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Classification and Existence of Non-Oscillatory Solutions of Second-Order Neutral Delay Difference Equations

Yong Zhou and B. G. Zhang

Abstract. In this paper, we give a classification of non-oscillatory solutions of a second-order neutral delay difference equation of the form

$$\Delta^2(x_n - c_n x_{n-\tau}) + f(n, x_{g_1(n)}, \ldots, x_{g_m(n)}) = 0 \qquad (n \geq n_0 \in \mathbb{N}).$$

Some existence results for each kind of non-oscillatory solutions are also established.

Keywords: *Neutral difference equations, non-oscillatory solutions*

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1. Introduction

Consider second-order nonlinear neutral difference equations of the form

$$\Delta^2(x_n - c_n x_{n-\tau}) + f(n, x_{g_1(n)}, \ldots, x_{g_m(n)}) = 0 \qquad (n \geq n_0 \in \mathbb{N}). \tag{1}$$

With respect to equations (1), throughout we shall assume the following:

(i) $\tau \in \mathbb{N}$, $\{c_n\} \subset \mathbb{R}_+$ and there exists $\delta \in (0,1]$ such that $c_n \leq 1 - \delta$ for $n \geq n_0$.

(ii) $g_j : \mathbb{N}_{n_0} \to \mathbb{N}_{n_0}$, $\mathbb{N}_{n_0} = \{n_0, n_0 + 1, \ldots\}$, and $\lim_{n\to\infty} g_j(n) = \infty$ $(j = 1, 2, \ldots, m)$.

(iii) $f : \mathbb{N}_{n_0} \times \mathbb{R}^m \to \mathbb{R}$ is continuous with respect to the last m arguments and $y_1 f(n, y_1, \ldots, y_m) > 0$ for $y_1 y_j > 0$ $(j = 2, \ldots, m)$. Moreover,

$$|f(n, x_1, \ldots, x_m)| \geq |f(n, y_1, \ldots, y_m)|$$

when $|y_j| \leq |x_j|$ and $x_j y_j > 0$ $(j = 1, 2, \ldots, m)$.

The forward difference Δ is defined as usual, i.e. $\Delta x_n = x_{n+1} - x_n$. Set

$$y_n = x_n - c_n x_{n-\tau}. \tag{2}$$

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In [3, 4], Agarwal, Manuel and Thandapani give a classification of all non-trivial solutions of certain second-order neutral delay difference equations according to the sign of $\{y_n \Delta y_n\}$ and established the existence of solutions in some classes.

In this paper, we study the existence and asymptotic behaviour of non-oscillatory solutions of (1). More precisely, we give a classification of non-oscillatory solutions of (1) according to their asymptotic behaviour. Moreover, we establish some existence results for each kind of non-oscillatory solutions of (1). Especially, we obtain two necessary and sufficient conditions for existence of non-oscillatory solutions of (1).

2. Results

First we show some lemmas which will be useful for the main results.

Lemma 1. *Let $\{x_n\}$ be an eventually positive or eventually negative solution of (1). If $\lim_{n\to\infty} x_n = 0$, then $\{y_n\}$ is eventually negative or eventually positive, respectively, and $\lim_{n\to\infty} y_n = 0$. If $\lim_{n\to\infty} x_n = 0$ fails, then $\{y_n\}$ is eventually positive or eventually negative.*

Proof. Let $\{x_n\}$ be an eventually positive solution of (1). Then $\Delta^2 y_n < 0$ eventually. Thus Δy_n is decreasing and $\Delta y_n > 0$ or $\Delta y_n < 0$ eventually. Also, $y_n > 0$ or $y_n < 0$ eventually. If $\lim_{n\to\infty} x_n = 0$, from (2) we have $\lim_{n\to\infty} y_n = 0$. Since $\{y_n\}$ is monotonic, so $\lim_{n\to\infty} \Delta y_n = 0$, which implies that $\Delta y_n > 0$. Therefore, $y_n < 0$ eventually. If $\lim_{n\to\infty} x_n = 0$ fails, then $\limsup_{n\to\infty} x_n > 0$. We show that $y_n > 0$ eventually. If not, then $y_n < 0$ eventually. If $\{x_n\}$ is unbounded, then there exists a sequence $\{n_k\}$ such that $\lim_{k\to\infty} n_k = \infty$, $x_{n_k} = \max_{n_0 \le n \le n_k}\{x_n\}$ and $\lim_{k\to\infty} x_{n_k} = \infty$. From (2), we have

$$y_{n_k} = x_{n_k} - c_{n_k} x_{n_k - \tau} \ge x_{n_k}(1 - c_{n_k}). \tag{3}$$

Thus $\lim_{k\to\infty} y_{n_k} = \infty$, which is a contradiction. If $\{x_n\}$ is bounded, then there exists a sequence $\{n_k\}$ such that $\lim_{k\to\infty} n_k = \infty$ and $\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$. Since the sequences $\{c_{n_k}\}$ and $\{x_{n_k-\tau}\}$ are bounded, there exist convergent subsequences. Without loss of generality, we may assume that $\lim_{k\to\infty} x_{n_k-\tau}$ and $\lim_{k\to\infty} c_{n_k}$ exist. Hence

$$0 \ge \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} (x_{n_k} - c_{n_k} x_{n_k-\tau}) \ge \limsup_{n\to\infty} x_n \Big(1 - \lim_{k\to\infty} c_{n_k}\Big) > 0$$

which is a contradiction again. Therefore, $y_n > 0$ eventually. A similar proof can be given if $x_n < 0$ eventually ∎

Lemma 2. *Assume that $\lim_{n\to\infty} c_n = c \in [0, 1)$ and $\{x_n\}$ is an eventually positive or eventually negative solution of (1). If $\lim_{n\to\infty} y_n = a \in \mathbb{R}$, then $\lim_{n\to\infty} x_n = \frac{a}{1-c}$. If $\lim_{n\to\infty} y_n = \pm\infty$, then $\lim_{n\to\infty} x_n = \pm\infty$, respectively.*

Proof. Let $\{x_n\}$ be an eventually positive solution of (1). Then $x_n \ge y_n$ eventually. If $\lim_{n\to\infty} = y_n = \infty$, then $\lim_{n\to\infty} x_n = \infty$. Now we consider the case that $\lim_{n\to\infty} y_n = a \in \mathbb{R}$. Thus $\{y_n\}$ is bounded which implies that $\{x_n\}$ is bounded (see (3)). Therefore, there exists a sequence $\{n_k\}$ such that $\lim_{k\to\infty} n_k = \infty$ and

$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$. As before, without loss of generality we may assume that $\lim_{k\to\infty} c_{n_k}$ and $\lim_{k\to\infty} x_{n_k-\tau}$ exist. Hence

$$a = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} x_{n_k} - \lim_{k\to\infty} c_{n_k} \lim_{k\to\infty} x_{n_k-\tau} \geq \limsup_{n\to\infty} x_n(1-c),$$

i.e.

$$\frac{a}{1-c} \geq \limsup_{n\to\infty} x_n. \tag{4}$$

On the other hand, there exists a sequence $\{n'_k\}$ such that $\lim_{k\to\infty} x_{n'_k} = \liminf_{n\to\infty} x_n$. Without loss of generality we assume that $\lim_{k\to\infty} c_{n'_k}$ and $\lim_{k\to\infty} x_{n'_k-\tau}$ exist. Hence

$$a = \lim_{k\to\infty} y_{n'_k} = \lim_{k\to\infty} x_{n'_k} - \lim_{k\to\infty} c_{n'_k} \lim_{k\to\infty} x_{n'_k-\tau} \geq \liminf_{n\to\infty} x_n(1-c)$$

or

$$\frac{a}{1-c} \leq \liminf_{n\to\infty} x_n. \tag{5}$$

Combining (4) and (5) we obtain $\lim_{n\to\infty} x_n = \frac{a}{1-c}$. A similar proof can be given if $x_n < 0$ ∎

We are now ready to prove the following results.

Theorem 1. *Assume that* $\lim_{n\to\infty} c_n = c \in [0,1)$*, denote by* S *the set of all non-oscillatory solutions of* (1) *and define the following subsets:*

$$\begin{aligned}
S(0,0,0) &= \Big\{\{x_n\} \in S:\ \lim_{n\to\infty} x_n = 0,\ \lim_{n\to\infty} y_n = 0,\ \lim_{n\to\infty} \Delta y_n = 0\Big\}\\
S(b,a,0) &= \Big\{\{x_n\} \in S:\ \lim_{n\to\infty} x_n = b = \frac{a}{1-c},\ \lim_{n\to\infty} y_n = a,\ \lim_{n\to\infty} \Delta y_n = 0\Big\}\\
S(\infty,\infty,0) &= \Big\{\{x_n\} \in S:\ \lim_{n\to\infty} x_n = \infty,\ \lim_{n\to\infty} y_n = \infty,\ \lim_{n\to\infty} \Delta y_n = 0\Big\}\\
S(\infty,\infty,d) &= \Big\{\{x_n\} \in S:\ \lim_{n\to\infty} x_n = \infty,\ \lim_{n\to\infty} y_n = \infty,\ \lim_{n\to\infty} \Delta y_n = d \neq 0\Big\}.
\end{aligned}$$

Then

$$S = S(0,0,0) \cup S(b,a,0) \cup S(\infty,\infty,0) \cup S(\infty,\infty,d).$$

Proof. Without loss of generality, let $\{x_n\}$ be an eventually positive solution of (1). If $\lim_{n\to\infty} x_n = 0$, by Lemma 1, $\lim_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} \Delta y_n = 0$, i.e. $x_n \in S(0,0,0)$. If $\lim_{n\to\infty} x_n = 0$ fails, then by Lemma 2 $y_n > 0$ eventually, and it is easy to see that $\Delta y_n > 0$ and $\Delta^2 y_n < 0$ eventually. If $\lim_{n\to\infty} y_n = a > 0$ exists, then $\lim_{n\to\infty} \Delta y_n = 0$, by Lemma 2, and we have $\lim_{n\to\infty} x_n = \frac{a}{1-c} = b$, i.e. $x_n \in S(b,a,0)$. If $\lim_{n\to\infty} y_n = \infty$, then by Lemma 2 $\lim_{n\to\infty} x_n = \infty$. Since $\Delta^2 y_n < 0$ and $\Delta y_n > 0$, we have $\lim_{n\to\infty} \Delta y_n = d$, where $d = 0$ or $d > 0$. Then either $\{x_n\} \in S(\infty,\infty,0)$ or $\{x_n\} \in S(\infty,\infty,d)$ ∎

In the following we shall show some existence results for each kind of non-oscillatory solution of (1). For this, denote by X the Banach space $l_\infty^{n_0}$ of all bounded real sequences $x = \{x_n\}_{n\geq n_0}$ with norm $\|x_n\| = \sup_{n\geq n_0} |x_n|$.

Theorem 2. *Assume that* $\lim_{n\to\infty} c_n = c \in [0,1)$. *Then* (1) *has a non-oscillatory solution* $\{x_n\} \in S(b,a,0)$ $(a \neq 0$ *and* $b \neq 0)$ *if and only if*

$$\sum_{j=n_0}^{\infty} j|f(j,b_1,\ldots,b_1)| < \infty \tag{6}$$

for some $b_1 \neq 0$.

Proof. *Necessity.* Without loss of generality, let $\{x_n\} \in S(b,a,0)$ be an eventually positive solution of (1). By Theorem 1 we know that $b > 0$ and $a > 0$. From (1) and (2) we have

$$\Delta^2 y_n = -f(n, x_{g_1(n)}, \ldots, x_{g_m(n)}).$$

Summing both sides of this equalty from $s \geq n_0$ to ∞ we get

$$\Delta y_s = \sum_{j=s}^{\infty} f(j, x_{g_1(j)}, \ldots, x_{g_m(j)}).$$

Summing both sides of the equality from $N \geq n_0$ to $n-1 > N$ we get

$$\begin{aligned} y_n = y_N &+ \sum_{j=N}^{n-1} (j-N+1) f\big(j, x_{g_1(j)}, \ldots, x_{g_m(j)}\big) \\ &+ \sum_{j=n}^{\infty} (n-N) f\big(j, x_{g_1(j)}, \ldots, x_{g_m(j)}\big). \end{aligned} \tag{7}$$

Since $\lim_{j\to\infty} x_{g_i(j)} = b > 0$ $(i = 1,2,\ldots,m)$, there exists an $N \geq n_0$ such that $x_{g_i(j)} \geq \frac{b}{2}$ for $j \geq N$. Hence from (7) we have

$$\sum_{j=N}^{n-1} (j-N+1)|f(j, \tfrac{b}{2}, \ldots, \tfrac{b}{2})| < y_n - y_N$$

which implies that (6) holds.

Sufficiency. Set $b_1 > 0$ and $A > 0$ so that $A < (1-c)b_1$. From (6) there exists a sufficiently large $N \geq n_0$ so that for $n \geq N$ we have $n - \tau \geq n_0$ and $g_i(n) \geq n_0$ $(i = 1,2,\ldots,m)$, and

$$\frac{A}{b_1} + c_n + \frac{1}{b_1} \sum_{j=N}^{\infty} j f(j, b_1, \ldots, b_1) \leq 1. \tag{8}$$

Define a set Ω by

$$\Omega = \Big\{ \{x_n\} \in X \,|\, 0 \leq x_n \leq b_1 \quad (n \geq n_0) \Big\}$$

and an operator T on Ω by

$$Tx_n = \begin{cases} A + c_n x_{n-\tau} + \displaystyle\sum_{j=N}^{n-1} j f\big(j, x_{g_1(j)}, \ldots, x_{g_m(j)}\big) & \\ \qquad\qquad + \displaystyle\sum_{j=n}^{\infty} (n-1) f\big(j, x_{g_1(j)}, \ldots, x_{g_m(j)}\big) & \text{if } n \geq N+1 \\ Tx_{N+1} & \text{if } n_0 \leq n < N+1. \end{cases} \tag{9}$$

Clearly, for $\{x_n\} \in \Omega$,

$$\begin{aligned} Tx_n &\le A + c_n b_1 + \sum_{j=N}^{n-1} j f(j, b_1, \ldots, b_1) + \sum_{j=n}^{\infty} (n-1) f(j, b_1, \ldots, b_1) \\ &\le A + c_n b_1 + \sum_{j=N}^{\infty} j f(j, b_1, \ldots, b_1) \qquad (n \ge N+1) \\ &\le b_1 \end{aligned}$$

and

$$Tx_n = Tx_{N+1} \le b_1 \qquad (n_0 \le n \le N+1),$$

i.e. $T\Omega \subset \Omega$.

Define a series of sequences $\{x_n^{(k)}\}$ $(k \in \mathbb{N}_{n_0})$ as

$$\left.\begin{aligned} &x_n^{(0)} = 0 \\ &x_n^{(k)} = Tx_n^{(k-1)} \quad (k \in \mathbb{N}) \end{aligned}\right\} \qquad (n \ge n_0). \tag{10}$$

By induction, we can prove that

$$0 \le x_n^{(k-1)} \le x_n^{(k)} \le b_1 \qquad (n \ge n_0, k \in \mathbb{N}).$$

Then there exists $\{x_n\} \in \Omega$ such that $\lim_{k\to\infty} x_n^{(k)} = x_n$ $(n \ge n_0)$.

In the following, we shall show that

$$\lim_{k\to\infty} \sum_{j=n}^{\infty} (n-1) f\big(j, x_{g_1(j)}^{(k)}, \ldots, x_{g_m(j)}^{(k)}\big) = \sum_{j=n}^{\infty} (n-1) f\big(j, x_{g_1(j)}, \ldots, x_{g_m(j)}\big).$$

In fact, by (6), for any $\varepsilon > 0$ there exists $N_1 \ge n_0$ such that

$$\sum_{j=N_1}^{\infty} j f(j, b_1, \ldots, b_1) < \varepsilon.$$

Thus, for $N_2 \ge N_1$ we get

$$\begin{aligned} &\left| \sum_{j=n}^{N_2} (n-1) f\big(j, x_{g_1(j)}^{(k)}, \ldots, x_{g_m(j)}^{(k)}\big) - \sum_{j=n}^{\infty} (n-1) f\big(j, x_{g_1(j)}^{(k)}, \ldots, x_{g_m(j)}^{(k)}\big) \right| \\ &\qquad = \left| \sum_{j=N_2+1}^{\infty} (n-1) f\big(j, x_{g_1(j)}^{(k)}, \ldots, x_{g_m(j)}^{(k)}\big) \right| \\ &\qquad \le \sum_{j=N_2+1}^{\infty} j f\big(j, x_{g_1(j)}^{(k)}, \ldots, x_{g_m(j)}^{(k)}\big) \\ &\qquad \le \sum_{j=N_2+1}^{\infty} j f(j, b_1, \ldots, b_1) \\ &\qquad < \varepsilon. \end{aligned}$$

Hence

$$\sum_{j=n}^{N_2}(n-1)f\big(j,x^{(k)}_{g_1(j)},\dots,x^{(k)}_{g_m(j)}\big) \;\longrightarrow\; \sum_{j=n}^{\infty}(n-1)f\big(j,x^{(k)}_{g_1(j)},\cdots,x^{(k)}_{g_m(j)}\big)$$

uniformly for $k\in\mathbb{N}$ as $N_2\to\infty$. Therefore

$$\begin{aligned}
&\lim_{k\to\infty}\sum_{j=n}^{\infty}(n-1)f\big(j,x^{(k)}_{g_1(j)},\cdots,x^{(k)}_{g_m(j)}\big)\\
&\quad=\lim_{k\to\infty}\lim_{N_2\to\infty}\sum_{j=n}^{N_2}(n-1)f\big(j,x^{(k)}_{g_1(j)},\dots,x^{(k)}_{g_m(j)}\big)\\
&\quad=\lim_{N_2\to\infty}\lim_{k\to\infty}\sum_{j=n}^{N_2}(n-1)f\big(j,x^{(k)}_{g_1(j)},\dots,x^{(k)}_{g_m(j)}\big)\\
&\quad=\lim_{N_2\to\infty}\sum_{j=n}^{N_2}(n-1)f\big(j,x_{g_1(j)},\dots,x_{g_m(j)}\big)\\
&\quad=\sum_{j=n}^{\infty}(n-1)f\big(j,x_{g_1(j)},\dots,x_{g_m(j)}\big).
\end{aligned}$$

Let $k\to\infty$. Then (10) gives

$$x_n=\begin{cases} A+c_nx_{n-\tau}+\displaystyle\sum_{j=N}^{n-1}jf\big(j,x_{g_1(j)},\dots,x_{g_m(j)}\big) & \\ \qquad\qquad +\displaystyle\sum_{j=n}^{\infty}(n-1)f\big(j,x_{g_1(j)},\dots,x_{g_m(j)}\big) & \text{if } n\ge N+1\\ x_{N+1} & \text{if } n_0\le n<N+1.\end{cases}$$

Clearly, $x_n>0$ for $n\ge n_0$. Therefore, $\{x_n\}$ is a positive solution of (1). Since $0<A\le x_n\le b_1$, from Theorem 1, $\{x_n\}\in S(b,a,0)$ ∎

Theorem 3. *Assume that* $\lim_{n\to\infty}c_n=c\in[0,1)$. *Then* (1) *has a non-oscillatory solution* $\{x_n\}\in S(\infty,\infty,d)$ $(d\neq 0)$ *if and only if*

$$\sum_{j=n_0}^{\infty}\big|f\big(j,hg_1(j),\dots,hg_m(j)\big)\big|<\infty \tag{11}$$

for some $h\neq 0$.

Proof. *Necessity.* Without loss of generality, let $\{x_n\}\in S(\infty,\infty,d)$ be an eventually positive solution of (1). From Theorem 1, we have $d>0$. From (1) and (2) we have

$$\Delta^2y_n+f\big(n,x_{g_1(n)},\dots,x_{g_m(n)}\big)=0\qquad (n\ge n_0).$$

Summing both sides of this equality from $n_1 \geq n_0$ to $n-1 > n_1$ we get

$$\Delta y_n - \Delta y_{n_1} + \sum_{j=n_1}^{n-1} f\big(j, x_{g_1(j)}, \ldots, x_{g_m(j)}\big) = 0.$$

Since $\lim_{n\to\infty} \Delta y_n = d > 0$, we have

$$\sum_{j=n_1}^{\infty} f\big(j, x_{g_1(j)}, \ldots, x_{g_m(j)}\big) < \infty \tag{12}$$

and there exist $d_1 > 0$ and $n_2 \geq n_1$ such that $y_n \geq d_1 n$ for $n \geq n_2$. Therefore

$$\begin{aligned}\sum_{j=n_1}^{\infty} f\big(j, x_{g_1(j)}, \ldots, x_{g_m(j)}\big) &\geq \sum_{j=n_1}^{\infty} f\big(j, y_{g_1(j)}, \ldots, y_{g_m(j)}\big) \\ &\geq \sum_{j=n_1}^{\infty} f\big(j, d_1 g_1(j), \ldots, d_1 g_m(j)\big).\end{aligned} \tag{13}$$

Choosing $h = d_1$ and combining (12) and (13), we get

$$\sum_{j=n_1}^{\infty} f\big(j, hg_1(j), \ldots, hg_m(j)\big) < \infty. \tag{14}$$

Sufficiency. Set $h > 0$, $d > 0$ and $B > 0$. From (14) there exists a sufficiently large $N \geq n_0$ so that for $n \geq N$ we have $n - \tau \geq n_0$ and $g_j(n) \geq n_0 \ \ (j = 1, 2, ..., m)$ and

$$\frac{d}{h} + \frac{B}{nh} + c_n + \frac{1}{nh} \sum_{j=N}^{\infty} f\big(j, hg_1(j), \ldots, hg_m(j)\big) < 1. \tag{15}$$

Define a set Ω by

$$\Omega = \Big\{\{z_n\} \in X : d \leq z_n \leq h \ \ (n \geq n_0)\Big\}$$

and an operator T on Ω by

$$Tz_n = \begin{cases} d + \dfrac{B}{n} + c_n \dfrac{n-\tau}{n} z_{n-\tau} & \\ \quad + \dfrac{1}{n} \displaystyle\sum_{j=N}^{n-1} j f\big(j, g_1(j) z_{g_1(j)}, \ldots, g_m(j) z_{g_m(j)}\big) & \text{if } n \geq N+1 \\ \quad + \dfrac{n-1}{n} \displaystyle\sum_{j=n}^{\infty} f\big(j, g_1(j) z_{g_1(j)}, \ldots, g_m(j) z_{g_m(j)}\big) & \\ Tz_{N+1} & \text{if } n_0 \leq n < N+1. \end{cases} \tag{16}$$

Clearly, for $\{z_n\} \in \Omega$

$$\begin{aligned} Tz_n &\le d + \frac{B}{n} + c_n h + \frac{1}{n}\sum_{j=N}^{n-1} jf\big(j, hg_1(j), \dots, hg_m(j)\big) \\ &\quad + \frac{1}{n}\sum_{j=n}^{\infty}(n-1)f\big(j, hg_1(j), \dots, hg_{m(j)}\big) \qquad (n \ge N+1) \\ &\le d + \frac{B}{n} + c_n h + \frac{1}{n}\sum_{j=N}^{\infty} jf\big(j, hg_1(j), \dots, hg_m(j)\big) \\ &< h \end{aligned}$$

and

$$Tz_n = Tz_{N+1} \le \delta \qquad (n_0 \le n < N+1).$$

It is easy to see that $Tz_n \ge d$ for $n \ge n_0$. Hence, $T\Omega \subset \Omega$. Define a series of sequences $\{z_n^{(k)}\}$ $(k \in \mathbb{N}_0)$ by

$$\left.\begin{aligned} z_n^{(0)} &= d \\ z_n^{(k)} &= Tz_n^{(k-1)} \quad (k \in \mathbb{N}) \end{aligned}\right\} \qquad (n \ge n_0).$$

We can prove that

$$d \le z_n^{(k)} \le z_n^{(k+1)} \le h \qquad (n \ge n_0, k \in \mathbb{N}_0).$$

Then there exists $\{z_n\} \in \Omega$ such that $\lim_{k\to\infty} z_n^{(k)} = z_n$ and $d \le z_n \le h$ $(n \ge n_0)$. Clearly, $z_n = Tz_n$ $(n \ge n_0)$, i.e.

$$z_n = \begin{cases} d + \frac{B}{n} + c_n \frac{n-\tau}{n} z_{n-\tau} \\ \quad + \frac{1}{n}\sum_{j=N}^{n-1} jf\big(j, g_1(j)z_{g_1(j)}, \dots, g_m(j)z_{g_m(j)}\big) & \text{if } n \ge N+1 \\ \quad + \frac{1}{n}\sum_{j=n}^{\infty}(n-1)f\big(j, g_1(j)z_{g_1(j)}, \dots, g_m(j)z_{g_m(j)}\big) \\ z_{N+1} & \text{if } n_0 \le n < N+1. \end{cases}$$

Let $x_n = nz_n$ $(n \ge n_0)$. Then we have

$$x_n = \begin{cases} d_n + B + c_n x_{n-\tau} \\ \quad + \sum_{j=N}^{n-1} jf\big(j, x_{g_1(j)}, \dots, x_{g_m(j)}\big) & \text{if } n \ge N+1 \\ \quad + \sum_{j=n}^{\infty}(n-1)f\big(j, x_{g_1(j)}, \dots, x_{g_m(j)}\big) \\ x_{N+1} & \text{if } n_0 \le n < N+1. \end{cases} \tag{18}$$

Hence, $\{x_n\}$ is a positive solution of (1). On the other hand, $x_n \geq y_n \geq dn + B$. Hence $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} y_n = \infty$. From (18), we have

$$\begin{aligned}\Delta y_n &= d + \sum_{j=n}^{\infty} f\big(j, x_{g_1(j)}, \ldots, x_{g_m(j)}\big)\\ &= d + \sum_{j=n}^{\infty} f\big(j, g_1(j) z_{g_1(j)}, \ldots, g_m(j) z_{g_m(j)}\big)\\ &\leq d + \sum_{j=n}^{\infty} f\big(j, hg_1(j), \ldots, hg_m(j)\big).\end{aligned}$$

Hence, $\lim_{n\to\infty} \Delta y_n = d$. Therefore, $\{x_n\} \in S(\infty, \infty, d)$ ∎

Theorem 4. *Assume that* $\lim_{n\to\infty} c_n = c \in [0,1)$. *Further, assume that*

$$\sum_{j=n_0}^{\infty} \big|f\big(j, hg_1(j), \ldots, hg_m(j)\big)\big| < \infty \tag{19}$$

and

$$\sum_{j=n_0}^{\infty} j|f(j, b_1, \ldots, b_1)| = \infty \tag{20}$$

for some $h \neq 0$ *and* $b_1 \neq 0$, *respectively. Then* (1) *has a non-oscillatory solution* $\{x_n\} \in S(\infty, \infty, 0)$.

Proof. Without loss of generality, assume that $h > 0$ and $b_1 > 0$. From (19) there exists a sufficiently large $N > n_0$ so that for $n \geq N$ we have $n - \tau \geq n_0$ and $g_j(n) \geq n_0 \ \ (j = 1, 2, \ldots, m)$ and

$$\frac{b_1}{nh} + c_n + \frac{1}{h} \sum_{j=N}^{\infty} f\big(j, hg_1(j), \ldots, hg_m(j)\big) < 1. \tag{21}$$

Define a set Ω by

$$\Omega = \Big\{\{z_n\} \in X : 0 \leq z_n \leq h \ \ (n \geq n_0)\Big\}$$

and an operator T on Ω by

$$Tz_n = \begin{cases} \dfrac{b_1}{n} + c_n \dfrac{n-\tau}{n} z_{n-\tau} \\ \qquad + \dfrac{1}{n} \displaystyle\sum_{j=N}^{n-1} j f\big(j, g_1(j) z_{g_1(j)}, \ldots, g_m(j) z_{g_m(j)}\big) & \text{if } n \geq N+1 \\ \qquad + \dfrac{1}{n} \displaystyle\sum_{j=n}^{\infty} (n-1) f\big(j, g_1(j) z_{g_1(j)}, \ldots, g_m(j) z_{g_m(j)}\big) \\ Tz_{N+1} & \text{if } n_0 \leq n < N+1. \end{cases}$$

Clearly, for $\{z_n\} \in \Omega$

$$\begin{aligned} Tz_n &\le \frac{b_1}{n} + c_n h + \frac{1}{n}\sum_{j=N}^{n-1} j f\big(j, hg_1(j), \ldots, hg_m(j)\big) \\ &\quad + \frac{1}{n}\sum_{j=n}^{\infty}(n-1) f\big(j, hg_1(j), \ldots, hg_m(j)\big) \qquad (n \ge N+1) \\ &\le \frac{b_1}{n} + c_n h + \sum_{j=N}^{\infty} f\big(j, hg_1(j), \ldots, hg_m(j)\big) \\ &\le h \end{aligned}$$

and

$$Tz_n = Tz_{N+1} \le h \qquad (n_0 \le n < N+1),$$

i.e. $T\Omega \subset \Omega$.

Define a series of sequences $\{z_n^{(k)}\}$ $(k \in \mathbb{N}_0)$ by

$$\left.\begin{aligned} z_n^{(0)} &= 0 \\ z_n^{(k)} &= Tz_n^{(k-1)} \quad (k \in \mathbb{N}) \end{aligned}\right\} \qquad (n \ge n_0). \tag{22}$$

By induction, we can prove that

$$0 \le z_n^{(k)} \le z_n^{(k+1)} \le h \qquad (n \ge n_0, k \in \mathbb{N}_0).$$

Then there exists $\{z_n\} \in \Omega$ such that $\lim_{k\to\infty} z_n^{(k)} = z_n$ $(n \ge n_0)$.

Clearly, $z_n = Tz_n$ $(n \ge n_0)$, i.e.

$$z_n = \begin{cases} \dfrac{b_1}{n} + c_n \dfrac{n-\tau}{n} z_{n-\tau} \\ \qquad + \dfrac{1}{n}\displaystyle\sum_{j=N}^{n-1} j f\big(j, g_1(j) z_{g_1(j)}, \ldots, g_m(j) z_{g_m(j)}\big) & \text{if } n \ge N+1 \\ \qquad + \dfrac{1}{n}\displaystyle\sum_{j=n}^{\infty} (n-1) f\big(j, g_1(j) z_{g_1(j)}, \ldots, g_m(j) z_{g_m(j)}\big) \\ z_{N+1} & \text{if } n_0 \le n < N+1. \end{cases}$$

Let $x_n = n z_n$ $(n \ge n_0)$. Then we have

$$x_n = \begin{cases} b_1 + c_n x_{n-\tau} + \displaystyle\sum_{j=N}^{n-1} j f\big(j, x_{g_1(j)}, \ldots, x_{g_m(j)}\big) \\ \qquad\qquad + \displaystyle\sum_{j=n}^{\infty} (n-1) f\big(j, x_{g_1(j)}, \ldots, x_{g_m(j)}\big) & \text{if } n \ge N+1 \\ x_{N+1} & \text{if } n_0 \le n < N+1. \end{cases} \tag{23}$$

Hence, $\{x_n\}$ is a positive solution of (1). On the other hand, from (23) we have $x_n \geq b_1$ and

$$x_n \geq y_n = x_n - c_n x_{n-\tau} \geq \sum_{j=N}^{n-1} j f(j, b_1, \ldots, b_1)$$

for $n \geq n_0$ which along with (20) implies $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} y_n = \infty$. By (23) we get

$$\begin{aligned}\Delta y_n &= \sum_{j=n}^{\infty} f\big(j, x_{g_1(j)}, \ldots, x_{g_m(j)}\big)\\ &= \sum_{j=n}^{\infty} f\big(j, g_1(j) z_{g_1(j)}, \ldots, g_m(j) z_{g_m(j)}\big)\\ &\leq \sum_{j=n}^{\infty} f\big(j, hg_1(j), \ldots, hg_m(j)\big).\end{aligned}$$

Hence

$$0 \leq \lim_{n\to\infty} \Delta y_n \leq \lim_{n\to\infty} \sum_{j=n}^{\infty} f\big(j, hg_1(j), \ldots, hg_m(j)\big) = 0,$$

i.e. $\lim_{n\to\infty} \Delta y_n = 0$. Therefore, $\{x_n\} \in S(\infty, \infty, 0)$ ∎

Theorem 5. *Assume that* $\lim_{n\to\infty} c_n = c \in [0,1)$. *Further, assume that there exists* $d > 0$ *such that*

$$\sum_{j=n_0}^{\infty} f(j, d_1, \ldots, d_1) = \infty \qquad \text{for any } d_1 \in (0, d]. \tag{24}$$

Then every solution $\{x_n\}$ *of* (1) *either oscillates or* $\{x_n\} \in S(0,0,0)$.

Proof. Let $\{x_n\}$ be an eventually positive solution of (1). By Lemma 1, if $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} y_n = 0$ and so $\lim_{n\to\infty} \Delta y_n = 0$. Hence, $\{x_n\} \in S(0,0,0)$. If $\lim_{n\to\infty} x_n = 0$ fails, then $y_n > 0$ eventually. Since $\Delta^2 y_n < 0$, we have $\Delta y_n > 0$. Therefore, there exists $\overline{d} \in (0, d]$ such that $x_n \geq y_n \geq \overline{d}$. From (1) and (2) we have

$$\Delta^2 y_n = -f\big(n, x_{g_1(n)}, \ldots, x_{g_m(n)}\big).$$

Summing both sides of the equation from n_0 to $n-1$ we obtain

$$\Delta y_n - \Delta y_{n_0} = -\sum_{j=n_0}^{n-1} f\big(j, x_{g_1(j)}, \ldots, x_{g_m(j)}\big) \leq -\sum_{j=n_0}^{n-1} f(j, \overline{d}, \ldots, \overline{d}).$$

Let $n \to \infty$. Then we get $\sum_{j=n_0}^{\infty} f(j, \overline{d}, \ldots, \overline{d}) < \infty$ which contradicts (24) and completes the proof ∎

3. Examples

In the following we shall give three examples which demonstrate the applicability and the importance of the results obtained in Section 2.

Example 1. Consider the difference equation

$$\Delta^2(x_n - \tfrac{1}{2}x_{n-2}) + \tfrac{2^{-n-2}}{(1+2^{-n+1})^3}\, x_{n-1}^3 = 0 \qquad (n \geq 2) \tag{25}$$

for which condition (6) of Theorem 2 is satisfied. In fact, the sequence $\{x_n\} = \{1 + \frac{1}{2^n}\}$ is a non-oscillatory solution of (25) which belongs to the class $S(1, \frac{1}{2}, 0)$.

Example 2. Consider the difference equation

$$\Delta^2(x_n - \tfrac{1}{4}x_{n-1}) + \tfrac{2^{-n-3}}{(n-1-2^{-n+1})^5}\, x_{n-1}^5 = 0 \qquad (n \geq 2) \tag{26}$$

for which condition (11) of Theorem 3 is satisfied. In fact, the sequence $\{x_n\} = \{n - \frac{1}{2^n}\}$ is a non-oscillatory solution of (26) which belongs to the class $S(\infty, \infty, \frac{3}{4})$.

Example 3. Consider the difference equation

$$\Delta^2(x_n - \tfrac{1}{2}x_{n-2}) + 2^{2n-5}x_{n-1}^3 = 0 \qquad (n \geq 2) \tag{27}$$

for which condition (24) of Theorem 5 is satisfied. In fact, the sequence $\{x_n\} = \{\frac{1}{2^n}\}$ is a non-oscillatory solution of (27) which belongs to the class $S(0, 0, 0)$.

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