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On minimal surfaces bounded by two convex curves in parallel planes

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Abstract. We prove that a compact minimal surface bounded by two closed convex curves in parallel planes close enough to each other must be topologically an annulus.

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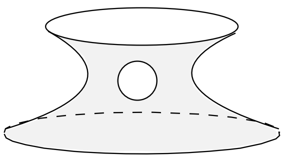
1. Introduction

Let Γ_1 and Γ_2 be two closed convex curves in parallel planes in euclidean space, and let M be a minimal annulus with boundary Γ_1 and Γ_2. In a celebrated paper [13], B. Shiffman proved that M is foliated by convex curves in planes parallel to the planes of Γ_1 and Γ_2. Moreover, if Γ_1 and Γ_2 are circles, then M is foliated by circles in parallel planes, and is therefore a piece of a catenoid or a Riemann minimal example.

It is natural to ask whether one can relax the hypothesis that M is an annulus, or if other topological types are possible:

Can two convex curves in parallel planes bound a compact minimal surface of genus ≥ 1?



W. Meeks has conjectured that the answer to this question is no. Here is what is known about this conjecture. Without loss of generality we may assume that Γ_1 and Γ_2 are in horizontal planes. R. Schoen [12] has proven that the conjecture is true (so the answer to the question is no) if Γ_1 and Γ_2 are both symmetric with respect to the vertical planes $x_1 = 0$ and $x_2 = 0$, using the Alexandrov moving plane technique.

A. Ros [11] has proven that the conjecture is true if Γ_2 is a vertical translate of Γ_1, using the Lopez–Ros deformation.

Even in the case of two circles with different axes, the conjecture seems to be open. Also using the bridge principle, one can construct examples of non-convex curves in parallel planes bounding a minimal surface of genus one.

In this paper, we study this problem in the case of two parallel planes close to each other. The question can be formulated more precisely as follows: let γ_1 and γ_2 be two convex curves in the horizontal plane $x_3 = 0$.

Is it true that if T is a small enough vertical translation, then $\gamma_1 \cup T(\gamma_2)$ does not bound any minimal surface of genus $k \geq 1$?

How small T must be should depend in some way on the given curves γ_1 and γ_2, because of the invariance by scaling of the minimal surface equation. The main result of the paper is the following

Theorem 1. *Let γ_1 and γ_2 be two smooth convex Jordan curves in the horizontal plane $x_3 = 0$, bounding respectively the convex domains Ω_1 and Ω_2. Fix some integer $k \geq 0$. Let $(M_n)_n$ be a sequence of compact, connected minimal surfaces of genus k with boundary γ_1 and $T_n(\gamma_2)$, where $(T_n)_n$ is a sequence of vertical translations. If $k = 0$, further assume that M_n is not the stable annulus.*

(1) (*Compactness*) *If $T_n \to T \neq 0$, then a subsequence of $(M_n)_n$ converges smoothly to a compact minimal surface of genus k bounded by γ_1 and $T(\gamma_2)$.*

(2) (*Concentration*) *If $T_n \to 0$, then there exists $k+1$ distinct points $p_1, \dots, p_{k+1}$ in $\Omega_1 \cap \Omega_2$ and a subsequence, still denoted $(M_n)_n$, such that the curvature of $(M_n)_n$ concentrates at $p_1, \dots, p_{k+1}$, in the following sense: for any small $\rho > 0$ it holds that*

$$\lim_{n\to\infty} C(M_n \cap B(p_i, \rho)) = 4\pi \quad \text{for all } i,$$

$$\lim_{n\to\infty} C\Big(M_n \setminus \bigcup_{i=1}^{k+1} B(p_i, \rho)\Big) = 0,$$

*where $B(p, \rho)$ denotes the euclidean ball and $C(U) = \int_U |K| dA$ denotes the total curvature of U. Moreover, the configuration $p_1, \dots, p_{k+1}$ is **balanced**, in an electrostatic sense which we explain in the next section.*

We will see that near a point of concentration, the surface looks in fact like a small catenoid, which explains the 4π mass of curvature.

In Section 2.3, we will prove that there are no balanced configurations in the genus one case ($k = 1$), so $T_n \to 0$ is impossible in this case. Hence, there exists $\varepsilon > 0$ (depending on γ_1 and γ_2) such that if $\|T\| < \varepsilon$, $\gamma_1 \cup T(\gamma_2)$ bounds no minimal surface of genus one. We will also give several partial results in the higher genus case, under various assumptions.

It is of course desirable to know how ε depends on γ_1 and γ_2. For this, one has to allow the curves γ_1 and γ_2 to depend on n. We will prove a more general result in this case, see Theorem 2.

Remark 1. If $\gamma_1 \cup T(\gamma_2)$ bounds a (connected) compact minimal surface M, then $\Omega_1 \cap \Omega_2$ cannot be empty. Also M is embedded, see Proposition 3 for these two facts.

Aknowledgements. I had interesting discussions about this problem with L. Hauswirth, D. Hoffman, L. Mazet and M. Wolf. I am especially grateful to L. Mazet for proving that there are no balanced configurations in the case $k \geq 1$, $\Omega_1 = \Omega_2$, see Proposition 2. I would like to thank R. Mazzeo for pointing out the fact that the Robin function of a convex domain is convex, which plays an important role in Section 2. Finally, I would like to thank the referee for a very careful reading of the paper, finding two errors, and suggesting reference [5].

2. Balanced configurations

Let Ω_1 and Ω_2 be two bounded domains in the plane with non-empty intersection. Let $G_{i,p}(z)$ denotes the Green function of Ω_i. Recall that $G_{i,p}(z)$ is harmonic in $\Omega_i \setminus \{p\}$ with zero boundary value and a logarithmic singularity at p. One can write

$$G_{i,p}(z) = \log|z-p| + H_{i,p}(z),$$

where the regular part $H_{i,p}(z)$ is harmonic in Ω_i. It is known that $G_{i,p}(z)$ is a symmetric function of (z, p).

Given $k+1$ distinct points $\{p_1, \dots, p_{k+1}\}$ in $\Omega_1 \cap \Omega_2$, let us define forces by

$$F_i = \nabla H_{1,p_i}(p_i) + \nabla H_{2,p_i}(p_i) + \sum_{j \neq i} \left(\nabla G_{1,p_j}(p_i) + \nabla G_{2,p_j}(p_i)\right).$$

Definition 1. We say the configuration $\{p_1, \dots, p_{k+1}\}$ is balanced if $F_i = 0$ for $i = 1, \dots, k+1$.

When $\Omega_1 = \Omega_2 = \Omega$, one can interpret F_i as 2-dimensional electrostatic forces. The physical model is the following: we have a 2-dimensional vacuum chamber Ω, whose boundary is made of a conductor metal. We put inside some unit positive charges at $p_1 \dots, p_{k+1}$. These charges induce a continuous charge on the boundary. Then F_i is the force resulting of the interaction of p_i with the other particles and with the boundary.

Conjecture 1. If Ω_1 and Ω_2 are convex domains and $k \geq 1$, there are no balanced configuration with $k+1$ points.

We will prove that the conjecture is true in the case $k = 1$, and give some partial results in the case $k \geq 2$. If we relax the convexity condition, then balanced configurations are possible. We will see an example in Section 2.5.

2.1. Facts about the Green function of a convex domain. In this section we collect several results about the Green function $G_p(z)$ of a bounded, convex domain Ω. We write $G_p(z) = \log|z-p| + H_p(z)$, where $H_p(z)$ is the regular part of the Green function. The Robin function of Ω is defined by

$$\mathrm{Rob}(z) = H_z(z).$$

The critical points of the Robin function are called the *harmonic centers* of Ω. Since the Robin function goes to $+\infty$ on the boundary, any bounded domain has at least one harmonic center (a minimum). A very useful fact is the following:

The Robin function of a convex domain is convex.

This has been proven by various authors, see [2] and the references therein. The referee pointed out that in fact the Robin function of a bounded convex domain is strictly convex, see [5]. Therefore, a bounded convex domain has a unique harmonic center.

If $f : \mathbb{D} \to \Omega$ is a conformal representation of a domain Ω on the unit disk, one can compute the Green function of Ω, its regular part and the Robin function in term of f:

$$\begin{aligned} G_{f(p)}(f(z)) &= \log|z-p| - \log|1-\bar{p}z|, \\ H_{f(p)}(f(z)) &= -\log\left|\frac{f(z)-f(p)}{z-p}\right| - \log|1-\bar{p}z|, \\ \mathrm{Rob}(f(z)) &= -\log|f'(z)| - \log(1-|z|^2). \end{aligned}$$

Another fact about the Green function of a convex domain which we will use is the following

Lemma 1. *Let Ω be a convex domain. Then for any $p \in \Omega$, the level lines of G_p are convex curves.*

Proof. This is very likely well known, but I could not find a reference in the literature, so I provide a proof. Fix some point $p \in \Omega$. Let $f : \mathbb{D} \to \Omega$ be a conformal representation of Ω such that $f(0) = p$. Then $G_p(f(z)) = \log|z|$, so f sends the circles centered at the origin to the level lines of G_p. Fix some $r \in (0,1)$ and let $\gamma_r(t) = f(re^{\mathrm{i}t})$. The image of γ_r is convex if $\arg \gamma_r'(t)$ is increasing. We have

$$\begin{aligned}(\arg \gamma_r'(t))' &= (\operatorname{Im} \log \gamma_r'(t))' \\ &= \operatorname{Im}\left(\frac{\gamma''(t)}{\gamma'(t)}\right) \\ &= \operatorname{Im}\left(\mathrm{i}\,\frac{f''(re^{\mathrm{i}t})}{f'(re^{\mathrm{i}t})} re^{\mathrm{i}t}\right) + 1 := g(re^{\mathrm{i}t}),\end{aligned}$$

where the function g is harmonic in $\mathbb{D}$, since f' does not vanish. When $r = 1$, $\arg \gamma_1'(t)$ is increasing because Ω is convex. Hence g is non-negative on the unit circle. By the maximum principle, g is positive in the disk, so the image of γ_r is strictly convex if $r < 1$. □

2.2. Genus zero. In this section we discuss the case $k = 0$, so there is only one point p_1. We write $\mathrm{Rob}_i(z)$ for the Robin function of Ω_i. By symmetry of the Green function, $\nabla \mathrm{Rob}_i(z) = 2\nabla H_{i,z}(z)$, so

$$F_1 = \frac{1}{2}(\nabla \mathrm{Rob}_1(p_1) + \nabla \mathrm{Rob}_2(p_1)).$$

The configuration is balanced if p_1 is a critical point of $\mathrm{Rob}_1 + \mathrm{Rob}_2$. Now the function $\mathrm{Rob}_1 + \mathrm{Rob}_2$ is strictly convex on $\Omega_1 \cap \Omega_2$, so it has a unique critical point (a minimum).

Returning to minimal surfaces, it is known that two convex curves in parallel planes bound at most two minimal annuli, one stable and one unstable [8]. Our result describes what happens to the unstable annulus when the distance between the planes goes to zero: the curvature concentrates at the minimum of the function $\mathrm{Rob}_1 + \mathrm{Rob}_2$.

2.3. Genus one

Proposition 1. *Let $k \geq 1$. If Ω_1 and Ω_2 are convex, then there are no balanced configurations with $k+1$ points, all on the same line L.*

Proof. We may assume that the points $p_1, \dots, p_{k+1}$ are in this order on L. Let $R = \frac{1}{2}(\mathrm{Rob}_1 + \mathrm{Rob}_2)$. This is a convex function in $\Omega_1 \cap \Omega_2$. Hence the maximum value of R at the points $p_1, \dots, p_{k+1}$ is either achieved at p_1 or p_{k+1}, let us say p_1. We have

$$F_1 = \nabla R(p_1) + \sum_{j>1} \nabla G_{1,p_j}(p_1) + \nabla G_{2,p_j}(p_1).$$

The point p_2 is inside the convex domain $R(z) \leq R(p_1)$ so $\langle \nabla R(p_1), \overrightarrow{p_2 p_1} \rangle \geq 0$. Regarding the other terms, since p_j lies inside the domain $G_{1,p_j}(z) < G_{1,p_j}(p_1)$

which is convex by Lemma 1, we have $\langle \nabla G_{1,p_j}(p_1), \overrightarrow{p_j p_1}\rangle > 0$, and a similar statement holds for the Green function of Ω_2. Now all vectors $\overrightarrow{p_j p_1}$ are proportional to $\overrightarrow{p_2 p_1}$, with a positive coefficient, so we get $\langle F_1, \overrightarrow{p_2 p_1}\rangle > 0$. Hence the configuration cannot be balanced. □

In the case $k = 1$, since two points are always on a line, there are no balanced configurations. This gives:

Corollary 1 (Genus one case). *Given two smooth convex Jordan curves γ_1 and γ_2, there exists $\varepsilon > 0$ (depending on γ_1 and γ_2), such that for any vertical translation T with $\|T\| < \varepsilon$, $\gamma_1 \cup T(\gamma_2)$ cannot bound any compact minimal surface of genus one.*

2.4. Higher genus. In the case $k \geq 2$, we have a result under an additional symmetry assumption to ensure that the points $p_1, \dots, p_{k+1}$ are on a line:

Corollary 2. *Given two smooth convex Jordan curves γ_1 and γ_2, both symmetric with respect to a given line L, and some integer $k \geq 2$, there exists ε (depending on k, γ_1 and γ_2) such that for any vertical translation T with $\|T\| < \varepsilon$, $\gamma_1 \cup T(\gamma_2)$ cannot bound any compact minimal surface of genus k.*

Note that this corollary applies in particular to the interesting case of two circles.

Proof. Indeed, by a theorem of R. Schoen [12] (using Alexandrov's moving plane method), any minimal surface M with boundary $\gamma_1 \cup T(\gamma_2)$ will be symmetric with respect to the vertical plane P through L. Moreover, the part of M on each side of P is a graph over P. Hence if we have a sequence of minimal surfaces $(M_n)_n$ of genus k, with boundary $\gamma_1 \cup T_n(\gamma_2)$ with $T_n \to 0$, the curvature will concentrate at points $p_1, \dots, p_{k+1}$, all on the line L (this is because in a neighborhood of p_i, M_n looks like a small catenoid, as we shall see). By Proposition 1, we get a contradiction. □

Next we present a result which was discovered by L. Mazet.

Proposition 2. *Assume that Ω_1 and Ω_2 have the same harmonic center. Then there are no balanced configurations with two or more points.*

Note that the proposition applies in particular to the case where $\Omega_1 = \Omega_2$. (Of course, in this particular case, the Meeks conjecture is known to be true by the work of A. Ros [11], so we do not get a new result, regarding minimal surfaces.)

Proof. Let $f_i : \mathbb{D} \to \Omega_i$ be a conformal representation. We transport the hyperbolic metric $2|dz|/(1 - |z|^2)$ on the disk to get a hyperbolic metric $\lambda_i |dz|$ on Ω_i.

Explicitely,

$$\lambda_i(z) = \frac{1}{|f_i'(f_i^{-1}(z))|(1-|f_i(z)|^2)} = 2\exp(\mathrm{Rob}_i(z)), \quad z \in \Omega_i.$$

The hyperbolic distance d_{Ω_i} on Ω_i and the Green function are related by

$$G_{i,p}(z) = \log\tanh\frac{d_{\Omega_i}(z,p)}{2}.$$

This comes from the fact that the hyperbolic distance on the disk is given by

$$d_{\mathbb{D}}(z,p) = 2\mathrm{arctanh}\left|\frac{z-p}{1-\bar{p}z}\right|.$$

Without loss of generality, we may assume that 0 is the harmonic center of Ω_1 and Ω_2, and that $|p_1| \geq |p_j|$ for all j (so $p_1 \neq 0$). Since $\mathrm{Rob}_i(0) < \mathrm{Rob}_i(p_1)$ and the Robin function is strictly convex, we have

$$\langle \nabla \mathrm{Rob}_i(p_1), p_1 \rangle > 0.$$

Let us fix some indices $i = 1, 2$ and $j \geq 2$ and consider the geodesic γ from p_j to p_1 for the hyperbolic metric on Ω_i. We know that this geodesic is minimizing. Let τ be the tangent vector to this geodesic at p_1. I claim that $\langle \tau, p_1 \rangle \geq 0$. Indeed, if this is false, then since $|p_j| \leq |p_1|$, there exists a point $p \neq p_1$ on γ such that $|p| = |p_1|$, and $|z| > |p_1|$ on the sub-arc γ' of γ delimited by p and p_1. Then consider the radial projection π from γ' to the circle $C(0, |p_1|)$. By convexity, the Robin function, hence the conformal factor λ_i, is increasing on the segment $[0, z]$. Hence $\lambda_i(\pi(z)) < \lambda_i(z)$. Since the projection makes euclidean length smaller, the hyperbolic length of the circular arc from p to p_1 is smaller than the hyperbolic length of γ', which contradicts the fact that γ is minimizing. Now the gradient of $G_{i,p_j}(p_1)$ is proportional to τ, hence

$$\langle \nabla G_{i,p_j}(p_1), p_1 \rangle \geq 0.$$

This implies that $\langle F_1, p_1 \rangle > 0$, so the configuration cannot be balanced. □

2.5. Explicit computations. When we have an explicit conformal representation $f : \mathbb{D} \to \Omega$ of a domain Ω, we can compute explicitly the forces using the formulae in Section 2.1. It is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ and use complex notations, so $\nabla = 2\frac{\partial}{\partial \bar{z}}$. The Robin function of the domain Ω satisfies

$$\frac{\partial \mathrm{Rob}}{\partial z}(f(z)) \times f'(z) = -\frac{f''(z)}{2f'(z)} + \frac{\bar{z}}{1-|z|^2}.$$

Take $\Omega_1 = \Omega_2 = \Omega$ and consider a configuration $p_1, \dots, p_{k+1} \in \Omega$. Writing $p_i = f(z_i)$, $z_i \in \mathbb{D}$, the forces are given by

$$\overline{F_i} \times f'(z_i) = -\frac{f''(z_i)}{f'(z_i)} + 2\sum_{j \neq i} \frac{1}{z_i - z_j} - 2\sum_j \frac{1}{z_i - \frac{1}{\overline{z_j}}}.$$

We use these formula to provide counterexamples in the case of a non-convex domain. Consider for example

$$f(z) = \frac{1}{z-a} + \frac{1}{z+a},$$

where a is some real number. Provided $a > 1$ this is a conformal representation on the unit disk $\mathbb{D}$. When a is close enough to 1, the image $\Omega = f(\mathbb{D})$ is a non convex domain. Figure 1 shows this domain in the case $a = 5/4$.

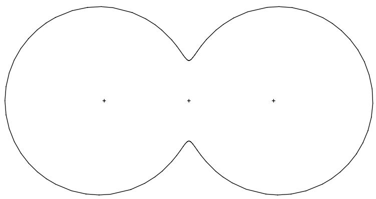


Figure 1. A non-convex domain admitting three harmonic centers.

Assume that z is real. Then using the above formula, $f(z)$ is a harmonic center if $z^3(1-3a^2) + z(3a^2 - a^4) = 0$. Solving for z and taking $f(z)$ gives three harmonic centers. These points are represented in Figure 1 when $a = 5/4$. With a little more computations, it is possible to check that there are no other harmonic centers (namely, $z \notin \mathbb{R}$).

We can also compute a balanced configuration with two points, assuming the following symmetry: $z_2 = -z_1 \in \mathbb{R}$. The balancing condition boils down to a degree four equation, which gives two balanced configurations. One of them is represented in Figure 2, still in the case $a = 5/4$. I do not know if there are other balanced configurations.

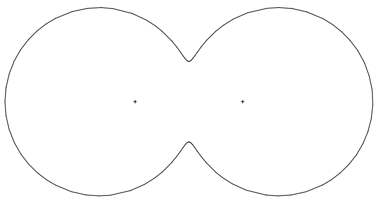


Figure 2. A balanced configuration with two points.

3. Proof of Theorem 1

3.1. Preliminaries. Throughout the paper we use the following notations. M is a compact embedded minimal surface of genus k, with boundary $\Gamma = \gamma_1 \cup T(\gamma_2)$, where T is a vertical translation of vector $(0, 0, t)$ and γ_1, γ_2 are two convex Jordan curves in the horizontal plane $x_3 = 0$. Ω_1 and Ω_2 denote the convex domains in the plane with boundary respectively γ_1 and γ_2. In case we have a sequence of minimal surfaces $(M_n)_n$, we label $\Gamma_n = \partial M_n$, $T_n = (0, 0, t_n)$ and $\gamma_{i,n} = \partial\Omega_{i,n}$ the corresponding quantities. (The genus k will always be fixed.)

The following proposition collects several elementary facts about minimal surfaces bounded by two convex curves in parallel planes.

Proposition 3. *Let M be a compact, connected minimal surface of genus k bounded by two convex curves γ_1 and $T(\gamma_2)$. Then the following holds.*

(1) *The total curvature $C(M)$ of M is at most $4\pi(k + 1)$.*

(2) *M is embedded, and for any ball $B(p, R)$, the area of $M \cap B(p, R)$ is less than $2\pi R^2$.*

(3) *$\Omega_1 \cap \Omega_2$ is not empty.*

(4) *M is contained in the intersection of the tubular neighborhood of radius t of $(\Omega_1 \cup \Omega_2) \times \mathbb{R}$ with the horizontal slab $0 < x_3 < t$.*

(5) *If M is not a stable annulus, then for any disk D of radius $\geq t$ included in $\Omega_1 \cap \Omega_2$, M intersects the vertical cylinder $D \times \mathbb{R}$.*

Proof. By the Gauss–Bonnet formula,

$$\int_M K + \int_{\partial M} \kappa_g = 2\pi\chi(M) = 2\pi(2 - 2k - 2).$$

This gives

$$C(M) = -\int_M K = 4\pi k + \int_{\partial M} \kappa_g.$$

Now it is well known that $|\kappa_g| \leq |\kappa|$, where κ denotes the curvature of the boundary. As each γ_i is a convex planar curve, $\int_{\gamma_i} |\kappa| = 2\pi$. This proves the first point. The second point is proven in [4], using the monotonicity formula for minimal surfaces with boundary. (Indeed, the boundary has total curvature 4π, and the density at p of the cone with vertex p generated by the boundary is less than 2. The fact that the boundary is not connected is not a problem, see Section 6 in [4].)

Regarding point 3, let us assume by contradiction that $\Omega_1 \cap \Omega_2 = \emptyset$. Let P be a vertical plane separating Ω_1 and Ω_2. Let M' be the symmetric of M with respect to P. Let us translate M' horizontally in the direction of P. Since M is connected,

M and M' will eventually intersect (maybe from the very beginning). First assume that $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$. Then the boundary of M and M' never intersect, nor does the boundary of one intersect the interior of the other, since the interiors are in the slab delimited by the two horizontal planes. Hence at a last contact point, M and M' are tangent, contradicting the maximum principle. If $\overline{\Omega}_1$ and $\overline{\Omega}_2$ intersect at some boundary point, one can slightly rotate M' about the horizontal line contained in P, so that the boundaries of M and M' do not intersect. The convex hull property guarantees that the boundaries will not intersect the interiors, and the same argument applies.

To prove point 4, let C be a horizontal circle of radius t in the horizontal plane $x_3 = 0$. There exists a catenoid A bounded by $C \cup T(C)$. (The radius t is not the smallest radius such that such a catenoid exists: the smallest value is about $0.754439t$. The constants in the proposition are not optimal.) If the circle C is disjoint from the convex hull of $\Omega_1 \cap \Omega_2$ then A does not intersect M. One can then slide C horizontally. As long as C remains disjoint from $\Omega_1 \cup \Omega_2$, A does not intersect M by the maximum principle. This proves point 4.

To prove point 5, assume by contradiction that there exists a disk $D \subset \Omega_1 \cap \Omega_2$ of radius t such that M does not intersect the vertical cylinder $D \times \mathbb{R}$. Let C be the boundary of the disk D. Let us foliate each $\Omega_i \setminus D$, $i = 1, 2$, by convex curves $\gamma_{i,s}$, $s \in [0, 1]$, so that $\gamma_{i,0} = C$ and $\gamma_{i,1} = \partial\Omega_i$. From the existence of a catenoid bounded by C and $T(C)$ and Lemma 2.1 in [8], there exists, for each $s \in [0, 1]$, a unique stable annulus A_s bounded by $\gamma_{1,s} \cup T(\gamma_{2,s})$. Moreover, as A_s is stable and unique, it depends continuously on s by standard results (namely, curvature estimates for stable minimal surfaces). By the maximum principle, M is disjoint from A_s for all $s \in [0, 1)$. By point 1 of Lemma 2.1 in [8], M is contained in the compact domain bounded by A_1, Ω_1 and $T(\Omega_2)$. Hence $M = A_1$. □

3.2. Main theorem. In this section we state a slightly more general result than Theorem 1, allowing the domains to depend on n.

Let $(\gamma_{1,n})_n$ and $(\gamma_{2,n})_n$ be two sequences of smooth convex Jordan curves in the plane, bounding the domains $\Omega_{1,n}$ and $\Omega_{2,n}$ respectively. Let T_n be a sequence of vertical translations and $(M_n)_n$ be a sequence of minimal surfaces of fixed genus k with boundary $\gamma_{1,n} \cup T_n(\gamma_{2,n})$. If $k = 0$, assume further that M_n is not a stable minimal annulus. By point 3 of Proposition 3, each $\Omega_{1,n} \cap \Omega_{2,n}$ is non-empty. We assume that the in-radius of $\Omega_{1,n} \cap \Omega_{2,n}$ is greater than $r > 0$, for some r independent of n. We also assume that $\Omega_{1,n}$ and $\Omega_{2,n}$ are included in the disk $D(0, R)$ for some R independent of n. Finally, we assume that the curvature of $\gamma_{1,n}$ and $\gamma_{2,n}$ is bounded by some constant independent of n. Passing to a subsequence, $(\gamma_{1,n})_n$ and $(\gamma_{2,n})_n$ converge to two convex Jordan curves γ_1 and γ_2, bounding respectively two convex domains Ω_1 and Ω_2 with non-empty intersection (thanks to the hypothesis on the in-radius).

Theorem 2. *In the above setup:*

(1) (*Compactness*) *If* $T_n \to T \neq 0$, *then a subsequence of* $(M_n)_n$ *converges smoothly to a compact minimal surface of genus* k *bounded by* γ_1 *and* $T(\gamma_2)$.

(2) (*Concentration*) *If* $T_n \to 0$, *then there exists* $k+1$ *distinct points* $p_1, \dots, p_{k+1}$ *in* $\Omega_1 \cap \Omega_2$ *and a subsequence, still denoted* $(M_n)_n$, *such that the curvature of* $(M_n)_n$ *concentrates at* $p_1, \dots, p_{k+1}$, *in the sense of Theorem* 1. *Moreover, the configuration* $p_1, \dots, p_{k+1}$ *is* balanced, *in the sense of Definition* 1.

As a consequence, the constant ε in Corollaries 1 and 2 depends on the following quantities: the genus k of M, a bound on the curvature of γ_1 and γ_2, a bound on their diameter, and a lower bound on the in-radius of $\Omega_1 \cap \Omega_2$.

Proof of point 1 *of Theorem* 2. By points 1 and 2 of Proposition 3, we have uniform area and total curvature estimates. By a standard compactness result, (namely by points 1, 2, 3 of Theorem 3 in [14]), there exists a subsequence of $(M_n)_n$, still denoted $(M_n)_n$, and a finite set S in $\mathbb{R}^3$, such that M_n converges on compact subsets of $\mathbb{R}^3 \setminus S$ to an embedded minimal surface M with boundary included in $\Gamma = \gamma_1 \cup T(\gamma_2)$. Moreover, M must be connected, else M_n is not connected for n large enough. If M is flat, then its boundary lies in a plane, so M is either in the plane $x_3 = 0$ or $x_3 = t$. Since $t \neq 0$, this contradicts the fact than $\partial M_n = \gamma_{1,n} \cup T_n(\gamma_{2,n})$. So M is not flat. Let us see that the multiplicity of the limit $M_n \to M$ is one. The multiplicity is well defined and constant in each component of $M \setminus \Gamma$. Let U be a component of $M \setminus \Gamma$ where the multiplicity m is maximal, and assume that $m \geq 2$. Let p be a point on $\partial U \subset \Gamma$. For small $r > 0$, $B(p,r) \cap M_n$ has m components. One of them meets ∂M_n. The others do not, and are graphs over $T_p M$ of functions which converge uniformly to the function which expresses locally M as a graph over $T_p M$. Since M_n lies in the horizontal slab $0 \leq x_3 \leq t_n$, $T_p M$ must be horizontal. By the boundary maximum principle, since M lies in the slab $0 \leq x_3 \leq t$, M is flat, a contradiction. Hence $M_n \to M$ with multiplicity one. By the proof of point 4 in Theorem 3 in [14], the singular set S is empty. This proves point 1 of Theorem 2. □

The remaining of the paper is devoted to the proof of point 2 of Theorem 2.

3.3. Limits under scaling. Let $(M_n)_n$ be a sequence of minimal surfaces as in the paragraph before Theorem 2. Let $(h_n)_n$ be a sequence of homotheties of $\mathbb{R}^3$, with ratio diverging to ∞ as $n \to \infty$, and let $\widetilde{M}_n = h_n(M_n)$. The goal of this section is to prove that the limit of $(\widetilde{M}_n)_n$ is either flat or a catenoid.

Let $\tilde{\gamma}_{1,n} = h_n(\gamma_{1,n})$, $\tilde{\gamma}_{2,n} = h_n(T_n(\gamma_{2,n}))$ and $\widetilde{\Gamma}_n = \partial \widetilde{M}_n = \tilde{\gamma}_{1,n} \cup \tilde{\gamma}_{2,n}$. Note that since the curvature of $\gamma_{i,n}$ is uniformly bounded, the curvature of $\tilde{\gamma}_{1,n}$ goes to zero as $n \to \infty$. If $\tilde{\gamma}_{i,n}$ has an accumulation point, then a subsequence of $(\tilde{\gamma}_{i,n})_n$ converges on compact subsets of $\mathbb{R}^3$ to a horizontal line L_i. Hence passing to a subsequence, $(\widetilde{\Gamma}_n)_n$ converges to a set $\widetilde{\Gamma}$ which consists of zero, one or two horizontal lines. (When $\widetilde{\Gamma} = \emptyset$ this means that for any $R > 0$, $\widetilde{\Gamma}_n$ is outside the ball $B(0, R)$ for n large enough.)

By Theorem 3 in [14], there exists a finite set S in $\mathbb{R}^3$ and a subsequence of $(\widetilde{M}_n)_n$, still denoted the same, which converges on compact subsets of $\mathbb{R}^3 \setminus (S \cup \widetilde{\Gamma})$ to a minimal surface $\widetilde{M}$ with boundary included in $\widetilde{\Gamma}$. Note that $\widetilde{M}$ can be disconnected.

Proposition 4. *If $\widetilde{M}$ has a non-flat component, then S and $\widetilde{\Gamma}$ are empty and $\widetilde{M}$ is a catenoid.*

Proof. There are three cases, depending on whether $\widetilde{\Gamma}$ is empty, one line or two lines.

First case: $\widetilde{\Gamma}$ is empty. Then one component of $\widetilde{M}$ is a complete, embedded, non-flat minimal surface with finite total curvature. From the area estimate, point 2 of Proposition 3, it has at most two ends. Therefore it is a catenoid by the Theorem of R. Schoen [12]. By embeddedness $\widetilde{M}$ has no other component. Since the catenoid is unstable, the multiplicity of the limit $\widetilde{M}_n \to \widetilde{M}$ is one by a standard argument (Proposition 4.2.1 in [9]), and the singular set S is empty (Proposition 1.0.1 in [9], see also the end of the proof of Theorem 4.3.2).

Second case: $\widetilde{\Gamma}$ consists of one line L. Since M_n lies in the slab $0 \le x_3 \le t_n$, $\widetilde{M}$ lies in a half space bounded by the horizontal plane Π containing L. Extending $\widetilde{M}$ by reflection in L, we obtain a non-flat, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature. By Theorem 2.2.1 in [9], a non-flat, embedded minimal surface of finite total curvature cannot intersect a plane along a line, so we get a contradiction. (This theorem uses the argument of J. Choe and M. Soret in [3].)

Third case: $\widetilde{\Gamma}$ consists of two lines L_1 and L_2. Since M_n lies in the slab $0 \le x_3 \le t_n$, $\widetilde{M}$ lies in the slab bounded by the horizontal planes containing L_1 and L_2. Since $\widetilde{M}$ is non-flat, these two lines do not lie in the same horizontal plane. Hence we may assume that L_1 lies in the plane $x_3 = 0$ and L_2 lies in the plane $x_3 = 1$. The horizontal projections of $\tilde{\gamma}_{1,n}$ and $\tilde{\gamma}_{2,n}$ bound some convex domains $\widetilde{\Omega}_{1,n}$ and $\widetilde{\Omega}_{2,n}$, let $H_i = \lim \widetilde{\Omega}_{i,n}$. Then H_1 and H_2 are half planes, whose boundary lines are the horizontal projections of L_1 and L_2. By point 4 of Proposition 3 applied to $\widetilde{M}_n$ and letting $n \to \infty$, $\widetilde{M}$ is inside the tubular neighborhood of radius one of $(H_1 \cup H_2) \times \mathbb{R}$. By point 5 of the same proposition, for any disk D of radius 1 contained in $H_1 \cap H_2$, $\widetilde{M}$ intersects $D \times \mathbb{R}$. Let me call these two properties, respectively, property A and property B. Roughly speaking, property A means that the horizontal projection of $\widetilde{M}$ is contained in $H_1 \cup H_2$, and property B means that it contains $H_1 \cap H_2$ (although not quite). As we shall see, properties A and B severely restrict the possibilities for the limit $\widetilde{M}$.

Note that in case L_1 and L_2 are parallel, the boundaries of Ω_1 and Ω_2 are tangent at some point p. Since Ω_1 and Ω_2 are convex with non-empty intersection, they lie on the same side of this tangent line. Therefore, $H_1 \subset H_2$ or $H_2 \subset H_1$ (in other words, $H_1 \cap H_2$ is not a strip).

Let θ be the angle between L_1 and L_2. Extending $\widetilde{M}$ by reflection in L_1 and L_2, we obtain an embedded minimal surface $\widehat{M}$ in $\mathbb{R}^3/S_{2\theta}$, where $S_{2\theta}$ is a vertical screw motion of angle 2θ if $\theta \neq 0$, and a translation (maybe not vertical) in case $\theta = 0$. Since $\widehat{M}$ has finite total curvature, a theorem of W. Meeks and H. Rosenberg [7] says that its ends are all simultaneously of type Scherk, helicoid or planar. We deal with each case separately.

First case: $\widehat{M}$ has Scherk type ends. Then the horizontal projection of $\widetilde{M}$ stays at bounded distance from a finite set of half-lines, contradicting property B.

Second case: $\widehat{M}$ has helicoidal ends. In the case $\theta = 0$, the period must be vertical, since $\widetilde{M}$ lies in a horizontal slab. Each asymptotic half-helicoid intersects the horizontal plane $x_3 = 0$ along a half-line. Since $\widehat{M}$ intersects the plane $x_3 = 0$ along the line L_1, it has precisely two helicoidal ends. Outside of a vertical cylinder, $\widetilde{M}$ has two components, each asymptotic to a piece of a helicoid, and having two half-lines on its boundary. Let us write $L_1 = L_1' \cup L_1''$ and $L_2 = L_2' \cup L_2''$, where L_1', L_2', L_1'' and L_2'' are half-lines defined as in Figure 3. Let E_1 and E_2 be the two pieces of helicoid that $\widetilde{M}$ is asymptotic to, labeled so that E_1 has L_1' on its boundary. Note that by property A, none of them can make a full turn, and by property B, both cover $H_1 \cap H_2$. If E_1 climbs from L_1' to L_2', then E_2 must climb from L_1'' to L_2'' (this can happen only in the non-parallel case). But then none of them covers $H_1 \cap H_2$. Hence E_1 must climb from L_1' to L_2'', and E_2 from L_1'' to L_2'. But then E_1 and E_2 intersect, which contradicts embeddedness. Therefore, $\widehat{M}$ cannot have helicoidal ends.

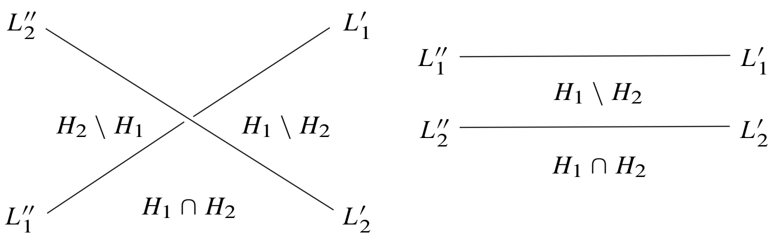


Figure 3. Definition of the four half-lines in the case where L_1 and L_2 are not parallel (left) or parallel (right).

Third case: $\widehat{M}$ has planar ends. Since $\widetilde{M}$ lies in the slab $0 \leq x_3 \leq 1$, the ends must be asymptotic to horizontal planes. By a theorem of Y. Choe and M. Soret [3], $\theta = 0$, so the lines L_1 and L_2 are parallel. We may assume without loss of generality that H_1 is the half-plane $x_1 > 0$, then H_2 is the half-plane $x_1 > a$ for some a. So $\widetilde{M}$ is asymptotic to the half planes $x_3 = 0$, $x_1 > 0$ and $x_3 = 1$, $x_1 > a$. Note that because of this, $\widehat{M}$ cannot be a Riemann minimal example: indeed the part of a Riemann minimal example between two consecutive horizontal lines is asymptotic to two half horizontal planes pointing into opposite directions.

To obtain a contradiction, we use the argument of Choe and Soret, as explained in [9]. We may assume that the stereographically projected Gauss map g takes on the value 0 at the end at height $x_3 = 0$. Then by embeddedness, it must take on the value ∞ at the other end. Note that g is real on L_1 and L_2.

By the boundary maximum principle for $\widetilde{M}$, $g \neq 0, \infty$ on L_1, so g has constant sign along L_1. Close to the end, $\widetilde{M}$ lies in $x_2 > 0$, $x_3 > 0$, so we have $g > 0$ on L_1. By a similar argument, g is also positive on L_2.

Arguing as in [9], for $\varepsilon > 0$ small enough, the intersection of $\widetilde{M}$ with $0 < x_3 < \varepsilon$ is conformally an annulus $1 < |z| < r$ for some $r > 1$, with $x_3 = 0$ on $|z| = 1$ and $x_3 = \varepsilon$ on $|z| = r$. Since x_3 is harmonic, it follows that $x_3 = \lambda \log |z|$ with $\lambda = \varepsilon / \log r$, and

$$\phi_3 = 2\frac{\partial x_3}{\partial z} dz = \lambda \frac{dz}{z}.$$

(In [9], the authors claim that $\lambda = 1$, but this is only the case after a suitable scaling of the surface.) If γ is a closed curve on $\widehat{M}$, let us define

$$F(\gamma) = \mathrm{i} \int_\gamma \overline{g^{-1}\phi_3} = \mathrm{i} \int_\gamma g\phi_3.$$

(These two integrals are equal because γ is closed. $F(\gamma)$ represents the horizontal part of the flux along γ, seen as a complex number.) Let γ_s be the curve $x_3 = s$ on $\widehat{M}$, oriented as a boundary of $x_3 < s$. Then in the conformal representation, γ_ε is the circle $|z| = r$, with the positive orientation. Since g is holomorphic in $1 \leq |z| \leq r$,

$$F(\gamma_\varepsilon) = \mathrm{i} \int_{|z|=r} g\phi_3 = \mathrm{i} \int_{|z|=1} g\phi_3 = \mathrm{i} \int_{\theta=0}^{2\pi} g(e^{i\theta})\lambda \mathrm{i}\, d\theta < 0.$$

In the same way, we can represent conformally the intersection of $\widetilde{M}$ with $1 - \varepsilon < x_3 < 1$ with an annulus $1 < |z| < r$ for some other $r > 1$, with $x_3 = 1 - \lambda \log |z|$ and $\phi_3 = -\lambda dz/z$. The level curve $\gamma_{1-\varepsilon}$ corresponds to the circle $|z| = r$, with the negative orientation:

$$F(\gamma_{1-\varepsilon}) = -\mathrm{i} \int_{|z|=r} \overline{g^{-1}\phi_3} = -\mathrm{i} \int_{|z|=1} \overline{g^{-1}\phi_3} = -\mathrm{i} \int_{\theta=0}^{2\pi} (g(e^{i\theta}))^{-1}\lambda \mathrm{i}\, d\theta > 0.$$

However, $F(\gamma_\varepsilon) = F(\gamma_{1-\varepsilon})$ because the two curves are homologous. Hence we have a contradiction. □

From Proposition 4 we get the following

Proposition 5. *Let $(M_n)_n$ be a sequence of minimal surfaces as in the paragraph before Theorem 2. Let $(h_n)_n$ be a sequence of homotheties of $\mathbb{R}^3$, and let $\widetilde{M}_n = h_n(M_n)$. There exists a finite set S and a subsequence of $(\widetilde{M}_n)_n$ (still denoted $(\widetilde{M}_n)_n$) such that the curvature of $(\widetilde{M}_n)_n$ is uniformly bounded on the compacts of $\mathbb{R}^3 \setminus S$. Moreover, for any $p \in S$ and any $r > 0$, it holds that*

$$\limsup C(\widetilde{M}_n \cap B(p,r)) \geq 4\pi.$$

The point of this proposition is that for any point of concentration of curvature, the amount of total curvature which concentrates at this point is always at least 4π. This is well known for interior points of concentration (see for instance the proof of Theorem 3 in [14] or Theorem 4.3.1 in [9]), but wrong in general for boundary points of concentration. For example, if γ_1 and γ_2 are two convex curves which intersect at a finite number of points and A_n is the *stable* annulus bounded by $\gamma_1 \cup T_n(\gamma_2)$ with $T_n \to 0$, then the curvature concentrates at the intersection points of γ_1 and γ_2. The mass of curvature that concentrates at each point is equal to twice the angle between the curves at this point. (A blow-up would produce pieces of helicoids.)

This proposition can be proven exactly as Theorem 4.3.1 in [9], using a standard blowup argument. By Proposition 4, the only limits which can appear are catenoids, whence the 4π. We omit the details.

3.4. Weak limit. In this section, we adapt the weak compactness result of A. Ros [10] (in the case of complete embedded minimal surfaces of finite total curvature) to our case, namely when there is a boundary.

Proposition 6. *Let $(M_n)_n$ be as in the paragraph before Theorem 2 and assume that $T_n \to 0$. There exists a subsequence, still denote $(M_n)_n$, and $k+1$ sequences of homotheties $(h_{i,n})_n$, $1 \leq i \leq k+1$, such that the following is true:*

1) *$h_{i,n}(M_n)$ converges smoothly on compact subsets of $\mathbb{R}^3$ to a vertical catenoid, with multiplicity one.*
2) *For any small $\varepsilon > 0$, there exists $R > 0$, independent of n, such that if we let $B_{i,n} = h_{i,n}^{-1}(B(0,R))$, then $C(M_n \setminus \bigcup_i B_{i,n}) \leq \varepsilon$.*
3) *For n large enough, the balls $B_{i,n}$ are disjoint and $M_n \setminus \bigcup B_{i,n}$ has two components $U_{1,n}$ and $U_{2,n}$. Each $U_{j,n}$ is a graph over $\Omega_{j,n}$ minus $k+1$ small convex disks, for $j = 1, 2$.*

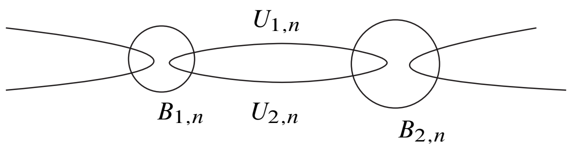


Figure 4. Weak limit, genus one.

Remark 2. This proposition implies that $\lim C(M_n) = 4\pi(k+1)$.

Proof. We follow the main lines of the argument of A. Ros, adapted to the case of minimal surfaces with boundary. Passing to a subsequence, we may assume that $\lim C(M_n)$ exists. We write $\lim C(M_n) = 4\pi\ell + \alpha$ with $\ell \in \mathbb{N}$ and $0 \leq \alpha < 4\pi$. We first prove the following partial statement:

Claim 1. *In the above setup, $\alpha = 0$ and there exists ℓ sequences of homotheties $(h_{i,n})_n$, such that for $1 \leq i \leq \ell$, $(h_{i,n}(M_n))_n$ converges on compact subsets of $\mathbb{R}^3$ to a catenoid, with multiplicity one.*

Proof. The idea of A. Ros to detect where the curvature concentrates is to look at balls B such that $C(M_n \cap B) = 2\pi$, and to select the smallest such ball. It turns out that the value 2π is not important for this argument: any *fixed* value $\mu \in (0, 4\pi)$ works fine. We choose μ as follows: if $\alpha = 0$, we take $\mu = 2\pi$. If $\alpha > 0$, we take $\mu = \alpha/2$, and we want to get a contradiction. (In what follows, when we use the word "small", it means: "small compared to μ". It is therefore important that μ is fixed once for all.)

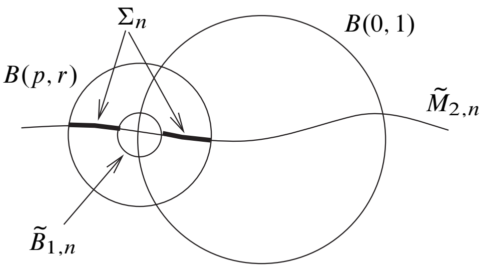
First step. If $\ell = \alpha = 0$, then the claim is trivially true. Else $\lim C(M_n) > \mu$, hence for n large enough, the family of balls B such that $C(M_n \cap B) = \mu$ is non-empty. Let $B'_{1,n}$ be a ball of minimum radius in this family. Let $h_{1,n}$ be the homothety such that $h_{1,n}(B'_{1,n}) = B(0,1)$ and let $\widetilde{M}_{1,n} = h_{1,n}(M_n)$. Note that $C(\widetilde{M}_{1,n} \cap B(0,1)) = \mu$ and that $B(0,1)$ is a smallest ball with this property. By Proposition 5, passing to a subsequence, there exists a finite set S such that $(\widetilde{M}_{1,n})_n$ converges to a minimal surface $\widetilde{M}_1$ on compact subsets of $\mathbb{R}^3 \setminus S$. If $p \in S$, then by Proposition 5, $C(\widetilde{M}_{1,n} \cap B(p, \frac{1}{2})) > \mu$ for n large enough. As this contradicts the choice of $B'_{1,n}$, S must be empty. Since $C(\widetilde{M}_{1,n} \cap B(0,1)) = \mu > 0$, $\widetilde{M}_1$ cannot be flat. If the ratio of $h_{1,n}$ were bounded, then since $T_n \to 0$, $\widetilde{M}_1$ would be included in the horizontal plane, hence flat. Hence the ratio of $h_{1,n}$ is not bounded. By Proposition 4, $\widetilde{M}_1$ is a catenoid and the multiplicity of the limit is one. Given $\varepsilon > 0$, there exists $R_1 > 0$ such that $|C(\widetilde{M}_1 \cap B(0,R_1)) - 4\pi| \leq \varepsilon/2$. From the smooth convergence of $(\widetilde{M}_{1,n})_n$ to $\widetilde{M}_1$ on $B(0,R_1)$, we get $|C(\widetilde{M}_{1,n} \cap B(0,R_1)) - 4\pi| \leq \varepsilon$ for n large enough. Let $B_{1,n} = h_{1,n}^{-1}(B(0,R_1))$. Then $|C(M_n \cap B_{1,n}) - 4\pi| \leq \varepsilon$. In particular $\lim C(M_n) \geq 4\pi$ and $\ell \geq 1$. This concludes the first step of the weak limit process.

Second step. If $\ell = 1$ and $\alpha = 0$ then we are done. Else $\lim C(M_n) > 4\pi + \mu$. By taking ε small enough, for n large enough, the family of balls B such that $C([M_n \setminus B_{1,n}] \cap B) = \mu$ is non-empty. Let $B'_{2,n}$ be a ball of minimum radius in this family. Let $h_{2,n}$ be the homothety such that $h_{2,n}(B'_{2,n}) = B(0,1)$ and let $\widetilde{M}_{2,n} = h_{2,n}(M_n)$. Let $\widetilde{B}_{1,n} = h_{2,n}(B_{1,n})$. By construction, the radius of $B'_{1,n}$ is at most the radius of $B'_{2,n}$. Hence $\widetilde{B}_{1,n}$ is a ball of radius at most R_1. Passing to a subsequence, the center of $\widetilde{B}_{1,n}$ either converges, or goes to infinity. We treat each case separately.

First case: The center of $\widetilde{B}_{1,n}$ diverges. Then we can argue as in the first step and conclude that $(\widetilde{M}_{2,n})_n$ converges on compact subsets of $\mathbb{R}^3$ to a catenoid $\widetilde{M}_2$. There exists $R_2 > 0$ such that $|C(M_n \cap B_{2,n}) - 4\pi| \leq \varepsilon$, where $h_{2,n}(B_{2,n}) = B(0, R_2)$. For n large enough, $B_{1,n}$ and $B_{2,n}$ are disjoint. Hence $\lim C(M_n) \geq 8\pi$.

Second case: The center of $\widetilde{B}_{1,n}$ converges to a point p. In this case we want to obtain a contradiction. Passing to a subsequence, the radius of $\widetilde{B}_{1,n}$ has a limit r. If $r > 0$, then from the convergence of $\widetilde{M}_{1,n}$ to a catenoid, we obtain that $C(M_n \cap B'_{2,n}) \leq \varepsilon$. This contradicts the definition of $B'_{2,n}$. Hence $r = 0$ and the sequence of balls $(\widetilde{B}_{1,n})_n$ collapses into the point p. The sequence $(\widetilde{M}_{2,n})_n$ converges to $\widetilde{M}_2$ with singular set S. Clearly $p \in S$, and in fact $S = \{p\}$, else we contradict the choice of $B'_{2,n}$ as in the first step. Since S is non-empty, all components of $\widetilde{M}_2$ are flat by Proposition 4. If $p \notin \overline{B(0,1)}$, then from the convergence of $(\widetilde{M}_{2,n})$ to a flat minimal surface on compact subsets of $\mathbb{R}^3 \setminus \{p\}$, $C(\widetilde{M}_{2,n} \cap B(0,1)) \to 0$. This contradicts the choice of $B'_{2,n}$. Hence $p \in \overline{B(0,1)}$.

Fix a small $r > 0$. For n large enough, $\widetilde{B}_{1,n} \subset B(p,r)$. Let $\Sigma_n = \widetilde{M}_{2,n} \cap B(p,r) \setminus \widetilde{B}_{1,n}$. From the smooth convergence of $(\widetilde{M}_{2,n})_n$ to a flat limit on $B(0,1) \setminus B(p,r)$, we have $\lim C(\widetilde{M}_{2,n} \cap B(0,1) \setminus B(p,r)) = 0$. Hence $\lim C(\Sigma_n \cap B(0,1)) = \mu$. If $\lim C(\Sigma_n) > \mu$, then since $r < 1$ we contradict the minimality of $B(0,1)$. Hence $\lim C(\Sigma_n) = \mu$.



By looking at the Gauss image of Σ_n, we shall see that $\lim C(\Sigma_n)$ is a multiple of 4π, thus obtaining a contradiction. The boundary of Σ_n is included in the union of the boundaries of $\widetilde{B}_{1,n}$, $B(p,r)$ and $\widetilde{M}_{2,n}$. On each component of $\partial\Sigma_n \cap \partial\widetilde{B}_{1,n}$, we have from the convergence to a catenoid that the Gauss map is close to a constant value (in fact arbitrarily close, by taking R_1 large enough). On each component of $\partial\Sigma_n \cap \partial B(p,r)$, the Gauss map is close to a a constant value: this follows from the convergence to a flat limit on compact subsets of $\mathbb{R}^3 \setminus \{p\}$. Finally, we need to understand the Gauss map on $\partial\Sigma_n \cap \partial\widetilde{M}_{2,n}$, in case this is not empty. On the boundary of $\widetilde{M}_{2,n}$, the argument of the Gauss map is equal to the argument of the horizontal vector normal to the boundary. Since the curvature of the boundary of $\widetilde{M}_{2,n}$ is bounded, the *argument* of the Gauss map on $\partial\widetilde{M}_{2,n} \cap B(p,r)$ is close to a constant value (arbitrarily close, by taking $r > 0$ small enough). We conclude that the image by the Gauss map of each component of $\partial\Sigma_n$ is either a small disk, or a star-shaped curve bounding a small area on the sphere. Since the Gauss map is open, the image of Σ_n has area close to a multiple of 4π. This contradicts the fact that $C(\Sigma_n)$ is close to μ, and concludes the second step of our weak limit process.

We iterate this process ℓ times and produce ℓ sequences of homotheties $(h_{i,n})_n$ and balls $(B_{i,n})_n$ as wanted. Moreover, $\lim C(M_n) \geq 4\pi\ell$. If $\alpha > 0$, then by taking $\varepsilon > 0$ small enough, we have that for n large enough, the family of balls B such that $C([M_n \setminus \bigcup B_{i,n}] \cap B) = \mu$ is non-empty. So we can do one more step and conclude that $\lim C(M_n) \geq 4\pi(\ell+1)$, a contradiction. Therefore $\alpha = 0$. This proves the claim. □

For n large enough, the balls $B_{i,n}$ are disjoint, hence

$$C(M_n \setminus \bigcup_{i=1}^{\ell} B_{i,n}) \leq \ell\varepsilon$$

which proves point 2 of Proposition 6 (replacing ε by ε/ℓ).

Claim 2. *The Gauss map converges to the vertical on each component of ∂M_n, in the following sense:*

$$\lim_{n\to\infty} \min_{x\in\partial M_n} |N_3(x)| = 1.$$

Proof. We prove the claim for the bottom component of ∂M_n, the proof for the top component is similar. Let x_n be a point on $\gamma_{1,n}$ such that $|N_3(x_n)|$ is minimum. Let $p_{i,n}$ be the center of $B_{i,n}$. Let $d_n = \min_i d(x_n, p_{i,n})$. Passing to a subsequence, $\lim \frac{d_n}{t_n} \in [0,\infty]$ exists.

First case: $\lim \frac{d_n}{t_n} > 0$ (possibly infinite). Let h_n be the homothety of ratio $1/t_n$ which maps x_n to 0. Let $\widetilde{M}_n = h_n(M_n)$. By Proposition 5, $(\widetilde{M}_n)_n$ converges to a minimal surface $\widetilde{M}$ with singular set S (possibly empty). Moreover, $0 \notin S$, because else $\lim \frac{d_n}{t_n} = 0$. Let $\widetilde{\Gamma} = \lim \partial\widetilde{M}_n$, then $\widetilde{\Gamma}$ is a horizontal line L_1 through the origin, and possibly a line L_2 in the horizontal plane $x_3 = 1$. Since $\widetilde{\Gamma}$ is not empty, all components of $\widetilde{M}$ are flat by Proposition 4. If $\widetilde{\Gamma} = L_1$ or L_1 and L_2 are not parallel, then all components of $\widetilde{M}$ must be planes or half-planes. Since $\widetilde{M}_n$ lies in the horizontal slab $0 \leq x_3 \leq 1$, all must be horizontal. Since $\widetilde{M}_n \to \widetilde{M}$ smoothly in a neighborhood of 0, we conclude that $N_3(x_n)$ converges to a vertical vector. If L_1

and L_2 are parallel, then one component U of $\widetilde{M}$ might be the (non-horizontal) strip bounded by L_1 and L_2. By property B of the proof of Proposition 4, there must be another component. This other component cannot be a half-plane, because L_1 and L_2 already bound U. Hence it must be a horizontal plane, but then we contradict property A of the proof of Proposition 4. We conclude that again all components of $\widetilde{M}$ are horizontal planes and half-planes.

Second case: $\lim \frac{d_n}{t_n} = 0$. In this case, let h_n be the homothety of ratio $1/d_n$ which maps x_n to 0. Let $\widetilde{M}_n = h_n(M_n)$. By Proposition 4, $(\widetilde{M}_n)_n$ converges to a minimal surface $\widetilde{M}$ with singular set $S \neq \emptyset$, and $d(0, S) = 1$. Since $S \neq \emptyset$, all components of $\widetilde{M}$ are flat by Proposition 4. Note that $\lim \partial \widetilde{M}_n$ is a horizontal line L containing the origin. Let us assume that there exists a component U of $\widetilde{M}$ which is not horizontal. Then since $\widetilde{M}$ lies in the half space $x_3 \geq 0$, U must be a half-plane with boundary L, and its multiplicity is one, so $p \notin U$. The component of $\widetilde{M}$ containing p must be a horizontal plane $x_3 = a$, and its multiplicity is at least 2. If $a > 0$, then we contradict embeddedness. If $a = 0$, then the density of $\widetilde{M}$ at the origin is greater than or equal to $\frac{5}{2}$, so we contradict point 2 of Proposition 3. Hence all components of $\widetilde{M}$ are horizontal, so $N_3(x_n)$ converges to a vertical vector. This proves the claim. □

It remains to prove that all catenoids are vertical, the third statement of Proposition 6, and that $\ell = k + 1$.

Let U be a component of $M_n \setminus \bigcup B_{i,n}$. Since the balls $B_{i,n}$ do not intersect $\Gamma_n = \partial M_n$, each component of ∂U is either a component of Γ_n, or a small circle included in some $\partial B_{i,n}$ (one of the two boundary components of the inside catenoid). By the previous claim or convergence to a catenoid, on each boundary component, the Gauss map is close to a constant. Since $C(U)$ is small, the Gauss map is close to a constant a on U. Let $P = a^{\perp}$ and let $\pi : U \to P$ be the projection. Then π is a local diffeomorphism so π is open. Consider a component of ∂U of the second type, namely a small circle γ included in some $\partial B_{i,n}$. From the convergence to a catenoid, we can glue a disk along γ in such a way that π remains a local diffeomorphism. Perform this surgery for all such boundary circles γ and call $\widetilde{U}$ the result. Then $\pi : \widetilde{U} \to P$ is a local diffeomorphism hence open. If ∂U does not intersect Γ_n then $\widetilde{U}$ is compact without boundary, but then $\pi : U \to P$ cannot be a local diffeomorphism. Hence ∂U has a component equal to $\Gamma_{1,n}$ or $\Gamma_{2,n}$, so there are at most two such components U. Since the gauss map is close to a vertical constant on $\Gamma_{1,n}$ and $\Gamma_{2,n}$, we conclude that P is the horizontal plane and all catenoids are vertical.

If $M_n \setminus \bigcup B_{i,n}$ has only one component U, then $\ell = 0$. (Indeed, if $\ell \geq 1$, the Gauss map is close to a constant on the boundary of $B_{1,n} \cap M_n$, but this contradicts the convergence to a catenoid inside.) Since π has no critical point on M_n, M_n is an annulus. Since the Gauss map is close to a constant on M_n, M_n is stable by the Barbosa do Carmo criterium. This is a contradiction since M_n is not the stable annulus by hypothesis.

Hence $M_n \setminus \bigcup B_{i,n}$ has precisely two components $U_{1,n}$ and $U_{2,n}$, with $\Gamma_{i,n} \subset \partial U_{i,n}$. Gluing disks as above, the projection π from $\widetilde{U}_{i,n}$ to the horizontal plane is open, and is one to one on $\partial\widetilde{U}_{i,n} = \Gamma_{i,n}$, so $\pi\colon \widetilde{U}_{i,n} \to \Omega_{i,n}$ is a diffeomorphism. This proves the third point of Proposition 6. Finally, the genus of M_n is $\ell - 1$, so $\ell = k + 1$. □

3.5. Flux. To make further progress we need the notion of flux. Let γ be a curve on an oriented minimal surface M, and let ν be the co-normal along γ, chosen so that the basis $\{\nu, \gamma'\}$ of the tangent plane is direct (so if γ is the oriented boundary of some domain, ν is the exterior co-normal). The flux along γ is the vector $\int_\gamma \nu ds$. This is a homology invariant vector. If we denote by $X^* = (X_1^*, X_2^*, X_3^*)$ the conjugate minimal immersion, then $\text{flux}(\gamma) = \int_\gamma dX^*$. If M is the graph of a function $u(x, y)$, and is oriented by the upwards pointing normal, then one has the following formulae for the conjugate minimal immersion:

$$\begin{aligned} dX_1^* &= \frac{u_x u_y dx + [1 + (u_y)^2]dy}{\sqrt{1 + (u_x)^2 + (u_y)^2}}, \\ dX_2^* &= \frac{-[1 + (u_x)^2]dx - u_x u_y dy}{\sqrt{1 + (u_x)^2 + (u_y)^2}}, \\ dX_3^* &= \frac{u_x dy - u_y dx}{\sqrt{1 + (u_x)^2 + (u_y)^2}}. \end{aligned} \tag{1}$$

When ∇u is small, these formulae give the following expansions, with $z = x + \mathrm{i}\, y$:

$$dX_1^* - \mathrm{i}\, dX_2^* = \mathrm{i}\, d\bar{z} + 2\mathrm{i} \left(\frac{\partial u}{\partial z}\right)^2 dz + o(|\nabla u|^2), \tag{2}$$

$$dX_3^* = \operatorname{Im}\left(2\frac{\partial u}{\partial z} dz\right) + o(|\nabla u|). \tag{3}$$

3.6. Limit rescaled graph. As we have seen, outside $k + 1$ small balls, M_n has two components $U_{1,n}$ and $U_{2,n}$. Each component $U_{i,n}$ is the graph over $\Omega_{i,n}$ minus small disks of a function which we call $u_{i,n}$. We have $u_{1,n} = 0$ on $\partial\Omega_{1,n}$ and $u_{2,n} = t_n$ on $\partial\Omega_{2,n}$. In this section, we prove that after suitable scaling, these functions converge to explicit harmonic functions u_1 and u_2, each having $k + 1$ logarithmic singularities.

Without loss of generality, we may assume (by changing the homotheties $h_{i,n}$) that all the limit catenoids are the standard catenoid $\cosh^2 x_3 = x_1^2 + x_2^2$. Let $\lambda_{i,n}$ be the ratio of $h_{i,n}$ and $\lambda_n = \min \lambda_{i,n}$. Passing to a subsequence, we may assume that $\lambda_n = \lambda_{i_0,n}$ for some index i_0. Passing again to a subsequence, the following limit exists:

$$c_i = \lim_{n\to\infty} \frac{\lambda_n}{\lambda_{i,n}} \in [0, 1].$$

Note that $c_{i_0} = 1$, so at least one c_i is non-zero. Let $p_{i,n} \in \mathbb{R}^2$ be the horizontal projection of the center of $B_{i,n}$. Passing to a subsequence, $p_i = \lim p_{i,n} \in \overline{\Omega_1 \cap \Omega_2}$ exists. Note that at this point, we do not know that the points $p_1, \dots, p_{k+1}$ are distinct.

Proposition 7. *The following limits exist:*

$$u_1 := \lim_{n\to\infty} \lambda_n u_{1,n} = -\sum_{i=1}^{k+1} c_i G_{1,p_i},$$

$$u_2 := \lim_{n\to\infty} \lambda_n (u_{2,n} - t_n)(z) = \sum_{i=1}^{k+1} c_i G_{2,p_i},$$

where $G_{i,p}$ denotes the Green function of Ω_i. The convergence is the smooth convergence on compact subsets of $\Omega_i \setminus \{p_1, \dots, p_{k+1}\}$.

Note that in this proposition, the points p_i do not need to be distinct, and may also be on the boundary. If $p \in \partial\Omega_i$, $G_{i,p}$ should be understood as zero. Note that if p converges to a boundary point q of Ω_i, then $G_{i,p}$ converges uniformly to 0 on compact subsets of $\overline{\Omega_i} \setminus \{q\}$ (this is easy to check by explicit formula for the disk, so is true for any bounded convex domain by conformal invariance of the Green function). This makes this definition natural.

Proof of the proposition. We orient M_n so that the normal points up in $U_{1,n}$ and down on $U_{2,n}$. Let $\gamma_{1,i,n}$ and $\gamma_{2,i,n}$ denote the top and bottom boundary components of $M_n \cap B_{i,n}$ (oriented as boundaries). From the convergence to catenoids we have

$$\lim_{n\to\infty} \lambda_n \mathrm{flux}(\gamma_{2,i,n}) = -\lim_{n\to\infty} \lambda_n \mathrm{flux}(\gamma_{1,i,n}) = (0, 0, 2\pi c_i).$$

This gives

$$\lim_{n\to\infty} \lambda_n \mathrm{flux}(\gamma_{1,n}) = -\lim_{n\to\infty} \lambda_n \mathrm{flux}(\gamma_{2,n}) = \sum_{i=1}^{k+1} (0, 0, 2\pi c_i).$$

Now the third coordinate of the co-normal ν has constant sign on each curve $\gamma_{1,i,n}$ and $\gamma_{2,i,n}$ (from the convergence to catenoids), and on $\gamma_{1,n}$ and $\gamma_{2,n}$ (from the convex hull property). Hence for $i = 1, 2$ we have the estimate

$$\int_{\partial U_{i,n}} \lambda_n |dX_3^*| \le C$$

for some uniform constant C. Since the normal is close to be vertical on each $U_{i,n}$, we have $\sqrt{1+|\nabla u_{i,n}|^2}\leq 2$ for n large enough, hence from equation (1), we have

$$\int_{\partial U_{i,n}} \lambda_n |\nabla u_{i,n}| \leq 2C. \tag{4}$$

From this integral estimate, we must conclude the convergence of a subsequence of $(\lambda_n u_{i,n})_n$. If $u_{i,n}$ were harmonic, this would be quite elementary. So we make a conformal representation of $U_{i,n}$ onto a planar domain. Via this representation, $u_{i,n}$ becomes harmonic and we can conclude.

We shall only consider $u_{1,n}$, the proof for $u_{2,n}$ is entirely similar. By Koebe's theorem on uniformization of planar domains, there exists a conformal representation f_n of $U_{1,n}$ onto the unit disk minus $k+1$ circular disks, such that f_n maps $\gamma_{1,n}$ to the unit circle. Such a conformal representation is unique up to a Möbius transform of the disk. Let $\pi_n\colon U_{1,n}\to\Omega_{1,n}$ be the projection on the horizontal plane and let $\tilde{f}_n = f_n\circ\pi_n^{-1}$. Using a Möbius transform of the disk, we may normalize $\tilde{f}_n$ by $\tilde{f}_n(z_0)=0$ and $\frac{\partial \tilde{f}_n}{\partial z}(z_0)>0$, where z_0 is a fixed point of Ω_1, away from $p_1,\dots,p_{k+1}$. Note that $\tilde{f}_n$ is defined on compact subsets of $\Omega_1\setminus\{p_1,\dots,p_{k+1}\}$ for n large enough, and is κ_n-quasi conformal with $\kappa_n\to 1$ as $n\to\infty$ (because f_n is conformal and π_n is κ_n-quasi conformal, since the Gauss map converges to a vertical vector). Since $(\tilde{f}_n)_n$ is bounded, by a standard normal family result ([6], Theorem 5.1, page 73), passing to a subsequence, $(\tilde{f}_n)_n$ converges on compact subsets of $\Omega_1\setminus\{p_1,\dots,p_{k+1}\}$ to a 1-quasi conformal (hence holomorphic) function f. Moreover, $f(z_0)=0$ and $f'(z_0)\geq 0$. By Riemann's theorem, f extends holomorphically to $p_1,\dots,p_{k+1}$. Let $q_i=f(p_i)$, $i=1,\dots,k+1$. By [6], Theorem 5.5, page 78, f is either a diffeomorphism, or a constant function onto a boundary point, which is not possible since $f(z_0)=0$. Hence f is the unique conformal representation of Ω_1 onto the unit disk such that $f(z_0)=0$, $f'(z_0)>0$. Since the limit is uniquely determined, the whole sequence $(\tilde{f}_n)_n$ converges to f.

Let $\Omega'_n = f_n(U_{1,n})$ and

$$v_n = \lambda_n X_{3,n}\circ f_n^{-1} = \lambda_n u_n\circ\pi_n\circ f_n^{-1}, \quad \phi_n = \frac{\partial v_n}{\partial z},$$

where $X_{3,n}\colon M_n\to\mathbb{R}$ denotes the third coordinate of the immersion. Since M_n is minimal and f_n is conformal, v_n is a harmonic function so ϕ_n is a holomorphic function on Ω'_n. From equation (4) we have

$$\int_{\partial\Omega'_n} |\phi_n| \leq C.$$

Fix a small $\varepsilon>0$ and let U_ε be the set of points in $D(0,1)$ which are at distance greater than ε from $\partial D(0,1)$ and $q_1,\dots,q_{k+1}$. Observe that for n large enough,

$U_\varepsilon \subset \Omega'_n$ and if $z \in U_\varepsilon$, $d(z, \partial\Omega'_n) \geq \varepsilon/2$. By Cauchy's theorem we have, for n large enough and $z \in U_\varepsilon$,

$$|\phi_n(z)| = \frac{1}{2\pi}\left|\int_{\partial\Omega'_n} \frac{\phi_n(w)}{w-z} dw\right| \leq \frac{1}{2\pi}\int_{\partial\Omega'_n} \frac{|\phi_n|}{\varepsilon/2} \leq \frac{C}{\pi\varepsilon}.$$

Hence $(\phi_n)_n$ is bounded on U_ε. By the theorem on normal families, a subsequence of $(\phi_n)_n$ converges on compact subsets of $D(0,1) \setminus \{q_1, \dots, q_{k+1}\}$ to a holomorphic function ϕ. From the above estimate, ϕ has at most simple poles at each $q_1, \dots, q_{k+1}$. Since $v_n = 2\mathrm{Re}\int \phi_n$, we obtain that (v_n) converges to a harmonic function v which has at most logarithmic singularities at $q_1, \dots, q_{k+1}$ and vanishes on $\partial D(0,1)$. (To see that $v = 0$ on the unit circle, we must ensure the convergence of $(\phi_n)_n$ on the boundary. This can be done as follows: since v_n is zero on the unit circle, the 1-form $\omega_n = \phi_n dz$ is pure imaginary on the unit circle. By the Schwartz reflection principle, one can extend the holomorphic one form ω_n by reflection in the circle namely, by $\sigma^*\omega_n = -\overline{\omega_n}$, where $\sigma(z) = 1/\bar{z}$. Fix some $r < 1$ close to 1. Then $(\omega_n)_n$ is bounded on the circles $|z| = r$ and $|z| = \frac{1}{r}$, so by the maximum principle, it is bounded in the annular region $r < |z| < \frac{1}{r}$. Hence, passing to a subsequence, the convergence holds up to $\partial D(0,1)$.)

Since $\lambda_n u_n = v_n \circ f_n \circ \pi_n^{-1}$, $(\lambda_n u_n)_n$ converges to a harmonic function u_1 which is zero on $\partial\Omega_1$ and has at most logarithmic singularities at $p_1, \dots, p_{k+1}$. By formula (3), the principal part of u_1 at p_i is $-c_i \log|z - p_i|$. This proves the proposition. □

3.7. The balancing condition. In this section, we compute the limit of the horizontal part of the flux, scaled by $(\lambda_n)^2$, on $\partial B_{i,n} \cap M_n = \gamma_{1,i,n} \cup \gamma_{2,i,n}$. Writing that this flux is zero will give the balancing condition. We assume that the configuration $p_1, \dots, p_{k+1}$ is regular, in the following sense:

(1) the points $p_1, \dots, p_{k+1}$ are distinct,

(2) $p_i \in \Omega_1 \cap \Omega_2$ for all i.

A configuration is singular when several points are equal, or when some points are on the boundary of $\Omega_1 \cap \Omega_2$. The case of singular configurations will be studied in Section 3.9. Let us define

$$F(\gamma) = \mathrm{flux}_1(\gamma) - \mathrm{i}\,\mathrm{flux}_2(\gamma) = \int_\gamma dX_1^* - \mathrm{i}\, dX_2^*.$$

By formula (2), we have

$$F(\gamma_{1,i,n}) = 2\mathrm{i}\int_{C(p_i,\varepsilon)} \left(\frac{\partial u_{1,n}}{\partial z}\right)^2 dz + o(|\nabla u_{1,n}|^2),$$

$$\lim_{n\to\infty}(\lambda_n)^2 F(\gamma_{1,i,n}) = 2\mathrm{i}\int_{C(p_i,\varepsilon)}\left(\frac{\partial u_1}{\partial z}\right)^2 dz = -4\pi\,\mathrm{Res}_{p_i}\left(\frac{\partial u_1}{\partial z}\right)^2.$$

Now

$$\frac{\partial u_1}{\partial z} = -\frac{c_i}{2(z-p_i)} - c_i\frac{\partial H_{1,p_i}}{\partial z} - \sum_{j\neq i} c_j\frac{\partial G_{1,p_j}}{\partial z}.$$

This gives, expanding the square and computing the residue,

$$\lim_{n\to\infty}(\lambda_n)^2 F(\gamma_{1,i,n}) = -4\pi\left(c_i^2\frac{\partial H_{1,p_i}}{\partial z}(p_i) + \sum_{j\neq i} c_i c_j\frac{\partial G_{1,p_j}}{\partial z}(p_i)\right).$$

We have the same formula for $F(\gamma_{2,i,n})$, replacing H_{1,p_i} by H_{2,p_i} and G_{1,p_j} by G_{2,p_j}. (Regarding orientations: the normal points down in $U_{2,n}$, so there is a minus sign in front of the formulae for dX^*, and we must give $C(p_i,\varepsilon)$ the negative orientation, which gives another minus sign in front of the residue. These two minus signs compensate.) Since $\gamma_{1,i,n}+\gamma_{2,i,n}$ bounds $M_n\cap B_{i,n}$, the sum of the two fluxes is zero, so we obtain, for all $i=1,\dots,k+1$,

$$c_i^2\left(\frac{\partial H_{1,p_i}}{\partial z}(p_i) + \frac{\partial H_{2,p_i}}{\partial z}(p_i)\right) + \sum_{j\neq i} c_i c_j\left(\frac{\partial G_{1,p_j}}{\partial z}(p_i) + \frac{\partial G_{2,p_j}}{\partial z}(p_i)\right) = 0.$$

This is not quite the balancing condition yet. We still must prove that all the c_i are equal to one, which is the goal of the next section.

Remark 3. To prove that balanced configurations do not exist in Sections 2.3 and 2.4, we do not really need that all c_i are equal to one: we could very well use the above balancing condition, provided that all c_i are positive. However, the simplest way to prove that no c_i vanishes seems to prove that all are in fact equal to one.

3.8. Equal neck-sizes

Proposition 8. *Assume the configuration is non-singular (in the sense explained at the beginning of Section 3.7). Then all c_i are equal to one.*

Proof. For each neck, we use catenoidal barriers to estimate the height t_n between the boundary curves as a function of $\lambda_{i,n}$. From this estimate we conclude that all $c_i = \lim\frac{\lambda_{i,n}}{\lambda_n}$ are equal.

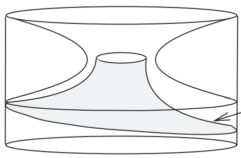
Given $0<r<R$, let $\mathcal{C}(r,R)$ be the part of the catenoid of waist radius r defined by

$$\sqrt{x_1^2+x_2^2} = r\cosh(x_3/r), \qquad \sqrt{x_1^2+x_2^2} < R,$$

so that $\mathcal{C}(r,R)$ is bounded by two horizontal circles of radius R at height $\pm r$ argcosh $\frac{R}{r}$. Let $\mathcal{C}^+(r,R)$ and $\mathcal{C}^-(r,R)$ denote the upper half (in $x_3 > 0$) and lower half (in $x_3 < 0$) of $\mathcal{C}(r,R)$.

Let $p_{i,n} \in \mathbb{R}^2$ and $\eta_{i,n} \in (0,t_n)$ be respectively the horizontal projection and the third coordinate of the center of $B_{i,n}$, so $p_{i,n} \to p_i$ and $\eta_{i,n} \to 0$. Since the configuration is non-singular, there exists $\varepsilon > 0$ such that for n large enough, the disks $D(p_{i,n},\varepsilon)$, $i = 1,\dots,k+1$ are disjoint and inside $\Omega_{1,n} \cap \Omega_{2,n}$. From the convergence of $(\lambda_n u_{1,n})_n$ to u_1 on compact subsets of $\Omega_1 \setminus \{p_1,\dots,p_{k+1}\}$, we have $|\lambda_n u_{1,n}| \le C$ on the circles $C(p_{i,n},\varepsilon)$ for some uniform constant C.

Upper bound for $\eta_{i,n}$. Fix some $\alpha > 1$ close to one. Let $\Sigma_{i,n}$ be the part of M_n inside the vertical cylinder $D(p_{i,n},\varepsilon) \times (0,\eta_{i,n})$. By convergence of $h_{i,n}(M_n)$ to a catenoid, the horizontal projection of the top component of $\partial\Sigma_{i,n}$ is a curve close to a circle of radius $1/\lambda_{i,n}$, so it is inside the disk $D(p_{i,n},\alpha/\lambda_{i,n})$. (Here we assume, without loss of generality, that all limit catenoids are centered at the origin.) Consider the catenoid $\mathcal{C}(\alpha/\lambda_{i,n},\varepsilon)$. Translate it horizontally so that its axis is the vertical line through $p_{i,n}$. Translate it vertically up so that it is disjoint from $\Sigma_{i,n}$, and then move it down.



By the maximum principle, the first contact point will occur when the bottom circle touches the lower boundary component of $\partial\Sigma_{i,n}$, so its height will be at most C/λ_n. In this situation, the catenoid will be above $\Sigma_{i,n}$. The intersection of $\Sigma_{i,n}$ with $x_3 = \eta_{i,n} - 1/\lambda_{i,n}$ is close to a circle of radius $\cosh(1)/\lambda_{i,n}$, which is greater than the waist radius of the catenoid, so $\eta_{i,n} - 1/\lambda_{i,n}$ must be less than the height of the waist of the catenoid. Using that $\operatorname{argcosh}(x) \le \log(2x)$ for $x \ge 1$, this gives the estimate

$$\eta_{i,n} \le \frac{C}{\lambda_n} + \frac{\alpha}{\lambda_{i,n}} \operatorname{argcosh} \frac{\varepsilon\lambda_{i,n}}{\alpha} \le \frac{C'}{\lambda_n} + \alpha\frac{\log\lambda_{i,n}}{\lambda_{i,n}}$$

for some uniform constant C'. By the same argument, we have the same upper bound for $t_n - \eta_{i,n}$. Adding the two estimates gives

$$t_n \le 2\alpha\frac{\log\lambda_{i,n}}{\lambda_{i,n}} + \frac{2C'}{\lambda_n}. \tag{5}$$

Lower bound for $\eta_{i,n}$. Fix some $\beta < 1$ close to one. Consider the lower half-catenoid $\mathcal{C}^-(\beta/\lambda_{i,n},\varepsilon)$. Translate it horizontally so that its axis is the vertical line through

$p_{i,n}$. Translate it vertically down so that it is disjoint from M_n and then up. By the maximum principle, the first contact point will occur when the bottom circle of the half-catenoid touches the boundary of M_n, and the part of M_n inside the cylinder $D(p_{i,n}, \varepsilon) \times (0, t_n)$ will be above the catenoid. Using that $\operatorname{argcosh}(x) \geq \log(x)$ for $x \geq 1$, this gives the estimate

$$\eta_{i,n} \geq \beta \frac{\log \lambda_{i,n}}{\lambda_{i,n}} - \frac{C''}{\lambda_n}$$

for some uniform constant C''. By the same argument, we have the same lower bound for $t_n - \eta_{i,n}$. Adding the two estimate and taking $i = i_0$ (recall that $\lambda_n = \lambda_{i_0,n}$ by definition), we obtain

$$t_n \geq 2\beta \frac{\log \lambda_n}{\lambda_n} - \frac{2C''}{\lambda_n}. \tag{6}$$

Combining (6) and (5), we obtain

$$\alpha \frac{\log \lambda_{i,n}}{\lambda_{i,n}} \geq \beta \frac{\log \lambda_n}{\lambda_n} - \frac{C' + C''}{\lambda_n}$$

which holds for any $\alpha > 1$ and $\beta < 1$, both close to one, and for n large enough. From this we get

$$\alpha \frac{\lambda_n}{\lambda_{i,n}} \log\left(\frac{\lambda_{i,n}}{\lambda_n}\right) \geq \left(\beta - \alpha \frac{\lambda_n}{\lambda_{i,n}}\right) \log \lambda_n - C' - C''.$$

The left hand side has a finite limit when $n \to \infty$, so $\beta - \alpha \frac{\lambda_n}{\lambda_{i,n}} \leq 0$ for n large enough, else the right hand side goes to $+\infty$. This gives $c_i \geq \frac{\beta}{\alpha}$. The conclusion follows by letting α and β go to one. □

3.9. The singular case. Let us introduce some terminology. Let p_i be a point of the configuration. If $p_j \neq p_i$ for all $j \neq i$ then we say that p_i is a *simple* point, else that p_i is a *multiple* point. If p_i is not on the boundary of $\Omega_1 \cap \Omega_2$ we say that p_i is *interior*. If $c_i = 0$, then we say that p_i is *evanescent*. Evanescent points correspond to catenoidal necks which collapse too fast. Multiple points correspond to catenoidal necks which collapse to the same point. We want to prove that the configuration is non-singular, namely all points of the configuration are simple and interior.

If $(\varphi_n)_n$ is a sequence of homotheties of the plane with ratio $\mu_n \to \infty$, we define $\widetilde{\Omega}_{i,n} = \varphi_n(\Omega_{i,n})$, $\tilde{p}_{i,n} = \varphi_n(p_{i,n})$ and $\tilde{u}_{i,n} = u_{i,n} \circ \varphi_n^{-1}$. Passing to a subsequence, $\tilde{p}_i = \lim \tilde{p}_{i,n}$ exists in $\mathbb{C} \cup \{\infty\}$, and $\widetilde{\Omega}_{i,n}$ converges to either a half-plane H_i or the whole plane. We have the following generalization of Proposition 7 to this setup:

Proposition 9. *If* $\lim \widetilde{\Omega}_{1,n}$ *is a half-plane* H_1, *then*

$$\lim_{n\to\infty} \lambda_n \tilde{u}_{1,n}(z) = -\sum_{i=1}^{k+1} c_i \log \left| \frac{z-\tilde{p}_i}{z-\sigma_1(\tilde{p}_i)} \right|,$$

where σ_1 *denotes the symmetry with respect to the boundary line of* H_1. *If* $\lim \widetilde{\Omega}_{1,n}$ *is the whole plane, then*

$$\lim_{n\to\infty} \lambda_n (\tilde{u}_{1,n}(z) - \tilde{u}_{1,n}(z_0)) = -\sum_{i=1}^{k+1} c_i \log \left| \frac{z-\tilde{p}_i}{z_0-\tilde{p}_i} \right|.$$

The convergence is on compact subsets of H_1 *or* $\mathbb{C}$ *minus* $\tilde{p}_1, \dots, \tilde{p}_{k+1}$. *In case* $\tilde{p}_i = \infty$, *the corresponding term in the above formulae should be understood as zero. A similar statement holds for* $\tilde{u}_{2,n}$.

We start by proving:

Proposition 10. *The points which are not evanescent are interior and simple amongst non-evanescent points* (*which simply means that they are distinct*).

Proof. the proof is by contradiction. If this is not true, then by making blow-ups, we obtain a balanced configuration as before, with the domains Ω_1 and Ω_2 replaced by half-planes (in the case of a boundary point) or the whole plane (in the case of a multiple point). We obtain a contradiction by proving that balanced configuration are impossible in these cases. Note that forces won't see evanescent points, which is why we only get information about non-evanescent points. We will rule out evanescent points in the next proposition.

Without loss of generality, we may assume that the points which are not evanescent are $p_1, \dots, p_r$, for some $r \geq 1$. Let

$$\delta_n = \min\big(\{d(p_{i,n}, \partial(\Omega_{1,n} \cap \Omega_{2,n})), 1 \leq i \leq r\} \cup \{d(p_{i,n}, p_{j,n}), 1 \leq i < j \leq r\}\big).$$

We want to prove that $\inf \delta_n > 0$. Assume by contradiction that $\inf \delta_n = 0$. Then we can find a subsequence such that $\lim \delta_n = 0$. Passing to a subsequence, and maybe changing indices, δ_n is always equal to the distance of $p_{1,n}$ to the boundary or to $d(p_{1,n}, p_{2,n})$. Let φ_n be the homothety of ratio $\mu_n = 1/\delta_n$ in the plane which maps $p_{1,n}$ to the origin. Then passing to a subsequence and using the notations before Proposition 9, $\tilde{p}_i = \lim \tilde{p}_{i,n} \in \mathbb{C} \cup \{\infty\}$ and $\widetilde{\Omega}_\ell = \lim \widetilde{\Omega}_{\ell,n}$ exist, $\ell = 1, 2$. Moreover, $\tilde{p}_1 = 0$, and the points $\tilde{p}_1, \dots, \tilde{p}_r$ are at distance at least one from each other and from the boundary. We may assume that the points $\tilde{p}_i$ which are finite are $\tilde{p}_1, \dots, \tilde{p}_s$ for some $s \geq 1$.

If $\widetilde{\Omega}_\ell = \mathbb{C}$ then arguing as in Section 3.7 and using Proposition 9, we have

$$\lim_{n\to\infty} \frac{\lambda_n^2}{\mu_n^2} F(\gamma_{\ell,i,n}) = -4\pi \operatorname{Res}_{\tilde{p}_i}\Big(\frac{\partial}{\partial z}\sum_{j=1}^{s} c_j \log|z-\tilde{p}_j|\Big)^2 = -2\pi \sum_{j\neq i} \frac{c_i c_j}{\tilde{p}_i - \tilde{p}_j}.$$

If $\widetilde{\Omega}_\ell = H_\ell$ is a half-plane then we have ($\sigma_\ell(z)$ denotes again the symmetry with respect to the boundary of H_ℓ)

$$\lim_{n\to\infty} \frac{\lambda_n^2}{\mu_n^2} F(\gamma_{\ell,i,n}) = -4\pi \operatorname{Res}_{\tilde{p}_i}\Big(\frac{\partial}{\partial z}\sum_{j=1}^{s} c_j \log|z-\tilde{p}_j| - c_j \log|z-\sigma_\ell(\tilde{p}_j)|\Big)^2$$
$$= -2\pi\Big(\sum_{j\neq i} \frac{c_i c_j}{\tilde{p}_i - \tilde{p}_j} - \sum_{j=1}^{s} \frac{c_i c_j}{\tilde{p}_i - \sigma_\ell(\tilde{p}_j)}\Big).$$

First case: Both $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are the whole plane $\mathbb{C}$. Then necessarily $s \geq 2$, and the above formulae give the balancing formula:

$$2\sum_{j\neq i} \frac{c_i c_j}{\tilde{p}_i - \tilde{p}_j} = 0 \quad \text{for all } i \leq s.$$

Since $s \geq 2$, it is straightforward to see that there are no balanced configurations $\tilde{p}_1, \dots, \tilde{p}_s$. (Simply consider an extremal point, namely which is not in the convex hull of the others. The force on such a point cannot vanish.)

Second case: $\widetilde{\Omega}_1 = H_1$ is a half-plane and $\widetilde{\Omega}_2$ is the whole plane. By rotation and translation, we may assume that H_1 is the half plane $\operatorname{Im}(z) > 0$, so $\sigma_1(z) = \bar{z}$. The above formulae give the balancing condition:

$$\sum_{j\neq i} \frac{c_i c_j}{\tilde{p}_i - \tilde{p}_j} + \sum_{j=1}^{s} \frac{-c_i c_j}{\tilde{p}_i - \overline{\tilde{p}_j}} = 0 \quad \text{for all } i \leq s.$$

If $\tilde{p}_i$ has the smallest imaginary part amongst $\tilde{p}_1, \dots, \tilde{p}_s$, then all terms in the first sum have non-negative imaginary part, and all terms in the second have positive imaginary part, hence the force cannot be zero. The case where $\widetilde{\Omega}_1$ is the whole plane and $\widetilde{\Omega}_2$ is a half-plane is identical.

Third case: $\widetilde{\Omega}_1 = H_1$ and $\widetilde{\Omega}_2 = H_2$ are both half-planes. Note that $H_1 \cap H_2$ cannot be a strip, because this would contradict the fact that $\Omega_{1,n} \cap \Omega_{2,n}$ contains a disk of fixed radius. So by translation and rotation we may assume that for $\ell = 1, 2$, H_ℓ is the half-plane $y > \tan(\alpha_\ell)x$ for some $\alpha_\ell \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We obtain the balancing condition:

$$\sum_{\ell=1}^{2}\Big(\frac{-c_i^2}{\tilde{p}_i - \sigma_\ell(\tilde{p}_i)} + \sum_{j\neq i}\Big[\frac{c_i c_j}{\tilde{p}_i - \tilde{p}_j} - \frac{c_i c_j}{p_i - \sigma_\ell(\tilde{p}_j)}\Big]\Big) = 0 \quad \text{for all } i \leq s. \qquad (7)$$

Fix some $\ell = 1, 2$, and write $H = H_\ell$, $\sigma = \sigma_\ell$. If z, w are points in H then we clearly have $\mathrm{Im}(\sigma(z)) < \mathrm{Im}(z)$ and $|\sigma(w) - z| > |w - z|$. If $\mathrm{Im}(z) \le \mathrm{Im}(\sigma(w))$ then we have

$$\begin{aligned}\mathrm{Im}\left(\frac{1}{z-w} - \frac{1}{z-\sigma(w)}\right) &= \frac{\mathrm{Im}(w-z)}{|z-w|^2} + \frac{\mathrm{Im}(z-\sigma(w))}{|z-\sigma(w)|^2}\\ &\ge \frac{\mathrm{Im}(w-z)}{|z-w|^2} + \frac{\mathrm{Im}(z-\sigma(w))}{|z-w|^2} = \frac{\mathrm{Im}(w-\sigma(w))}{|z-w|^2} > 0.\end{aligned}$$

If $\mathrm{Im}(\sigma(w)) < \mathrm{Im}(z) \le \mathrm{Im}(w)$ then the same conclusion holds (this time both terms are positive).

Now consider the point $\tilde{p}_i$ which has the smallest imaginary part. It follows from what we have just seen that the first term and all brackets in (7) have positive imaginary part. So the force on $\tilde{p}_i$ cannot vanish. This proves the proposition. □

Proposition 11. *The configuration is non-singular.*

Proof. If we look at the proof of Proposition 8, we see that to get the upper bound for $\eta_{i,n}$, we only need that the point p_i is simple, while for the lower bound of $\eta_{i,n}$, we only need that the point p_i is interior. Since $c_{i_0} = 1$, Proposition 10 says that p_{i_0} is interior. Hence the lower bound for t_n, equation (6), holds.

Let p_i be a point of the configuration. If p_i is simple, then as we observed above, we can obtain an upper bound for $\eta_{i,n}$ and conclude that $c_i = 1$ as in Section 3.8, so p_i is interior by Proposition 10. Therefore, to prove the proposition, we only have to prove that all points are simple.

Assume by contradiction that there exists a multiple point. By changing indices, we may assume that $p_1 = p_2 = \cdots = p_m$ for some integer $m \ge 2$. Passing to a subsequence and changing indices, we may assume that $\lambda_{1,n} = \min\{\lambda_{i,n} : 1 \le i \le m\}$ for all n. Our first goal is to prove that $c_1 > 0$ by obtaining an upper bound for $\eta_{1,n}$. We estimate the height $\eta_{1,n}$ using an extremal length argument, which is more flexible than the use of a catenoidal barrier, although it gives a cruder result.

Let Γ be a family of curves in the plane. The extremal length $\lambda(\Gamma)$ of Γ is defined as follows (see Ahlfors' book [1]):

$$\lambda(\Gamma) = \sup_\rho \frac{L(\rho)^2}{A(\rho)};$$

$$L(\rho) = \inf_{\gamma\in\Gamma} L_\gamma(\rho), \quad L_\gamma(\rho) = \int_\gamma \rho|dz|, \quad A(\rho) = \iint \rho^2 dx\,dy.$$

Here ρ is any measurable non-negative function in the plane, such that $A(\rho) \ne 0, \infty$. If Ω is an annulus and Γ is the set of curves which connect its two boundary components, then $\lambda(\Gamma)$ is called the modulus of Ω. The modulus is a conformal invariant and is monotonous, namely $\Omega \subset \Omega' \Rightarrow \mathrm{mod}(\Omega) \leq \mathrm{mod}(\Omega')$. The modulus of the annulus $D(0,R) \setminus D(0,r)$ is $\frac{1}{2\pi}\log\frac{R}{r}$.

There exists $\varepsilon > 0$ such that all points of the configuration are either equal to p_1 or at distance greater than 2ε from p_1. By Proposition 7, we have $|\lambda_n u_{1,n}| < C$ on the circle $C(p_{1,n},\varepsilon)$ for some uniform constant C. Let $a_n = \frac{C}{\lambda_n}$. Consider the subset of $\pi(U_{1,n}) \subset \Omega_{1,n}$ defined by $u_{1,n} > a_n$. Let Σ_1 the component which has $\pi(\gamma_{1,1,n})$ on its boundary, see Figure 5. (By slightly perturbing a_n, the level line $u_{1,n} = a_n$ consists of a finite number of regular Jordan curves.) The boundary of Σ_1 consists of a Jordan curve α_1 on which $u_{1,n} = a_n$, and one or several small convex curves $\pi(\gamma_{1,i,n})$ with $i \leq m$, on which $u_{1,n} > a_n$. (The fact that $i \leq m$ can be ensured by taking the constant C large enough.)

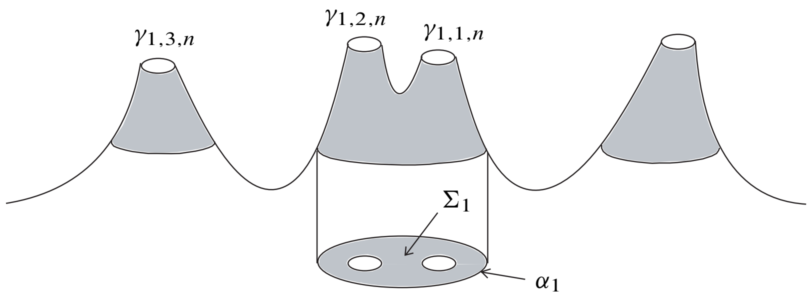


Figure 5. Definition of Σ_1 and α_1, here $m = 2$.

Let ρ_1 be the function which is equal to $|\nabla u_{1,n}|$ on Σ_1, and zero elsewhere. We first estimate the area $A(\rho_1)$ by the following interesting computation, writing $u = u_{1,n}$ and ν the unit exterior co-normal along $\partial\Sigma_1$:

$$\begin{aligned}
A(\rho_1) &= \iint_{\Sigma_1} |\nabla u|^2 \leq \sqrt{2}\iint_{\Sigma_1} \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \\
&= \sqrt{2}\iint_{\Sigma_1} \mathrm{div}\left((u-a_n)\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \\
&= \sqrt{2}\int_{\partial\Sigma_1} (u-a_n)\frac{\frac{\partial u}{\partial \nu}}{\sqrt{1+|\nabla u|^2}} \\
&\leq 4\pi \sum_i \frac{\eta_{i,n}-a_n}{\lambda_{1,n}}.
\end{aligned}$$

On the first line we have used $|\nabla u| \leq 1$. On the second line we have used the minimal surface equation. On the third line, the divergence theorem. For the last line, we estimate each boundary term: the term along α_1 vanishes since $u = a_n$. Along each small convex curve $\pi(\gamma_{1,i,n})$, we have $u \leq \eta_{i,n}$ and $\frac{\partial u}{\partial \nu} > 0$, so the integral can be estimated by the flux along this curve, which is close to $\frac{2\pi}{\lambda_{i,n}} \leq \frac{2\pi}{\lambda_{1,n}}$. The sum is on all indices i such that $\pi(\gamma_{1,i,n})$ lies on the boundary of Σ_1. There are at most m terms. We do the same argument for the function $u_{2,n}$, considering the set $u_{2,n} > t_n - a_n$ and writing $\rho_2 = |\nabla u_{2,n}|$, and we obtain

$$A(\rho_2) \leq 4\pi \sum_i \frac{t_n - \eta_{i,n} - a_n}{\lambda_{1,n}}.$$

Adding the two estimates gives

$$A(\rho_1) + A(\rho_2) \leq 4\pi m \frac{t_n - 2a_n}{\lambda_{1,n}}.$$

Let $\mathcal{A}$ be the annulus bounded by α_1 and $\pi(\gamma_{1,1,n})$. We consider the family Γ of curves in $\mathcal{A}$ which connect the two boundary components. By definition of $L(\rho_1)$, there exists a curve γ in Γ such that $L_\gamma(\rho_1)$ is less than say $L(\rho_1) + \frac{1}{\lambda_{1,n}}$. If γ happens to enter one of the small disks bounded by $\pi(\gamma_{1,i,n})$ (with $2 \leq i \leq m$), then we replace the portion of γ inside this disk (where $\rho_1 = 0$) by an arc on the boundary of the disk (where $\rho_1 = |\nabla u|$). This increases $L(\rho_1)$ by an amount less than the flux along this curve. We may also assume that γ enters each disk at most once by shunting all unnecessary circuits. This way, we obtain a curve γ which stays inside Σ_1 and such that $L_\gamma(\rho_1) \leq L(\rho_1) + \frac{C'}{\lambda_{1,n}}$ for some uniform constant C'. Then we write

$$\eta_{1,n} - a_n = \int_\gamma du \leq \int_\gamma |\nabla u| = L_\gamma(\rho_1) \leq L(\rho_1) + \frac{C'}{\lambda_{1,n}}.$$

We do the same thing for $u_{2,n}$ and add the two estimates, we obtain

$$t_n - 2a_n \leq L(\rho_1) + L(\rho_2) + \frac{2C'}{\lambda_{1,n}} \leq 2(L(\rho_1) + L(\rho_2)).$$

To obtain the last inequality, we observe that if $L(\rho_1) + L(\rho_2) \leq \frac{2C'}{\lambda_{1,n}}$, then $t_n \leq \frac{2C+4C'}{\lambda_n}$, but this is impossible since $t_n \gg \frac{1}{\lambda_n}$ by equation (6).

To estimate the modulus of the annulus $\mathcal{A}$, we observe that we can find a uniform R such that $\mathcal{A}$ is contained in the annulus $D(p_{1,n}, R) \setminus D(p_{1,n}, \frac{1}{\lambda_{1,n}})$. By monotonicity of the modulus, the modulus of $\mathcal{A}$ is bounded by $\frac{1}{2\pi} \log(R\lambda_{1,n})$, which we can safely bound by $\log \lambda_{1,n}$ for n large enough. All this gives, using the definition of the modulus as an extremal length,

$$\begin{aligned}(t_n - 2a_n)^2 &\leq 4(L(\rho_1) + L(\rho_2))^2 \leq 8(L(\rho_1)^2 + L(\rho_2)^2) \\ &\leq 8\log(\lambda_{1,n})(A(\rho_1) + A(\rho_2)) \\ &\leq 32\pi m(t_n - 2a_n)\frac{\log \lambda_{1,n}}{\lambda_{1,n}}.\end{aligned}$$

It follows that

$$t_n \leq 32\pi m \frac{\log \lambda_{1,n}}{\lambda_{1,n}} + \frac{2C}{\lambda_n}.$$

This upper bound for t_n is similar to (5), although the constant is not as good. Using the lower bound, equation (6), and arguing as in the last paragraph of the proof of Proposition 8, we obtain that $c_1 \geq \frac{\beta}{16\pi m} > 0$. This implies in particular that the point p_1 is interior by Proposition 10.

Let $\delta_n = \min\{d(p_{1,n}, p_{j,n}) \ : \ 2 \leq j \leq m\}$. Observe that $\delta_n \lambda_{1,n} \to \infty$. Passing to a subsequence and changing indices, we may assume that $\delta_n = d(p_{1,n}, p_{2,n})$. We want to prove that $c_2 > 0$. Proposition 10 then implies that $p_1 \neq p_2$, a contradiction.

Let q_n be the middle point of $p_{1,n}, p_{2,n}$. Fix as before some $\alpha > 1$ and $\beta < 1$ close to 1. Using the catenoidal barrier $\mathcal{C}(\frac{\beta}{\lambda_{1,n}}, \frac{\delta_n}{2})$ as in the proof of Proposition 8, we can estimate $\eta_{1,n} - u_{1,n}(q_n)$ and $u_{2,n}(q_n) - \eta_{1,n}$. Adding the two estimates gives the lower bound

$$u_{2,n}(q_n) - u_{1,n}(q_n) \geq \frac{2\beta}{\lambda_{1,n}} \operatorname{argcosh}\left(\frac{\delta_n \lambda_{1,n}}{2\beta}\right) - \frac{C}{\lambda_n} \geq \frac{2\beta}{\lambda_{1,n}} \log(\delta_n \lambda_{1,n}) - \frac{C'}{\lambda_n}.$$

Let φ_n be the homothety of ratio $\mu_n = 1/\delta_n$ which maps $p_{1,n}$ to 0. Let $\tilde{p}_{i,n} = \varphi_n(p_{i,n})$. Passing to a subsequence, $\lim \tilde{p}_{i,n} = \tilde{p}_i$ exists (possibly infinite), with $\tilde{p}_1 = 0$ and $|\tilde{p}_2| = 1$. If all other $\tilde{p}_j$ are distinct from $\tilde{p}_2$, then there exists $\varepsilon \in (0,1)$ such that the disk $D(\tilde{p}_2, \varepsilon)$ contains no other point $\tilde{p}_j$. Going back to the original scale, we can use the catenoidal barrier $\mathcal{C}(\frac{\alpha}{\lambda_{2,n}}, \delta_n \varepsilon)$ to estimate $\eta_{2,n} - u_{1,n}(a_n)$ and $u_{2,n}(a_n) - \eta_{2,n}$. Adding the two estimates gives the upper bound

$$u_{2,n}(a_n) - u_{1,n}(a_n) \leq \frac{2\alpha}{\lambda_{2,n}} \log(\delta_n \lambda_{2,n}) + \frac{C'}{\lambda_n}$$

for some uniform constant C'. Combining the two estimates, we obtain after elementary operations

$$\left(\beta - \alpha \frac{\lambda_{1,n}}{\lambda_{2,n}}\right) \log(\delta_n \lambda_{1,n}) \leq C' \frac{\lambda_{1,n}}{\lambda_n} + \alpha \frac{\lambda_{1,n}}{\lambda_{2,n}} \log\left(\frac{\lambda_{2,n}}{\lambda_{1,n}}\right).$$

The right member has a finite limit (for the first term, this is because $c_1 > 0$). Since $\delta_n \lambda_{1,n} \to \infty$, we must have $\beta \leq \alpha \frac{\lambda_{1,n}}{\lambda_{2,n}}$ for n large enough. Hence $\beta \frac{\lambda_n}{\lambda_{1,n}} \leq \alpha \frac{\lambda_n}{\lambda_{2,n}}$. Passing to the limit gives $\beta c_1 \leq \alpha c_2$. Since we already know that $c_1 > 0$, we obtain $c_2 > 0$, hence by Proposition 10, $p_1 \neq p_2$, a contradiction. In case there are several points $\tilde{p}_j$ equal to $\tilde{p}_2$, we use instead the above extremal length argument to estimate $u_{2,n}(a_n) - u_{1,n}(a_n)$, and conclude again that $c_2 > 0$. This proves the proposition. □

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