 |y - \tilde{y}|^{-3} dy d\tilde{y} \Bigg) \Bigg\}^{\frac{1}{2}}. \end{aligned} \tag{2.5}$$

In (2.5) integration over S_0 and S_1 is performed separately in order to avoid confusion. The reason is that for large r the boundary $\partial\Omega$ consists of two non-intersecting parts and the distance in $\mathbb{R}^3$ between two points y and $\tilde{y}$ located one above the other on S_0 and S_1 is equal to 1, while the distance between them on $\partial\Omega$ is $O(|y|)$. Interpretating the symbol $|y - \tilde{y}|$ appropriately one can delete the first sum over j in (2.5) and replace $S_j \setminus B_1$ by $\partial\Omega \setminus B_1$.

2.2 Auxiliary propositions. Below we make use of basic properties of the spaces $\mathcal{V}^l_{\beta,\kappa}(\Omega)$ which we collect in this section.

Lemma 2.2 (see [15]). *Let* $v \in \mathcal{V}^l_{\beta,\kappa}(\Omega)$ $(l \geq 1, 0 \leq \kappa \leq l-1, \beta \in \mathbb{R})$. *Then* $\partial_y v \in \mathcal{V}^{l-1}_{\beta+1,\kappa-1}(\Omega)$ *and* $\partial_z v \in \mathcal{V}^{l-1}_{\beta,\kappa}(\Omega)$. *There holds the inequality*

$$\|\partial_y v; \mathcal{V}^{l-1}_{\beta+1,\kappa-1}(\Omega)\| + \|\partial_z v; \mathcal{V}^{l-1}_{\beta,\kappa}(\Omega)\| \leq c\,\|v; \mathcal{V}^l_{\beta,\kappa}(\Omega)\|.$$

Lemma 2.3.

(i) *The embeddings*

$$\mathcal{V}^l_{\beta,\kappa}(\Omega) \hookrightarrow \mathcal{V}^{l-1}_{\beta,\kappa}(\Omega) \qquad (l \geq 1, 0 \leq \kappa \leq l-1) \tag{2.6}$$

$$\mathcal{V}^l_{\beta_1,\kappa}(\Omega) \hookrightarrow \mathcal{V}^l_{\beta,\kappa}(\Omega) \qquad (l \geq 0, 0 \leq \kappa \leq l, \beta_1 > \beta) \tag{2.7}$$

are continuous.

(ii) *If* $l \geq 1$, $0 \leq \kappa \leq l-1$ *and* $\varepsilon > 0$, *then the embedding*

$$\mathcal{V}^l_{\beta,\kappa}(\Omega) \hookrightarrow \mathcal{V}^{l-1}_{\beta-\varepsilon,\kappa}(\Omega) \tag{2.8}$$

is compact.

Proof. Continuity of the embeddings (2.6) - (2.7) follows from the definition of the norm (2.1). Moreover,

$$\big\|u;\, \mathcal{V}^{l-1}_{\beta-\varepsilon,\kappa}(\Omega \setminus B_{2R})\big\| \leq cR^{-\varepsilon}\big\|u;\, \mathcal{V}^l_{\beta,\kappa}(\Omega \setminus B_R)\big\|.$$

Since $\mathcal{V}^l_{\beta,\kappa}(\Omega \cap B_{2R})$ coincides with $W^{l,2}(\Omega \cap B_{2R})$ algebraically and topologically, well known properties of Sobolev spaces show that the embedding operator (2.8) can be represented as sum of a small operator (as $R \to \infty$) and a compact one. Thus (2.8) is compact ∎

Let us prove one simple interpolation result.

Lemma 2.4. *Let* $v \in [\mathcal{V}^1_{\beta,0}(\Omega)]^*$, *where* $[\mathcal{V}^1_{\beta,0}(\Omega)]^*$ *is the dual space to* $\mathcal{V}^1_{\beta,0}(\Omega)$ *with respect to the scalar product in* $L^2(\Omega)$. *Suppose that* $\nabla v \in L^2_{-\beta}(\Omega)$. *Then* $v \in L^2_{-\beta}(\Omega)$ *and*

$$\|v;\, L^2_{-\beta}(\Omega)\|^2 \leq c\Big(\|v;\, [\mathcal{V}^1_{\beta,0}(\Omega)]^*\|^2 + \|\nabla v;\, L^2_{-\beta}(\Omega)\|^2\Big).$$

Proof. Let us cover the domain Ω by the infinite union of "cubes"

$$Q_{s,k} = \big\{x \in \Omega :\, |x_1 - s|, |x_2 - k| \leq \tfrac{1}{2}\big\} \qquad (s, k \in \mathbb{Z}).$$

By [17 : Chapter 3/Lemma 7.1], for any function $v \in W^{-1,2}(Q_{s,k})$ with $\nabla v \in L^2(Q_{s,k})$ there holds the inclusion $v \in L^2(Q_{s,k})$ and the estimate

$$\|v;\, L^2(Q_{s,k})\|^2 \leq c\Big(\|v;\, W^{-1,2}(Q_{s,k})\|^2 + \|\nabla v;\, L^2(Q_{s,k})\|^2\Big)$$

with constant c independent of $s, k \in \mathbb{Z}$. Let us multiply the last inequalities by $(1 + (s^2 + k^2))^{-\beta}$ and sum them over all $s, k \in \mathbb{Z}$. Taking into account that $(1 + r^2)$ is equivalent to $(1 + (s^2 + k^2))$ in $Q_{s,k}$, we obtain

$$\|v;\, L^2_{-\beta}(\Omega)\|^2 \leq c\Bigg(\sum_{k,s\in\mathbb{Z}} \big(1 + (s^2 + k^2)\big)^{-\beta}\|v;\, W^{-1,2}(Q_{s,k})\|^2 + \|\nabla v;\, L^2_{-\beta}(\Omega)\|^2\Bigg).$$

Further, the equivalency of the norms $\|\eta(1+r^2)^{\beta/2};\, W^{1,2}(\Omega)\|$ and $\|\eta;\, \mathcal{V}^1_{\beta,0}(\Omega)\|$ gives the inequality

$$\sum_{k,s\in\mathbb{Z}} \big(1+(s^2+k^2)\big)^{-\beta} \|v;\, W^{-1,2}(Q_{s,k})\|^2 \le c\,\|v;\, [\mathcal{V}^1_{\beta,0}(\Omega)]^*\|^2$$

which competes the proof of the lemma ∎

2.3 Space $\mathcal{D}^l_\beta(\Omega)$ - the domain of the Stokes operator. We fix some weight and regularity indeces, i.e. numbers $\beta\in\mathbb{R}$ and $l\in\mathbb{N}_0$ and denote by $\mathcal{D}^l_\beta(\Omega)$ the space of vector functions $(\mathbf{u},p)$ satisfying the inclusions

$$\mathbf{u}'\in\mathcal{V}^{l+1}_{\beta+1,l}(\Omega) \qquad u_3\in\mathcal{V}^{l+1}_{\beta+2,l-1}(\Omega) \tag{2.9}$$

$$p\in\mathcal{V}^l_{\beta,l}(\Omega) \qquad \partial_z p\in\mathcal{V}^{l-1}_{\beta+2,l-1}(\Omega). \tag{2.10}$$

The norm in $\mathcal{D}^l_\beta(\Omega)$ is given by the formula

$$\begin{aligned}&\|(\mathbf{u},p);\mathcal{D}^l_\beta(\Omega)\|\\ &\quad=\|\mathbf{u}';\mathcal{V}^{l+1}_{\beta+1,l}(\Omega)\|+\|u_3;\mathcal{V}^{l+1}_{\beta+2,l-1}(\Omega)\|+\|p;\mathcal{V}^l_{\beta,l}(\Omega)\|+\|\partial_z p;\mathcal{V}^{l-1}_{\beta+2,l-1}(\Omega)\|.\end{aligned} \tag{2.11}$$

Such definition of the space $\mathcal{D}^l_\beta(\Omega)$ has been used in the paper [15]. For purporses of this paper it is more convenient to employ the following equivalent definition. Let us represent the pressure function p as sum

$$p(x)=p_\perp(y,z)+\overline{p}(y) \tag{2.12}$$

where

$$\overline{p}(y)=\int_0^1 p(y,z)\,dz$$

is the mean value of p with respect to $z\in(0,1)$. The projection $p_\perp$ obviously has zero mean value:

$$\overline{p}_\perp(y,z)=\overline{p}(y,z)-\overline{\overline{p}}(y)=\overline{p}(y)-\overline{p}(y)=0.$$

Moreover,

$$\overline{\partial_y p_\perp(y,z)}=\overline{\partial_y p(y,z)}-\overline{\partial_y\overline{p}(y)}=\partial_y\overline{p}(y)-\partial_y\overline{p}(y)=0.$$

Hence by the one-dimensional Poincare inequality we obtain $p_\perp\in L^2_{\beta+2}(\Omega)$, $\partial_y p_\perp\in L^2_{\beta+3}(\Omega)$ and

$$\|p_\perp;\, L^2_{\beta+2}(\Omega)\|\le c\,\|\partial_z p_\perp;\, L^2_{\beta+2}(\Omega)\|=c\,\|\partial_z p;\, L^2_{\beta+2}(\Omega)\|$$

$$\|\partial_y p_\perp;\, L^2_{\beta+3}(\Omega)\|\le c\,\|\partial_z\partial_y p_\perp;\, L^2_{\beta+3}(\Omega)\|.$$

Thus $p_\perp\in\mathcal{V}^l_{\beta+2,l}(\Omega)$ and

$$\|p_\perp;\, \mathcal{V}^l_{\beta+2,l}(\Omega)\|\le c\,\|\partial_z p;\, \mathcal{V}^{l-1}_{\beta+2,l-1}(\Omega)\|.$$

For the mean value $\overline{p}$ we get the inclusion $\overline{p} \in V^l_{\beta+l}(\mathbb{R}^2)$ and the estimate

$$\|\overline{p};\, V^l_{\beta+l}(\mathbb{R}^2)\| \le c\, \|p;\, \mathcal{V}^l_{\beta,l}(\Omega)\|.$$

Therefore the space $\mathcal{D}^l_\beta(\Omega)$ may be redefined as space of all vector functions $(\mathbf{u}, p)$ such that $\mathbf{u}$ satisfies inclusions (2.9) and p admits representation (2.12) with

$$\left.\begin{aligned} p_\perp &\in \mathcal{V}^l_{\beta+2,l}(\Omega)\\ \overline{p} &\in V^l_{\beta+l}(\mathbb{R}^2)\end{aligned}\right\}. \tag{2.13}$$

An equivalent norm in $\mathcal{D}^l_\beta(\Omega)$ is given by the formula

$$\begin{aligned} &\|(\mathbf{u}, p); \mathcal{D}^l_\beta(\Omega)\|\\ &\quad = \|\mathbf{u}'; \mathcal{V}^{l+1}_{\beta+1,l}(\Omega)\| + \|u_3; \mathcal{V}^{l+1}_{\beta+2,l-1}(\Omega)\| + \|p_\perp; \mathcal{V}^l_{\beta+2,l}(\Omega)\| + \|\overline{p}; V^l_{\beta+l}(\mathbb{R}^2)\|.\end{aligned} \tag{2.14}$$

2.4 Space $\mathcal{R}^l_\beta(\Omega; \partial\Omega)$ – the range of the Stokes operator. The space $\mathcal{R}^l_\beta(\Omega; \partial\Omega)$ ($l \ge 1$) consists of triples $(\mathbf{f}, g, \mathbf{h})$ such that

$$\left.\begin{aligned} g &\in \mathcal{V}^l_{\beta+2,l-1}(\Omega)\\ \mathbf{h}' &\in \mathcal{V}^{l+\frac{1}{2}}_{\beta+1,l}(\partial\Omega)\\ h_3 &\in \mathcal{V}^{l+\frac{1}{2}}_{\beta+2,l-1}(\partial\Omega)\end{aligned}\right\} \tag{2.15}$$

while $\mathbf{f}$ admits the representation

$$\mathbf{f} = \mathbf{f}_0 + \partial_z \mathbf{f}_1 + \nabla\psi \tag{2.16}$$

with

$$\left.\begin{aligned} \mathbf{f}_0 &\in \mathcal{V}^{l-1}_{\beta+2,l-1}(\Omega)\\ \mathbf{f}_1' &\in \mathcal{V}^l_{\beta+1,l}(\Omega)\\ f_{13} &\in \mathcal{V}^l_{\beta+2,l-1}(\Omega)\\ \psi_\perp &\in \mathcal{V}^l_{\beta+2,l}(\Omega)\\ \overline{\psi} &\in V^l_{\beta+l}(\mathbb{R}^2)\end{aligned}\right\}. \tag{2.17}$$

The norm in $\mathcal{R}^l_\beta(\Omega; \partial\Omega)$ is given by

$$\begin{aligned} &\|(\mathbf{f}, g, \mathbf{h}); \mathcal{R}^l_\beta(\Omega; \partial\Omega)\|\\ &\quad = \inf\Big\{\|\mathbf{f}_0; \mathcal{V}^{l-1}_{\beta+2,l-1}(\Omega)\| + \|\mathbf{f}_1'; \mathcal{V}^l_{\beta+1,l}(\Omega)\|\\ &\qquad + \|f_{13}; \mathcal{V}^l_{\beta+2,l-1}(\Omega)\| + \|\psi_\perp; \mathcal{V}^l_{\beta+2,l}(\Omega)\| + \|\overline{\psi}; V^l_{\beta+l}(\mathbb{R}^2)\|\Big\}\\ &\qquad + \|g; \mathcal{V}^l_{\beta+2,l-1}(\Omega)\| + \|\mathbf{h}'; \mathcal{V}^{l+\frac{1}{2}}_{\beta+1,l}(\partial\Omega)\| + \|h_3; \mathcal{V}^{l+\frac{1}{2}}_{\beta+2,l-1}(\partial\Omega)\|\end{aligned} \tag{2.18}$$

where the infimum is taken over all representations (2.16). From Lemmata 2.2 and 2.3 we derive the following assertions.

Lemma 2.5. *The embeddings*

$$\left.\begin{aligned} \mathcal{R}^l_\beta(\Omega;\partial\Omega) &\hookrightarrow \mathcal{R}^{l-1}_\beta(\Omega;\partial\Omega) \\ \mathcal{R}^l_{\beta_1}(\Omega;\partial\Omega) &\hookrightarrow \mathcal{R}^l_\beta(\Omega;\partial\Omega) \end{aligned}\right\} \qquad (l \geq 1,\ \beta_1 > \beta)$$

are continuous.

Theorem 2.1. *The operator $\mathcal{S}^l_\beta$ of problem* (1.2) – (1.3),

$$\mathcal{D}^l_\beta(\Omega) \ni (\mathbf{u},p) \longmapsto \mathcal{S}^l_\beta(\mathbf{u},p) = (\mathbf{f},g,\mathbf{h}) \in \mathcal{R}^l_\beta(\Omega;\partial\Omega) \tag{2.19}$$

is continuous.

2.5 Coercive estimate for the solution of problem (1.2) - (1.3). The following result is proved in [15].

Theorem 2.2. *Let* $(\mathbf{u},p) \in L^2_\beta(\Omega) \times L^2_\beta(\Omega)$ *be the solution of problem* (1.2) – (1.3) *with right-hand side* $(\mathbf{f},g) \in \mathcal{R}^l_\beta(\Omega;\partial\Omega)$ $(l \geq 1, \beta \in \mathbb{R})$. *Then* $(\mathbf{u},p) \in \mathcal{D}^l_\beta(\Omega)$ *and*

$$\begin{aligned} &\|(\mathbf{u},p);\mathcal{D}^l_\beta(\Omega)\| \\ &\quad \leq c\Big(\|(\mathbf{f},g,\mathbf{h});\mathcal{R}^l_\beta(\Omega;\partial\Omega)\| + \|\mathbf{u};L^2_\beta(\Omega)\| + \|p_\perp;L^2_\beta(\Omega)\| + \|\overline{p};L^2_\beta(\mathbb{R}^2)\|\Big). \end{aligned} \tag{2.20}$$

In order to prove the Fredholm property of mapping (2.19) we need to transform estimate (2.20) into

$$\|(\mathbf{u},p);\mathcal{D}^l_\beta(\Omega)\| \leq c\Big(\|(\mathbf{f},g,\mathbf{h});\mathcal{R}^l_\beta(\Omega;\partial\Omega)\| + \|K(\mathbf{u},p);\mathcal{D}^l_\beta(\Omega)\|\Big) \tag{2.21}$$

where K is a compact operator in $\mathcal{D}^l_\beta(\Omega)$. As shown in [15], the function $\overline{p} \in L^2_\beta(\mathbb{R}^2) \cap W^{l,2}_{loc}(\mathbb{R}^2)$ satisfies the Poisson equation

$$-\tfrac{1}{6}\Delta'_y\overline{p}(y) = \mathcal{F}(y) \qquad (y \in \mathbb{R}^2) \tag{2.22}$$

where

$$\begin{aligned} \mathcal{F}(y) &= \mathcal{F}^{(1)}(y) + \operatorname{div}'_y \mathcal{F}^{(2)}(y) + \Delta'_y\mathcal{F}^{(3)}(y) + \Delta'_y\mathcal{F}^{(0)}(y) \\ \mathcal{F}^{(0)}(y) &= \int_0^1 \partial_z p(y,z)\big(\tfrac{1}{6}z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3\big)dz \\ \mathcal{F}^{(1)}(y) &= 2\nu\int_0^1 g(y,z)\,dz \\ \mathcal{F}^{(2)}(y) &= -\int_0^1 \mathbf{f}'(y,z)z(z-1)\,dz \\ \mathcal{F}^{(3)}(y) &= -\nu\int_0^1 \operatorname{div}'_y\mathbf{u}'(y,z)z(z-1)\,dz. \end{aligned}$$

The inclusion $(\mathbf{f}, g, \mathbf{h}) \in \mathcal{R}^l_\beta(\Omega; \partial\Omega)$ furnishes $\mathbf{f}' \in L^2_{\beta+1}(\Omega)$, $\operatorname{div}'_y \mathbf{f}' \in L^2_{\beta+2}(\Omega)$ and $g \in L^2_{\beta+2}(\Omega)$. Hence, $\mathcal{F}^{(1)} \in L^2_{\beta+2}(\mathbb{R}^2)$, $\operatorname{div}'_y \mathcal{F}^{(2)} \in L^2_{\beta+2}(\mathbb{R}^2)$ and

$$\|\mathcal{F}^{(1)}; L^2_{\beta+2}(\mathbb{R}^2)\| + \|\operatorname{div}'_y \mathcal{F}^{(2)}; L^2_{\beta+2}(\mathbb{R}^2)\| \le c\|(\mathbf{f}, g, \mathbf{h}); \mathcal{R}^l_\beta(\Omega; \partial\Omega)\|.$$

Further, $(\mathbf{u}, p) \in \mathcal{D}^l_\beta(\Omega)$ so that

$$\begin{aligned} &\mathbf{u}' \in \mathcal{V}^{l+1}_{\beta+1,l}(\Omega) \qquad && \Delta'_y \operatorname{div}'_y \mathbf{u}' \in L^2_{\beta+3}(\Omega) \subset L^2_{\beta+2}(\Omega) \\ &\partial_z p \in L^2_{\beta+2}(\Omega) && \Delta'_y(\partial_z p) \in L^2_{\beta+4}(\Omega) \subset L^2_{\beta+2}(\Omega). \end{aligned}$$

This implies $\Delta'_y \mathcal{F}^{(0)} \in L^2_{\beta+2}(\mathbb{R}^2)$, $\Delta'_y \mathcal{F}^{(3)} \in L^2_{\beta+2}(\mathbb{R}^2)$ and

$$\begin{aligned} &\big\|\Delta'_y \mathcal{F}^{(0)};\, L^2_{\beta+2}(\mathbb{R}^2)\big\| + \big\|\Delta'_y \mathcal{F}^{(3)};\, L^2_{\beta+2}(\mathbb{R}^2)\big\| \\ &\qquad \le c\Big(\big\|\Delta'_y \operatorname{div}'_y \mathbf{u}';\, L^2_{\beta+2}(\Omega)\big\| + \big\|\Delta'_y(\partial_z p);\, L^2_{\beta+2}(\Omega)\big\|\Big). \end{aligned}$$

Thus,

$$\mathcal{F} = \mathcal{F}^{(1)} + \operatorname{div}'_y \mathcal{F}^{(2)} + \Delta'_y\big(\mathcal{F}^{(0)} + \mathcal{F}^{(3)}\big) \in L^2_{\beta+2}(\mathbb{R}^2)$$

and

$$\begin{aligned} &\|\mathcal{F}; L^2_{\beta+2}(\mathbb{R}^2)\| \\ &\quad \le c\Big(\|(\mathbf{f}, g, \mathbf{h}); \mathcal{R}^l_\beta(\Omega)\| + \|\Delta'_y \operatorname{div}'_y \mathbf{u}'; L^2_{\beta+2}(\Omega)\| + \|\Delta'_y(\partial_z p);\, L^2_{\beta+2}(\Omega)\|\Big). \end{aligned} \tag{2.23}$$

The punctured space $\mathbb{R}^2 \setminus \{0\}$ might be interpreted as two-dimensional cone (a complete one) in $\mathbb{R}^2$ so that $\mathbb{R}^2$ is a domain with conical outlet to infinity. Therefore general theorems on elliptic problems in such domains can be applied while treating the solution $\overline{p}$ of equation (2.22). It is known (see [1, 2, 12]) that such problems have the Fredholm property in the scale of Kondratie spaces $V^l_\gamma(\mathbb{R}^2)$ if and only if every power solution $w(y) = r^{-\lambda}\Psi(\varphi)$ of the corresponding homogeneous problem is trivial, provided that λ lies on the line $\{\lambda \in \mathbb{C} :\, \operatorname{Re}\lambda = \gamma - l + 1\}$ ((r, φ) are polar coordinates in $\mathbb{R}^2$). For the Laplace operator (2.22) all power solutions consist of harmonic polynomials of orders $m \in \mathbb{N}_0$ and derivatives of the fundamental solution $\Gamma(y) = -\frac{1}{2\pi}\ln|y|$. This information together with the general results (see [1, 2, 12]) and estimate (2.23) gives

Lemma 2.6. *Let $\overline{p} \in L^2_\beta(\mathbb{R}^2) \cap W^{l,2}_{loc}(\mathbb{R}^2)$ $(l \ge 2, \beta \notin \pm\mathbb{N}_0)$ be the solution of the Poisson equation* (2.22)*. Then $\overline{p} \in V^2_{\beta+2}(\mathbb{R}^2)$ and there holds the inequality*

$$\begin{aligned} \|\overline{p}\ V^2_{\beta+2}(\mathbb{R}^2)\| &\le c\Big(\|\mathcal{F};\, L^2_{\beta+2}(\mathbb{R}^2)\| + \|\mathcal{K}_1\overline{p};\, V^2_{\beta+2}(\mathbb{R}^2)\|\Big) \\ &\le c\Big(\|(\mathbf{f}, g, \mathbf{h}); \mathcal{R}^l_\beta(\Omega; \partial\Omega)\| + \|\Delta'_y \operatorname{div}'_y \mathbf{u}'; L^2_{\beta+2}(\Omega)\| \\ &\qquad + \|\Delta'_y(\partial_z p);\, L^2_{\beta+2}(\Omega)\| + \|\mathcal{K}_1\overline{p};\, V^2_{\beta+2}(\mathbb{R}^2)\|\Big) \end{aligned} \tag{2.24}$$

where $\mathcal{K}_1$ is a compact operator in $V^2_{\beta+2}(\mathbb{R}^2)$.

Remark 2.1. Lemma 2.6 remains valid also for $l = 1$ and $l = 0$. However, because of the shortage of the regularity in these cases the Poisson equation (2.22) for $\overline{p}$ should be understood in the sence of distributions, i.e. the solution $\overline{p} \in L^2_\beta(\mathbb{R}^2)$ satisfies the integral identity

$$\begin{aligned} &-\tfrac{1}{6}\int_{\mathbb{R}^2} \overline{p}(y)\Delta'_y\eta(y)\,dy \\ &\quad= \int_{\mathbb{R}^2}\Big(\mathcal{F}^{(1)}(y)\eta(y) - \mathcal{F}^{(2)}(y)\cdot\nabla'_y\eta(y) + \big(\mathcal{F}^{(0)}(y)+\mathcal{F}^{(3)}(y)\big)\Delta'_y\eta(y)\Big)dy \end{aligned} \tag{2.25}$$

for all $\eta \in C_0^\infty(\mathbb{R}^2)$ where

$$\begin{aligned} \mathcal{F}^{(0)} &\in L^2_{\beta+2}(\mathbb{R}^2) \subset L^2_{\beta+1}(\mathbb{R}^2) \\ \mathcal{F}^{(1)} &\in L^2_{\beta+2}(\mathbb{R}^2) \subset L^2_{\beta+1}(\mathbb{R}^2) \\ \mathcal{F}^{(2)} &\in L^2_{\beta+1}(\mathbb{R}^2) \\ \mathcal{F}^{(3)} &\in L^2_{\beta+2}(\mathbb{R}^2) \subset L^2_{\beta+1}(\mathbb{R}^2). \end{aligned}$$

Since results analogous to Lemma 2.6 are true for the solution $\overline{p} \in L^2_\beta(\mathbb{R}^2)$ of the Poisson identity (2.25) (e.g. [2]: Section 6.3] and [12: Theorems 3.5.7 and 4.2.4]), we conclude the estimate

$$\begin{aligned} \|\overline{p}\, L^2_\beta(\mathbb{R}^2)\| \le c\Big(&\|(\mathbf{f},g,\mathbf{h});\mathcal{R}^l_\beta(\Omega;\partial\Omega)\| + \|\mathrm{div}'_y\,\mathbf{u}'; L^2_{\beta+1}(\Omega)\| \\ &+ \|\partial_z p;\ L^2_{\beta+1}(\Omega)\| + \|\tilde{\mathcal{K}}_1\overline{p};\ L^2_\beta(\mathbb{R}^2)\|\Big) \end{aligned} \tag{2.26}$$

where $\tilde{\mathcal{K}}_1$ is a compact operator in $L^2_\beta(\mathbb{R}^2)$ ∎

First, let $l \ge 2$ and $\beta \notin \pm\mathbb{N}_0$. Using inequality (2.24) we can rewrite estimate (2.20) in the form

$$\begin{aligned} \|(\mathbf{u},p);\mathcal{D}^l_\beta(\Omega)\| \le c\Big(&\|(\mathbf{f},g,\mathbf{h});\mathcal{R}^l_\beta(\Omega;\partial\Omega)\| + \|\mathbf{u}; L^2_\beta(\Omega)\| \\ &+ \|p_\perp; L^2_\beta(\Omega)\| + \|\Delta'_y\mathrm{div}'_y\,\mathbf{u}'; L^2_{\beta+2}(\Omega)\| \\ &+ \|\Delta'_y(\partial_z p);\ L^2_{\beta+2}(\Omega)\| + \|\mathcal{K}_1\overline{p};\ V^2_{\beta+2}(\mathbb{R}^2)\|\Big). \end{aligned} \tag{2.27}$$

By Lemma 2.2, $\Delta'_y\mathrm{div}'_y\ \mathbf{u}' \in \mathcal{V}^{l-2}_{\beta+4,l-3}(\Omega)$ and $\Delta'_y(\partial_z p) \in \mathcal{V}^{l-3}_{\beta+4,l-3}(\Omega)$. Moreover, by virtue of Lemma 2.3 the embeddings

$$\begin{aligned} \mathcal{V}^{l-2}_{\beta+4,l-3}(\Omega) &\hookrightarrow L^2_{\beta+2}(\Omega) \\ \mathcal{V}^{l-3}_{\beta+4,l-3}(\Omega) &\hookrightarrow L^2_{\beta+2}(\Omega) \\ \mathcal{V}^{l+1}_{\beta+1,l}(\Omega) &\hookrightarrow L^2_{\beta}(\Omega) \\ \mathcal{V}^{l+1}_{\beta+2,l-2}(\Omega) &\hookrightarrow L^2_{\beta}(\Omega) \\ \mathcal{V}^{l}_{\beta+2,l}(\Omega) &\hookrightarrow L^2_{\beta}(\Omega) \end{aligned}$$

are compact. Hence, there hold the inequalities

$$\begin{aligned}
\|\Delta'_y \mathrm{div}'_y\,\mathbf{u}'; L^2_{\beta+2}(\Omega)\| &\le c\,\|\mathcal{K}_2\mathbf{u}'; \mathcal{V}^{l+1}_{\beta+1,l}(\Omega)\| \\
\|\Delta'_y(\partial_z p);\ L^2_{\beta+2}(\Omega)\| &\le c\,\|\mathcal{K}_3 p_\perp;\ \mathcal{V}^{l}_{\beta+2,l}(\Omega)\| \\
\|(\mathbf{u}', u_3);\ L^2_\beta(\Omega)\times L^2_\beta(\Omega)\| &\le c\,\|\mathcal{K}_4(\mathbf{u}', u_3);\ \mathcal{V}^{l+1}_{\beta+1,l}(\Omega)\times \mathcal{V}^{l+1}_{\beta+2,l-1}(\Omega)\| \\
\|p_\perp;\ L^2_\beta(\Omega)\| &\le c\,\|\mathcal{K}_5 p_\perp;\ \mathcal{V}^{l}_{\beta+2,l}(\Omega)\|
\end{aligned}$$

where $\mathcal{K}_i$ $(i = 2, 3, 4, 5)$ are compact operators. Therefore from (2.27) estimate (2.21) follows. In the cases $l = 0$ and $l = 1$ we analogously get estimate (2.21) using inequality (2.26) instead of (2.24). Thus, we have proved

Theorem 2.3. *Let* $(\mathbf{u}, p) \in \mathcal{D}^l_\beta(\Omega)$ *be the solution of problem* (1.2) – (1.3) *with right-hand side* $(\mathbf{f}, g, \mathbf{h}) \in \mathcal{R}^l_\beta(\Omega; \partial\Omega)$ $(l \ge 1, \beta \in \mathbb{R}\setminus\{\pm\mathbb{N}_0\})$. *Then estimate* (2.21) *holds with* $\mathcal{K}$ *being a compact operator in* $\mathcal{D}^l_\beta(\Omega)$.

2.6 Asymptotic representation of the solution. Let us formulate a result concerning the asymptotic behavior of the solution $(\mathbf{u}, p)$ of problem (1.2) - (1.3).

Theorem 2.4 (see [15]). *Assume that*

$$(\mathbf{f}, g, \mathbf{h}) \in \mathcal{R}^l_{\beta+k}(\Omega; \partial\Omega) \qquad (l \ge 1,\ \beta \notin \pm\mathbb{N}_0,\ k \in \mathbb{N}). \tag{2.28}$$

Then the solution

$$(\mathbf{u}, p) \in L^2_\beta(\Omega) \times L^2_\beta(\Omega) \tag{2.29}$$

of problem (1.2) – (1.3) *admits the asymptotic representation*

$$\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix} = \chi(r) \sum_{-\beta-k-1<m<-\beta-1} \begin{pmatrix} c^+_m\mathbf{u}^+_m(y,z) + c^-_m\mathbf{u}^-_m(y,z)\\ c^+_m p^+_m(y) + c^-_m p^-_m(y)\end{pmatrix} + \begin{pmatrix}\tilde{\mathbf{u}}\\ \tilde{p}\end{pmatrix} \tag{2.30}$$

where χ *is a smooth cut-off function with* $\chi(r) = 1$ *for* $r \ge 2$ *and* $\chi(r) = 0$ *for* $r \le 1$,

$$\mathbf{u}^{\pm\prime}_m(y,z) = \tfrac{1}{2\nu} z(z-1)\nabla'_y p^\pm_m(y), \quad u^\pm_{3m}(y,z) = 0, \quad p^+_0(y) = 1, \quad p^-_0(y) = -\tfrac{1}{2\pi}\ln r$$

$$\left.\begin{aligned} p^+_m(y) &= (2\pi|m|)^{-\frac12} r^m \cos(m\varphi)\\ p^-_m(y) &= (2\pi|m|)^{-\frac12} r^m \sin(|m|\varphi)\end{aligned}\right\} \tag{2.31}$$

$c^\pm_m$ $(m \in \pm\mathbb{N}_0)$ *are constants and* $(\tilde{\mathbf{u}}, \tilde{p}) \in \mathcal{D}^l_{\beta+k}(\Omega)$. *There holds the estimate*

$$\begin{aligned}
&\|(\tilde{\mathbf{u}}, \tilde{p}); \mathcal{D}^l_{\beta+k}(\Omega)\| + \sum_{-\beta-k-1<m<-\beta-1} (|c^+_m| + |c^-_m|)\\
&\le c\Big(\|(\mathbf{f}, g, \mathbf{h}); \mathcal{R}^l_{\beta+k}(\Omega;\partial\Omega)\| + \|\mathbf{u}; L^2_\beta(\Omega)\| + \|p_\perp; L^2_\beta(\Omega)\| + \|\overline{p}; L^2_\beta(\mathbb{R}^2)\|\Big).
\end{aligned} \tag{2.32}$$

Remark 2.2. Analogous asymptotic formulae were obtained also for second order scalar elliptic operators (see [9, 11]) and for the Lame operator (see [6 - 8, 10]).

2.7 Green' formula. Let $(\mathbf{u},p)\in\mathcal{D}^l_\beta(\Omega)$ and $(\mathbf{v},q)\in C_0^\infty(\overline{\Omega})$. Then for the Stokes problem (1.2) – (1.3) there holds Green' formula

$$\begin{aligned}\int_\Omega(-\nu\Delta\mathbf{u}+\nabla p)\cdot\mathbf{v}\,dx-\int_\Omega q\,\mathrm{div}\mathbf{u}\,dx+\int_{\partial\Omega}\mathbf{u}\cdot(\mathbf{n}q-\nu\partial_n\mathbf{v})\,ds\\ =\int_\Omega(-\nu\Delta\mathbf{v}+\nabla q)\cdot\mathbf{u}\,dx-\int_\Omega p\,\mathrm{div}\,\mathbf{v}\,dx+\int_{\partial\Omega}\mathbf{v}\cdot(\mathbf{n}p-\nu\partial_n\mathbf{u})\,ds.\end{aligned}\tag{2.33}$$

Here $\mathbf{n}$ is the unit vector of the outward normal to $\partial\Omega$ and $\partial_n=\frac{\partial}{\partial\mathbf{n}}$ denotes the derivative with respect to $\mathbf{n}$. Note that all integrals in (2.33) are finite since $(\mathbf{v},q)$ is identically zero for large $|x|$. It is not difficult to verify that the integrals in (2.33) remain finite if $(\mathbf{v},q)\in\mathcal{D}^l_{-\beta-2}(\Omega)$. Therefore by continuity we conclude the following assertion.

Lemma 2.7. *Green' formula* (2.33) *holds true for any pairs* $(\mathbf{u},p)\in\mathcal{D}^l_\beta(\Omega)$ *and* $(\mathbf{v},q)\in\mathcal{D}^l_{-\beta-2}(\Omega)$.

3. The Fredholm property

In this section we prove the main result of the paper: the Fredholm property of the Stokes operator $\mathcal{S}^l_\beta$, i.e. we prove that the range $\mathcal{S}^l_\beta\mathcal{D}^l_\beta(\Omega)$ is a closed subspace of $\mathcal{R}^l_\beta(\Omega;\partial\Omega)$ and that

$$\dim\ker\mathcal{S}^l_\beta<\infty$$
$$\dim\mathrm{coker}\,\mathcal{S}^l_\beta<\infty.$$

Theorem 3.1. *The operator* $\mathcal{S}^l_\beta$ $(l\geq1)$ *of the Stokes problem* (1.2) – (1.3) *is of Fredholm type, if* $\beta\notin\mathbb{Z}$. *If* $\beta\in\mathbb{Z}$, *then the range of* $\mathcal{S}^l_\beta$ *is not closed.*

Proof. The finite-dimensionality of $\ker\mathcal{S}^l_\beta$ and the closedness of the range $\mathcal{S}^l_\beta\mathcal{D}^l_\beta(\Omega)$ follow from estimate (2.21) (see Theorem 2.3) and a lemma by J. Peetre (see [18] or [3: Lemma 2.5.1]).

Let us prove the finite-dimensionality of $\mathrm{coker}\,\mathcal{S}^l_\beta$. We show that the subspace $\ker(\mathcal{S}^l_\beta)^*=\mathrm{coker}\,\mathcal{S}^l_\beta$ admits the representation

$$\mathrm{coker}\,\mathcal{S}^l_\beta=\Big\{\big(\mathbf{v},q,(\mathbf{n}q-\nu\partial_n\mathbf{v})\big|_{\partial\Omega}\big)\,:\,(\mathbf{v},q)\in\ker\mathcal{S}^l_{-\beta-2}\Big\}.\tag{3.1}$$

Let us consider the bounded linear functional $F_{(\mathbf{v},q)}$ given on $\mathcal{R}^l_\beta(\Omega;\partial\Omega)$ by the formula

$$\begin{gathered}F_{(\mathbf{v},q)}(\mathbf{f},g,\mathbf{h})=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,dx-\int_\Omega g\,q\,dx+\int_{\partial\Omega}\mathbf{h}\cdot(\mathbf{n}q-\nu\partial_n\mathbf{v})\,ds\\ (\mathbf{v},q)\in\mathcal{D}^l_{-\beta-2}(\Omega).\end{gathered}\tag{3.2}$$

If $(\mathbf{f},g,\mathbf{h})\in\mathcal{S}^l_\beta\mathcal{D}^l_\beta(\Omega)$ and $(\mathbf{v},q)\in\ker\mathcal{S}^l_{-\beta-2}$, then from Green's formula (2.33) it follows that $F_{(\mathbf{v},q)}(\mathbf{f},g,\mathbf{h})=0$. Thus $F_{(\mathbf{v},q)}$ is orthogonal to $\mathcal{S}^l_\beta\mathcal{D}^l_\beta(\Omega)$ and therefore $F_{(\mathbf{v},q)}\in\ker(\mathcal{S}^l_\beta)^*$. Hence we have proved the inclusion

$$\Big\{\big(\mathbf{v},q,(\mathbf{n}q-\nu\partial_n\mathbf{v})|_{\partial\Omega}\big)\,:\,(\mathbf{v},q)\in\ker\mathcal{S}^l_{-\beta-2}\Big\}\subset\ker(\mathcal{S}^l_\beta)^*.\tag{3.3}$$

In order to prove the inverse inclusion we first consider the case $l = 1$ and introduce the operator $\mathcal{S}_\beta^*$ adjoint to $\mathcal{S}_\beta$ (with respect to the scalar product in $L^2(\Omega)^4 \times L^2(\partial\Omega)^3$). For brevity we write $\mathcal{S}_\beta$, $\mathcal{D}_\beta(\Omega)$ etc., omitting the regularity index $l = 1$. We mention as well known fact (see, e.g., [3, 19]) that the operator $\mathcal{S}_\beta^*$ acts on the space of distributions by the formula

$$\mathcal{R}_\beta(\Omega;\partial\Omega)^* \ni (\mathbf{v}, q, \mathbf{w}) \longmapsto \mathcal{S}_\beta^*(\mathbf{v}, q, \mathbf{w}) = S(\pi_\Omega \mathbf{v}, \pi_\Omega q) + \mathbf{w} \otimes \delta_{\partial\Omega}.$$

Here $\pi_\Omega \mathbf{v}$ and $\pi_\Omega q$ are the extensions of $\mathbf{v}$ and q, respectively, by zero from Ω to the entire $\mathbb{R}^3$, $\delta_{\partial\Omega}$ is the Dirac function concentrated on $\partial\Omega$ so that $\mathbf{w} \otimes \delta_{\partial\Omega}$ is the distribution defined by the formula

$$(\mathbf{w} \otimes \delta_{\partial\Omega}, \varphi)_{\mathbb{R}^3} = (\mathbf{w}, \varphi)_{\partial\Omega} \qquad \big(\varphi \in C_0^\infty(\mathbb{R}^3)\big)$$

where $(\cdot,\cdot)_{\partial\Omega}$ denotes the scalar product in $L^2(\partial\Omega)$, and

$$S(\pi_\Omega \mathbf{v}, \pi_\Omega q) = \big(-\nu\Delta\pi_\Omega \mathbf{v} + \nabla\pi_\Omega q; -\mathrm{div}\pi_\Omega \mathbf{v}\big)$$

is the Stokes operator (1.2). Note that due to Green's formula (2.33) this operator is formally self-adjoint.

Let $\omega, \widehat{\omega}$ be two neighbourhoods of a point in $\overline{\Omega}$ and $\overline{\omega} \subset \widehat{\omega}$. If the right-hand side $\mathbf{U} = (U_1, U_2, U_3, U_4)$ of the equation

$$\mathcal{S}_\beta^*(\mathbf{v}, q, \mathbf{w}) = \mathbf{U} \in \mathcal{D}_\beta(\Omega)^* \tag{3.4}$$

belongs to $H^s(\Omega \cap \widehat{\omega})^3 \times H^{s+1}(\Omega \cap \widehat{\omega})$, then first $(\mathbf{v}, q)$ belongs to $H^{s+2}(\Omega \cap \omega)^3 \times H^{s+1}(\Omega \cap \omega)$, second it satisfies the relations $S(\mathbf{v}, q) = \mathbf{U}$ in $\Omega \cap \omega$ and $\mathbf{v} = 0$ on $\partial\Omega \cap \omega$, and third $\mathbf{w}$ coincides with the trace of $(\mathbf{n}q - \nu\partial_n \mathbf{v})$ on $\partial\Omega \cap \omega$ (see [19] and [3: Chapter 2.5.3]). Since $\ker \mathcal{S}_\beta^*$ contains the solutions $(\mathbf{v}, q, \mathbf{w}) \in \mathcal{R}_\beta(\Omega;\partial\Omega)^*$ of the homogeneous equation (3.4) (i.e. $\mathbf{U} = 0$), we conclude that $(\mathbf{v}, q) \in C_{loc}^\infty(\overline{\Omega})$ solves the homogeneous Stokes problem (1.2) - (1.3) and $\mathbf{w}$ is the trace of $(\mathbf{n}q - \nu\partial_n \mathbf{v})$ on $\partial\Omega$. Further, by definition $\mathcal{R}_\beta(\Omega;\partial\Omega)$ contains the subspace

$$\mathbf{R} = L^2_{\beta+2}(\Omega)^3 \times \mathcal{V}^1_{\beta+2,0}(\Omega) \times \mathcal{V}^{\frac{3}{2}}_{\beta+1,1}(\partial\Omega)^2 \times \mathcal{V}^{\frac{3}{2}}_{\beta+2,0}(\partial\Omega)$$

(we assume that $\mathbf{f}_1 = 0$ and $\psi = 0$ in representation (2.16) for $\mathbf{f}$, i.e. $\mathbf{f} = \mathbf{f}_0$). Consequently, $\mathcal{R}_\beta(\Omega;\partial\Omega)^* \subset \mathbf{R}^*$. The first two factors in $\mathbf{R}^*$ coinside with $L^2_{-\beta-2}(\Omega)^3 \times [\mathcal{V}^1_{\beta+2,0}(\Omega)]^*$ and hence we have $\mathbf{v} \in L^2_{-\beta-2}(\Omega)^3$ and $q \in [\mathcal{V}^1_{\beta+2,0}(\Omega)]^*$.

Let us show that q belongs to $L^2_{-\beta-2}(\Omega)$. Denote by ζ_ρ the smooth cut-off function with $\zeta_\rho(r) = 1$ for $r \le \rho$, $\zeta_\rho(r) = 0$ for $r \ge 2\rho$ and

$$\left.\begin{aligned} |\nabla\zeta_\rho(r)| &\le c\,(1 + r^2)^{-\frac{1}{2}} \\ |\nabla\nabla\zeta_\rho(r)| &\le c\,(1 + r^2)^{-1} \end{aligned}\right\} \tag{3.5}$$

with constant c independent of ρ and r. We multiply the homogeneous Stokes equations (1.2) by $\zeta_\rho(r)^2(1+r^2)^{-\beta-1}\mathbf{v}(x)$ and integrate by parts in Ω:

$$\begin{aligned}\nu\int_\Omega &\zeta_\rho(r)^2(1+r^2)^{-\beta-1}|\nabla\mathbf{v}(x)|^2dx\\ &=\int_\Omega q\,\mathbf{v}(x)\cdot\nabla[\zeta_\rho(r)^2(1+r^2)^{-\beta-1}]\,dx\\ &\quad-\nu\int_\Omega\nabla\mathbf{v}(x)\cdot\mathbf{v}(x)\nabla[\zeta_\rho(r)^2(1+r^2)^{-\beta-1}]\,dx\\ &=I_1+I_2.\end{aligned}\tag{3.6}$$

Using (3.5) it is easy to show that

$$|I_2|\le\tfrac{\nu}{4}\int_\Omega\zeta_\rho(r)^2(1+r^2)^{-\beta-1}|\nabla\mathbf{v}(x)|^2dx+c(\nu)\int_\Omega(1+r^2)^{-\beta-2}|\mathbf{v}(x)|^2dx.\tag{3.7}$$

For the first summand I_1 we get

$$\begin{aligned}|I_1|&\le\big\|q;\,[\mathcal{V}^1_{\beta+2,0}(\Omega)]^*\big\|\,\big\|\mathbf{v}\nabla[\zeta_\rho(r)^2(1+r^2)^{-\beta-1}];\,\mathcal{V}^1_{\beta+2,0}(\Omega)\big\|\\ &\le c\,\big\|q;\,[\mathcal{V}^1_{\beta+2,0}(\Omega)]^*\big\|\\ &\quad\times\Big(\int_\Omega(1+r^2)^{-\beta-2}|\mathbf{v}|^2dx+\nu\int_\Omega\zeta_\rho^2(1+r^2)^{-\beta-1}|\nabla\mathbf{v}|^2dx\Big)^{\frac12}\\ &\le\tfrac{\nu}{4}\int_\Omega\zeta_\rho^2(1+r^2)^{-\beta-1}|\nabla\mathbf{v}|^2dx\\ &\quad+c(\nu)\Big(\big\|q;\,[\mathcal{V}^1_{\beta+2,0}(\Omega)]^*\big\|^2+\int_\Omega(1+r^2)^{-\beta-2}|\mathbf{v}|^2dx\Big).\end{aligned}\tag{3.8}$$

Substituting (3.7), (3.8) into (3.6) we derive the estimate

$$\begin{aligned}\int_\Omega\zeta_\rho^2(1+r^2)^{-\beta-1}|\nabla\mathbf{v}|^2dx&\le c\Big(\|q;\,[\mathcal{V}^1_{\beta+2,0}(\Omega)]^*\|^2+\int_\Omega(1+r^2)^{-\beta-2}|\mathbf{v}|^2dx\Big)\\ &<\infty\end{aligned}\tag{3.9}$$

with constant c independent of ρ. Passing in (3.9) $\rho\to\infty$, we get $\nabla\mathbf{v}\in L^2_{-\beta-1}(\Omega)$. Since the solution $(\mathbf{v},p)$ is smooth, from local estimates it follows (see [15: Proof of Lemma 3.1]) that $\nabla q\in L^2_{-\beta-1}(\Omega)\subset L^2_{-\beta-2}(\Omega)$ and

$$\|\nabla q;\,L^2_{-\beta-1}(\Omega)\|\le c\,\|\nabla\mathbf{v};\,L^2_{-\beta-1}(\Omega)\|.$$

By Lemma 2.4 we conclude that $q\in L^2_{-\beta-2}(\Omega)$ and

$$\|q;\,L^2_{-\beta-2}(\Omega)\|\le c\Big(\|q;\,[\mathcal{V}^1_{\beta+2,0}(\Omega)]^*\|+\|\nabla q;\,L^2_{-\beta-2}(\Omega)\|\Big)<\infty.$$

Thus the solution $(\mathbf{v}, p)$ of the homogeneous Stokes problem (1.2) - (1.3) belongs to $L^2_{-\beta-2}(\Omega)^3 \times L^2_{-\beta-2}(\Omega)$. By Theorem 2.2, $(\mathbf{v}, p)$ belongs to $\mathcal{D}_{-\beta-2}(\Omega)$ and hence

$$\ker \mathcal{S}^*_\beta \subset \Big\{ \big(\mathbf{v}, q, (\mathbf{n}q - \nu\partial_n \mathbf{v})|_{\partial\Omega}\big) : (\mathbf{v}, q) \in \ker \mathcal{S}_{-\beta-2} \Big\}. \tag{3.10}$$

Formulae (3.3) and (3.10) prove representation (3.1) of $\operatorname{coker} \mathcal{S}_\beta$. Since the numbers β and $-\beta-2$ belong to the prohibited set $\mathbb{Z}$ simultaneously, $\dim \ker \mathcal{S}_{-\beta-2} < \infty$ and the finite-dimensionality of $\operatorname{coker} \mathcal{S}_\beta$ is proved. Moreover, from (3.2) and Green's formula (2.33) we derive the following compatibility conditions for the Stokes problem (1.2) - (1.3):

$$\int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx - \int_\Omega g\, q\, dx + \int_{\partial\Omega} \mathbf{h} \cdot (\mathbf{n}q - \nu\partial_n \mathbf{v})\, ds = 0 \tag{3.11}$$

for all $(\mathbf{v}, p) \in \ker \mathcal{S}_{-\beta-2}$.

Let us consider the case $l > 1$. Assume that $(\mathbf{f}, g, \mathbf{h}) \in \mathcal{R}^l_\beta(\Omega; \partial\Omega) \subset \mathcal{R}^1_\beta(\Omega; \partial\Omega)$ with $\beta \notin \mathbb{Z}$. If the right-hand side $(\mathbf{f}, g, \mathbf{h})$ satisfies the compatibility conditions (3.11), then there exists a solution $(\mathbf{u}, p) \in \mathcal{D}^1_\beta(\Omega)$ of problem (1.2) - (1.3). By virtue of Theorem 2.2 we get $(\mathbf{u}, p) \in \mathcal{D}^l_\beta(\Omega)$. This means that $(\mathbf{f}, g, \mathbf{h})$ is orthogonal to $\ker [\mathcal{S}^l_\beta]^*$. By the Hahn-Banach theorem this gives

$$\ker [\mathcal{S}^l_\beta]^* \subset \Big\{ \big(\mathbf{v}, q, (\mathbf{n}q - \nu\partial_n \mathbf{v})|_{\partial\Omega}\big) : (\mathbf{v}, q) \in \ker \mathcal{S}^1_{-\beta-2} \Big\}.$$

Since by Theorem 2.2 $\ker \mathcal{S}^1_{-\beta-2} = \ker \mathcal{S}^l_{-\beta-2}$, the last relation together with (3.3) furnishes

$$\ker [\mathcal{S}^l_\beta]^* = \Big\{ \big(\mathbf{v}, q, (\mathbf{n}q - \nu\partial_n \mathbf{v})|_{\partial\Omega}\big) : (\mathbf{v}, q) \in \ker \mathcal{S}^l_{-\beta-2} \Big\}. \tag{3.12}$$

Thus in the case $\beta \notin \mathbb{Z}$

$$\dim \ker [\mathcal{S}^l_\beta]^* = \dim \ker \mathcal{S}^l_{-\beta-2} < \infty.$$

This proves the Fredholm property for $\mathcal{S}^l_\beta$ with $l > 1$ and $\beta \notin \mathbb{Z}$.

Consider now the case $\beta \in \mathbb{Z}$. Since $\mathcal{D}^l_\beta(\Omega) \subset \mathcal{D}^l_{\beta-\varepsilon}(\Omega)$ and $\mathcal{R}^l_\beta(\Omega; \partial\Omega) \subset \mathcal{R}^l_{\beta-\varepsilon}(\Omega; \partial\Omega)$ for all $\varepsilon > 0$, it follows that

$$\ker \mathcal{S}^l_\beta \subset \ker \mathcal{S}^l_{\beta-\varepsilon}$$
$$\operatorname{coker} \mathcal{S}^l_\beta \subset \operatorname{coker} \mathcal{S}^l_{\beta+\varepsilon}.$$

Consequently, the subspaces $\ker \mathcal{S}^l_\beta$ and $\operatorname{coker} \mathcal{S}^l_\beta$ are finite-dimensional for all $\beta \in \mathbb{R}$. We shall show that for $\beta \in \mathbb{Z}$ the range $\operatorname{Im} \mathcal{S}^l_\beta$ is not closed and hence $\mathcal{S}^l_\beta$ looses the Fredholm property.

Let $\beta = -m - 1$ $(m \in \mathbb{Z})$. Denote by χ the smooth cut-off function with $\chi(r) = 1$ for $r < 1$ and $\chi(r) = 0$ for $r > 2$ and let $\chi_R(r) = \chi(\frac{r}{R})$ $(R \geq 2)$. We take

$$\begin{aligned} p_0(y) &= -(2\pi)^{-1} \ln r \\ p_m(y) &= (2\pi|m|)^{-\frac{1}{2}} r^m \cos(m\varphi) \quad (m \neq 0) \\ \mathbf{u}_m(y, z) &= \tfrac{1}{2\nu} z(z-1) \nabla p_m(y) \end{aligned}$$

and put

$$(\widehat{\mathbf{u}}_m, \widehat{p}_m) = (1-\chi(r))\chi_R(r)(\mathbf{u}_m, p_m).$$

It is easy to compute that

$$\begin{aligned} &\big\|(\widehat{\mathbf{u}}_m, \widehat{p}_m);\ \mathcal{D}^l_{-m-1}(\Omega)\big\|^2 \\ &\quad\geq \big\|(\widehat{\mathbf{u}}_m, \widehat{p}_m);\ L^2_{-m}(\Omega)^3 \times L^2_{-m-1}(\Omega)\big\|^2 \\ &\quad\geq c\Big(1 + \int_2^R \big(r^{-2m} r^{2(m-1)} + r^{-2(m+1)} r^{2m}\big) r\, dr\Big) \\ &\quad\geq c\Big(1 + \ln\frac{R}{2}\Big). \end{aligned} \tag{3.13}$$

On the other hand, $(\mathbf{u}_m, p_m)$ satisfies the homogeneous Stokes problem (1.2) - (1.3) in $\Omega \setminus \{x :\ r = 0\}$. Therefore

$$\begin{aligned} -\nu\Delta\widehat{\mathbf{u}}_m + \nabla\widehat{p}_m &= [-\nu\Delta + \nabla,\ (1-\chi)\chi_R](\mathbf{u}_m, p_m) \equiv \mathbf{f}_m \qquad &(x \in \Omega) \\ \operatorname{div}\widehat{\mathbf{u}}_m &= [\operatorname{div},\ (1-\chi)\chi_R]\mathbf{u}_m \equiv g_m &(x \in \Omega) \\ \widehat{\mathbf{u}}_m &= 0 &(x \in \partial\Omega) \end{aligned}$$

where $[A, B]$ stands for the commutator of the operators A and B. The right-hand side $(\mathbf{f}_m, g_m)$ has a compact support lying in the union of the annuli $\{x \in \Omega :\ 1 < r < 2\}$ and $\{x \in \Omega :\ R < r < 2R\}$. Calculating the norm $\|(\mathbf{f}_m, g_m);\ \mathcal{R}^l_{-m-1}(\Omega; \partial\Omega)\|^2$, we find that it is bounded by the expression

$$c\Big(1 + \int_R^{2R} R^{-2} r^{-2m} r^{2m} r\, dr\Big) \leq \text{const} \tag{3.14}$$

where c is independent of $R \in (2, \infty)$. The range $\operatorname{Im}\mathcal{S}^l_{-m-1}$ is closed if and only if for every $(\mathbf{v}, q) \in \mathcal{D}^l_{-m-1}(\Omega) \ominus \ker\mathcal{S}^l_{-m-1}$ the estimate

$$\big\|(\mathbf{v}, q);\ \mathcal{D}^l_{-m-1}(\Omega)\big\| \leq c_* \big\|\mathcal{S}^l_{-m-1}(\mathbf{v}, q);\ \mathcal{R}^l_{-m-1}(\Omega; \partial\Omega)\big\|$$

holds true with constant c_* independent of $(\mathbf{v}, q)$. Letting $R \to \infty$ in formulae (3.14) and (3.13) we see that for $(\widehat{\mathbf{u}}_m, \widehat{p}_m)$ the last estimate fails, i.e. $\operatorname{Im}\mathcal{S}^l_{-m-1}$ is not closed. The theorem is proved ∎

Lemma 3.1. *If $\beta \geq -1$, then $\mathcal{S}^l_\beta$ is a monomorphism, and if $\beta < -1$, then $\mathcal{S}^l_\beta$ is an epimorphism.*

Proof. Let $\beta \geq -1$ and $(\mathbf{u}, p) \in \ker\mathcal{S}^l_\beta$. Multiplying the homogeneous equations (1.2) by $\mathbf{u}$ and integrating by parts in Ω, we derive

$$\nu \int_\Omega |\nabla\mathbf{u}(x)|^2 dx = 0. \tag{3.15}$$

(Note that by definition of the space $\mathcal{D}^l_\beta(\Omega)$ all the integrals involved converge for $\beta \geq -1$.) From (3.15) it follows $|\nabla\mathbf{u}(x)| = 0$ and hence $\mathbf{u}(x) = 0$. The Stokes equations (1.2) imply $\nabla p = 0$ in Ω, i.e. $p(x) = c$. If $c \neq 0$, then the integral $\int_\Omega (1 + r^2)^\beta |c|^2 dx$ diverges (recall that $\beta \geq -1$) what contradicts with the condition $p \in L^2_\beta(\Omega)$. Thus $c = 0$ and $\ker\mathcal{S}^l_\beta = \emptyset$ for $\beta \geq -1$. For $\beta < -1$ the relation $\dim\operatorname{coker}\mathcal{S}^l_\beta = 0$ follows from (3.12), since in this case $-2-\beta > -1$ and $\ker\mathcal{S}^l_{-2-\beta} = \emptyset$ ∎

4. Coefficients in the asymptotics and computation of the index

Let $(\mathbf{u}, p) \in \mathcal{D}^l_\beta(\Omega)$ $(\beta > -1)$ be a solution of the Stokes problem (1.2) - (1.3) with right-hand side $(\mathbf{f}, g, \mathbf{h}) \in \mathcal{R}^l_{\beta+k}(\Omega; \partial\Omega)$ $(k \in \mathbb{N})$. From Theorem 2.4 it follows that the solution $(\mathbf{u}, p)$ admits the asymptotic representation (2.30) - (2.31). On the other hand, by Lemma 3.1 we know that the operator $\mathcal{S}^l_\beta$ with $\beta > -1$ is a monomorphism, i.e. the solution is unique. Therefore, the coefficients $c^{\pm}_{-m}$ $(m \in \mathbb{N})$ in the asymptotic formulae (2.30) - (2.31) are uniquely determined by the right-hand side $(\mathbf{f}, g, \mathbf{h})$. In this section we find integral formulae for the coefficients c^-_0 and $c^{\pm}_{-m}$ $(m \in \mathbb{N})$.

We start with the computation of c^-_0.

Lemma 4.1. *Let* $(\mathbf{u}, p) \in \mathcal{D}^l_\beta(\Omega), \beta \in (-2, -1)$, *be a solution of problem* (1.2)−(1.3) *with right-hand side* $(\mathbf{f}, g, \mathbf{h}) \in \mathcal{R}^l_{\beta+1}(\Omega; \partial\Omega)$. *Then the coefficient* c^-_0 *in the asymptotic formula*

$$\begin{pmatrix} \mathbf{u}(x) \\ p(x) \end{pmatrix} = \chi(r) \begin{pmatrix} c^+_0 \mathbf{u}^+_0(y, z) + c^-_0 \mathbf{u}^-_0(y, z) \\ c^+_0 p^+_0(y) + c^-_0 p^-_0(y) \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{u}}(x) \\ \tilde{p}(x) \end{pmatrix} \tag{4.1}$$

where $(\tilde{\mathbf{u}}, \tilde{p}) \in \mathcal{D}^l_{\beta+1}(\Omega)$ *(see* (2.30)) *admits the integral representations*

$$c^-_0 = -12\nu\bigg(\int_{\partial\Omega} \mathbf{h} \cdot \mathbf{n}\, ds - \int_\Omega g\, dx\bigg). \tag{4.2}$$

Proof. Let us apply to the solutions $(\mathbf{u}, p)$ and $(\mathbf{u}^+_0, p^+_0) = (\mathbf{0}, 1)$ Green's formula in the domain $\Omega_R = \{x \in \Omega : r < R\ (R > 2)\}$:

$$\int_{\Omega_R} (-\nu\Delta\mathbf{u} + \nabla p) \cdot \mathbf{0}\, dx - \int_{\Omega_R} \operatorname{div} \mathbf{u}\, dx + \int_{\partial\Omega_R \cup S_R} \mathbf{u} \cdot \mathbf{n}\, ds = 0$$

where $\partial\Omega_R = \partial\Omega \cap \Omega_R$ and $S_R = \{x \in \Omega : r = R\}$. This furnishes

$$-\int_{\Omega_R} g\, dx + \int_{\partial\Omega_R} \mathbf{h} \cdot \mathbf{n}\, ds + \int_{S_R} \mathbf{u} \cdot \mathbf{n}\, ds = 0. \tag{4.3}$$

Taking into account representation (4.1) for $\mathbf{u}$, we compute

$$\begin{aligned} \int_{S_R} \mathbf{u} \cdot \mathbf{n}\, ds &= c^-_0 \int_{S_R} \mathbf{u}^-_0 \cdot \mathbf{n}\, ds + \int_{S_R} \tilde{\mathbf{u}} \cdot \mathbf{n}\, ds \\ &= -\tfrac{c^-_0}{4\nu\pi} \int_{S_R} z(z-1)\nabla \ln r \cdot \nabla r\, ds + \int_{S_R} \tilde{\mathbf{u}} \cdot \mathbf{n}\, ds \\ &= \tfrac{c^-_0}{12\nu} + \int_{S_R} \tilde{\mathbf{u}} \cdot \mathbf{n}\, ds. \end{aligned}$$

Since $\tilde{\mathbf{u}} \in L^2_{\beta+2}(\Omega), \beta \in (-2, -1)$, we get

$$\begin{aligned} \bigg|\int_{S_R} \tilde{\mathbf{u}} \cdot \mathbf{n}\, ds\bigg| &\le c\bigg(R^{-2(\beta+2)+1} \int_{S_R} (1+r)^{2(\beta+2)} |\tilde{\mathbf{u}}|^2 ds\bigg)^{\frac{1}{2}} \\ &\le c\bigg(R \int_{S_R} (1+r)^{2(\beta+2)} |\tilde{\mathbf{u}}|^2 ds\bigg)^{\frac{1}{2}} \\ &= o(R^{-1}) \to 0 \quad \text{as } R \to \infty \end{aligned}$$

(at least for some subsequence R_l). Substituting the last two formulae into (4.3) and passing to the limit as $R_l \to \infty$, we derive (4.2) ∎

In the previous lemma we have already used a special solution of the homogeneous Stokes problem $\boldsymbol{\zeta}_0^+(x) = \big(\mathbf{u}_0^+(y,z), p_0^+(y)\big)^T = (\mathbf{0},1)^T$. Let us construct special solutions $\boldsymbol{\zeta}_m^\pm = (\boldsymbol{\xi}_m^\pm, \eta_m^\pm)^T$ for $m \in \mathbb{N}$.

Lemma 4.2. *For every $m \in \mathbb{N}$ there exist solutions $\boldsymbol{\zeta}_m^\pm$ of the homogeneous Stokes problem* (1.2) – (1.3) *which admit the asymptotic forms*

$$\boldsymbol{\zeta}_m^\pm = \begin{pmatrix} \boldsymbol{\xi}_m^\pm(x) \\ \eta_m^\pm(x) \end{pmatrix} = \begin{pmatrix} \mathbf{u}_m^\pm(y,z) \\ p_m^\pm(y) \end{pmatrix} + \begin{pmatrix} \tilde{\boldsymbol{\xi}}_m^\pm(x) \\ \tilde{\eta}_m^\pm(x) \end{pmatrix} \qquad (m \in \mathbb{N}) \tag{4.4}$$

where $\big(\mathbf{u}_m^\pm(y,z), p_m^\pm(y)\big)$ [1)] *are given by* (2.31) *and* $(\tilde{\boldsymbol{\xi}}_m^\pm, \tilde{\eta}_m^\pm) \in \mathcal{D}_\gamma^l(\Omega)$ *with arbitrary γ satisfying the relation*

$$-1 < \gamma < 0. \tag{4.5}$$

Proof. We shall look for the solution $(\boldsymbol{\xi}_m^\pm, \eta_m^\pm)$ in form (4.4). Since $(\mathbf{u}_m^\pm, p_m^\pm)$ solve the homogeneous Stokes problem (1.2) - (1.3) in the layer Π, we obtain for $(\tilde{\boldsymbol{\xi}}_m^\pm, \tilde{\eta}_m^\pm)$ the non-homogeneous problem (1.2) - (1.3) with right-hand side $(\mathbf{0}, 0, \mathbf{h}_m^\pm)$ where $\mathbf{h}_m^\pm = -\mathbf{u}_m^\pm|_{\partial\Omega}$ has compact support contained in $\{x \in \partial\Omega : |x| < 1\}$. Thus, $(\mathbf{0}, 0, \mathbf{h}_m^\pm) \in \mathcal{R}_\gamma^l(\Omega;\partial\Omega) \subset \mathcal{R}_{\gamma-1}^l(\Omega;\partial\Omega)$. Since $(\gamma-1) \in (-2,-1)$, the operator $\mathcal{S}_{\gamma-1}^l$ is of Fredholm type (Theorem 3.1) and $\dim \operatorname{coker} \mathcal{S}_{\gamma-1}^l = 0$ (Lemma 3.1). Therefore, problem (1.2) - (1.3) is solvable in $\mathcal{D}_{\gamma-1}^l(\Omega)$ for all right-hand sides from $\mathcal{R}_{\gamma-1}^l(\Omega;\partial\Omega)$ and we find the remainder $(\tilde{\boldsymbol{\xi}}_m^\pm, \tilde{\eta}_m^\pm) \in \mathcal{D}_{\gamma-1}^l(\Omega)$. Moreover, $(\tilde{\boldsymbol{\xi}}_m^\pm, \tilde{\eta}_m^\pm)$ admits the asymptotic representation (4.1):

$$\begin{pmatrix} \tilde{\boldsymbol{\xi}}_m^\pm(x) \\ \tilde{\eta}_m^\pm(x) \end{pmatrix} = \chi(r) \begin{pmatrix} c_0^+\mathbf{u}_0^+(y,z) + c_0^-\mathbf{u}_0^-(y,z) \\ c_0^+ p_0^+(y) + c_0^- p_0^-(y) \end{pmatrix} + \begin{pmatrix} \widehat{\boldsymbol{\xi}}_m^\pm(x) \\ \widehat{\eta}_m^\pm(x) \end{pmatrix}$$

with $(\widehat{\boldsymbol{\xi}}_m^\pm, \widehat{\eta}_m^\pm) \in \mathcal{D}_\gamma^l(\Omega)$. We normalize $(\tilde{\boldsymbol{\xi}}_m^\pm, \tilde{\eta}_m^\pm)$ by the condition $\lim_{|x|\to\infty} \tilde{\eta}_m^\pm(x) = 0$ so that $c_0^+ = 0$. Since $\tilde{\boldsymbol{\xi}}_m^\pm|_{\partial\Omega} = -\mathbf{u}_m^\pm|_{\partial\Omega}$ on $\partial\Omega$, from (4.2) we get

$$c_0^- = 12\nu\int_{\partial\Omega} \mathbf{h}_m^\pm \cdot \mathbf{n}\, ds = 12\nu\int_\Omega \operatorname{div} \mathbf{u}_m^\pm(y,z)\, dx = 0 \qquad (m \in \mathbb{N}).$$

Thus we obtain $(\widehat{\boldsymbol{\xi}}_m^\pm, \widehat{\eta}_m^\pm) = (\tilde{\boldsymbol{\xi}}_m^\pm, \tilde{\eta}_m^\pm) \in \mathcal{D}_\gamma^l(\Omega)$ and this concludes the proof of the lemma ∎

Let us compute now the coefficients $c_{-m}^\pm$ $(m \in \mathbb{N})$.

Lemma 4.3. *Let $(\mathbf{u}, p) \in \mathcal{D}_\beta^l(\Omega)$ $(\beta > -1)$ be a solution of problem* (1.2) – (1.3) *with right-hand side $(\mathbf{f}, g, \mathbf{h}) \in \mathcal{R}_{\beta+k}^l(\Omega;\partial\Omega)$ $(k \in \mathbb{N})$. Then the coefficients $c_{-m}^\pm$ in the asymptotic formulae* (2.30) – (2.31) *admit the integral representations*

$$c_{-m}^\pm = -12\nu\Big(\int_\Omega \mathbf{f}\cdot\boldsymbol{\xi}_m^\pm dx - \int_\Omega g\,\eta_m^\pm dx + \int_{\partial\Omega} \mathbf{h}\cdot(\eta_m^\pm\mathbf{n} - \nu\partial_n\boldsymbol{\xi}_m^\pm)ds\Big) \tag{4.6}$$
$$\big(-\beta - k - 1 < -m < -\beta - 1\big)$$

[1)] Note that for $m \in \mathbb{N}$ the functions $p_m^\pm$ are harmonic polynomials and therefore $(\mathbf{u}_m^\pm, p_m^\pm) \in C^\infty(\bar{\Omega})$.

where $(\boldsymbol{\xi}_m^{\pm}, \eta_m^{\pm})$ *are the solutions of the homogeneous problem* (1.2) – (1.3) *constructed in Lemma* 4.2.

Proof. Let us apply to $(\mathbf{u}, p)$ and $(\boldsymbol{\xi}_m^{\pm}, \eta_m^{\pm})$ Green's formula in the domain $\Omega_R = \{x \in \Omega : r < R \ \ (R > 2)\}$:

$$\begin{aligned}&\int_{\Omega_R} (-\nu\Delta\mathbf{u} + \nabla p)\cdot\boldsymbol{\xi}_m^{\pm}dx - \int_{\Omega_R} \operatorname{div}\mathbf{u}\,\eta_m^{\pm}dx + \int_{\partial\Omega_R\cup S_R} \mathbf{u}\cdot(\mathbf{n}\eta_m^{\pm} - \nu\partial_n\boldsymbol{\xi}_m^{\pm})\,ds \\ &= \int_{\Omega_R} (-\nu\Delta\boldsymbol{\xi}_m^{\pm} + \nabla\eta_m^{\pm})\cdot\mathbf{u}\,dx - \int_{\Omega_R} \operatorname{div}\boldsymbol{\xi}_m^{\pm}\,p\,dx + \int_{\partial\Omega_R\cup S_R} \boldsymbol{\xi}_m^{\pm}\cdot(\mathbf{n}p - \nu\partial_n\mathbf{u})\,ds.\end{aligned} \tag{4.7}$$

Since $(\boldsymbol{\xi}_m^{\pm}, \eta_m^{\pm})$ fulfils the homogeneous equations (1.2) - (1.3), from (4.7) we derive

$$\begin{aligned}&\int_{\Omega_R} \mathbf{f}\cdot\boldsymbol{\xi}_m^{\pm}dx - \int_{\Omega_R} g\,\eta_m^{\pm}dx + \int_{\partial\Omega_R} \mathbf{h}\cdot(\mathbf{n}\eta_m^{\pm} - \nu\partial_n\boldsymbol{\xi}_m^{\pm})\,ds \\ &+ \int_{S_R} \mathbf{u}\cdot(\mathbf{n}\eta_m^{\pm} - \nu\partial_n\boldsymbol{\xi}_m^{\pm})\,ds = \int_{S_R} \boldsymbol{\xi}_m^{\pm}\cdot(\mathbf{n}p - \nu\partial_n\mathbf{u})\,ds.\end{aligned} \tag{4.8}$$

Let us calculate the right-hand side of (4.8). Taking account of the asymptotic representations (2.30) - (2.31) and (4.4) for $(\mathbf{u}, p)$ and $(\boldsymbol{\xi}_m^{\pm}, \eta_m^{\pm})$, respectively, we get

$$\begin{aligned}&\int_{S_R} \boldsymbol{\xi}_m^{\pm}\cdot(\mathbf{n}p - \nu\partial_n\mathbf{u})\,ds \\ &\quad= \int_{S_R} \tilde{\boldsymbol{\xi}}_m^{\pm}\cdot(\mathbf{n}p - \nu\partial_n\mathbf{u})\,ds \\ &\qquad+ \int_{S_R} \mathbf{u}_m^{\pm}\cdot \sum_{-\beta-k-1<-l<-\beta-1} \Big[\mathbf{n}\big(c_{-l}^{+}p_{-l}^{+} + c_{-l}^{-}p_{-l}^{-}\big) - \nu\big(c_{-l}^{+}\partial_n\mathbf{u}_{-l}^{+} + c_{-l}^{-}\partial_n\mathbf{u}_{-l}^{-}\big)\Big]ds.\end{aligned} \tag{4.9}$$

The first integral in the right-hand side here can be majorated by

$$\begin{aligned}&\Big(R\int_{S_R} |\tilde{\boldsymbol{\xi}}_m^{\pm}|^2(1+r^2)^{\gamma+1}ds\Big)^{\frac12}\Big(R\int_{S_R} |p|^2(1+r^2)^{\beta}R^{-2(\beta+\gamma+1)-2}ds + \\ &R\int_{S_R} |\mathbf{u}|^2(1+r^2)^{\beta+1}R^{-2(\beta+\gamma+1)-4}ds\Big)^{\frac12} \le c\Big(R\int_{S_R} |\tilde{\boldsymbol{\xi}}_m^{\pm}|^2(1+r^2)^{\gamma+1}ds\Big)^{\frac12} \\ &\times\Big(R\int_{S_R} |p|^2(1+r^2)^{\beta}ds + R^{-1}\int_{S_R} |\mathbf{u}|^2(1+r^2)^{\beta+1}ds\Big)^{\frac12}.\end{aligned} \tag{4.10}$$

Since $\tilde{\boldsymbol{\xi}}_m^{\pm} \in L^2_{\gamma+1}(\Omega)$, $\mathbf{u} \in L^2_{\beta+1}(\Omega)$, $p \in L^2_{\beta}(\Omega)$ (see the definition of the space $\mathcal{D}^l_{\beta}(\Omega)$), expresion (4.10) vanishes as $R \to \infty$ (at least, for some subsequence $R_j \to \infty$). Further, using the relations

$$\int_0^{2\pi} \cos(m\varphi)\sin(|l|\varphi)\,d\varphi = 0$$

$$\int_0^{2\pi} \sin(|m|\varphi)\sin(|l|\varphi)\,d\varphi = \int_0^{2\pi} \cos(m\varphi)\cos(l\varphi)\,d\varphi = \pi\delta_{m,l}$$

we find that

$$\begin{aligned}
&\int_{S_R} \mathbf{u}_m^{\pm} \cdot \sum_{-\beta-k<-l<-\beta-1} \Big[\mathbf{n}\big(c_{-l}^{+}p_{-l}^{+} + c_{-l}^{-}p_{-l}^{-}\big) - \nu\big(c_{-l}^{+}\partial_n \mathbf{u}_{-l}^{+} + c_{-l}^{-}\partial_n \mathbf{u}_{-l}^{-}\big)\Big]\, ds \\
&\qquad = \int_{S_R} \mathbf{u}_m^{\pm} \cdot \mathbf{n}\big(c_{-m}^{+}p_{-m}^{+} + c_{-m}^{-}p_{-m}^{-}\big)\, ds \\
&\qquad\quad - \nu \int_{S_R} \mathbf{u}_m^{\pm} \cdot \big(c_{-m}^{+}\partial_n \mathbf{u}_{-m}^{+} + c_{-m}^{-}\partial_n \mathbf{u}_{-m}^{-}\big)\, ds \\
&\qquad = c_{-m}^{\pm} \int_{S_R} (2\nu)^{-1} z(z-1)\partial_n p_m^{\pm} p_{-m}^{\pm}\, ds + R^{-2}c(m) \\
&\qquad = -\tfrac{1}{24\nu} c_{-m}^{\pm} + o(R^{-1}).
\end{aligned} \tag{4.11}$$

Analogously one can compute the integral

$$\int_{S_R} \mathbf{u} \cdot (\mathbf{n}\eta_m^{\pm} - \nu\partial_n \boldsymbol{\xi}_m^{\pm})\, ds = \tfrac{1}{24\nu} c_{-m}^{\pm} + o(R^{-1}). \tag{4.12}$$

Substituting formulae (4.9) - (4.12) into (4.8) and passing $R \to \infty$, we derive formula (4.6) ∎

Now we are in a position to compute the dimensions of $\ker \mathcal{S}_\beta^l$ and $\operatorname{coker} \mathcal{S}_\beta^l$.

Theorem 4.1.

(i) *If* $\beta \in (k-1, k)$ $(0 \le k \in \mathbb{Z})$, *then* $\dim \operatorname{coker} \mathcal{S}_\beta^l = 2k+1$.

(ii) *If* $\beta \in (q-1, q)$ $(\mathbb{Z} \ni q \le -1)$, *then* $\dim \ker \mathcal{S}_\beta^l = -2q-1$.

(iii) *If* $\beta \in (p, p+1)$ $(p \in \mathbb{Z})$, *then* $\operatorname{Ind} \mathcal{S}_\beta^l = -2p-1$.

Proof. Let $(\mathbf{f}, g, \mathbf{h}) \in \mathcal{R}_\beta^l(\Omega; \partial\Omega)$ $(\beta \in (k-1,k), k \ge 0)$. Then there exists a solution $(\mathbf{u}, p) \in \mathcal{D}_{\beta_1}^l(\Omega)$ $(\beta_1 = \beta - k - 1 \in (-2,-1))$ of problem (1.2) - (1.3). (Note that $\mathcal{R}_\beta^l(\Omega; \partial\Omega) \subset \mathcal{R}_{\beta_1}^l(\Omega; \partial\Omega)$ and by Lemma 3.1 the operator $\mathcal{S}_{\beta_1}^l$ $(\beta_1 \in (-2,-1))$ is an epimorphism.) For $(\mathbf{u}, p)$ there holds the asymptotic formula (2.30) where the constants c_0^- and $c_{-m}^{\pm}$ $(m = 1, \ldots, k)$ admit the integral representations (4.2) and (4.6), respectively. Hence under $2k+1$ compatibility conditions

$$\int_{\partial\Omega} \mathbf{h} \cdot \mathbf{n}\, ds - \int_\Omega g\, dx = 0$$

$$\int_\Omega \mathbf{f} \cdot \boldsymbol{\xi}_m^{\pm} dx - \int_\Omega g\, \eta_m^{\pm} dx + \int_{\partial\Omega} \mathbf{h} \cdot (\eta_m^{\pm}\mathbf{n} - \nu\partial_n \boldsymbol{\xi}_m^{\pm})\, ds = 0 \quad (m = 1, \ldots, k)$$

we obtain

$$\begin{pmatrix} \mathbf{u}(x) \\ p(x) \end{pmatrix} = c_0^{+} \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{u}}(x) \\ \tilde{p}(x) \end{pmatrix}$$

where $(\tilde{\mathbf{u}}, \tilde{p}) \in \mathcal{D}_\beta^l(\Omega)$. Normalizing this solution by the condition $\lim_{|x|\to\infty} p(x) = 0$ we get $(\mathbf{u}, p) = (\tilde{\mathbf{u}}, \tilde{p}) \in \mathcal{D}_\beta^l(\Omega)$. Thus assuming $2k+1$ compatibility conditions to be valid, we have proved the existence of the solution $(\mathbf{u}, p) \in \mathcal{D}_\beta^l(\Omega)$. Since for $\beta \in$

$(k-1,k)$ $(k\geq 0)$ the operator $\mathcal{S}^l_\beta$ is a Fredholm monomorphism (see Lemma 3.1), these conditions are necessary. Therefore, we conclude

$$\dim\operatorname{coker}\mathcal{S}^l_\beta = 2k+1.$$

Statement (ii) follows now from the fact that

$$\dim\ker\mathcal{S}^l_\beta = \dim\operatorname{coker}\mathcal{S}^l_{-\beta-2}.$$

Statement (iii) has become evident ∎

5. Asymptotic conditions at infinity

As follows from Lemma 3.1, there is no admissible β such that the operator $\mathcal{S}^l_\beta$ is of index zero. In order to compensate this lack we introduce function spaces with detached asymptotics and impose conditions at infinity. For $\beta<-1$ the operator $\mathcal{S}^l_\beta$ is an epimorphism, and for $\beta>-1$, $\mathcal{S}^l_\beta$ is a monomorphism (see Lemma 3.1). Let us take

$$\beta_\pm = -1\pm N\pm\delta \qquad \big(N\in\mathbb{N}_0, \delta\in(0,1)\big). \tag{5.1}$$

For simplicity we fix the regularity index l and omit it in notations. Moreover, we denote

$$\mathcal{S}^l_{\beta_\pm}=\mathcal{S}_\pm, \qquad \mathcal{D}^l_{\beta_\pm}(\Omega)=\mathcal{D}_\pm(\Omega), \qquad \mathcal{R}^l_{\beta_\pm}(\Omega;\partial\Omega)=\mathcal{R}_\pm(\Omega;\partial\Omega).$$

Let us consider the mapping $\mathcal{S}_-:\mathcal{D}_-(\Omega)\longmapsto\mathcal{R}_-(\Omega;\partial\Omega)$ and its preimage $\mathbb{D}_\pm(\Omega)$ of the lineal $\mathcal{R}_+(\Omega;\partial\Omega)\subset\mathcal{R}_-(\Omega;\partial\Omega)$ (since the preimage is related both to the indices " + " and " − ", we mark it by " ± "). Due to Theorem 2.4, $\mathbb{D}_\pm(\Omega)$ consists of vector functions $\mathbf{U}=(\mathbf{u},p)$ taking the asymptotic form

$$\mathbf{U}=\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix}=\sum_{-N\leq m\leq N}\chi\left[c^+_m\begin{pmatrix}\mathbf{u}^+_m\\ p^+_m\end{pmatrix}+c^-_m\begin{pmatrix}\mathbf{u}^-_m\\ p^-_m\end{pmatrix}\right]+\begin{pmatrix}\tilde{\mathbf{u}}\\ \tilde{p}\end{pmatrix} \tag{5.2}$$

where $\tilde{\mathbf{U}}=(\tilde{\mathbf{u}},\tilde{p})\in\mathcal{D}_+(\Omega)$ and $(\mathbf{u}^\pm_m,p^\pm_m)$ are given by (2.31). This means that $\mathbb{D}_\pm(\Omega)$ is formed by the sum of linear combinations of the special solutions $(\mathbf{u}^\pm_m,p^\pm_m)$ and the "rapidly" decaying remainder $\tilde{\mathbf{U}}=(\tilde{\mathbf{u}},\tilde{p})\in\mathcal{D}_+(\Omega)$. Furthermore, the quotient space $\mathbb{D}_\pm(\Omega)/\mathcal{D}_+(\Omega)$ can be identified with $\mathbb{R}^{4N+2}$ and we introduce in $\mathbb{D}_\pm(\Omega)$ the norm induced by the asymptotic representation (5.2)

$$\|\mathbf{U};\mathbb{D}_\pm(\Omega)\|=\Big(\|\tilde{\mathbf{U}};\mathcal{D}_+(\Omega)\|^2+\|\mathbf{a};\mathbb{R}^{2N+1}\|^2+\|\mathbf{b};\mathbb{R}^{2N+1}\|^2\Big)^{\frac{1}{2}}$$

where $\mathbf{a}$ and $\mathbf{b}$ are columns of height $2N+1$,

$$\begin{aligned}\mathbf{a}&=\big(c^-_0,c^+_{-1},c^-_{-1},\ldots,c^+_{-N},c^-_{-N}\big)^T\\ \mathbf{b}&=\big(c^+_0,c^+_1,c^-_1,\ldots,c^+_N,c^-_N\big)^T.\end{aligned} \tag{5.3}$$

Let $\mathfrak{S}_\pm$ be the restriction of $\mathcal{S}_-$ on $\mathbb{D}_\pm(\Omega)$. Due to estimate (2.32),

$$\|\mathbf{a};\mathbb{R}^{2N+1}\|+\|\mathbf{b};\mathbb{R}^{2N+1}\|\leq c\Big(\|\mathfrak{S}_\pm\mathbf{U};\mathcal{R}_+(\Omega;\partial\Omega)\|+\|(\mathbf{u},p);L^2_{\beta_-}(\Omega)\|\Big).$$

Therefore the operator

$$\mathfrak{S}_\pm:\mathbb{D}_\pm(\Omega)\longmapsto\mathcal{R}_+(\Omega;\partial\Omega) \tag{5.4}$$

of problem (1.2) - (1.3) is continuous. Moreover, in view of Theorems 3.1 and 4.1, it inherits properties of $\mathcal{S}_-$ and the following assertion is valid.

Theorem 5.1. *The mapping* (5.4) *is a Fredholm epimorphism and*

$$\dim\ker\mathfrak{S}_\pm = \dim\ker\mathcal{S}_- = 2N+1. \tag{5.5}$$

There appear the continuous projections

$$\begin{aligned}\mathbb{D}_\pm(\Omega) \ni \mathbf{U} &\longmapsto \pi_1\mathbf{U} = \mathbf{a} \in \mathbb{R}^{2N+1}\\ \mathbb{D}_\pm(\Omega) \ni \mathbf{U} &\longmapsto \pi_0\mathbf{U} = \mathbf{b} \in \mathbb{R}^{2N+1}.\end{aligned} \tag{5.6}$$

We also determine

$$\pi = \begin{pmatrix}\pi_1\\ \pi_0\end{pmatrix} : \mathbb{D}_\pm(\Omega) \longmapsto \mathbb{R}^{4N+2}.$$

We treat $\pi_0\mathbf{U}$, $\pi_1\mathbf{U}$ and $\pi\mathbf{U}$ as columns in $\mathbb{R}^{2N+1}$, $\mathbb{R}^{2N+1}$ and $\mathbb{R}^{4N+2}$, respectively.

Let us connect with Green's formula (2.33) the linear form

$$Q_\Omega(\mathbf{U},\mathbf{V}) = Q_\Omega(\mathbf{u},p;\,\mathbf{v},q)$$

defined by

$$\begin{aligned}Q_\Omega(\mathbf{U};\mathbf{V}) \equiv (-\nu\Delta\mathbf{u}+\nabla p,\,\mathbf{v})_\Omega + (-\operatorname{div}\mathbf{u},q)_\Omega + (\mathbf{u},q\mathbf{n}-\nu\partial_n\mathbf{v})_{\partial\Omega}\\ -(\mathbf{u},-\nu\Delta\mathbf{v}+\nabla q)_\Omega - (p,-\operatorname{div}\mathbf{v})_\Omega - (p\mathbf{n}-\nu\partial_n\mathbf{u},\mathbf{v})_{\partial\Omega}\end{aligned} \tag{5.7}$$

where $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{\partial\Omega}$ stand for extensions of the scalar products in $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively. Since $(\mathbf{u}_m^\pm, p_m^\pm)$ satisfy the homogeneous equations (1.2) - (1.3) in $\Pi\setminus\{x\in\mathbb{R}^3 : r=0\}$, for any $\mathbf{U},\mathbf{V}\in\mathbb{D}_\pm(\Omega)$ we get the inclusions (see (5.2))

$$\left.\begin{aligned}&\big(-\nu\Delta\mathbf{u}+\nabla p, -\operatorname{div}\mathbf{u}, \mathbf{u}|_{\partial\Omega}\big)\\ &\big(-\nu\Delta\mathbf{v}+\nabla q, -\operatorname{div}\mathbf{v}, \mathbf{v}|_{\partial\Omega}\big)\end{aligned}\right\} \in \mathcal{R}_+(\Omega,\partial\Omega)$$

and therefore all integrals in the left-hand side of (5.7) converge. Hence Q_Ω is a continuous antisymmetric form on $\mathbb{D}_\pm(\Omega)^2$,

$$Q_\Omega(\mathbf{V};\mathbf{U}) = -Q_\Omega(\mathbf{U};\mathbf{V}). \tag{5.8}$$

Due to Lemma 2.7,

$$Q_\Omega(\mathbf{V};\mathbf{U}) = Q_\Omega(\mathbf{U};\mathbf{V}) = 0 \tag{5.9}$$

for all $\mathbf{V}\in\mathbb{D}_+(\Omega)\subset\mathbb{D}_\pm(\Omega)$ and all $\mathbf{U}\in\mathbb{D}_\pm(\Omega)$. Thus Q_Ω can be naturally treated as a form defined on the quotient space

$$\big(\mathbb{D}_\pm(\Omega)/\mathcal{D}_+(\Omega)\big)^2 \approx \mathbb{R}^{4N+2}\times\mathbb{R}^{4N+2}.$$

Lemma 5.1. *If* $\mathbf{U},\mathbf{V}\in\mathbb{D}_\pm(\Omega)$, *then*

$$Q_\Omega(\mathbf{U};\mathbf{V}) = \langle\pi_0\mathbf{U},\pi_1\mathbf{V}\rangle_{2N+1} - \langle\pi_1\mathbf{U},\pi_0\mathbf{V}\rangle_{2N+1} \tag{5.10}$$

where $\langle\cdot,\cdot\rangle_K = 12\nu\,[\cdot,\cdot]_K$ *with* $[\cdot,\cdot]_K$ *being the scalar product in* $\mathbb{R}^K$.

Proof. According to the asymptotic form (5.2), we can represent $\mathbf{U}$ as sum

$$\begin{aligned}\mathbf{U} = \begin{pmatrix}\mathbf{u}\\ p\end{pmatrix} &= \sum_{1\le m\le N} \chi\left[c_0^+\begin{pmatrix}\mathbf{u}_0^+\\ p_0^+\end{pmatrix} + c_m^+\begin{pmatrix}\mathbf{u}_m^+\\ p_m^+\end{pmatrix} + c_m^-\begin{pmatrix}\mathbf{u}_m^-\\ p_m^-\end{pmatrix}\right]\\ &\quad + \sum_{-N\le m\le -1} \chi\left[c_0^-\begin{pmatrix}\mathbf{u}_0^-\\ p_0^-\end{pmatrix} + c_m^+\begin{pmatrix}\mathbf{u}_m^+\\ p_m^+\end{pmatrix} + c_m^-\begin{pmatrix}\mathbf{u}_m^-\\ p_m^-\end{pmatrix}\right] + \begin{pmatrix}\tilde{\mathbf{u}}\\ \tilde{p}\end{pmatrix}\\ &= \mathbf{U}_N + \mathbf{U}_{-N} + \widetilde{\mathbf{U}} \quad \big(\widetilde{\mathbf{U}} \in \mathcal{D}_+(\Omega)\big).\end{aligned}$$

Analogously,

$$\mathbf{V} = \mathbf{V}_N + \mathbf{V}_{-N} + \widetilde{\mathbf{V}} \qquad \big(\widetilde{\mathbf{V}} \in \mathcal{D}_+(\Omega)\big).$$

By virtue of (5.9), $Q_\Omega(\mathbf{U}, \widetilde{\mathbf{V}}) = Q_\Omega(\widetilde{\mathbf{U}}, \mathbf{V}) = 0$ so that

$$\begin{aligned}Q_\Omega(\mathbf{U},\mathbf{V}) - Q_\Omega(\mathbf{U}_{-N},\mathbf{V}_N) - Q_\Omega(\mathbf{U}_N,\mathbf{V}_{-N}) - Q_\Omega(\mathbf{U}_{-N},\mathbf{V}_{-N})\\ = Q_\Omega(\mathbf{U}_N,\mathbf{V}_N).\end{aligned} \tag{5.11}$$

Arguing as in the proof of Lemmata 4.1 and 4.3 and applying Green's formula in the truncated domain Ω_R, we find that

$$\begin{aligned}\lim_{R\to\infty}\Big(Q_{\Omega_R}(\mathbf{U}_{-N},\mathbf{V}_N) + Q_{\Omega_R}(\mathbf{U}_N,\mathbf{V}_{-N})\Big) &= \langle \pi_1\mathbf{U}, \pi_0\mathbf{V}\rangle_{2N+1} - \langle \pi_0\mathbf{U}, \pi_1\mathbf{V}\rangle_{2N+1}\\ \lim_{R\to\infty} Q_{\Omega_R}(\mathbf{U}_{-N},\mathbf{V}_{-N}) &= 0.\end{aligned} \tag{5.12}$$

Thus, the left-hand side of equality (5.11) is finite. The term $Q_{\Omega_R}(\mathbf{U}_N,\mathbf{V}_N)$ is equal to the sum $\sum_{j=1}^{2N}\alpha_j R^j$ where α_j are constants. Therefore, its limit as $R\to\infty$ can be finite only if $\alpha_j = 0$ $(j = 1,\dots,2N;$ arguing as in the proof of Lemma 4.3, one can compute directly that $\alpha_j = 0)$. Thus, we have got the equality $Q_\Omega(\mathbf{U}_N,\mathbf{V}_N) = 0$ which together with (5.11) - (5.12) implies (5.10) ∎

• We call (5.10) *the generalized Green's formula.*

Lemma 5.2. *Let*

$$\mathbb{X} = \begin{pmatrix}\mathbb{B}\\ \mathbb{S}\end{pmatrix} \qquad \text{and} \qquad \mathbb{Y} = \begin{pmatrix}-\mathbb{T}\\ \mathbb{Q}\end{pmatrix} \tag{5.13}$$

where $\mathbb{B}$, $\mathbb{T}$, $\mathbb{S}$, $\mathbb{Q}$ *are* $(2N+1)\times(4N+2)$*-matrices. Suppose that* $\mathbb{X}$ *and* $\mathbb{Y}$ *satisfy the relation*

$$\mathbb{Y}^*\mathbb{X} = \mathbb{J} \equiv \begin{pmatrix}\mathbb{O} & \mathbb{I}\\ -\mathbb{I} & \mathbb{O}\end{pmatrix}. \tag{5.14}$$

Then the generalized Green's formula (5.10) *may be rewritten as*

$$\begin{aligned}(-\nu\Delta\mathbf{u} + \nabla p,\, \mathbf{v})_\Omega + (-\operatorname{div}\mathbf{u}, q)_\Omega + (\mathbf{u}, T\mathbf{V})_{\partial\Omega} + \langle \mathbb{B}\pi\mathbf{U}, \mathbb{T}\pi\mathbf{V}\rangle_{2N+1}\\ = (\mathbf{u}, -\nu\Delta\mathbf{v} + \nabla q)_\Omega + (p, -\operatorname{div}\mathbf{v})_\Omega + (T\mathbf{U}, \mathbf{v})_{\partial\Omega} + \langle \mathbb{S}\pi\mathbf{U}, \mathbb{Q}\pi\mathbf{V}\rangle_{2N+1}\end{aligned} \tag{5.15}$$

where $T\mathbf{U} = (p\mathbf{n} - \nu\partial_n\mathbf{u})|_{\partial\Omega}$.

Proof. Simple algebraic manipulations with matrices turn (5.10) into (5.15) (cf. [12: Section 6.2.2] and [16: Lemma 6.2]) ∎

Remark 5.1.

1) From (5.14) it follows that $\det \mathbb{X} \neq 0$ and $\mathbb{Y} = (\mathbb{J}\mathbb{X}^{-1})^*$. Therefore, for any $(2N+1)\times(4N+2)$-matrix $\mathbb{B}$, the rank of which is equal to $2N+1$, there exist matrices $\mathbb{S}$, $\mathbb{T}$, $\mathbb{Q}$ such that (5.13) - (5.15) are fulfilled. If $\mathbb{S}$ is also fixed and $\det \binom{\mathbb{B}}{\mathbb{S}} \neq 0$, then $\mathbb{T}$ and $\mathbb{Q}$ are uniquely defined.

2) If $\mathbb{S} = \mathbb{T}$ and $\mathbb{Q} = \mathbb{B}$, Green's formula (5.15) takes the form

$$\begin{aligned}&(-\nu\Delta\mathbf{u} + \nabla p, \mathbf{v})_\Omega + (-\operatorname{div}\mathbf{u}, q)_\Omega + (\mathbf{u}, T\mathbf{V})_{\partial\Omega} + \langle \mathbb{B}\pi\mathbf{U}, \mathbb{T}\pi\mathbf{V}\rangle_{2N+1} \\ &\quad = (\mathbf{u}, -\nu\Delta\mathbf{v} + \nabla q)_\Omega + (p, -\operatorname{div}\mathbf{v})_\Omega + (T\mathbf{U}, \mathbf{v})_{\partial\Omega} + \langle \mathbb{T}\pi\mathbf{U}, \mathbb{B}\pi\mathbf{V}\rangle_{2N+1}.\end{aligned} \tag{5.16}$$

• We call (5.16) the *symmetric generalized Green's formula.*

Based on the generalized Green's formulae (5.15) and (5.16) and arguing in the same way as in [12, 16], we provide problem (1.2) - (1.3) with the additional conditions

$$\mathbb{B}\pi\mathbf{U} = \mathbf{H} \in \mathbb{R}^{2N+1}. \tag{5.17}$$

• *We call* (5.17) *the asymptotic conditions at infinity.*

We connect problem (1.2) - (1.3), (5.17) with the mapping

$$\mathbb{D}_\pm(\Omega) \ni \mathbf{U} \longmapsto \mathbb{A}\mathbf{U} = (\mathfrak{S}_\pm\mathbf{U}, \mathbb{B}\pi\mathbf{U}) \in \mathbb{R}_\pm(\Omega;\partial\Omega) \tag{5.18}$$

where $\mathbb{R}_\pm(\Omega;\partial\Omega) = \mathcal{R}_+(\Omega;\partial\Omega) \times \mathbb{R}^{2N+1}$. It is clear that $\mathbb{A}$ inherits the Fredholm property from $\mathfrak{S}_\pm$. Furthermore, in (5.18) we observe $2N+1$ additional conditions and therefore the difference of the indices of $\mathfrak{S}_\pm$ and $\mathbb{A}$ is equal to $2N+1$, i.e. $\operatorname{Ind}\mathbb{A} = 0$. Precisely, this equality follows from

$$\operatorname{Ind}\mathbb{A} = \operatorname{Ind}\big(\mathfrak{S}_\pm\big|_{\{\mathbf{U}\in\mathbb{D}_\pm(\Omega):\,\mathbb{B}\pi\mathbf{U}=0\}}\big) = \operatorname{Ind}\mathfrak{S}_\pm - (2N+1) = 0.$$

Theorem 5.2.

1) $\ker\mathbb{A} = \{\mathbf{V} \in \ker\mathfrak{S}_\pm : \mathbb{B}\pi\mathbf{V} = 0\}$.

2) *If the generalized Green's formula* (5.15) *is valid, then*

$$\operatorname{coker}\mathbb{A} = \Big\{\big(\mathbf{V}, T\mathbf{V}\big|_{\partial\Omega}, \mathbb{T}\pi\mathbf{V}\big) : \mathbf{V} \in \ker\mathfrak{S}_\pm,\ \mathbb{Q}\pi\mathbf{V} = 0\Big\}. \tag{5.19}$$

Proof. The first assertion follows from the inclusion $\ker\mathbb{A} \subset \ker\mathfrak{S}_\pm$, the second one has been proved in [12: Proposition 6.2.5] (see also [16: Theorem 6.5]) ∎

The subspace $\dim\ker\mathfrak{S}_\pm$ contains the solution $\boldsymbol{\zeta}_0^+ = (\mathbf{0}, 1)$ and the solutions $\boldsymbol{\zeta}_m^\pm = (\boldsymbol{\xi}_m^\pm, \eta_m^\pm)$ $(m = 1,\dots,N)$ of the homogeneous problem (1.2) - (1.3) (see Lemma 4.2). Since the dimension of $\ker\mathfrak{S}_\pm$ coincides with the number of linear independent solutions we have found that $\ker\mathfrak{S}_\pm$ becomes the linear hull of them:

$$\ker\mathfrak{S}_\pm = \mathcal{L}\big\{\boldsymbol{\zeta}_0^+, \boldsymbol{\zeta}_1^+, \boldsymbol{\zeta}_1^-, \dots, \boldsymbol{\zeta}_N^+, \boldsymbol{\zeta}_N^-\big\} \equiv \big\{\boldsymbol{\zeta} = \mathfrak{Z}\mathbf{c} : \mathbf{c} \in \mathbb{R}^{2N+1}\big\} \tag{5.20}$$

where $\mathfrak{Z} = \big(\boldsymbol{\zeta}_0^+, \boldsymbol{\zeta}_1^+, \boldsymbol{\zeta}_1^-, \dots, \boldsymbol{\zeta}_N^+, \boldsymbol{\zeta}_N^-\big)$ is a $4 \times (2N+1)$-matrix-function or, what is the same, a row of solutions. Due to Lemma 4.2, each element $\boldsymbol{\zeta} \in \ker\mathfrak{S}_\pm$ can be represented in the form

$$\boldsymbol{\zeta} = \mathfrak{Z}\mathbf{c} = \mathfrak{X}\mathbf{c} - \chi\mathfrak{Y}\mathfrak{M}\mathbf{c} + \widetilde{\mathfrak{U}}\mathbf{c} \tag{5.21}$$

where the solution rows $\mathfrak{X}$ and $\mathfrak{Y}$ are defined by

$$\mathfrak{X} = \left(\begin{pmatrix} \mathbf{u}_0^+ \\ p_0^+ \end{pmatrix}, \begin{pmatrix} \mathbf{u}_1^+ \\ p_1^+ \end{pmatrix}, \begin{pmatrix} \mathbf{u}_1^- \\ p_1^- \end{pmatrix}, \dots, \begin{pmatrix} \mathbf{u}_N^+ \\ p_N^+ \end{pmatrix}, \begin{pmatrix} \mathbf{u}_N^- \\ p_N^- \end{pmatrix} \right)$$

$$\mathfrak{Y} = \left(\begin{pmatrix} \mathbf{u}_0^- \\ p_0^- \end{pmatrix}, \begin{pmatrix} \mathbf{u}_{-1}^+ \\ p_{-1}^+ \end{pmatrix}, \begin{pmatrix} \mathbf{u}_{-1}^- \\ p_{-1}^- \end{pmatrix}, \dots, \begin{pmatrix} \mathbf{u}_{-N}^+ \\ p_{-N}^+ \end{pmatrix}, \begin{pmatrix} \mathbf{u}_{-N}^- \\ p_{-N}^- \end{pmatrix} \right),$$

$\mathfrak{M}$ is a constant $(2N+1) \times (2N+1)$-matrix and $\widetilde{\mathfrak{U}} \in \mathcal{D}_+(\Omega)^{2N+1}$. Note that

$$\left. \begin{aligned} \pi_0 \mathfrak{Z}\mathbf{c} &= \mathbf{c} \\ \pi_1 \mathfrak{Z}\mathbf{c} &= -\mathfrak{M}\mathbf{c} \end{aligned} \right\}. \tag{5.22}$$

• We call the matrix $\mathfrak{M}$ the *augmented flow polarization matrix.*

Theorem 5.3. *$\mathfrak{M}$ is a symmetric matrix.*

Proof. Let $\mathbf{c}, \mathbf{C}$ be arbitrary constant vectors in $\mathbb{R}^{2N+1}$. Since $\mathfrak{Z}\mathbf{c}$ and $\mathfrak{Z}\mathbf{C}$ are solutions of the homogeneous problem (1.2) - (1.3) we get $Q_\Omega(\mathfrak{Z}\mathbf{c}; \mathfrak{Z}\mathbf{C}) = 0$. On the other hand, from the generalized Green's formula (5.10) there follows that

$$\begin{aligned} Q_\Omega(\mathfrak{Z}\mathbf{c}; \mathfrak{Z}\mathbf{C}) &= \langle \pi_0 \mathfrak{Z}\mathbf{c}, \pi_1 \mathfrak{Z}\mathbf{C} \rangle_{2N+1} - \langle \pi_1 \mathfrak{Z}\mathbf{c}, \pi_0 \mathfrak{Z}\mathbf{C} \rangle_{2N+1} \\ &= \langle \mathfrak{M}\mathbf{c}, \mathbf{C} \rangle_{2N+1} - \langle \mathbf{c}, \mathfrak{M}\mathbf{C} \rangle_{2N+1} \\ &= \langle \mathbf{c}, (\mathfrak{M}^* - \mathfrak{M})\mathbf{C} \rangle_{2N+1} \\ &= 0. \end{aligned}$$

Thus, $\mathfrak{M} = \mathfrak{M}^*$ ∎

Remark 5.2. The matrix $\mathfrak{M}$ has the form $\mathfrak{M} = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0}^T & \mathbb{M} \end{pmatrix}$ where $\mathbf{0} = (0, \dots, 0)$ and $\mathbb{M}$ is a symmetric $2N \times 2N$-matrix. This follows from the fact that the solution $\boldsymbol{\zeta}_0^+$ has the form $\boldsymbol{\zeta}_0^+ = (\mathbf{0}, 1)^T$ and from the symmetry of $\mathfrak{M}$.

• We call the matrix $\mathbb{M}$ the *flow polarization matrix.*

Theorem 5.4. *Let $\mathfrak{B} = \mathbb{B}\,(-\mathfrak{M}, \mathbb{I})^T$ where $\mathbb{I}$ is the unit $(2N+1) \times (2N+1)$-matrix. Then*

$$\dim \ker \mathbb{A} = 2N + 1 - \operatorname{rank} \mathfrak{B}. \tag{5.23}$$

Proof. The elements $\boldsymbol{\zeta} \in \ker \mathfrak{S}_\pm$ admit the representation $\boldsymbol{\zeta} = \mathfrak{Z}\mathbf{c}$ ($\mathbf{c} \in \mathbb{R}^{2N+1}$; see (5.21)). Since $\pi_1 \boldsymbol{\zeta} = \mathbf{c}$, $\pi_0 \boldsymbol{\zeta} = -\mathfrak{M}\mathbf{c}$ and due to the symmetry of $\mathfrak{M}$, $\mathbb{B}\pi\boldsymbol{\zeta} = 0$ if and only if $\mathbb{B}\,(-\mathfrak{M}, \mathbb{I})^T \mathbf{c} = 0$. Therefore, owing to Theorem 5.2/(1) we conclude (5.23) ∎

Remark 5.3. In view of (5.19) the compatibility conditions for problem (1.2) - (1.3), (5.17) take the form

$$(\mathbf{f}, \mathbf{v})_\Omega + (g,\, q)_\Omega + (\mathbf{h}, T\mathbf{U})_{\partial\Omega} + \langle \mathbf{H}, \mathbb{T}\pi\mathbf{V} \rangle_{2N+1} = 0 \tag{5.24}$$

for all $\mathbf{V} = (\mathbf{v}, q) \in \ker \mathfrak{S}_\pm$ with $\mathbb{Q}\pi\mathbf{V} = 0$.

In accordance with (5.19), (5.24) it is very natural to say that problems (1.2) - (1.3), (5.17) and (1.2) - (1.3) with additional conditions

$$\mathbb{Q}\pi\mathbf{V} = \mathbf{K} \in \mathbb{R}^{2N+1} \tag{5.25}$$

are *adjoint with respect to the generalized Green's formula* (5.15). In the case when the symmetric generalized Green's formula (5.16) takes place, problem (1.2) - (1.3), (5.17) becomes *formally self-adjoint.*

Theorem 5.5.

1) *If* $\Omega = \Pi$, *then* $\mathbb{M} = \mathbb{O}$.

2) *If* $\Omega \neq \Pi$ *and* $\Omega \subset \Pi$, *then the matrix* $\mathbb{M}$ *is positive definite.*

Proof. Let $\mathbf{c} = (0, \mathbf{c}')$ with $\mathbf{c}' \in \mathbb{R}^{2N} \setminus \{0\}$ be arbitrary. We take

$$\mathbf{V} = (\mathbf{v}, q) = \mathfrak{Z}\mathbf{c} = \mathbf{V}^0 + \mathbf{V}^\# \in \ker \mathfrak{S}_\pm$$

where

$$\begin{aligned}\mathbf{V}^0 &= (\mathbf{v}^0, q^0) = \mathfrak{X}\mathbf{c} \\ \mathbf{V}^\# &= (\mathbf{v}^\#, q^\#) = -\chi\mathfrak{Y}\mathfrak{M}\mathbf{c} + \tilde{\mathfrak{U}}\mathbf{c} \in \mathcal{D}^l_\gamma(\Omega) \quad (\gamma \in (-1, 0))\end{aligned}$$

(see (5.21) and Lemma 4.2). By formula (4.6) and the definition of $\mathbb{M}$ we get

$$\langle \mathbb{M}\mathbf{c}', \mathbf{c}' \rangle_{2N} = \int_{\partial\Omega} \mathbf{v}^\# \cdot T(\mathbf{V})\, ds. \tag{5.26}$$

(Note that $-\nu\Delta\mathbf{v}^\# + \nabla q^\# = 0$ and $\operatorname{div} \mathbf{v}^\# = 0$.) If $\Omega = \Pi$, then $\mathbf{V}^0$ is the exact solution of the homogeneous problem (1.2) - (1.3). Hence $\mathbf{V}^\# = 0$ and $\mathbb{M} = \mathbb{O}$.

Since $\mathbf{v}^\# = -\mathbf{v}^0$ on $\partial\Omega$,

$$\int_{\partial\Omega} \mathbf{v}^\# \cdot T(\mathbf{V})\, ds = \int_{\partial\Omega} \mathbf{v}^\# \cdot T(\mathbf{V}^\#)\, ds - \int_{\partial\Omega} \mathbf{v}^0 \cdot T(\mathbf{V}^0)\, ds. \tag{5.27}$$

Integrating by parts in Ω and $\Pi \setminus \Omega$, we derive

$$\begin{aligned}\int_{\partial\Omega} \mathbf{v}^\# \cdot T(\mathbf{V}^\#)\, ds &= \int_\Omega |\nabla \mathbf{v}^\#|^2 dx \\ \int_{\partial\Omega} \mathbf{v}^0 \cdot T(\mathbf{V}^0)\, ds &= -\int_{\Pi\setminus\Omega} |\nabla \mathbf{v}^0|^2 dx.\end{aligned} \tag{5.28}$$

The sign " $-$ " in the second equality of (5.28) appears because of the oposite direction of the outward normal $\mathbf{n}$. The Dirichlet integral of $\mathbf{v}^\#$ is finite since $\mathbf{V}^\# \in \mathcal{D}^l_\gamma(\Omega)$ for $\gamma \in (-1, 0)$. The formula

$$\langle \mathbb{M}\mathbf{c}', \mathbf{c}' \rangle_{2N} = \int_\Omega |\nabla \mathbf{v}^\#|^2 dx + \int_{\Pi\setminus\Omega} |\nabla \mathbf{v}^0|^2 dx > 0$$

follows from (5.26) - (5.28) and completes the proof ∎

Example 5.1. Let $N = 0$ and $\mathbb{B} = (1, 0)$ is a matrix of size 1×2. Then the condition $\mathbb{B}\pi\mathbf{U} = \pi_1\mathbf{U} = c_0^-$ prescribes the total flux of the fluid over the surface S_R. The matrix $\mathfrak{Z}$ consists of one solution ζ_0^+. Hence $\dim \ker \mathfrak{S}_\pm = 1$, $\pi_1\mathfrak{Z}\mathbf{c} = 0$ for all $\mathbf{c}$ and $\mathfrak{M} = \mathbb{O}$ (see (5.22)). We have $\mathfrak{B} = \mathbb{B}(-\mathfrak{M}, \mathbb{I})^T = \mathbb{O}$ and, by Theorem 5.4, $\dim \ker \mathbb{A} = 1 - \operatorname{rank} \mathfrak{B} = 1$. Therefore the operator $\mathbb{A}$ is an epimorphism with one-dimentional kernel (constant pressure).

If $\mathbb{B} = (0, 1)$, then $\mathbb{B}\pi\mathbf{U} = \pi_0\mathbf{U} = c_0^+$ prescribes the limit of the pressure component as $r \to \infty$. We get $\pi_0\mathfrak{Z}\mathbf{c} = 1$, $\mathbb{M} = \mathbb{I}$ and $\mathfrak{B} = \mathbb{B}(-\mathfrak{M}, \mathbb{I})^T = \mathbb{I}$. By Theorem 5.4, $\dim\ker\mathbb{A} = 1 - \operatorname{rank}\mathfrak{B} = 0$ and the operator $\mathbb{A}$ is an isomorphism.

Example 5.2. Let $N = 1$ and

$$\mathbb{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix}.$$

We consider the condition $\mathbb{B}\pi\mathbf{U} = (H_1, H_2, 0)^T$ which prescribes the total flux H_1 over S_R and the linear flux H_2 of $\mathbf{u}$ in the direction $\mathbf{e}^\alpha = (\cos\alpha, \sin\alpha)$ (cf. [14]). We obtain $\mathfrak{Z} = \{\zeta_0^+, \zeta_1^+, \zeta_1^-\}$, $\dim\ker\mathfrak{S}_\pm = 3$ and

$$\mathfrak{B} = \mathbb{B}\,(-\mathfrak{M}, \mathbb{I})^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}.$$

Hence $\dim\ker\mathbb{A} = 3 - \operatorname{rank}\mathfrak{B} = 1$ and the operator $\mathbb{A}$ is an epimorphism.

If we prescribe instead of the total flux the limit H_1 of the pressure component as $r \to \infty$, we shall take

$$\mathbb{B} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

and consider the condition $\mathbb{B}\pi\mathbf{U} = (H_1, H_2, 0)^T$. In this case we get the unitary matrix

$$\mathfrak{B} = \mathbb{B}\,(-\mathfrak{M}, \mathbb{I})^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix},$$

$\dim\ker\mathbb{A} = 3 - \operatorname{rank}\mathfrak{B} = 0$ and the operator $\mathbb{A}$ is an isomorphism.

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Received 10.03.2000