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On Inequalities of Korn, Friedrichs, Magenes-Stampacchia-Nečas and Babuška-Aziz

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Abstract. The equivalence between the inequalities of Korn, Friedrichs, Magenes-Stampacchia-Nečas and Babuška-Aziz is derived using some elementary properties of the gradient, divergence and curl operators implied by these inequalities.

Keywords: *Korn's inequalities, continuum mechanics, equations of mathematical physics*

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1. Introduction

Horgan has shown in [6] that the Friedrichs' inequality and the second Korn's inequality are equivalent for a plane domain. Moreover, he found that the Friedrichs' and Korn's constants Γ and K are related by $K = 2(1+\Gamma)$. He used a variational approach, leading to the consideration of associated eigenvalue problems. These results were extended by Horgan and Payne [8] to include the relations between the inequalities of Friedrichs and Babuška-Aziz. Using a direct method, Horgan and Payne proved that also these two inequalities are equivalent. Besides, they found that the optimal constant C of the inequality of Babuška-Aziz is related to Γ by the simple arithmetic relation $\Gamma = C - 1$. More recently, Velte [14] has rederived the relation between these two inequalities from the properties of the solutions of certain eigenvalue problems in variational form which are directly associated to the inequalities of Friedrichs and Babuška-Aziz. He has also shown in [15] that these results have a counterpart in dimension three.

The author's purpose in the present paper is to give a simple equivalence proof of the inequalities of all these inequalities. For a connected bounded domain, we show that this equivalence is a direct consequence of some elementary properties (i.e., the closedness of their image) of the gradient, divergence and curl operators implied by all these inequalities. First we show the equivalence between the inequalities of Babuška-Aziz and Magenes-Stampacchia-Nečas, then that between the inequalities of Friedrichs and Magenes-Stampacchia-Nečas, and finally that between the inequalities of Korn and Magenes-Stampacchia-Nečas. This clearly suffices to prove that all the inequalities mentioned are equivalent.

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2. Notations and basic results

Let Ω be a bounded, Lipschitzian, simply connected domain of the two-dimensional Euclidean space $\mathbb{R}^2$. We denote by $L^2(\Omega)$ the space of square integrable functions on Ω, by $H^1(\Omega)$ the space of functions on Ω with square integrable gradient, by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$, the closure in $H^1(\Omega)$ of the space of infinitely differentiable functions with compact support contained in Ω. Moreover, we need the following subspaces of $L^2(\Omega)$: $L_0^2(\Omega)$, the space of functions orthogonal to the constants, and $\mathcal{H}$, the space of harmonic functions.

Spaces of vector fields on Ω will be denoted by superposing the dimension of the vector space on the symbols, enclosed between square brackets, of the analogous spaces of functions. Bold face letters will be used to indicate spaces of tensor fields on Ω. For example, $[L^2(\Omega)]^2$ and $\mathbf{L}^2(\Omega)$ are the spaces of square integrable vector and tensor fields on Ω, respectively.

Let us summarize some well known results about the divergence, gradient and curl operators (see, for example, Girault and Raviart [4], Temam [12] and Velte [14]). Because of the Poincaré inequality the expression $\langle \nabla \mathbf{u}, \nabla \mathbf{v} \rangle_{\mathbf{L}^2(\Omega)}$ is a scalar product on $[H_0^1(\Omega)]^2$, and the associated norm is equivalent to the norm of $[H_0^1(\Omega)]^2$. The space $[H_0^1(\Omega)]^2$ can be decomposed into the direct sum of orthogonal subspaces with respect to the scalar product $\langle \nabla \mathbf{u}, \nabla \mathbf{v} \rangle_{\mathbf{L}^2(\Omega)}$:

$$[H_0^1(\Omega)]^2 = \operatorname{Ker}\operatorname{div} \oplus \operatorname{Ker}\operatorname{curl} \oplus W. \tag{1}$$

Moreover, an application of the Stokes theorem yields [1)]

$$\langle \nabla \mathbf{u}, \nabla \mathbf{v} \rangle_{\mathbf{L}^2(\Omega)} = \langle \operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v} \rangle_{L^2(\Omega)} + \langle \operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v} \rangle_{L^2(\Omega)}. \tag{2}$$

The divergence operator maps W into the space $\mathcal{H}$ of harmonic functions in $L_0^2(\Omega)$ and $\operatorname{Ker}\operatorname{curl}$ into $\mathcal{H}^\perp$. Similarly, the curl operator maps W into $\mathcal{H}$ and $\operatorname{Ker}\operatorname{div}$ into $\mathcal{H}^\perp$.

Two functions f and g in $L^2(\Omega)$ are harmonic conjugate if and only if, for all vector fields $\mathbf{v} \in [H_0^1(\Omega)]^2$,

$$\langle f, \operatorname{curl} \mathbf{v} \rangle_{L^2(\Omega)} - \langle g, \operatorname{div} \mathbf{v} \rangle_{L^2(\Omega)} = 0 \tag{3}$$

or, equivalently, if and only if the vector field $\mathbf{u} = (f, g)$ satisfies the relations

$$\operatorname{div} \mathbf{u} = \operatorname{curl} \mathbf{u} = 0. \tag{4}$$

Let $\mathbf{u} \in H_0^1(\Omega)$ be the solution of the Dirichlet problem for the Laplace operator $\Delta \mathbf{u} = \nabla p$, with $p \in L_0^2(\Omega)$. Then, for all $\mathbf{v} \in [H^1(\Omega)]^2$,

$$\begin{aligned} \langle \nabla \mathbf{u}, \nabla \mathbf{v} \rangle_{L^2(\Omega)} &= \langle p, \operatorname{div} \mathbf{v} \rangle_{L^2(\Omega)} \\ \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)} &= \|\nabla p\|_{[H^{-1}(\Omega)]^2}. \end{aligned} \tag{5}$$

[1)] The operator curl : $[H_0^1(\Omega)]^2 \to L^2(\Omega)$ is the composition of the divergence operator div : $[H_0^1(\Omega)]^2 \to L^2(\Omega)$ with the clockwise rotation of amplitude $\frac{\pi}{2}$. In the standard base $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$, $\operatorname{curl} \mathbf{u} = u^2_{,1} - u^1_{,2}$, $\mathbf{u} = u^1 \mathbf{e}_1 + u^2 \mathbf{e}_2$.

It follows that the Laplace operator is an isomorphism between $\operatorname{Ker}\operatorname{curl} \oplus W$ and $L^2_0(\Omega)$, sending $\operatorname{Ker}\operatorname{curl}$ onto $L^2_0(\Omega) \cap \mathcal{H}^\perp$ and W onto $L^2_0(\Omega) \cap \mathcal{H}$. Moreover, we have $\operatorname{div}\mathbf{u} = p$ if and only if $p \in \mathcal{H}^\perp$.

Finally, let us note the following:

- The duals of the operators div and curl are, in the order, the operators $-\nabla : L^2(\Omega) \to [H^{-1}(\Omega)]^2$ and $\mathbf{curl} : L^2(\Omega) \to [H^{-1}(\Omega)]^2$, with $\mathbf{curl}\, f = (f_{,2}, -f_{,1})$.
- The kernel of the gradient operator $\nabla : L^2(\Omega) \to [H^{-1}(\Omega)]^2$ is the subspace of constant functions in $L^2(\Omega)$: $\operatorname{Ker}\nabla = L^2_0(\Omega)^\perp$.
- The subspace $(\operatorname{Ker}\operatorname{div})^\perp = \{\mathbf{l} \in [H^{-1}(\Omega)]^2 : \langle \mathbf{l}, \mathbf{u} \rangle = 0,\ \mathbf{u} \in \operatorname{Ker}\operatorname{div}\}$ is isometrically isomorphic to the dual space of $\operatorname{Ker}\operatorname{curl} \oplus W$.

3. Equivalence results

The title inequalities can be formulated as follows.

Korn's inequality (in the second case). There is a constant K such that, for all vector fields $\mathbf{u} \in [H^1(\Omega)]^2$ with $\mathbf{W}(\mathbf{u}) \in \mathbf{L}^2_0(\Omega)$,

$$\|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \leq K \|\mathbf{E}(\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)} \tag{6}$$

where $\mathbf{E}(\mathbf{u}) = \operatorname{sym}\nabla\mathbf{u}$ and $\mathbf{W}(\mathbf{u}) = \operatorname{skw}\nabla\mathbf{u}$ are the symmetric and skew parts of the gradient $\nabla\mathbf{u}$ of $\mathbf{u}$.

Friedrichs' inequality. Let $f \in L^2_0(\Omega)$ and $f^* \in L^2(\Omega)$ be conjugate harmonic functions. Then there exists a constant Γ such that

$$\|f\|^2_{L^2(\Omega)} \leq \Gamma \|f^*\|^2_{L^2(\Omega)}. \tag{7}$$

Magenes-Stampacchia-Nečas inequality. There exists a constant M such that, for all scalar fields $f \in L^2_0(\Omega)$,

$$\|f\|_{L^2(\Omega)} \leq M \|\nabla f\|_{[H^{-1}(\Omega)]^2}. \tag{8}$$

Babuška-Aziz inequality. Let $f \in L^2_0(\Omega)$. Then there exists a vector field $\mathbf{u} \in [H^1_0(\Omega)]^2$ and a constant C such that

$$\operatorname{div}\mathbf{u} = f, \qquad \|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \leq C \|f\|^2_{L^2(\Omega)}. \tag{9}$$

For the Korn and the Friedrichs inequalities (Korn [9], Friedrichs [3]) we refer to the exhaustive review article by Horgan [7] and the references cited therein. A proof of Korn's inequality using the Magenes-Stampacchia-Nečas inequality is given in the recent paper [13]. Inequality (8) was first established by Magenes and Stampacchia [10] for a regular domain, then generalized by Nečas [11] to a domain with Lipschitz continuous boundary. Inequality (9) was stated by Babuška and Aziz [1] in the course of their analysis of the finite element method. Proofs of (8) and (9) can be found also in the books by Girault and Raviart [4] and Temam [12].

First we show the equivalence between the Magenes-Stampacchia-Nečas inequality and the Babuška-Aziz inequality. Though a derivation of this equivalence can be given using the Closed Range Theorem (these inequalities express the closure of the images of the divergence and gradient operators), here, following Girault and Raviart [4] and Temam [12], we prefer a direct proof leading also to a relation between the constants of the two inequalities.

Proposition 1. *The inequalities of Magenes-Stampacchia-Nečas and Babuška-Aziz are equivalent. Moreover, the constants occurring in the two inequalities are related by* $M = \sqrt{C}$.

Proof. Assume the Babuška-Aziz inequality (9). Then, for all $f, g \in L^2_0(\Omega)$,

$$\frac{\langle f, g\rangle}{\|g\|_{L^2(\Omega)}} = \frac{\langle \nabla f, \mathbf{u}\rangle}{\|\operatorname{div}\mathbf{u}\|_{L^2(\Omega)}} \leq \sqrt{C}\|\nabla f\|_{[H^{-1}(\Omega)]^2} \tag{10}$$

with $\operatorname{div}\mathbf{u} = g$. Consequently, taking the supremum over $g \in L^2_0(\Omega)$, we obtain $\|f\|_{L^2(\Omega)} \leq \sqrt{C}\|\nabla f\|_{[H^{-1}(\Omega)]^2}$, that is (8) with $M = \sqrt{C}$.

Now assume the Magenes-Stampacchia-Nečas inequality (8). This inequality implies, in particular, that $\mathrm{Im}\nabla$ is closed in $[H^{-1}(\Omega)]^2$. Therefore, as the operators ∇ and div are one the dual of the other,

$$\begin{aligned}\operatorname{Im}\nabla &= (\operatorname{Ker}\operatorname{div})^{\perp}\\ \operatorname{Im}\operatorname{div} &= (\operatorname{Ker}\nabla)^{\perp} = L^2_0(\Omega).\end{aligned} \tag{11}$$

Then for any $f \in L^2_0(\Omega)$ there exists $\mathbf{u} \in \operatorname{Ker}\operatorname{curl} \oplus W$ such that $\operatorname{div}\mathbf{u} = f$. Using the fact that the dual of $\operatorname{Ker}\operatorname{curl} \otimes W$ is isometrically isomorphic to $(\operatorname{Ker}\operatorname{div})^{\perp} = \operatorname{Im}\nabla$, we get

$$\begin{aligned}\|\mathbf{u}\|_{[H^1(\Omega)]^2} &= \sup_{g\in L^2(\Omega)} \frac{\langle \nabla g, \mathbf{u}\rangle}{\|\nabla g\|_{[H^{-1}(\Omega)]^2}}\\ &= \sup_{g\in L^2(\Omega)} \frac{\langle g, \operatorname{div}\mathbf{u}\rangle}{\|\nabla g\|_{[H^{-1}(\Omega)]^2}}\\ &\leq M\|\operatorname{div}\mathbf{u}\|_{L^2(\Omega)},\end{aligned} \tag{12}$$

that is the Babuška-Aziz inequality (9) with $C = M^2$ ∎

We next show that the Friedrichs inequality is equivalent to the Babuška-Aziz inequality. The proof is a minor modification of that given by Horgan and Payne in [8].

Proposition 2. *Friedrichs' inequality is equivalent to the inequality of Babuška-Aziz.*

Proof. Let $f = L^2_0(\Omega)$ in the image of the divergence operator. We suppose without loss of generality that $f = \operatorname{div}\mathbf{u}$ with $\mathbf{u} = \mathbf{v} + \mathbf{w}$, $\mathbf{v} \in \operatorname{Ker}\operatorname{curl}$ and $\mathbf{w} \in W$. From (2) and the properties of the Laplace operator it follows that curl $\mathbf{w}$ is a harmonic function and

$$\begin{aligned} \|\nabla \mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} &= \|\operatorname{div} \mathbf{v}\|^2_{L^2(\Omega)} \\ \|f\|_{L^2(\Omega)} &= \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega)} + \|\operatorname{div} \mathbf{w}\|_{L^2(\Omega)}. \end{aligned} \tag{13}$$

Let $g^* \in L^2_0(\Omega)$ be conjugate harmonic to curl $\mathbf{w}$. Then from (3) it follows

$$\begin{aligned} \|\operatorname{curl} \mathbf{w}\|^2_{L^2(\Omega)} &= \langle \operatorname{curl} \mathbf{w}, \operatorname{curl} \mathbf{w} \rangle_{L^2(\Omega)} \\ &= \langle g^*, \operatorname{div} \mathbf{w} \rangle_{L^2(\Omega)} \\ &\le \|\operatorname{div} \mathbf{w}\|_{L^2(\Omega)} \|g^*\|_{L^2(\Omega)} \end{aligned} \tag{14}$$

so that the Friedrichs' inequality (7) together with (2) yields

$$\|\nabla \mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} \le (\Gamma + 1) \|\operatorname{div} \mathbf{w}\|^2_{L^2(\Omega)}. \tag{15}$$

Substitution of (15) into (13) yields the inequality of Babuška-Aziz for all $f \in \operatorname{Im} \operatorname{div}$:

$$\begin{aligned} \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)} &= \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \mathbf{w}\|_{\mathbf{L}^2(\Omega)} \\ &\le \|\operatorname{div} \mathbf{v}\|^2_{L^2(\Omega)} + (\Gamma + 1)\|\operatorname{div} \mathbf{w}\|^2_{L^2(\Omega)} \\ &\le (\Gamma + 1) \|f\|^2_{L^2(\Omega)}. \end{aligned} \tag{16}$$

By this inequality the image of the divergence operator is closed in $L^2(\Omega)$ and, therefore, it coincides with the orthogonal complement of the kernel of $-\nabla : L^2(\Omega) \to [H^{-1}(\Omega)]^2$, the adjoint of $\operatorname{div} : [H^1_0(\Omega)]^2 \to L^2_0(\Omega)$. But $\operatorname{Ker} \nabla$ is the subspace of constant functions, so that $\operatorname{Ker} \operatorname{div} = L^2_0(\Omega)$.

Assume now the inequality of Babuška-Aziz. Since this inequality implies, in particular, $\operatorname{Im} \operatorname{div} = L^2_0(\Omega)$, we have that to any harmonic function f in $L^2_0(\Omega)$ it corresponds a vector field, necessarily in W, such that $\operatorname{div} \mathbf{w} = f$. Let f^* any conjugate harmonic function to f. We have

$$\begin{aligned} \|f\|^4_{L^2(\Omega)} &= \langle f, \operatorname{div} \mathbf{w} \rangle^2_{L^2(\Omega)} \\ &= \langle f^*, \operatorname{curl} \mathbf{w} \rangle^2_{L^2(\Omega)} \\ &\le \|\operatorname{curl} \mathbf{w}\|^2_{L^2(\Omega)} \|f^*\|^2_{L^2(\Omega)} \\ &\le (C - 1) \|f\|^2_{L^2(\Omega)} \|f^*\|^2_{L^2(\Omega)}, \end{aligned} \tag{17}$$

that is the Friedrichs' inequality (7) with $\Gamma = C - 1$ ∎

We complete the proof of the equivalence of the title inequalities by showing that the inequalities of Korn and Magenes-Stampacchia-Nečas are equivalent. In proving the implication

Korn's inequality $\implies$ inequality of Magenes-Stampacchia-Nečas

we also prove the implication

Korn's inequality $\implies$ Friedrichs' inequality.

Compared with the proof given by Horgan in [6], our proof is simpler, but does not lead to the optimal relation between the constants of the two inequalities $2\Gamma = K - 2$. The arguments employed in the proof of the opposite implication

inequality of Magenes-Stampacchia-Nečas $\implies$ Korn's inequality

are essentially the same as in [13].

Proposition 3. *Korn's inequality is equivalent to the inequality of Magenes-Stampacchia-Nečas.*

Proof. Assume Korn's inequality and let $f, f^* \in L^2_0(\Omega)$ be a pair of conjugate harmonic functions. By relation (4), the vectors $\mathbf{u} \equiv (f, f^*)$ and $\mathbf{v} \equiv (-f^*, f)$, and hence the matrix $\mathbf{F} = \mathbf{e}_1 \otimes \mathbf{u} + \mathbf{e}_2 \otimes \mathbf{v}$ are irrotational. Moreover, $\operatorname{skw} \mathbf{F} = f^*(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1) \in \mathbf{L}^2_0(\Omega)$ because of $f^* \in L^2_0(\Omega)$. It follows that $\mathbf{F}$ is the gradient of a vector field that satisfies the hypotheses of the Korn inequalities (6). Then we arrive at

$$2\big(\|f\|^2_{L^2(\Omega)} + \|f^*\|^2_{L^2(\Omega)}\big) = \|\mathbf{F}\|^2_{\mathbf{L}^2(\Omega)} \le K \|\operatorname{sym} \mathbf{F}\|^2_{\mathbf{L}^2(\Omega)} = 2K \|f\|^2_{L^2(\Omega)} \tag{18}$$

which implies the Friedrichs' inequality

$$\|f^*\|^2_{L^2(\Omega)} \le \Gamma \|f\|^2_{L^2(\Omega)} \tag{19}$$

with $\Gamma = K - 1$.

Let us consider the Dirichlet problem for the Laplace operator $\Delta \mathbf{u} = \nabla f$ with $f \in L^2_0(\Omega)$. The solution $\mathbf{u} \in [H^1_0(\Omega)]^2$ can be written in the form $\mathbf{u} = \mathbf{v} + \mathbf{w}$ where $\mathbf{v} \in \operatorname{Ker} \operatorname{curl}$ and $\mathbf{w} \in W$ are the solutions of the Dirichlet problems

$$\Delta \mathbf{v} = \nabla g \qquad \text{and} \qquad \Delta \mathbf{w} = \nabla h \tag{20}$$

being $g \in \mathcal{H}$ and $h \in \mathcal{H}^\perp$ the projections of f onto $\mathcal{H}$ and $\mathcal{H}^\perp$, respectively. From (2) and (5) it follows

$$\|h\|_{L^2(\Omega)} = \|\operatorname{div} \mathbf{w}\|_{L^2(\Omega)} = \|\nabla \mathbf{w}\|_{[L^2(\Omega)]^2} = \|\nabla h\|_{[H^{-1}(\Omega)]^2}. \tag{21}$$

By (3), $p = \operatorname{curl} \mathbf{v}$ and $p^* = (\operatorname{div} \mathbf{v} - g)$ are conjugate harmonic functions in $L^2_0(\Omega)$ so that (19) yields $\|p^*\|_{L^2(\Omega)} \le (K-1)\|p\|_{L^2(\Omega)}$, that is, by the definitions of p and p^*,

$$\|\operatorname{div} \mathbf{v}\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)} - 2\langle g, \operatorname{div} \mathbf{v}\rangle_{L^2(\Omega)} \le (K-1)\|\operatorname{curl} \mathbf{v}\|^2_{L^2(\Omega)}. \tag{22}$$

Then, from (2) and (5) it follows

$$\|g\|^2_{L^2(\Omega)} \le K \|\nabla g\|^2_{[H^{-1}(\Omega)]^2} \tag{23}$$

which together with (21) gives the Magenes-Stampacchia-Nečas inequality:

$$\begin{aligned} \|f\|_{L^2(\Omega)} &= \|g\|_{L^2(\Omega)} + \|h\|_{L^2(\Omega)} \\ &\le \sqrt{K} \|\nabla g\|_{[H^{-1}(\Omega)]^2} + \|\nabla h\|_{[H^{-1}(\Omega)]^2} \\ &\le \sqrt{K} \|\nabla f\|_{[H^{-1}(\Omega)]^2}. \end{aligned} \tag{24}$$

Assume now the Magenes-Stampacchia-Nečas inequality. Let $\mathbf{u}$ be a vector field in $[H^1(\Omega)]^2$ such that $\mathbf{W} = \operatorname{skw} \nabla \mathbf{u} \in \mathbf{L}^2_0(\Omega)$. The symmetric tensor $\mathbf{E} = \operatorname{sym} \nabla \mathbf{u}$ and the rotation function [2] ω of $\mathbf{W}$ satisfy (see, for example, Gurtin [5]) the relations

$$\begin{aligned} \nabla \omega &= \operatorname{curl} \mathbf{E} \\ \|\mathbf{W}\|^2_{[L^2(\Omega)]^2} &= 2\|\omega\|^2_{[L^2(\Omega)]^2}. \end{aligned} \tag{25}$$

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[2] The rotation function ω associated with $\mathbf{W}$ is the unique function such that $\mathbf{W} = \omega \operatorname{skw} (\mathbf{e}_1 \oplus \mathbf{e}_2)$, with $\{\mathbf{e}_1, \mathbf{e}_2\}$ the standard base of $\mathbb{R}^2$.

Moreover, $\omega \in L^2_0(\Omega)$ because of $\mathbf{W} \in \mathbf{L}^2_0(\Omega)$. Relations (25) and the Magenes-Stampacchia-Nečas inequality (8) yield

$$\|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \le \|\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)} + 2M^2 \|\operatorname{curl} \mathbf{E}\|^2_{[H^{-1}(\Omega)]^2} \tag{26}$$

with

$$\|\operatorname{curl} \mathbf{E}\|_{[H^{-1}(\Omega)]^2} = \sup_{\mathbf{v} \in [H^1(\Omega)]^2} \frac{\langle \operatorname{curl} \mathbf{E}, \mathbf{v} \rangle}{\|\mathbf{v}\|_{[H^1(\Omega)]^2}}. \tag{27}$$

We now estimate the norm of $\operatorname{curl} \mathbf{E}$. For each $\mathbf{v} \in [H^1(\Omega)]^2$, let $\mathbf{v}^* \in [H^1(\Omega)]^2$ be the vector field defined by $\mathbf{v}^* = v^2 \mathbf{e}_1 - v^1 \mathbf{e}_2$, with v^1, v^2 the components of $\mathbf{v}$ in the standard orthonormal base of $\mathbb{R}^2$. The following relations hold:

$$\begin{aligned} \|\operatorname{sym}(\operatorname{curl} \mathbf{v})\|_{\mathbf{L}^2(\Omega)} &= \|\operatorname{sym}(\nabla \mathbf{v}^*)\|_{\mathbf{L}^2(\Omega)} \\ \|\nabla \mathbf{v}\|_{[L^2(\Omega)]^2} &= \|\nabla \mathbf{v}^*\|_{[L^2(\Omega)]^2}. \end{aligned} \tag{28}$$

Furthermore, since by Proposition 1 the Magenes-Stampacchia-Nečas inequality is equivalent to the Babuška-Aziz inequality, we have also

$$\|\nabla \mathbf{v}^*\|_{\mathbf{L}^2(\Omega)} \le M \|\operatorname{curl} \mathbf{v}^*\|_{L^2(\Omega)}$$

so that

$$\begin{aligned} \|\operatorname{sym} \nabla \mathbf{v}^*\|^2_{\mathbf{L}^2(\Omega)} &= \|\nabla \mathbf{v}^*\|^2_{\mathbf{L}^2(\Omega)} - \frac{1}{2} \|\operatorname{curl} \mathbf{v}^*\|^2_{L^2(\Omega)} \\ &\le \frac{(2M^2 - 1)}{2M^2} \|\nabla \mathbf{v}^*\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{29}$$

Using the Schwarz inequality and (29), we get

$$\begin{aligned} \frac{\langle \operatorname{curl} \mathbf{E}, \mathbf{v} \rangle}{\|\mathbf{v}\|_{[H^1(\Omega)]^2}} &\le \frac{\|\mathbf{E}\|_{\mathbf{L}^2(\Omega)} \|\operatorname{sym} \operatorname{curl} \mathbf{v}\|_{\mathbf{L}^2(\Omega)}}{\|\mathbf{v}\|_{[H^1(\Omega)]^2}} \\ &\le \frac{\|\mathbf{E}\|_{\mathbf{L}^2(\Omega)} \|\operatorname{sym} \nabla \mathbf{v}^*\|_{\mathbf{L}^2(\Omega)}}{\|\nabla \mathbf{v}^*\|_{\mathbf{L}^2(\Omega)}} \\ &\le \frac{(2M^2 - 1)}{2M^2} \|\mathbf{E}\|_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{30}$$

Consequently,

$$\|\operatorname{curl} \mathbf{E}\|_{[H^{-1}(\Omega)]^2} \le \frac{(2M^2 - 1)}{2M^2} \|\mathbf{E}\|_{\mathbf{L}^2(\Omega)}. \tag{31}$$

Substitution of (31) into (26) yields the Korn inequality (6)

$$\|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \le K \|\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)} \tag{32}$$

with $K = 2M^2 = 2C$ $\blacksquare$

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