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Conditional Stability of a Real Inverse Formula for the Laplace Transform

S. Saitoh, Vu Kim Tuan and M. Yamamoto

Abstract. We establish a conditional stability estimate of a real inverse formula for the Laplace transform of functions under the assumption that the Bergman-Selberg norms of the Laplace transform of those functions are uniformly bounded. The rate of the stability estimate is shown to be of logarithmic order.

Keywords: *Laplace transform, real inversion formulas, conditional stability, Bergman-Selberg space, error estimates, Mellin transform, Gauss formula, convolution, reproducing kernels*

AMS subject classification: 44 A 10, 30 C 40, 44 A 15, 33 B 15, 33 C 05

1. Introduction and main results

We are concerned with the Laplace transform

$$(\mathcal{L}F)(x) = \int_0^\infty F(t)e^{-xt}dt \qquad (x > 0).$$

Our main purpose is to get some estimates of $F(t)$ $(t > 0)$ by means of $\sup_{x\geq 0} |(\mathcal{L}F)(x)|$. In particular, we are interested in estimates of F that are small when $\sup_{x\geq 0} |(\mathcal{L}F)(x)|$ is small. This kind of estimates is called *stability estimate* for the inverse Laplace transform and, in general, we cannot expect such stability estimates because the Laplace transform $\mathcal{L}$ advances the regularity of F very much. For example, consider $F_n(t) = \sin nt$ $(n \in \mathbb{N})$. Then $(\mathcal{L}F_n)(x) = \frac{n}{x^2+n^2}$ $(x > 0)$ and $\sup_{x>0} |(\mathcal{L}F_n)(x)| = \frac{1}{n} \to 0$ as $n \to \infty$, but $\lim_{n\to\infty} \|F_n\|_{L^\infty(0,\infty)} \neq 0$.

The lack of stability implies the ill-posedness in taking the inverse of the Laplace transform if we choose L^∞-norms for functions under consideration. However, it is possible to obtain some stability estimates provided that we restrict ourselves to some

S. Saitoh: Gunma Univ., Dept. Math., Tenjin, Kiryu 376, Japan; partially supported by the Japanese Ministry of Education, Science, Sports and Culture; Grant-in-Aid Scientific Research, Kiban Kenkyuu (A)(1), 10304009.
Vu Kim Tuan: Kuwait Univ., Dept. Math. & Comp. Sci., P.O. Box 5969, Safat 13060, Kuwait; supported by the Kuwait University Research Administration under the grant SM 187.
M. Yamamoto: The Tokyo Univ., Dept. Math. Sci., Komaba, Meguro, Tokyo 153, Japan
ssaitoh@sv1.math.sci.gunmy-u.ac.jp; vu@sci.kuniv.edu.kw; myama@ms.u-tokyo.ac.jp

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reasonable set of functions. They are called *conditional stability estimates*, and there are many such estimates depending on the choice of norms and "reasonable" functions classes. In this paper, we establish such a conditional stability estimate in L^∞-norm for a subclass of Hölder continuous functions. The image of this space under the Laplace transform turns out to be a Bergman-Selberg space.

For $q > 0$, we can define a norm equivalent to the Bergman-Selberg norm $\|\cdot\|_{H_q(\mathbb{R}^+)}$ by

$$\|f\|^2_{H_q(\mathbb{R}^+)} = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n(xf'(x))|^2 x^{2n+2q-1}dx. \tag{1.1}$$

It is known (see, e.g., Saitoh [4: Chapter 5]) that

$$\|F\|_{L^2_q} \equiv \left(\int_0^\infty |F(t)|^2 t^{1-2q}dt\right)^{\frac{1}{2}} = \|\mathcal{L}F\|_{H_q(\mathbb{R}^+)}. \tag{1.2}$$

Equality (1.2) means that the Laplace transform is an isometry between the norms $\|\cdot\|_{L^2_q}$ and $\|\cdot\|_{H_q(\mathbb{R}^+)}$ for any fixed $q > 0$. The norm $\|\cdot\|_{H_q(\mathbb{R}^+)}$ specifies our choice of an admissible set.

We state our main results.

Theorem 1. *Let* $\frac{1}{4} < q < 1, M > 0,$ *and*

$$\max\left\{\frac{1}{2}, 2q-1\right\} < \alpha < \min\{1, 2q\}. \tag{1.3}$$

Set

$$\mathcal{U} = \left\{f :\ \|f\|_{H_q(\mathbb{R}^+)} \le M \quad \text{and} \quad \|x^\alpha f(\cdot)\|_{H_{q-\frac{\alpha}{2}}(\mathbb{R}^+)} \le M\right\}. \tag{1.4}$$

Then for $0 < t_0 < t_1 < \infty$ *and* $0 < \gamma < \frac{2\alpha-1}{4}$ *there exists a constant* $C = C(\mathcal{U}, t_0, t_1, \gamma) > 0$ *such that*

$$\|F\|_{L^\infty(t_0,t_1)} \le C\left(\frac{-1}{\log\|\mathcal{L}F\|_{L^\infty(0,\infty)}}\right)^\gamma \tag{1.5}$$

if $\mathcal{L}F \in \mathcal{U}$.

The right-hand side of (1.5) tends to 0 as $\|\mathcal{L}F\|_{L^\infty(0,\infty)} \to 0$, but with the logarithmic rate. So the conditional stability estimate is worse than any Hölder continuity.

The subset $\mathcal{U}$ is defined on the set of images of the Laplace transform with the Bergman-Selberg norm. Sometimes it is more desirable to have a characterization based on the original functions.

Theorem 2. *Let* $\alpha, \gamma, q, t_0, t_1, M, C$ *be defined as in Theorem* 1 *and set*

$$\mathcal{V} = \left\{F \in C^1[0,\infty) :\ F(0) = 0,\ \|F\|_{L^2_q} \le M,\ \|F'\|_{L^2_{\frac{\alpha}{2}+q-1}} \le \frac{M\Gamma\left(\frac{3}{2}-\alpha\right)}{\sqrt{\pi}}\right\}. \tag{1.6}$$

Then estimate (1.5) *holds for all* $F \in \mathcal{V}$.

In the next section Preliminaries we shall show that the condition $\alpha < 1$ in (1.3) is sharp. That is, this assumption is needed essentially in the paper [2], which is the base of Theorems 1 and 2.

2. Preliminaries and sharp condition for α

The keys to the proofs of Theorems 1 and 2 are the real inversion formula of the Laplace transform (Byun and Saitoh [3], Saitoh [4]) and the error estimate of this real inversion formula (Amano, Saitoh and Yamamoto [2]):

Proposition 1 (see [3, 4]). *Let $q > 0$ be fixed, $\|F\|_{L^2_q} < \infty$ and $f = \mathcal{L}F$. Then the inversion formula*

$$F(t) = s - \lim_{N\to\infty} \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)\,dx \qquad (t > 0)$$

is valid where the limit is taken in the space L^2_q and the polynomials $P_{N,q}$ are given by the formulas

$$P_{N,q}(\xi) = \sum_{0\le\nu\le n\le N} \frac{(-1)^{\nu+1}\Gamma(2n+2q)}{\nu!\,(n-\nu)!\,\Gamma(n+2q+1)\Gamma(n+\nu+2q)}\,\xi^{n+\nu+2q-1}$$
$$\times\left\{\frac{2(n+q)}{n+\nu+2q}\,\xi^2 - \left(\frac{2(n+q)}{n+\nu+2q} + 3n + 2q\right)\xi + n(n+\nu+2q)\right\}.$$

Moreover, the series

$$\sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n(xf'(x))|^2 x^{2n+2q-1}dx$$

converges and the inequality

$$\left\|F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)\,dx\right\|^2_{L^2_q}$$
$$\le \sum_{n=N+1}^{\infty} \frac{1}{n!\,\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n(xf'(x))|^2 x^{2n+2q-1}dx$$

holds.

Proposition 2 (see [2]). *Let (1.3) hold. Then for $f \in H_q(\mathbb{R}^+)$ there exists a constant $M_1 = M_1(q,\alpha) > 0$ such that*

$$\left|F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)\,dx\right| \le M_1\|x^\alpha f(\cdot)\|_{H_{q-\frac{\alpha}{2}}(\mathbb{R}^+)} \frac{t^{q-1+\frac{\alpha}{2}}}{N^{\frac{2\alpha-1}{4}}}. \tag{2.1}$$

Theorem 1 in [2] only asserts that for $N \to \infty$

$$\left|F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)\,dx\right| \le t^{q-1+\frac{\alpha}{2}}\,o\left(\frac{1}{N^{\frac{2\alpha-1}{4}}}\right). \tag{2.2}$$

However, from the proof given in [2] we easily specify the dependency of the coefficient at the right-hand side of (2.2) to obtain inequality (2.1).

In order to see the necessity of the restriction $\alpha < 1$ in condition (1.3), recall

Proposition 3 (see [2: Lemma]). *If $f \in C^\infty(0,\infty)$ and*

$$I_{q,\alpha}(f) := \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n (xf'(x))|^2 x^{2n+2q-1+\alpha} dx < \infty \tag{2.3}$$

for a fixed $\alpha > \max\left(\frac{1}{2}, 2q-1\right)$, then

$$\left| \sum_{n=N+1}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \partial_x^n (xf'(x)) \partial_x^n (x\partial_x(e^{-tx}))\, x^{2n+2q-1} dx \right| = t^{\frac{\alpha-2q}{2}} o\big(N^{\frac{1-2\alpha}{4}}\big)$$

as $N \to \infty$.

Proposition 2 was proved with the use of Proposition 3. We will analyze the relation of the restriction $\alpha < 1$ with condition (2.3). Set

$$F_N(t) = \int_0^\infty f(x) e^{-xt} P_{N,q}(xt)\, dx$$

for any $q > 0$ and $F \in L_q^2$. Then, as shown in [3, 4],

$$F_N(t) = \sum_{n=0}^{N} \frac{t^{2q-1}}{n!\,\Gamma(n+2q+1)} \int_0^\infty \partial_x^n (xf'(x))\, \partial_x^n (x\partial_x(e^{-tx}))\, x^{2n+2q-1} dx$$

and, by Proposition 1, $s - \lim_{N\to\infty} F_N = F$.

We examine now properties of functions f satisfying (2.3). For the Mellin transform $(Mf)(s) = \int_0^\infty f(x) x^{s-1} dx$ of f recall the identity

$$\frac{1}{2\pi} \int_{-\infty}^\infty \frac{|(Mf)(q-it)|^2}{|\Gamma(q-it)|^2} dt = \|f\|^2_{H_q(\mathbb{R}^+)}$$

and notice that

$$\int_{-\infty}^\infty |(Mf)(q-it)|^2 (q^2+t^2)^2 \{(q+1)^2+t^2\} \cdots \{(q+n-1)^2+t^2\} dt = 2\pi \int_0^\infty |\partial_x^n (xf'(x))|^2 x^{2n+2q-1} dx$$

(see [4: Page 207/Formula (28)]). Hence,

$$\begin{aligned} 2\pi \int_0^\infty & |\partial_x^n (xf'(x))|^2 x^{2n+2q+\alpha-1} dx \\ &= \int_{-\infty}^\infty \left|(Mf)\Big(q+\frac{\alpha}{2}-it\Big)\right|^2 \left\{\Big(q+\frac{\alpha}{2}\Big)^2+t^2\right\}^2 \\ &\quad \times \left\{\Big(q+\frac{\alpha}{2}+1\Big)^2+t^2\right\} \cdots \left\{\Big(q+\frac{\alpha}{2}+n-1\Big)^2+t^2\right\} dt \end{aligned}$$

and so

$$
\begin{aligned}
I_{q,\alpha}(f) &= \tfrac{1}{2\pi}\sum_{n=0}^{\infty}\frac{1}{n!\,\Gamma(n+2q+1)} \\
&\quad\times\int_{-\infty}^{\infty}\Big|(Mf)\Big(q+\frac{\alpha}{2}-it\Big)\Big|^2\Big\{\Big(q+\frac{\alpha}{2}\Big)^2+t^2\Big\}^2 \\
&\quad\times\Big\{\Big(q+\frac{\alpha}{2}+1\Big)^2+t^2\Big\}\cdots\Big\{\Big(q+\frac{\alpha}{2}+n-1\Big)^2+t^2\Big\}dt \qquad (2.4)\\
&= \tfrac{1}{2\pi\Gamma(2q+1)}\int_{-\infty}^{\infty}\Big|(Mf)\Big(q+\frac{\alpha}{2}-it\Big)\Big|^2\Big\{\Big(q+\frac{\alpha}{2}\Big)^2+t^2\Big\} \\
&\quad\times\sum_{n=0}^{\infty}\frac{\big(q+\frac{\alpha}{2}+it\big)_n\,\big(q+\frac{\alpha}{2}-it\big)_n}{(2q+1)_n\,n!}\,dt
\end{aligned}
$$

where $(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$. Applying the famous Gauss summation formula (see [1: Page 556/Formulas (15.1.20) and (15.1.1)])

$$
\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (\mathrm{Re}(c-a-b)>0, c\notin-\mathbb{N}_0) \qquad (2.5)
$$

and using the property $\Gamma(\overline{z}) = \overline{\Gamma(z)}$ we obtain

$$
\sum_{n=0}^{\infty}\frac{\big(q+\frac{\alpha}{2}+it\big)_n\,\big(q+\frac{\alpha}{2}-it\big)_n}{(2q+1)_n n!} = \frac{\Gamma(2q+1)\Gamma(1-\alpha)}{|\Gamma(q+1-\frac{\alpha}{2}+it)|^2}
$$

when $\alpha<1$. Hence

$$
\begin{aligned}
I_{q,\alpha}(f) &= \tfrac{\Gamma(1-\alpha)}{2\pi}\int_{-\infty}^{\infty}\Big|(Mf)\Big(q+\frac{\alpha}{2}-it\Big)\Big|^2\frac{(q+\frac{\alpha}{2})^2+t^2}{|\Gamma(q+1-\frac{\alpha}{2}+it)|^2}\,dt \\
&= \tfrac{\Gamma(1-\alpha)}{2\pi}\int_{-\infty}^{\infty}\frac{|(Mf)(q+\frac{\alpha}{2}-it)|^2}{|\Gamma(q-\frac{\alpha}{2}+it)|^2}\,\frac{(q+\frac{\alpha}{2})^2+t^2}{(q-\frac{\alpha}{2})^2+t^2}\,dt \\
&\le C\int_{-\infty}^{\infty}\frac{|(Mf)(q+\frac{\alpha}{2}-it)|^2}{|\Gamma(q-\frac{\alpha}{2}-it)|^2}\,dt.
\end{aligned}
$$

Note that

$$
(Mf)\Big(q+\frac{\alpha}{2}-it\Big) = \int_0^{\infty} f(x)x^{q+\frac{\alpha}{2}-it-1}dx = M(f(x)x^{\alpha})\Big(q-\frac{\alpha}{2}-it\Big).
$$

Hence,

$$
\begin{aligned}
I_{q,\alpha}(f) &\le C\int_{-\infty}^{\infty}\Big|(M(x^{\alpha}f(x)))\Big(q-\frac{\alpha}{2}-it\Big)\Big|^2\frac{1}{|\Gamma(q-\frac{\alpha}{2}-it)|^2}dt \\
&= C\|x^{\alpha}f(x)\|^2_{H_{q-\frac{\alpha}{2}}(\mathbb{R}^+)}.
\end{aligned}
$$

We see that if $x^\alpha f(x) \in H_{q-\frac{\alpha}{2}}(\mathbb{R}^+)$, then the function f satisfies condition (2.3). Thus we get Proposition 2 under the condition $\alpha < 1$.

The condition $\alpha < 1$ is sharp. In the case $\alpha \geq 1$ we shall show that (2.3), that is (2.4) does not converge for $f \not\equiv 0$. Indeed, from (2.4)

$$\begin{aligned}
I_{q,\alpha}(f) = \tfrac{1}{2\pi} \sum_{n=0}^{\infty} & \frac{1}{n!\,\Gamma(n+2q+1)} \\
& \times \int_{-\infty}^{\infty} \left|(Mf)\Big(q+\frac{\alpha}{2}-it\Big)\right|^2 \left\{\Big(q+\frac{\alpha}{2}\Big)^2+t^2\right\}^2 \\
& \times \left\{\Big(q+\frac{\alpha}{2}+1\Big)^2+t^2\right\} \cdots \left\{\Big(q+\frac{\alpha}{2}+n-1\Big)^2+t^2\right\} dt \\
\geq & \frac{1}{2\pi\Gamma(2q+1)} \int_{-\infty}^{\infty} \left|(Mf)\Big(q+\frac{\alpha}{2}-it\Big)\right|^2 dt \\
& \times \Big(q+\frac{\alpha}{2}\Big)^2 \sum_{n=0}^{\infty} \frac{(q+\frac{\alpha}{2})_n (q+\frac{\alpha}{2})_n}{(2q+1)_n\, n!}.
\end{aligned}$$

Since $(2q+1)-(q+\frac{\alpha}{2})-(q+\frac{\alpha}{2}) \leq 0$, the last series is divergent, and $I_{q,\alpha}(f)$ is finite only if

$$\int_{-\infty}^{\infty} \left|(Mf)\Big(q+\frac{\alpha}{2}-it\Big)\right|^2 dt = 0,$$

that is, if $f \equiv 0$.

3. Proof of Theorem 1

We divide the proof into two steps.

First Step. We set $f = \mathcal{L}F$ and

$$F_N(t) = \int_0^\infty f(x) e^{-xt} P_{N,q}(xt)\, dx \qquad (t > 0).$$

In this step, we will estimate $|F_N(t)| \quad (t \in [t_0, t_1])$. We have

$$\begin{aligned}
|F_N(t)| &\leq \|f\|_{L^\infty(0,\infty)} \int_0^\infty |e^{-xt} P_{N,q}(xt)|\, dx \\
&= \frac{1}{t} \|f\|_{L^\infty(0,\infty)} \int_0^\infty |e^{-\xi} P_{N,q}(\xi)|\, d\xi \qquad (3.1)\\
&\leq \frac{1}{t} \|f\|_{L^\infty(0,\infty)} S(N,q).
\end{aligned}$$

Here we set

$$
\begin{aligned}
S(N,q) = \sum_{0\le\nu\le n\le N} & \frac{\Gamma(2n+2q)}{\nu!\,(n-\nu)!\,\Gamma(n+2q+1)\,\Gamma(n+\nu+2q)} \\
& \times \int_0^\infty \left\{ \frac{2(n+q)}{n+\nu+2q}\,\xi^{n+\nu+2q+1}e^{-\xi} \right. \\
& + \left(\frac{2(n+q)}{n+\nu+2q} + 3n+2q\right)\xi^{n+\nu+2q}e^{-\xi} \\
& \left. + n(n+\nu+2q)\,\xi^{n+\nu+2q-1}e^{-\xi} \right\} d\xi.
\end{aligned}
$$

It is sufficient to estimate $S(N,q)$. Noting that

$$
\Gamma(n+\nu+2q+2) = (n+\nu+2q+1)(n+\nu+2q)\Gamma(n+\nu+2q)
$$

and

$$
\Gamma(n+\nu+2q+1) = (n+\nu+2q)\Gamma(n+\nu+2q)
$$

we obtain

$$
\begin{aligned}
S(N,q) &= \sum_{0\le\nu\le n\le N} \frac{\Gamma(2n+2q)(4(n+q)+(6n+4q)(n+\nu+2q))}{\nu!\,(n-\nu)!\,\Gamma(n+2q+1)} \\
&\le \sum_{n=0}^{N} \left(\sum_{\nu=0}^{n} \frac{n!}{\nu!\,(n-\nu)!}\right) \frac{1}{n!}\,\frac{\Gamma(2n+2q)}{\Gamma(n+2q+1)}\,\big(4(n+q)+(6n+4q)(2n+2q)\big) \\
&= \sum_{n=0}^{N} 2^n(4n+4q)(3n+2q+1)\,\frac{\Gamma(2n+2q)}{n!\,\Gamma(n+2q+1)} \\
&\le C4^N \sum_{n=0}^{N} \frac{\Gamma(2n+2q)}{n!\,\Gamma(n+2q+1)}.
\end{aligned}
$$

Here and henceforth $C>0$ denotes a generic constant dependent only on $M, q, \alpha, \beta, t_0, t_1$ and we note that $(4n+4q)(3n+2q+1) \le C2^N$ for $0 \le n \le N$.

Moreover, we have

$$
\Gamma(2n+2q) = \tfrac{1}{\sqrt{2\pi}} 2^{2n+2q-\frac{1}{2}}\Gamma(n+q)\Gamma\Big(n+q+\frac{1}{2}\Big)
$$

(see, e.g., Abramowitz and Stegun [1: p. 256]), and so

$$
\begin{aligned}
\sum_{n=0}^{N} \frac{\Gamma(2n+2q)}{n!\,\Gamma(n+2q+1)} &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{N} 2^{2n+2q-\frac{1}{2}} \frac{\Gamma(n+q)\Gamma(n+q+\frac{1}{2})}{n!\,\Gamma(n+2q+1)} \\
&\le C4^N \sum_{n=0}^{N} \frac{\Gamma(n+q)\Gamma(n+q+\frac{1}{2})}{n!\,\Gamma(n+2q+1)} \\
&\le C4^N \sum_{n=0}^{\infty} \frac{\Gamma(n+q)\Gamma(n+q+\frac{1}{2})}{n!\,\Gamma(n+2q+1)} \\
&= C4^N \frac{\Gamma(2q+1)\Gamma(\frac{1}{2})}{\Gamma(q+1)\Gamma(q+\frac{1}{2})}.
\end{aligned}
$$

At the last equality, we use the Gauss summation formula (2.5). Therefore we obtain $S(N,q) \le C\,16^N$, so that inequality (3.1) yields

$$|F_N(t)| \le \frac{C}{t}\,16^N \|f\|_{L^\infty(0,\infty)} \qquad (t>0).$$

Second Step. It is sufficient to prove (1.5) for sufficiently small $\|f\|_{L^\infty(0,\infty)}$. Let $0 < t_0 \le t \le t_1$. We have

$$\begin{aligned} |F(t)| &\le |F_N(t)| + |F(t) - F_N(t)| \\ &\le \frac{C}{t}\,16^N \|f\|_{L^\infty(0,\infty)} + M_1 \|x^\alpha f(\cdot)\|_{H_{q-\frac{\alpha}{2}}(\mathbb{R}^+)} \frac{t^{q-1+\frac{\alpha}{2}}}{N^{\frac{2\alpha-1}{4}}} \\ &\le C\left(16^N \|f\|_{L^\infty(0,\infty)} + \frac{1}{N^{\frac{2\alpha-1}{4}}}\right) \end{aligned} \tag{3.2}$$

for all $N \in \mathbb{N}$. Here we note that $f \in \mathcal{U}$ implies $\|x^\alpha f(\cdot)\|_{H_{q-\frac{\alpha}{2}}(\mathbb{R}^+)} \le M$. We set $\eta = \|f\|_{L^\infty(0,\infty)}$ for simplicity. Let $0 < \gamma < \frac{2\alpha-1}{4}$ be chosen arbitrarily. We fix $N \in \mathbb{N}$ such that

$$\left(\log\frac{1}{\eta}\right)^{\frac{4\gamma}{2\alpha-1}} \le N < 1 + \left(\log\frac{1}{\eta}\right)^{\frac{4\gamma}{2\alpha-1}}.$$

Then we can see that

$$\frac{1}{N^{\frac{2\alpha-1}{4}}} \le \left(\log\frac{1}{\eta}\right)^{-\gamma}. \tag{3.3}$$

Moreover, we have

$$16^N \|f\|_{L^\infty(0,\infty)} = \eta \exp((\log 16)N) \le \eta \exp\left((\log 16) + (\log 16)\left(\log\frac{1}{\eta}\right)^{\frac{4\gamma}{2\alpha-1}}\right).$$

Since $\frac{4\gamma}{2\alpha-1} < 1$, we can easily verify

$$\lim_{\eta\downarrow 0} \eta \exp\left((\log 16)\left(\log\frac{1}{\eta}\right)^{\frac{4\gamma}{2\alpha-1}}\right)\left(\log\frac{1}{\eta}\right)^{\gamma} = 0.$$

Consequently, we see that

$$16^N \|f\|_{L^\infty(0,\infty)} \le \frac{C}{\left(\log\frac{1}{\eta}\right)^{\gamma}}. \tag{3.4}$$

Application of (3.3) and (3.4) in (3.2) yields conclusion (1.5). Thus the proof of Theorem 1 is complete.

4. Proof of Theorem 2

It is sufficient to show that $\mathcal{L}\mathcal{V} \subset \mathcal{U}$. Let

$$(I_0^\alpha F)(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} F(\tau)\, d\tau \qquad (\alpha > 0)$$

be the fractional integral of order α. Recall the Hardy inequality for the fractional integral (see [5: Formula (5.46')])

$$\int_0^\infty t^{-2\alpha-2\gamma} |I_0^\alpha t^\gamma F(t)|^2 dt \le \tfrac{\pi}{\Gamma^2(\alpha+\frac{1}{2})} \int_0^\infty |F(t)|^2 dt.$$

By replacing

$$\begin{aligned} \alpha \quad &\text{by} \quad 1-\alpha \\ \gamma \quad &\text{by} \quad \frac{\alpha}{2} + q - \frac{3}{2} \\ F(t) \quad &\text{by} \quad t^{\frac{3}{2}-\frac{\alpha}{2}-q} F'(t) \end{aligned}$$

we obtain

$$\begin{aligned} \int_0^\infty t^{1-2(q-\frac{\alpha}{2})} |I_0^{1-\alpha} F'(t)|^2 dt &\le \tfrac{\pi}{\Gamma^2(\frac{3}{2}-\alpha)} \int_0^\infty x^{1-(\alpha+2q-2)} |F'(x)|^2 dx \\ &= \tfrac{\pi}{\Gamma^2(\frac{3}{2}-\alpha)} \|F'\|^2_{L^2_{\frac{\alpha}{2}+q-1}}. \end{aligned}$$

Hence, if $F \in \mathcal{V}$, then $F' \in L^2_{\frac{\alpha}{2}+q-1}$ and

$$\|I_0^{1-\alpha} F'\|_{L^2_{q-\frac{\alpha}{2}}} \le \tfrac{\sqrt{\pi}}{\Gamma(\frac{3}{2}-\alpha)} \|F'\|_{L^2_{\frac{\alpha}{2}+q-1}} \le M,$$

so the corresponding Bergman-Selberg norm of its Laplace transform is also bounded by M,

$$\|\mathcal{L} I_0^{1-\alpha} F'\|_{H^2_{q-\frac{\alpha}{2}}(\mathbb{R}^+)} \le M.$$

We have (see [5: Formula (7.14)])

$$(\mathcal{L} I_0^{1-\alpha} F')(x) = x^{\alpha-1} (\mathcal{L} F')(x).$$

Since $F(0) = 0$, it is clear that

$$(\mathcal{L} F')(x) = x (\mathcal{L} F)(x).$$

Hence,

$$(\mathcal{L} I_0^{1-\alpha} F')(x) = x^\alpha (\mathcal{L} F)(x).$$

Thus

$$\|x^\alpha (\mathcal{L} F)(x)\|_{H^2_{q-\frac{\alpha}{2}}(\mathbb{R}^+)} \le M.$$

As $F \in \mathcal{V}$, we also have $\|\mathcal{L} F\|_{H^2_q(\mathbb{R}^+)} = \|F\|_{L^2_q} \le M$. Consequently, $\mathcal{L}\mathcal{V} \subset \mathcal{U}$.

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