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A Discretised Nonlinear Eigenvalue Problem with Many Spurious Branches of Solutions

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Abstract. We treat an example of a nonlinear eigenvalue problem in $L^2(0,1)$ which can be solved explicitly. It has a single branch of non-trivial solutions. Discretisation reduces the problem to a finite-dimensional one having many branches of non-trivial solutions. We investigate the convergence of these approximate solutions.

Keywords: *Bifurcation, essential spectrum, approximation*

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1. Introduction

In this paper we consider an example in the theory of equations of the type

$$Su - N(u) = \lambda u \qquad \text{for } u \in H \setminus \{0\} \tag{1}$$

where $S : H \to H$ is a positive self-adjoint operator acting on a real Hilbert space H and $N : H \to H$ is a nonlinear operator with $N(0) = 0$. Our goal is to expose, through a particular example of (1), a problem which can occur when we consider finite-dimensional approximations of (1). In fact this example shows us that the approximation has many branches of solutions which have no relevence for the initial problem (1). We show moreover why these branches of solutions do not converge to solutions of the initial problem.

In Section 2 we introduce our basic example of equation (1). Then, in Section 3, we make a discretisation to reduce the initial problem to a finite-dimensional problem. In Sections 4 and 5 we show that only one branch of solutions of the approximate problem converges to a branch of solutions of the initial problem and that the solutions on the other branches do not converge to solutions of the initial problem. Finally, in Section 6, we consider the same problem as above in relation to the spectrum of S.

General results concerning the existence and bifurcation of solutions for equation (1) are given in [1 - 4] and variants of the example discussed below are used to illustrate the sharpness of the conclusions in [1, 4]. Our observations concerning discrete approximations apply to such variants too but we feel it is sufficient to exhibit them in the simplest context. More general situations are treated in [5].

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2. Example

Let H be the real Hilbert space $L^2(0,1)$ with the usual scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. A bounded, positive, self-adjoint operator, $S: H \to H$, is defined by

$$Su = fu \qquad \text{for } u \in H \tag{2}$$

where $f:[0,1]\to\mathbb{R}$ is the function defined by $f(x)=x^2$. If $\sigma(S)$ denotes the spectrum of S (and $\rho(S)=\mathbb{R}\setminus\sigma(S)$ denotes the resolvent set of S), it is easy to check that $\sigma(S)=\overline{\operatorname{Im} f}=[0,1]$ and that S has no eigenvalues. For $\sigma>0$, we define $N: H\to H$ by

$$N(u)(x) = |I(u)|^\sigma I(u) \qquad \text{for } x\in[0,1] \tag{3}$$

where $I(u)=\int_0^1 u(x)\,dx$. Since $|I(u)|\le\|u\|$, it follows that $N(u)\in H$. With the operators defined by (2) and (3), equation (1) becomes

$$x^2u(x) - |I(u)|^\sigma I(u) = \lambda u(x) \qquad \text{a.e. on } [0,1] \text{ for } (\lambda,u)\in\mathbb{R}\times[H\setminus\{0\}] \tag{4}$$

and as we now show, it can be solved explicitly.

Since $I(u)$ is constant on $[0,1]$, any solution of (4) must have the form

$$u(x) = cd_\lambda(x) \qquad \text{a.e. on } [0,1] \tag{5}$$

where $c\neq 0$ is a constant and $d_\lambda(x)=(x^2-\lambda)^{-1}$. Furthermore, a function satisfying (5) belongs to H if and only if $\lambda\in\rho(S)$. Returning to (4), we find that c must be chosen so that

$$1 = |c|^\sigma |I(d_\lambda)|^\sigma I(d_\lambda) \tag{6}$$

which means that $\lambda\in\rho(S)$ must be chosen so that $I(d_\lambda)>0$. In this case, it is clear that

$$\{\lambda\in\rho(S):\, I(d_\lambda)>0\} = (-\infty,0).$$

Let

$$\mathcal{C} = \{(\lambda,\pm u_\lambda):\, \lambda\in(-\infty,0)\}$$

where $u_\lambda(x) = I(d_\lambda)^{-(\sigma+1)/\sigma} d_\lambda(x)$ a.e. on $[0,1]$. It follows from (6) that

$$\{(\lambda,u)\in\mathbb{R}\times H\setminus\{0\}:\, (4) \text{ is satisfied}\} = \mathbb{C}. \tag{7}$$

From (7) we see that the only possible bifurcation point for (4) is 0. To determine whether or not 0 is a bifurcation point we must study $\lim\|u_\lambda\|$ as $\lambda\to 0_-$. We have

$$\begin{aligned}\|u_\lambda\| &= \left\{\int_0^1 \frac{1}{x^2-\lambda}\,dx\right\}^{-(\sigma+1)/\sigma}\left\{\int_0^1 \frac{1}{(x^2-\lambda)^2}\,dx\right\}^{\frac12}\\ &= \left\{\frac{|\lambda|^{1/\sigma}}{2(1+|\lambda|)\arctan(1/\sqrt{|\lambda|})^{2(\sigma+1)/\sigma}} + \frac{|\lambda|^{(2-\sigma)/2\sigma}}{2\arctan(1/\sqrt{|\lambda|})^{2(\sigma+1)/\sigma-1}}\right\}^{\frac12}\end{aligned} \tag{8}$$

from which it follows that

$$\lim_{\lambda \to 0_-} \|u_\lambda\| = 0 \quad \Longleftrightarrow \quad \sigma < 2. \tag{9}$$

Thus 0 is a bifurcation point of (4) if and only if $\sigma < 2$. The different bifurcation diagrams for equation (4) with different values of σ are shown in Figure 1.

Figure 1: $\|u_\lambda\|$ with different values of σ

3. Approximation of the example

Let $n \in \mathbb{N} = \{1, 2, \ldots\}$. We define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} (\frac{k}{n})^2 & \text{if } \frac{k}{n} \le x < \frac{k+1}{n} \ (k = 0, 1, \ldots, n-1) \\ (\frac{n-1}{n})^2 & \text{if } x = 1. \end{cases}$$

Next we define a subspace of H by

$$Y_n = \left\{ \varphi \in H : \varphi \text{ is constant a.e. on } \left(\frac{k}{n}, \frac{k+1}{n} \right) \text{ for } k = 0, 1, \ldots, n-1 \right\}.$$

Clearly, $\dim Y_n = n$. We observe that $f_n \in Y_n$ and that $f_n \to f$ uniformly on $[0,1]$ as $n \to \infty$.

Now we can define the approximate problem in dimension n, or n-approximate problem, by

$$S_n u - N(u) = \mu u \qquad \text{with } (\mu, u) \in \mathbb{R} \times [Y_n \setminus \{0\}] \tag{10}$$

where $S_n : Y_n \to Y_n$ is the positive, self-adjoint operator defined by

$$S_n \varphi = f_n \varphi \qquad \text{for } \varphi \in Y_n. \tag{11}$$

Clearly, we have

$$\sigma(S_n) = \overline{\operatorname{Im} f_n} = \left\{ \left(\frac{k}{n}\right)^2 : 0 \le k \le n-1 \right\},$$

each point $(\frac{k}{n})^2$ is a simple eigenvalue of S_n and the bifurcation points of (10) are all the elements of $\sigma(S_n)$. With the operator defined by (11), equation (10) becomes

$$f_n(x)\varphi(x) - |I(\varphi)|^\sigma I(\varphi) = \mu \varphi(x) \quad \text{ a.e. on } [0,1] \text{ for } (\mu, \varphi) \in \mathbb{R} \times [Y_n \setminus \{0\}]. \tag{12}$$

Once again it is possible to solve this equation explicitly. Following the same way as above, we find that μ must be chosen so that $\mu \in \rho(S_n)$ and $I([f_n - \mu]^{-1}) > 0$. We note that

$$I([f_n - \mu]^{-1}) = \frac{1}{n} \sum_{j=0}^{n-1} \left[\left(\frac{j}{n}\right)^2 - \mu \right]^{-1}$$

and so one can check that, for each given $k = 1, \ldots, n-1$ and for $\mu \in \big((\frac{k-1}{n})^2, (\frac{k}{n})^2\big)$, $I([f_n - \mu]^{-1})$ increases from the limit $-\infty$ (as $\mu \to (\frac{k-1}{n})^2_+$) to $+\infty$ (as $\mu \to (\frac{k}{n})^2_-$). Therefore, for each $k = 1, \ldots, n-1$, there exists $\xi_k^n \in](\frac{k-1}{n})^2, (\frac{k}{n})^2[$ such that, for $\mu \in](\frac{k-1}{n})^2, (\frac{k}{n})^2[$, we have

$$I([f_n - \mu]^{-1}) \begin{cases} > 0 & \text{if } \mu > \xi_k^n \\ = 0 & \text{if } \mu = \xi_k^n \\ < 0 & \text{if } \mu < \xi_k^n. \end{cases}$$

Then we have

$$\big\{\mu \in \rho(S_n) : I([f_n - \mu]^{-1}) > 0\big\} = (-\infty, 0) \bigcup \cup_{k=1}^{n-1} \big(\xi_k^n, (\tfrac{k}{n})^2\big).$$

Let $J_0^n = (-\infty, 0)$ and $J_k^n = \big(\xi_k^n, (\frac{k}{n})^2\big)$ for $k = 1, 2, \ldots, n-1$. Then, for $\mu \in \cup_{k=0}^{n-1} J_k^n$, we set

$$u_\mu^n(x) = I([f_n - \mu]^{-1})^{-(\sigma+1)/\sigma} (f_n(x) - \mu)^{-1} \tag{13}$$

for $x \in [0,1]$. Then, for $k = 0, 1, \ldots, n-1$,

$$\mathcal{C}_k^n = \big\{(\mu, \pm u_\mu^n) : \mu \in J_k^n\big\}$$

is the branch of solutions of the n-approximate problem (10) which bifurcates from the point $\big((\frac{k}{n})^2, 0\big)$ in $\mathbb{R} \times Y_n$. Furthermore,

$$\bigcup_{k=0}^{n-1} \mathcal{C}_k^n = \Big\{(\mu, \varphi) \in \mathbb{R} \times [Y_n \setminus \{0\}] : \text{(12) is satisfied}\Big\}. \tag{14}$$

Note that $\mu \to \|u^n_\mu\|$ is continuous on J^n_k for $k = 0, 1, \ldots, n-1$. In fact,

$$\|u^n_\mu\| = \frac{\left(\frac{1}{n}\sum_{j=0}^{n-1}\left[(\frac{j}{n})^2 - \mu\right]^{-2}\right)^{1/2}}{\left(\frac{1}{n}\sum_{j=0}^{n-1}\left[(\frac{j}{n})^2 - \mu\right]^{-1}\right)^{(\sigma+1)/\sigma}}.$$

As μ approches ξ^n_k from above, it is clear that $\|u^n_\mu\| \to \infty$ since the denominator tends to zero by the definition of ξ^n_k, whereas the numerator remains bounded away from zero.

As μ approches $(\frac{k}{n})^2$ from below, we claim that $\|u^n_\mu\| \to 0$. To see this, let $t = (\frac{k}{n})^2 - \mu$ and observe that

$$\|u^n_\mu\| = \frac{\left(\frac{1}{n}t^{-2} + \alpha(\mu)\right)^{1/2}}{\left(\frac{1}{n}t^{-1} + \beta(\mu)\right)^{(\sigma+1)/\sigma}}$$

where

$$\alpha(\mu) = \frac{1}{n}\sum_{j=0, j\neq k}^{n-1}\left[\left(\frac{j}{n}\right)^2 - \mu\right]^{-2} \qquad \text{and} \qquad \beta(\mu) = \frac{1}{n}\sum_{j=0, j\neq k}^{n-1}\left[\left(\frac{j}{n}\right)^2 - \mu\right]^{-1}.$$

Thus

$$\|u^n_\mu\| = \frac{\left(\frac{1}{n}t^{2/\sigma} + t^{2(\sigma+1)/\sigma}\alpha(\mu)\right)^{1/2}}{\left(\frac{1}{n} + t\beta(\mu)\right)^{(\sigma+1)/\sigma}}$$

where $t^{2(\sigma+1)/\sigma}\alpha(\mu) \to 0$ and $t\beta(\mu) \to 0$ as $t \to 0_+$. It follows that $\|u^n_\mu\| \to 0$ as μ approches $(\frac{k}{n})^2$ from below.

Remark. If we solve equation (12) in the set H instead of Y_n we verify that the solutions belong to Y_n and are exactly the same as those obtained by solving the n-approximate problem and the bifurcation points are also the same. Thus we can directly consider the n-approximate problem in H.The bifurcation diagram of the n-approximate problem is drawn in Figure 2.

Figure 2: Bifurcation diagram of $S_n u - N(u) = \mu u$

Observing Figures 1 and 2 we want to show the two following results when $n \to +\infty$:

1. The branches $\mathcal{C}_0^n$ bifurcating from $\mu = 0$ converge to the branch $\mathcal{C}$ of exact solutions of (4) as $n \to \infty$. However, $\mathcal{C}$ bifurcates from 0 if and only if $0 < \sigma < 2$.
2. Solutions on the other branches do not converge.

4. The first result

We begin by studying the behaviour of the branches $\mathcal{C}_0^n$ as $n \to \infty$. We fix $\mu \in (-\infty, 0)$. By (14) we know that u_μ^n is a solution of (12) for all $n \geq 1$. We check that, in the notation of (7),

$$\lim_{n\to\infty} u_\mu^n(x) = I(d_\mu)^{-(\sigma+1)/\sigma} d_\mu(x) = u_\mu(x) \qquad \text{a.e. on } [0,1] \tag{15}$$

and the convergence is uniform on $[0,1]$ since $(f_n - \mu)^{-1} \to d_\mu$ uniformly on $[0,1]$ and $I([f_n - \mu]^{-1}) \to I(d_\mu)$. Thus $(\mu, u_\mu^n) \in \mathcal{C}_0^n$ converges to the solution $(\mu, u_\mu) \in \mathcal{C}$ of (4) for all $\mu \in (-\infty, 0)$. Now we have already shown that

$$\lim_{\mu\to 0_-} \|u_\mu\| = 0 \quad \Longleftrightarrow \quad \sigma < 2.$$

Figure 3: $\mathcal{C}_0^n$ with $n = 5, 10, 15$ and $\sigma = \frac{1}{5}$ (a), 1 (b), 2 (c) 6 (d)

Note however that, for each $n \in \mathbb{N}$, $\|u_\mu^n\| \to 0$ as $\mu \to 0_-$ for all $\sigma > 0$. Figure 3 shows the form of $\mathcal{C}_0^n$ for $n = 5, 10, 15$ and $\sigma = \frac{1}{5}, 1, 2, 6$ (in fact we represent in these figures $\{(\mu, \|u_\mu^n\|) : (\mu, \pm u_\mu^n) \in \mathcal{C}_0^n\}$ and the thick curve represents $\{(\mu, \|u_\mu\|) : (\mu, \pm u_\mu) \in \mathcal{C}\}$ for the respective σ).

5. The second result

The second phenomenon we want to understand is why the other branches of solutions for the n-approximate problem do not converge to a branch of solutions for the initial problem as $n \to \infty$. To do that we consider the sequence of the $2n$-approximate problems since the point $\frac{1}{4} = (\frac{n}{2n})^2$ is the bifurcation point for all the branches $\mathcal{C}_n^{2n}$ and we consider the limit when $n \to \infty$. It is clear that the following reasoning can be applied for any bifurcation point except 0 of an approximate problem of any dimension n.

For all $R > 0$ we can define a sequence

$$(\lambda_n, v_n) \in \mathcal{C}_n^{2n} \qquad \text{with } \|v_n\| = R. \tag{16}$$

Thus we want to show that the limit of the branches of solutions does not exist when $n \to \infty$. To do this we show that $(v_n)_{n\geq 1}$ converges weakly to 0 when $n \to \infty$. By (13) we have

$$v_n(x) = \left(\frac{1}{2n} \sum_{k=0}^{2n-1} \left[\left(\frac{k}{2n}\right)^2 - \lambda_n \right]^{-1} \right)^{-(\sigma+1)/\sigma} \big(f_{2n}(x) - \lambda_n\big)^{-1} \quad \text{a.e. on } [0,1]. \tag{17}$$

Lemma 5.1. *For any $\delta \in (0, \frac{1}{2})$ and any $R > 0$, the sequence $\{(\lambda_n, v_n)\}$ defined by (16) has the properties that $\lambda_n \to \frac{1}{4}$ and $v_n(x) \to 0$ uniformly on $[0, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, 1]$.*

Proof. Let

$$a_n = \frac{1}{2n} \sum_{k=0}^{2n-1} \left[\left(\frac{k}{2n}\right)^2 - \lambda_n \right]^{-2} \qquad \text{and} \qquad b_n = \frac{1}{2n} \sum_{k=0}^{2n-1} \left[\left(\frac{k}{2n}\right)^2 - \lambda_n \right]^{-1}.$$

Since $\lambda_n \in J_n^{2n}$ we know that $b_n = I([f_{2n} - \lambda_n]^{-1}) > 0$ and that $\lambda_n \in \big((\frac{n-1}{2n})^2, \frac{1}{4}\big)$. Hence $\lambda_n \to \frac{1}{4}$. Furthermore, from (17) it follows that $\|v_n\| = a_n^{1/2} / b_n^{(\sigma+1)/\sigma}$ and so

$$a_n^{1/2} = R\, b_n^{(\sigma+1)/\sigma} \qquad \text{for all } n \in \mathbb{N}. \tag{18}$$

Let us show that $a_n \to \infty$ as $n \to \infty$. In fact, for $0 \leq k \leq n-1$,

$$\left[\left(\frac{k}{2n}\right)^2 - \lambda_n \right]^2 \leq \left(\frac{k}{2n} - \sqrt{\lambda_n} \right)^2 \left(\frac{k}{2n} + \sqrt{\lambda_n} \right)^2 \leq \left(\sqrt{\lambda_n} - \frac{k}{2n} \right)^2$$

since $\frac{n-1}{2n} < \sqrt{\lambda_n} < \frac{1}{2}$. But $\left(\sqrt{\lambda_n} - \frac{k}{2n}\right)^2 \leq \left(\frac{1}{2} - \frac{k}{2n}\right)\frac{1}{2}$ for $0 \leq k \leq n-1$ and so

$$a_n \geq \frac{1}{2n} \sum_{k=0}^{n-1} \frac{4}{1 - \frac{k}{n}} = 2 \sum_{k=0}^{n-1} \frac{1}{n-k} = 2 \sum_{i=1}^{n} \frac{1}{i} \to \infty \qquad \text{as } n \to \infty.$$

From (18) we see that $b_n^{(\sigma+1)/\sigma} \to \infty$ as $n \to \infty$. For $\delta \in (0, \frac{1}{2})$ there exists $n_\delta > 0$ such that $\sqrt{\lambda_n} \in [\frac{1}{2} - \frac{\delta}{2}, \frac{1}{2})$ for all $n \geq n_\delta$ and so

$$\begin{aligned} |f(x) - \lambda_n| &= |x^2 - \lambda_n| \\ &= |x - \sqrt{\lambda_n}|(x + \sqrt{\lambda_n}) \\ &\geq \frac{\delta}{2}(x + \sqrt{\lambda_n}) \\ &\geq \frac{\delta}{2}\Big(\frac{1}{2} - \frac{\delta}{2}\Big) \\ &= \frac{\delta(1-\delta)}{4} \end{aligned}$$

for all $x \in [0, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, 1]$.

Furthermore, since $f_{2n} \to f$ uniformly on $[0,1]$, there exists $N(\delta) \geq n_\delta$ such that

$$|f_{2n}(x) - f(x)| \leq \frac{\delta(1-\delta)}{8} \qquad \text{for all } x \in [0,1] \text{ and all } n \geq N(\delta).$$

Hence for $n \geq N(\delta)$ and $x \in [0, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, 1]$,

$$\begin{aligned} \frac{\delta(1-\delta)}{4} &\leq |f(x) - \lambda_n| \\ &\leq |f(x) - f_{2n}(x)| + |f_{2n}(x) - \lambda_n| \\ &\leq \frac{\delta(1-\delta)}{8} + |f_{2n}(x) - \lambda_n| \end{aligned}$$

and consequently

$$|f_{2n}(x) - \lambda_n| \geq \frac{\delta(1-\delta)}{8}.$$

Finally, for $n \geq N(\delta)$ and $x \in [0, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, 1]$, we have shown that

$$|v_n(x)| \leq b_n^{-(\sigma+1)/\sigma} \frac{8}{\delta(1-\delta)} = \frac{R}{\sqrt{a_n}} \frac{8}{\delta(1-\delta)}$$

where $a_n \to \infty$ as $n \to \infty$. Thus $v_n \to 0$ uniformly on $[0, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, 1]$ ∎

We can now show that $(v_n)_{n \geq 1}$ converges weakly to 0 which implies that $(v_n)_{n\geq 1}$ does not converge in H.

Lemma 5.2. *For any $R > 0$, the sequence $(v_n)_{n\geq 1}$ defined by* (16) *converges weakly to* 0 *in H but contains no strongly convergent subsequence.*

Proof. Fix $\varphi \in H$ and $\epsilon > 0$. To show that $v_n \rightharpoonup 0$ weakly in H it is enough to show

$$\limsup_{n\to\infty} |\langle v_n, \varphi \rangle| \leq \epsilon.$$

For this we choose $\delta \in (0, \frac{1}{2})$ small enough so that

$$\int_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} \varphi^2(x)\,dx \le \Big(\frac{\epsilon}{R}\Big)^2.$$

Then

$$\begin{aligned}
|\langle v_n, \varphi\rangle| &\le \int_{|x-\frac{1}{2}|\le\delta} |v_n(x)|\,|\varphi(x)|\,dx + \max_{|x-\frac{1}{2}|\ge\delta} |v_n(x)| \int_{|x-\frac{1}{2}|\ge\delta} |\varphi(x)|\,dx \\
&\le \Big\{\int_{|x-\frac{1}{2}|\le\delta} v_n^2 dx\Big\}^{\frac{1}{2}} \Big\{\int_{|x-\frac{1}{2}|\le\delta} \varphi^2 dx\Big\}^{\frac{1}{2}} + \max_{|x-\frac{1}{2}|\ge\delta} |v_n(x)| \int_0^1 |\varphi|\,dx \\
&\le R\,\frac{\epsilon}{R} + \|\varphi\| \max_{|x-\frac{1}{2}|\ge\delta} |v_n(x)|
\end{aligned}$$

and Lemma 5.1 shows that $\limsup_{n\to\infty} |\langle v_n, \varphi\rangle| \le \epsilon$ as required.

If $\{v_{n_i}\}$ is a subsequence of $\{v_n\}$ such that $\|v_{n_i} - v\| \to 0$ as $i \to \infty$ for some $v \in H$, then $v_{n_i} \rightharpoonup v$ as $i \to \infty$ and so $v = 0$, showing that $\|v_{n_i}\| \to 0$ as $i \to \infty$. But $\|v_n\| = R > 0$ for all $n \in \mathbb{N}$ and so there is no such subsequence ∎

Remark. With an analogous reasoning it is possible to extend these results to a function f which is more general.

6. The linear problem

In this section we consider the linearization of (1)

$$Su = \lambda u \qquad \text{with } (\lambda, u) \in \mathbb{R} \times [H \setminus \{0\}]. \tag{19}$$

Since S has no eigenvalues it is clear that this equation has no solution (λ, u) with $u \neq 0$. We also consider the n-approximate problem of (19)

$$S_n u = \mu u \qquad \text{with } (\mu, u) \in \mathbb{R} \times [Y_n \setminus \{0\}] \tag{20}$$

where S_n and Y_n are defined as above. In this case S_n has the n eigenvalues

$$\Big\{\mu_k^n = \Big(\frac{k}{n}\Big)^2 : 0 \le k \le n-1\Big\}.$$

Each eigenvalue is simple and the eigenspace of μ_k^n is

$$E_k^n = \operatorname{span}\{X_k^n\}$$

where X_k^n is the characteristic function of $[\frac{k}{n}, \frac{k+1}{n})$. The bifurcation diagram is then represented in Figure 4.

Figure 4: Bifurcation diagram of $S_n u = \mu u$

Let us consider what happens to the branches of solutions as $n \to \infty$. To do that we consider once again the sequence of the $2n$-approximate problems. As before, $\frac{1}{4} = \mu_n^{2n}$ is an eigenvalue for all these problems. Once again we fix $R > 0$ and we define a sequence $(v_n)_{n\geq 1}$ where $v_n \in E_n^{2n}$ is such that $\|v_n\| = R$. It is easy to show that this sequence $(v_n)_{n\geq 1}$ converges weakly to 0 and thus does not contain a subsequence converging strongly in H. Finally, let us observe that for the linear problem this reasoning is also valid for the point 0.

References

[1] Heinz, H.-P. and C. A. Stuart: *Solvability of nonlinear equations in spectral gaps of the linearization.* Nonlin. Anal.: Theory, Methods & Appl. 19 (1992), 145 – 165.

[2] Heinz, H.-P., Küpper, T. and C. A. Stuart: *Existence and Bifurcation of solutions for nonlinear perturbations of the periodic Schrödinger equation.* J. Diff. Equ. 100 (1992), 341 – 354.

[3] Küpper, T. and C. A. Stuart: *Bifurcation into gaps in the essential spectrum.* J. reine & angew. Math. 409 (1990), 1 – 34.

[4] Küpper, T. and C. A. Stuart: *Necessary and sufficient conditions for gap-bifurcation.* Nonlin. Anal.: Theory, Methods & Appl. 18 (1992), 893 – 903.

[5] Vuillaume, G.: *Bifurcation du spectre essentiel: Théorie et exemples.* Diploma's work. Lausanne: Ecole Polytechnique Fédérale 1999 - 2000.

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