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On C^1-Regularity of Functions that Define G-Closure

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Abstract. In this paper we show that the functions which are used in the characterization of the G-closure or the G_θ-closure of sets of matrices are continuously differentiable. These regularity results are based on the observation by Ball, Kirchheim and Kristensen [1] that separate convexity and upper semidifferentiability imply continuous differentiability.

Keywords: *Homogenization, G-closure, quasiconvexity*

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1. Introduction

Recently Ball, Kirchheim and Kristensen [1] established a remarkable result that a great deal of quasiconvex envelopes are continuously differentiable provided that the original function is upper semidifferentiable and satisfies some rather mild growth conditions. This result is based on the facts that upper semidifferentiablity is preserved under the procedure of taking the infimum over families of uniformly upper semidifferentiable functions and upper semidifferentiablity and separate convexity imply continuous differentiablity.

In this paper we use the results from Ball, Kirchheim and Kristensen [1] to show that various functionals, which arise from an optimal material layout problem governed by a system of elliptic equations, are continuously differentiable. Especially, we concentrate on the functions which are used in the evaluation of G-closure or G_θ-closure of sets of matrices.

Let us first recall the formulation of the problem. By an optimal material layout problem we mean the following one:

$$\left.\begin{aligned} & I(u) \to \min \\ & \operatorname{div}\big(\mathcal{A}(x)\nabla u(x)\big) = f(x) \quad \text{in } \Omega \\ & u = (u_1, ..., u_m) \in H_0^1(\Omega, \mathbb{R}^m),\ \mathcal{A} \in \mathcal{M} \end{aligned}\right\} \tag{1}$$

where Ω is an open subset of $\mathbb{R}^n$, I is weakly continuous (with respect to H_0^1-topology) and $f \in L^2(\Omega, \mathbb{R}^m)$ is given. The control set $\mathcal{M}$ is defined as

$$\mathcal{M} = \Big\{\mathcal{A} \text{ measurable} :\ \mathcal{A}(x) \in M \text{ for a.e. } x \in \Omega\Big\}$$

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in which M is a bounded set of constant uniformly positive definite symmetric $mn \times mn$-matrices. The matrices $A \in M$ represent different material properties like conductivities of given phases. If we, moreover, have volume restrictions for phases, the control set $\mathcal{M}$ is replaced by

$$\mathcal{M}_\theta(\Omega) = \left\{ \mathcal{A} \text{ measurable} \left| \begin{array}{l} \text{meas}\,\big\{x \in \Omega : \mathcal{A}(x) \in M_s\big\} = \theta_s \\ s = 1, ..., s_0;\ \theta_1 + ... + \theta_{s_0} = |\Omega| \end{array} \right. \right\}$$

where M_s are given bounded pairwise disjoint sets of constant uniformly positive definite symmetric $mn \times mn$-matrices. In literature problems of type (1) are often treated as optimal design problems (see, e.g., [4, 13]).

It is well-known that the problem (1) has no, in general, a solution (see, e.g., [6]). Therefore, we have to find a proper extension of (1). To do that we replace $\mathcal{M}$ or $\mathcal{M}_\theta(\Omega)$ by its G-closure or G_θ-closure, respectively. Then the solvability of (1) follows.

Recall that the *G-closure* of the set $\mathcal{M}$ is defined as a larger set $G\mathcal{M}$ of all measurable symmetric $mn \times mn$-matrices $\mathcal{B}$ such that for a chosen $\mathcal{B} \in G\mathcal{M}$ there exists a sequence $\{\mathcal{A}_k\} \subset \mathcal{M}$ with the following property: For every fixed $f \in L^2(\Omega, \mathbb{R}^{nm})$ from

$$\begin{aligned} \text{div}\,\big(\mathcal{A}_k(x)\nabla u_k(x)\big) &= f(x) \ \text{ in } \Omega \qquad \big(u_k \in H_0^1(\Omega, \mathbb{R}^m), k \in \mathbb{N}\big) \\ \text{div}\,\big(\mathcal{B}(x)\nabla u_0(x)\big) &= f(x) \ \text{ in } \Omega \qquad \big(u_0 \in H_0^1(\Omega, \mathbb{R}^m)\big) \end{aligned}$$

it follows

$$\left. \begin{array}{ll} u_k \rightharpoonup u_0 & \text{weakly in } H_0^1(\Omega, \mathbb{R}^m) \\ \mathcal{A}_k \nabla u_k \rightharpoonup \mathcal{B}\nabla u_0 & \text{weakly in } L^2(\Omega, \mathbb{R}^{nm}) \end{array} \right\} \qquad (k \to \infty).$$

The set $G\mathcal{M}$ has the following description (see, e.g., [8 - 10]). Let $K \subset \mathbb{R}^n$ be the unit cube and let

$$V^\# = \left\{ v \in L^2(K, \mathbb{R}^{nm}) \,\middle|\, v = \nabla u,\ u \in H^1_{loc}(\mathbb{R}^n, \mathbb{R}^m),\ u \text{ is } K\text{-periodic} \right\}$$

$$N^\# = \text{cl} \left\{ \eta \in L^2(K, \mathbb{R}^{nm}) \left| \begin{array}{l} \eta = (\eta^1, ..., \eta^m), \eta^i = \big(\sum_{j=1}^n \frac{\partial}{\partial x_j} u^i_{1j}, ..., \sum_{j=1}^n \frac{\partial}{\partial x_j} u^i_{nj}\big) \\ u^i_{rj} = -u^i_{jr},\ u^i_{rj} \in H^1_{loc}(\mathbb{R}^n),\ u^i_{rj} \text{ is } K\text{-periodic} \\ \big(r, j = 1, ..., n;\ i = 1, ..., m\big) \end{array} \right. \right\}$$

$$V^0 = \left\{ v \in L^2(K, \mathbb{R}^{nm}) : v = \nabla u,\ u \in H^1_0(K, \mathbb{R}^m) \right\}$$

$$N^0 = \text{cl} \left\{ \eta \in L^2(K, \mathbb{R}^{nm}) \left| \begin{array}{l} \eta = (\eta^1, ..., \eta^m), \eta^i = \big(\sum_{j=1}^n \frac{\partial}{\partial x_j} u^i_{1j}, ..., \sum_{j=1}^n \frac{\partial}{\partial x_j} u^i_{nj}\big) \\ u^i_{rj} = -u^i_{jr}, u^i_{rj} \in H^1_0(K) \\ \big(r, j = 1, ..., n;\ i = 1, ..., m\big) \end{array} \right. \right\}.$$

In these notations (without loss of generality we can assume that $K \subset \Omega$)

$$G\mathcal{M} = \left\{ \mathcal{B} \text{ measurable} :\ \mathcal{B}(x) \in GM \text{ for a.e. } x \in \Omega \right\}$$

where

$$GM = \left\{ B \in \mathbb{R}^{nm\times nm} \;\middle|\; \begin{array}{l} B \text{ symmetric and for all } \xi,\zeta \in \mathbb{R}^{nm\times nm} \\ \sum_{i=1}^{nm}\big[(B\xi_i,\xi_i) + (B^{-1}\zeta_i,\zeta_i)\big] \ge F_0(\xi,\zeta) \end{array} \right\}$$

$$\xi = (\xi_1,\dots,\xi_{nm}),\ \zeta = (\zeta_1,\dots,\zeta_{nm}) \ \ \big(\xi_i,\zeta_i \in \mathbb{R}^{nm} \quad (i=1,\dots,nm)\big) \tag{2}$$

$$F_0(\xi,\zeta) = \inf_{\mathcal{A}\in\mathcal{M}} \sum_{i=1}^{nm} \inf_{\substack{v_i\in V\#\\ \eta_i\in N\#}} \int_K \Big\{\big(\mathcal{A}(x)(v_i+\xi_i), v_i+\xi_i\big) + \big(\mathcal{A}^{-1}(x)(\eta_i+\zeta_i), \eta_i+\zeta_i\big)\Big\}dx.$$

In turn, G_θ-closure of the set $\mathcal{M}_\theta(\Omega)$ is the set $G\mathcal{M}_\theta(\Omega)$ defined by

$$G\mathcal{M}_\theta(\Omega) = \mathrm{cl}\left\{ \mathcal{B} \text{ measurable} \;\middle|\; \begin{array}{l} \mathcal{B}(x) \in G_{\alpha(x)}M \text{ for a.e. } x\in\Omega, \alpha \text{ measurable} \\ \alpha(x) = \big(\alpha_1(x),\dots,\alpha_{s_0}(x)\big), \alpha \text{ piecewise constant} \\ 0 \le \alpha_s(x) \le 1, \int_\Omega \alpha_s(x)\,dx = \theta_s|\Omega|,\ s=1,\dots,s_0 \end{array} \right\}$$

where for a given $\theta = (\theta_1,\dots,\theta_{s_0})$ with $\theta_s \ge 0\ \ (s=1,\dots,s_0)$ and $\theta_1+\dots+\theta_{s_0} = 1$

$$G_\theta M = \left\{ B \in \mathbb{R}^{nm\times nm} \;\middle|\; \begin{array}{l} B \text{ symmetric and for all } \xi,\zeta \in \mathbb{R}^{nm\times nm} \\ \sum_{i=1}^{nm}\big[(B\xi_i,\xi_i) + (B^{-1}\zeta_i,\zeta_i)\big] \ge F_\theta(\xi,\zeta) \end{array} \right\}$$

$$\mathcal{M}_\theta = \left\{ \mathcal{A} \;\middle|\; \begin{array}{l} K \to \mathbb{R}^{mn\times mn} \text{ measurable} \\ \mathrm{meas}\big\{x\in K : \mathcal{A}(x)\in M_s\big\} = \theta_s\ (s=1,\dots,s_0) \end{array} \right\} \tag{3}$$

$$F_\theta(\xi,\zeta) = \inf_{\mathcal{A}\in\mathcal{M}_\theta} \sum_{i=1}^{nm} \inf_{\substack{v_i\in V\#\\ \eta_i\in N\#}} \int_K \Big\{\big(\mathcal{A}(x)(v_i+\xi_i), v_i+\xi_i\big) + \big(\mathcal{A}^{-1}(x)(\eta_i+\zeta_i), \eta_i+\zeta_i\big)\Big\}dx$$

and cl stands for the strong closure in the topology of $L^2(\Omega,\mathbb{R}^{nm\times nm})$.

The aim of this paper is to show that F_0 and F_θ are continuously differentiable. From this it follows that also $F_0(\cdot,0), F_0(0,\cdot)$ and $F_\theta(\cdot,0), F_\theta(0,\cdot)$, which are often used for the estimates of G-closed sets (see, e.g., [5]), are continuously differentiable.

2. Sufficient conditions for upper semidifferentiability

In this section we describe some sufficient conditions on a family of functions $\{f(\alpha,\cdot)\}$, α being a parameter, which ensure that the function

$$F(z) = \inf_\alpha f(\alpha,z) \qquad (z\in\mathbb{R}^N) \tag{4}$$

is upper semidifferentiable provided that the functions $f(\alpha,\cdot)$ are upper semidifferentiable uniformly with respect to α.

Definition 1. A function $F:\mathbb{R}^N\to\mathbb{R}$ is *upper semidifferentiable* at a point z_0 if there exists an element $a\in\mathbb{R}^N$ such that

$$\limsup_{z\to 0} \frac{F(z_0+z) - F(z_0) - (a,z)}{|z|} \le 0.$$

If F is upper semidifferentiable at every point z_0 of an open subset Q of $\mathbb{R}^N$, we say F is upper semidifferentiable *on* Q.

Definition 2. A function $F : \mathbb{R}^N \to \mathbb{R}$ is *locally separately convex* on an open subset $Q \subset \mathbb{R}^N$ if for every $z_0 \in Q$ there exists a ball $B(z_0) \subset Q$ with the center at z_0 such that for every canonical basis vector $e_i \in \mathbb{R}^N$ the function

$$\varphi(t) = F(z_0 + te_i)$$

is convex for $z_0 + te_i \in B(z_0)$.

Let S be a set of parameters α of general nature, $N \geq 2$ an integer, $Q \subset \mathbb{R}^N$ an open subset and

$$f : S \times Q \to \mathbb{R}, \qquad f = f(\alpha, z).$$

We introduce the following hypotheses on the family $\{f(\alpha, \cdot)\}$:

(H1) For every $\alpha \in S$ and every $z_0 \in Q$ there exists an element $a(\alpha, z_0) \in \mathbb{R}^N$ such that

$$\limsup_{z \to 0} \frac{f(z_0 + z) - f(z_0) - (a(\alpha, z_0), z)}{|z|} \leq 0.$$

(H2) For every $z_0 \in Q$ the set $\{f(\alpha, z_0) : \alpha \in S\}$ is bounded from below.

(H3) For every fixed $z_0 \in Q$ the set

$$\mathfrak{A}(z_0) = \Big\{ a(\alpha, z_0) : \alpha \in S,\ f(\alpha, z_0) \leq \inf_{\alpha' \in S} f(\alpha', z_0) + 1 \Big\}$$

is bounded.

(H4) For every fixed $z_0 \in Q$ there exists a continuous function $\gamma(z_0, \cdot) : \mathbb{R} \to \mathbb{R}$ with $\gamma(z_0, 0) = 0$ such that

$$f(\alpha, z_0 + z) - f(\alpha, z_0) - (a(\alpha, z_o), z) \leq \gamma(z_0, |z|)|z|$$

for all $\alpha \in \mathfrak{A}(z_0)$ and all $z_0 + z \in Q$ with $|z| < 1$.

Lemma 1. *Let the family of functions $\{f(\alpha, \cdot) : \alpha \in S\}$ satisfy hypotheses* (H1) - (H4)*. Then the function $F : Q \to \mathbb{R}$ given by*

$$F(z) = \inf_{\alpha \in S} f(\alpha, z)$$

is upper semidifferentiable at every point $z_0 \in Q$, and for every $z_0 \in Q$ there exists an element $a(z_0) \in \mathbb{R}^N$ such that

$$F(z_0 + z) - F(z_0) - (a(z_0), z) \leq \gamma(z_0, |z|)|z|$$

for all $z_0 + z \in Q$ with $|z| < 1$ where the function $\gamma(z_0, \cdot)$ is the same as in hypothesis (H4)*.*

Proof. Let $z_0 \in Q$ be fixed and $\{\alpha_k\} \subset S$ be a minimizing sequence for $f(\cdot, z_0)$, i.e.

$$F(z_0) = \lim_{k \to \infty} f(\alpha_k, z_0).$$

Then $\alpha_k \in \mathfrak{A}(z_0)$ for k large enough, and by virtue of hypothesis (H3) we can assume that the sequence $\{a(\alpha_k, z_0)\}$ converges to some element $a(z_0) \in \mathbb{R}^N$. Denote by δ_k the difference

$$\delta_k = f(\alpha_k, z_0) - F(z_0) \qquad (k \in \mathbb{N}).$$

Obviously, $\delta_k \to 0$ as $k \to \infty$. Because of hypothesis (H4) we have that for $z_0 + z \in Q$ with $|z| < 1$

$$\begin{aligned}
&F(z_0 + z) - F(z_0) - (a(z_0), z) \\
&\quad = \inf_{\alpha \in S} f(\alpha, z_0 + z) - \inf_{\alpha \in \mathfrak{A}(z_0)} f(\alpha, z_0) - (a(z_0), z) \\
&\quad \le f(\alpha_k, z_0 + z) - f(\alpha_k, z_0) + \delta_k - (a(\alpha_k, z_0), z) + \big(a(\alpha_k, z_0) - a(z_0), z\big) \\
&\quad \le \gamma(z_0, |z|)|z| + |\delta_k| + \big|a(\alpha_k, z_0) - a(z_0)\big|\, |z|.
\end{aligned}$$

Since this estimate is valid for all k large enough and

$$\left.\begin{aligned} \delta_k &\to 0 \\ |a(\alpha_k, z_o) - a(z_0)| &\to 0 \end{aligned}\right\} \qquad (k \to \infty),$$

then

$$F(z_0 + z) - F(z_0) - (a(z_0), z) \le \gamma(z_0, |z|)|z|$$

for all $z_0 + z \in Q$ with $|z| < 1$ ∎

From Lemma 1 and the results by Ball, Kirchheim and Kristensen [1] one can easily obtain the following

Corollary 1. *Let the family* $\{f(\alpha, \cdot) : \alpha \in S\}$ *satisfy hypotheses* (H1) - (H4) *and let the function*

$$F : Q \to \mathbb{R}, \qquad F(z) = \inf_{\alpha \in S} f(\alpha, z)$$

be locally separately convex on Q. Then F is continuously differentiable on Q.

Before proving the corollary, for the convenience of the readers we first recall the needed results from Ball, Kirchheim and Kristensen [1].

Theorem 1 (see [1: Corollary 2.3]). *Let $B \subset \mathbb{R}^N$ be an open ball and $F : B \to \mathbb{R}$ be separately convex. Denote by $D \subseteq B$ the set where F is differentiable. Then F is continuously differentiable on D.*

Theorem 2 (see [1: Corollary 2.5]). *Let $B(z_0) \subset \mathbb{R}^N$ be an open ball with the center at a point z_0. Suppose that $G : B(z_0) \to \mathbb{R}$ is separately convex, $F : B(z_0) \to \mathbb{R}$ is upper semidifferentiable at z_0, $G \le F$ and $G(z_0) = F(z_0)$. Then F and G are differentiable at z_0 and $F'(z_0) = G'(z_0)$.*

Proof of Corollary 1. Due to Lemma 1 the function F is upper semidifferentiable at every point $z_0 \in Q$. Since F is also locally separately convex, then Theorems 1 and 2 immediately give the continuously differentiablity of F on Q ∎

Remark 1. Under the assumptions of Corollary 1 the element $a(z_0)$ from the proof of Lemma 1 is equal to the derivative $F'(z_0)$ of F at z_0.

If one considers the upper semidifferentiability of functions of the kind (2) or (3), then $(\mathcal{A}, v_1, ..., v_{nm}, \eta_1, ..., \eta_{nm})$ play the role of the parameters $\alpha \in S$. For this case the upper semidifferentiability of $f(\alpha, \cdot)$ is rather obvious. The validity of hypotheses (H3) and (H4) is not so evident. Therefore, some growth conditions on integrands with respect to $v_i \in V^\#$ and $\eta_i \in N^\#$ must be imposed (for (2) and (3) they are given by uniform boundedness and positive definiteness of matrices $\mathcal{A}$).

In order to remain within the framework of G-closure and G_θ-closure problems we only consider integrands of the type

$$g : \mathbb{R}^{N'} \times K \times \mathbb{R}^N \to \mathbb{R}, \qquad g = g(y, x, z)$$

where K is the unit cube of $\mathbb{R}^n$, $N = (2nm)^{nm}$ and N' is an integer. Let $B \subset L^1(K, \mathbb{R}^{N'})$ be the set of admissible parameters β and let the following hypotheses be satisfied:

(H5) The function g is measurable in x, continuous in (y, z), and there exist constants $c_1, c_2 > 0$ and functions $h_1, h_2 \in L^1(K)$ such that

$$-h_1(x) + c_1|z|^2 \le g(\beta(x), x, z) \le h_2(x) + c_2|z|^2$$

for all $\beta \in B$, all $x \in K$ and all $z \in \mathbb{R}^N$.

(H6) For every $(y, x) \in \mathbb{R}^{N'} \times K$ the function $g(y, x, \cdot)$ is differentiable on $\mathbb{R}^N$, and the derivative g'_z is measurable in $x \in K$ and continuous in $(y, z) \in \mathbb{R}^{N'} \times \mathbb{R}^N$.

(H7) There exist a constant c_3, a function $h_3 \in L^2(K)$ and a continuous function $\gamma : \mathbb{R} \to \mathbb{R}$ with $\gamma(0) = 0$ such that

$$|g'_z(\beta(x), x, z)| \le h_3(x) + c_3|z|$$

$$|g'_z(\beta(x), x, z_0 + z) - g'_z(\beta(x), x, z_0)| \le (|h_3(x)| + c_3|z_0|)\gamma(|z|)$$

for all $\beta \in B$, all $x \in K$ and all $z_0, z \in \mathbb{R}^N$.

Lemma 2. *Let the function g satisfy hypotheses* (H5) - (H7). *Then the function*

$$f(\alpha, z) = \int_K g\big(\beta(x), x, z + w(x)\big)dx$$

with $\alpha \in S = B \times V$, $\alpha = (\beta, w)$, $Q = \mathbb{R}^N$ and V being a subspace of $L^2(K, \mathbb{R}^N)$ satisfies hypotheses (H1) - (H4).

Proof. From the representation

$$g\big(\beta(x), x, z_0 + z + w(x)\big) - g\big(\beta(x), x, z_0 + w(x)\big) = \int_0^1 \Big(g'_z\big(\beta(x), x, z_0 + w(x) + \lambda z\big), z\Big)d\lambda$$

it follows immediately that f satisfies hypotheses (H1) and (H4) with

$$a(\alpha, z_0) = \int_K g'_z\big(\beta(x), x, z_0 + w(x)\big)dx.$$

Hypothesis (H5) gives the lower bound for $f(\cdot, z_0)$ and hypothesis (H7) gives the boundedness of the sets $\mathfrak{A}(z_0)$ ∎

Remark 2. We do not impose here the most general conditions on the integrands g. For instance, one can consider the spaces $L^p(K, \mathbb{R}^N)$ with $1 < p < \infty$ instead of $p = 2$ or the weaker growth conditions.

3. Continuous differentiability of functions F_0 and F_θ

We first recall the following definition of $\mathbb{A}$-quasiconvexity from Fonseca and Müller (see [3]):

Definition 3. A function $f : \mathbb{R}^N \to \mathbb{R}$ is said to be $\mathbb{A}$-*quasiconvex* if

$$f(z) \le \int_K f(z + w(x))\, dx$$

for all $z \in \mathbb{R}^N$ and all K-periodic $w \in C^\infty(K, \mathbb{R}^N)$ such that $\mathbb{A}w = 0$ and $\int_K w(x)\, dx = 0$.

By $\mathbb{A}$ we denote a vectorial linear partial differential operator with constant coefficients (cf. the notion of *compensated compactness* [7, 12]). Typically $\mathbb{A}$ is a first order differential operator like curl or div.

Definition 4. Let f be a Borel function from $\mathbb{R}^N$ to $\mathbb{R}$. Its $\mathbb{A}$-*quasiconvex envelope* is defined by

$$Q_{\mathbb{A}} f(z) = \inf \left\{ \int_K f(z + w(x))\, dx \;\middle|\; \begin{array}{l} w \in C^\infty(K, \mathbb{R}^N) \cap \ker \mathbb{A} \\ w \text{ is } K\text{-periodic, } \int_K w(x)\, dx = 0 \end{array} \right\}.$$

The following result holds.

Proposition 1 [3: Propsition 3.4]. *If $\mathbb{A}$ has constant rank and if $f : \mathbb{R}^N \to \mathbb{R}$ is upper semicontinuous, then $Q_{\mathbb{A}} f$ is $\mathbb{A}$-quasiconvex and upper semicontinuous. Moreover, the restriction of $Q_{\mathbb{A}} f$ to each cone $z + \Lambda$ $(z \in \mathbb{R}^N)$ is convex, i.e.*

$$Q_{\mathbb{A}} f\big(ty + (1-t)z\big) \le t Q_{\mathbb{A}} f(y) + (1-t) Q_{\mathbb{A}} f(z)$$

for all $t \in (0,1)$ and $y, z \in \mathbb{R}^N$ such that $y - z \in \Lambda$ where $\Lambda = \cup_{w \in \mathbb{R}^n, |w|=1} \ker \mathbb{A}(w)$.

Let us return to our problem. First we justify that F_0 is a $\mathbb{A}$- quasiconvex envelope of the function

$$f_0\big(\xi_1, ..., \xi_{nm}, \zeta_1, ..., \zeta_{nm}\big) = \inf_{A \in M} \sum_{i=1}^{nm} \big\{ (A\xi_i, \xi_i) + (A^{-1}\zeta_i, \zeta_i) \big\}$$

which is the infimum over $A \in M$ of the integrand in (2). For this case the differential operator $\mathbb{A}$ is $(\mathrm{curl}, \mathrm{div})^{m \times nm}$ and $\mathbb{A}$ has constant rank. For the sake of simplicity of expressions only, we assume that $N = 2nm$ and $\xi, \zeta \in \mathbb{R}^{nm}$. Then $\mathbb{A} = (\mathrm{curl}, \mathrm{div})^m$ and

$$\begin{aligned} F_0 &= F_0(\xi, \zeta) \\ &= \inf_{\mathcal{A} \in \mathcal{M}} \inf_{\substack{v \in V^\# \\ \eta \in N^\#}} \int_K \Big\{ \big(\mathcal{A}(x)(v + \xi), v + \xi\big) + \big(\mathcal{A}^{-1}(x)(\eta + \zeta), \eta + \zeta\big) \Big\} dx \\ f_0 &= f_0(\xi, \zeta) = \inf_{A \in M} \Big\{ (A\xi, \xi) + (A^{-1}\zeta, \zeta) \Big\}. \end{aligned} \tag{5}$$

The matrices $A \in M$ are uniformly positive definite and the set M is bounded, hence the function f_0 is continuous with quadratic growth. In its turn, the set $\mathcal{M}$ is decomposable, i.e. if $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{M}$, then for every measurable $E \subset \Omega \quad (E \subset K)$ the matrix $\mathcal{A}_0$ defined by

$$\mathcal{A}_0(x) = \chi_E(x)\mathcal{A}_1(x) + (1 - \chi_E(x))\mathcal{A}_2(x)$$

belongs to $\mathcal{M}$, too (here χ_E is the characteristic function of E).These properties are sufficient for the equalities

$$\begin{aligned}
&\inf_{\mathcal{A}\in\mathcal{M}} \int_K \Big\{\big(\mathcal{A}(x)(v(x)+\xi), v(x)+\xi\big) + \big(\mathcal{A}^{-1}(x)(\eta(x)+\zeta), \eta(x)+\zeta\big)\Big\}dx \\
&\qquad = \int_K \inf_{A\in M} \Big\{\big(\mathcal{A}(x)(v(x)+\xi), v(x)+\xi\big) + \big(\mathcal{A}^{-1}(x)(\eta(x)+\zeta), \eta(x)+\zeta\big)\Big\}dx \\
&\qquad = \int_K f_0\big(v(x)+\xi, \eta(x)+\zeta\big)dx \qquad \forall v \in V^\#,\ \forall \eta \in N^\# \\
&F_0(\xi,\zeta) = \inf_{\substack{v\in V^\# \\ \eta\in N^\#}} \int_K f_0\big(v+\xi, \eta+\zeta\big)dx.
\end{aligned}$$

Indeed, the first equality is obvious for piecewise constant $(v, \eta) \in V^\# \times N^\#$. Since piecewise constant elements are dense in $V^\# \times N^\#$, then a simple continuity argument gives that this equality holds for all $(v, \eta) \in V^\# \times N^\#$. From this the second equality also follows.

The characteristic cone Λ for $\mathbb{A}$ is (see, e.g., Dagorogna [2] or Fonseca and Müller [3])

$$\Lambda = \bigcup_{\substack{w\in\mathbb{R}^n \\ |w|=1}} \left\{ z = (\xi,\zeta) \in \mathbb{R}^N \;\middle|\; \begin{array}{l} \xi = (\xi^1, ..., \xi^m),\ \zeta = (\zeta^1, ..., \zeta^m) \\ \xi^i = \alpha_i w,\ (\zeta^j, w) = 0,\ \alpha_i \in \mathbb{R} \ \ (i,j = 1, ..., m) \end{array} \right\}.$$

From [14] it follows that

$$L^2(K, \mathbb{R}^{nm}) = V^\# \oplus N^\# \oplus \mathbb{R}^{nm}$$

and that

$$(V^\# \oplus \mathbb{R}^{nm}) \times (N^\# \oplus \mathbb{R}^{nm}) = \ker \mathbb{A}$$

in the sense of distributions. Then by Definition 4 the function F_0 is the $\mathbb{A}$-quasiconvex envelope of f_0 and by Proposition 1 it is $\mathbb{A}$-quasiconvex and Λ-convex. Because Λ contains all canonical basis vectors e of $\mathbb{R}^{nm}$, the function F_0 is also separately convex.

Next we show that F_0 is upper semidifferentiable. Indeed, using the notations of Lemma 2 we have $B = \mathcal{M}$, $\alpha = (\mathcal{A}, v, \eta)$, $V = V^\# \times N^\#$ and

$$g = g\big(\mathcal{A}(\cdot), (\xi,\zeta)\big) = (\mathcal{A}(\cdot)\xi, \xi) + (\mathcal{A}^{-1}(\cdot)\zeta, \zeta).$$

Then, because obviously g satisfies hypotheses (H5) - (H7), Lemmas 1 and 2 imply that F_0 is upper semidifferentiable. Finally, due to Corollary 1 the functional F_0 is continuously differentiable on $\mathbb{R}^N$.

For the function F_θ the corresponding procedure is not so straightforward, because now it is not possible (by virtue of the integral restrictions in (3)) to bring the infimum over $\mathcal{A} \in \mathcal{M}_\theta$ inside the integral. Again, for the sake of simplicity of expressions only, we suppose that $N = 2nm$ and

$$\begin{aligned} F_\theta &= F_\theta(\xi,\zeta) \\ &= \inf_{\mathcal{A}\in\mathcal{M}_\theta} \inf_{\substack{v\in V^{\#} \\ \eta\in N^{\#}}} \int_K \Big\{ \big(\mathcal{A}(x)(v+\xi), v+\xi\big) + \big(\mathcal{A}^{-1}(x)(\eta+\zeta), \eta+\zeta\big) \Big\} dx. \end{aligned} \tag{6}$$

The upper semidifferentiability of F_θ follows in a similar way as for F_0. The only difference is that now $B = \mathcal{M}_\theta$. But for the separate convexity (or the $\mathbb{A}$-quasiconvexity) we can not apply Proposition 1. Below we present a direct proof for the separate convexity.

Let e be a given basis vector in $\mathbb{R}^N$. There are two cases: The first one where the non-zero entry of e corresponds to some entry of ξ, the second one where the non-zero entry of e corresponds to some entry of ζ. Both cases can be treated analogously, hence we will consider only one of them, say the case where the non-zero entry corresponds to the first entry of ζ.

Let $\zeta_0 = (1, 0, ..., 0) \in \mathbb{R}^{nm}$. We must show that

$$F_\theta(\xi, \zeta + \lambda\zeta_0) + F_\theta(\xi, \zeta - \lambda\zeta_0) \geq 2_\theta(\xi,\zeta)$$

for every fixed pair $(\xi,\zeta) \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$, which is equal to the convexity of F_θ in the direction e. It is clear that the element η_0,

$$\eta_0(x) = \begin{cases} \lambda\zeta_0 & \text{if } 0 < x_2 < \frac{1}{2} \\ -\lambda\zeta_0 & \text{if } \frac{1}{2} \leq x_2 < 1 \end{cases}$$

belongs to $N^{\#}$ (here $x = (x_1, ..., x_n) \in \mathbb{R}^n$). Indeed, since

$$L^2(K, \mathbb{R}^{nm}) = V^{\#} \oplus N^{\#} \oplus \mathbb{R}^{nm}$$

then it is sufficient to show that

$$\int_K (\eta_0, v)\, dx = 0 \qquad \forall v \in V^{\#}.$$

By the construction of η_0 and by definition of $V^{\#}$

$$(\eta_0, v)(x) = \begin{cases} \lambda u_{1x_1}(x) & \text{if } 0 < x_2 < \frac{1}{2} \\ -\lambda u_{1x_1}(x) & \text{if } \frac{1}{2} \leq x_2 < 1 \end{cases}$$

with some $u_1 \in H^1_{loc}(\mathbb{R}^n)$, u_1 K-periodic, which gives the needed relationship.

Let $\varepsilon > 0$ be given and let $\mathcal{A}_\pm \in \mathcal{M}_\theta$, $v_\pm \in V^{\#}$ and $\eta_\pm \in N^{\#}$ be such that

$$\begin{aligned} &F_\theta(\xi, \zeta \pm \lambda\xi_0) \geq \\ &\quad \int_K \Big\{ \big(\mathcal{A}_\pm(x)(v_\pm+\xi), v_\pm+\xi\big) + \big(\mathcal{A}_\pm^{-1}(x)(\eta_\pm+\zeta\pm\lambda\zeta_0), \eta_\pm+\zeta\pm\lambda\zeta_0\big) \Big\} - \varepsilon. \end{aligned} \tag{7}$$

In the first step we will show that estimates (7) hold for some $v_\pm \in V^0$ and $\eta_\pm \in N^0$, too. Extend the functions $\mathcal{A}_\pm$, $v_\pm$ and $\eta_\pm$ via K-periodicity to the whole $\mathbb{R}^n$ and define

$$\mathcal{A}^s_+(x) = \mathcal{A}_+(sx) \qquad (s \in \mathbb{N}),$$

and in a similar way $\mathcal{A}^s_-$, $v^s_\pm$ and $\eta^s_\pm$. By construction, $\mathcal{A}^\pm_s \in \mathcal{M}_\theta$, $v^s_\pm \in V^\#$, $\eta^s_\pm \in N^\#$ and

$$F_\theta(\xi, \zeta \pm \lambda\xi_0) \geq \int_K \Big\{ \big(\mathcal{A}^s_\pm(x)(v^s_\pm + \xi), v^s_\pm + \xi\big) + \big((\mathcal{A}^s_\pm)^{-1}(x)(\eta^s_\pm + \zeta \pm \lambda\zeta_0), \eta^s_\pm + \zeta \pm \lambda\zeta_0\big) \Big\} - \varepsilon. \tag{8}$$

From the definition of the space $V^\#$ we see that the element $v_+ \in V^\#$ has the representation $v_+(x) = \nabla u_+(x)$ for some $u_+ \in H^1_{loc}(\mathbb{R}^n, \mathbb{R}^m)$, u_+ K-periodic. We have

$$v^s_+(x) = \nabla\Big(\frac{1}{s} u_+(s\cdot)\Big)(x)$$

and by using standard cut-off functions $\varphi \in H^1_0(K)$ with $|\nabla\varphi| \leq \sqrt{s}$ we obtain that there exists an element $\tilde{v}^s_+ \in V^0$ such that

$$\|v^s_+ - \tilde{v}^s_+\|_{L^2(K, \mathbb{R}^{nm})} \leq c\, s^{-\frac{1}{4}}$$

where c does not depend on s. Analogous reasoning is valid for v^s_- and $\eta^s_\pm$. Since the integrals in (8) are continuous with respect to $v, \eta \in L^2(K, \mathbb{R}^{nm})$, then estimates (7) are valid for some $\mathcal{A}_\pm \in \mathcal{M}_\theta$, $v_\pm \in V^0$ and $\eta_\pm \in N^0$.

Extend again the functions $\mathcal{A}_\pm, v_\pm$ and $\eta_\pm$ via K-periodicity to the whole $\mathbb{R}^n$ (now $v_\pm \in V^0$ and $\eta_\pm \in N^0$). Define on

$$2K = \Big\{ x \in \mathbb{R}^n \,\Big|\, x = (x_1, ..., x_n) \text{ with } 0 < x_i < 2 \ \ (i = 1, ..., n) \Big\}$$

the matrix $\tilde{\mathcal{A}}$ and elements $\tilde{v}, \tilde{\eta}, \tilde{\eta}_0$ as

$$\begin{aligned}
\tilde{\mathcal{A}}(x) &= \mathcal{A}_*(2x), & \mathcal{A}_*(x) &= \begin{cases} \mathcal{A}_+(x) & \text{if } 0 < x_2 < 1 \\ \mathcal{A}_-(x) & \text{if } 1 \leq x_2 < 2 \end{cases} \\
\tilde{v}(x) &= v_*(2x), & v_*(x) &= \begin{cases} v_+(x) & \text{if } 0 < x_2 < 1 \\ v_-(x) & \text{if } 1 \leq x_2 < 2 \end{cases} \\
\tilde{\eta}(x) &= \eta_*(2x), & \eta_*(x) &= \begin{cases} \eta_+(x) & \text{if } 0 < x_2 < 1 \\ \eta_-(x) & \text{if } 1 \leq x_2 < 2 \end{cases} \\
\tilde{\eta}_0(x) &= \eta_0(x), & \eta_{0*}(x) &= \eta_0\Big(\frac{x}{2}\Big).
\end{aligned}$$

By construction, $\tilde{\mathcal{A}} \in \mathcal{M}_\theta$, $\tilde{v} \in V^\#$, $\tilde{\eta}, \tilde{\eta}_0 \in N^\#$ and

$$\begin{aligned}
&F_\theta(\xi, \zeta + \lambda\zeta_0) + F_\theta(\xi, \zeta - \lambda\zeta_0) \\
&\geq \frac{2}{2^n} \int_{2K} \Big\{ \big(\mathcal{A}_*(x)(v_* + \xi), v_* + \xi\big) + \big(\mathcal{A}_*^{-1}(x)(\eta_* + \zeta + \eta_{0*}), \eta_* + \zeta + \eta_{0*}\big) \Big\} dx - 2\varepsilon \\
&= 2 \int_K \Big\{ \big(\tilde{\mathcal{A}}(x)(\tilde{v} + \xi), \tilde{v} + \xi\big) + \big(\tilde{\mathcal{A}}^{-1}(x)(\tilde{\eta} + \tilde{\eta}_0 + \xi), \tilde{\eta} + \tilde{\eta}_0 + \zeta\big) \Big\} dx - 2\varepsilon \\
&\geq 2F_\theta(\xi, \zeta) - 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then

$$F_\theta(\xi, \zeta + \lambda\zeta_0) + F_\theta(\xi, \zeta - \lambda\zeta_0) \geq 2F_\theta(\xi, \zeta) \qquad \forall\lambda \in \mathbb{R}.$$

Thus, the function F_θ is separately convex and from Corollary 1 it follows that F_θ is continuously differentiable.

It is clear that the same reasoning is valid for the functions F_0 and F_θ defined by (2) and (3), respectively. Therefore, we have proved the following main result of this paper:

Theorem 3. *The functions F_0 and F_θ defined by* (2) *and* (3) *are continuously differentiable.*

Remark 3. Instead of integrands of the type $(A\xi, \xi) + (A^{-1}\xi, \xi)$ in (2) and (3) one can consider more general integrands $g : \mathbb{R}^{N'} \times \mathbb{R}^N \to \mathbb{R}$ which satisfy hypotheses (H5) - (H7) and the following additional hypothesis:

(H8) For every $\beta_0 \in B$ and every integer $s \in \mathbb{N}$ there exists $\beta_s \in B$ such that

$$g(\beta_s(x), z) = g(\tilde{\beta}_0(sx), z)$$

for all $(x, z) \in K \times \mathbb{R}^N$ where $\tilde{\beta}_0$ is the K-periodic extension of β_0 to the whole $\mathbb{R}^n$.

For simplicity of expressions only, we take $N = 2nm$. Then the function

$$F(\xi, \zeta) = \inf_{\beta \in B} \inf_{\substack{v \in V^\# \\ \eta \in N^\#}} \int_K g\Big(\beta(x), \xi + v(x), \zeta + \eta(x)\Big) dx \tag{9}$$

is upper semidifferentiable due to Lemmas 1 and 2. The proof of separate convexity of F is exactly the same as for F_θ. Moreover, this proof gives that in the definition of F by (9) (or in the definition of F_0 and F_θ by (2) and (3), respectively) the spaces $V^\#$ and $N^\#$ can be replaced by V^0 and N^0, respectively.

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