

# Some Series over the Product of Two Trigonometric Functions and Series Involving Bessel Functions

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**Abstract.** The sum of the series

$$S_\alpha = S_\alpha(s, a, b, f(y), g(x)) = \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)y) g((an-b)x)}{(an-b)^\alpha}$$

involving the product of two trigonometric functions is obtained using the sum of the series

$$\sum_{n=1}^{\infty} \frac{(s)^{(n-1)} f((an-b)x)}{(an-b)^\alpha} = \frac{c\pi}{2\Gamma(\alpha) f(\frac{\pi\alpha}{2})} x^{\alpha-1} + \sum_{i=0}^{\infty} (-1)^i \frac{F(\alpha-2i-\delta)}{(2i+\delta)!} x^{2i+\delta}$$

whose terms involve one trigonometric function. The first series is represented as series in terms of the Riemann zeta and related functions, which has a closed form in certain cases. Some applications of these results to the summation of series containing Bessel functions are given. The obtained results also include as special cases formulas in some known books. We further show how to make use of these results to obtain closed form solutions of some boundary value problems in mathematical physics.

**Keywords:** *Bessel functions, Riemann zeta and related functions*

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## 1. Introduction and preliminaries

In the paper [7] author considered trigonometric functions series containing coefficients which are reciprocal powers of  $n$  or  $2n-1$  where  $n \in \mathbb{N}$ . The representations of those series are given in [10] in the general form

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x)}{(an-b)^\alpha} = \frac{c\pi}{2\Gamma(\alpha) f(\frac{\pi\alpha}{2})} x^{\alpha-1} + \sum_{i=0}^{\infty} (-1)^i \frac{F(\alpha-2i-\delta)}{(2i+\delta)!} x^{2i+\delta} \quad (1)$$

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where  $\alpha \in \mathbb{R}^+$ ,  $a = \{\frac{1}{2}\}$ ,  $b = \{\frac{0}{1}\}$ ,  $s = 1$  or  $s = -1$ ,  $f = \{\frac{\sin}{\cos}\}$ ,  $\delta = \{\frac{1}{0}\}$  and where all relevant parameters are given in Table I wherein  $\zeta, \eta, \lambda$  and  $\beta$  are the Riemann zeta function and other sums of reciprocal powers defined by

$$\begin{aligned} \zeta(\alpha) &= \sum_{k=1}^{\infty} k^{-\alpha} \\ \eta(\alpha) &= \sum_{k=1}^{\infty} (-1)^{k-1} k^{-\alpha} = (1 - 2^{1-\alpha})\zeta(\alpha) \\ \lambda(\alpha) &= \sum_{k=0}^{\infty} (2k + 1)^{-\alpha} = (1 - 2^{-\alpha})\zeta(\alpha) \\ \beta(\alpha) &= \sum_{k=0}^{\infty} (-1)^k (2k + 1)^{-\alpha} \end{aligned}$$

(see [1, 3]). Note that when  $f(x) = \sin x$  and  $\alpha \rightarrow 2m$  or  $f(x) = \cos x$  and  $\alpha \rightarrow 2m + 1$  ( $m \in \mathbb{N}_0$ ), the limiting value of the right-hand side of (1) should be taken into account [5, 7]. In some cases, listed in Table II, when the right-hand side series truncate due to vanishing of  $F$  functions, representation (1) takes the closed form

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)x)}{(an - b)^\alpha} \\ &= (-1)^{\frac{\alpha-\delta}{2}} \frac{c\pi}{2(\alpha - 1)!} x^{\alpha-1} + \sum_{i=0}^M (-1)^i \frac{F(\alpha - 2i - \delta)}{(2i + \delta)!} x^{2i+\delta}, \end{aligned} \tag{2}$$

where  $\alpha \in \mathbb{N}$ ,  $M = \frac{\alpha-1}{2}$  for  $\alpha$  odd and  $M = \frac{\alpha}{2} - \delta$  for  $\alpha$  even. This is a generalization of the results in [8]. Some particular cases of (2) can be found in [1, 4, 6, 13].

Using (1) and (2), in [9, 10] the sums of some series over Bessel functions are given, expressed in terms of the Riemann zeta numbers and other sums of reciprocal powers.

Table I: Corresponding  $F$  and  $c$

Table II: Closed form cases

In this paper, for  $\alpha \in \mathbb{R}^+$ , we evaluate and represent series over the product of trigonometric functions

$$S_\alpha = S_\alpha(s, a, b, f(y), g(x)) = \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)y) g((an - b)x)}{(an - b)^\alpha} \tag{3}$$

where the parameters  $a, b, s$  are as in Table I, and  $f$  and  $g$  denote sin or cos, as series in terms of the Riemann zeta and related functions, which become closed form formulas

under some restrictions. Further, we illustrate applications of these results both to the summation of series over the product of Bessel and trigonometric functions and to obtaining closed form solutions of some boundary value problems in mathematical physics.

## 2. Outline of the method and a general result

We explain the procedure of determining the sum of the series (3) for  $\alpha = 2m - r$  ( $r = 0$  or  $r = 1$ ,  $m \in \mathbb{N}$ ), by using the series

$$S_{2m-1} = S_{2m-1}(1, 1, 0, \sin y, \cos x) = \sum_{n=1}^{\infty} \frac{\sin ny \cos nx}{n^{2m-1}} \quad (m \in \mathbb{N}). \quad (4)$$

We start with  $m = 1$ , i.e. the series  $S_1 = S_1(1, 1, 0, \sin y, \cos x)$ ,

$$S_1 = \sum_{n=1}^{\infty} \frac{\sin ny \cos nx}{n}.$$

Using the relation  $\sin ny \cos nx = \frac{1}{2}(\sin(n(y-x)) + \sin(n(y+x)))$  we obtain

$$S_1 = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(n(y-x))}{n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(n(y+x))}{n}.$$

On the basis of (2) we have

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n} = \frac{1}{2}(\pi - t) \quad (0 < t < 2\pi) \quad (5)$$

and find

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin(n(y-x))}{n} &= \frac{1}{2}(\pi - (y-x)) \quad (0 < y-x < 2\pi) \\ \sum_{n=1}^{\infty} \frac{\sin(n(y+x))}{n} &= \frac{1}{2}(\pi - (y+x)) \quad (0 < y+x < 2\pi). \end{aligned}$$

Therefore, in the domain

$$K_1 = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 < y \pm x < 2\pi \right\}$$

i.e.

$$K_1 = \left\{ (x, y) \in \mathbb{R}^2 \mid -\pi < x < \pi \text{ and } |x| < y < 2\pi - |x| \right\}$$

we have  $S_1 = \frac{1}{2}(\pi - y)$ . Setting  $t = y$  in (5) we obtain

$$S_1 = \sum_{n=1}^{\infty} \frac{\sin ny \cos nx}{n} = \sum_{n=1}^{\infty} \frac{\sin ny}{n} \quad ((x, y) \in K_1). \quad (6)$$

Integrating this equality with respect to  $x$  in  $K_1$ , interchanging the order of integration and summation, i.e.

$$\sum_{n=1}^{\infty} \frac{\sin ny}{n} \int_0^x \cos nx \, dx = \sum_{n=1}^{\infty} \frac{\sin ny}{n} \int_0^x dx,$$

we have

$$\sum_{n=1}^{\infty} \frac{\sin ny \sin nx}{n^2} = x \sum_{n=1}^{\infty} \frac{\sin ny}{n} \quad ((x, y) \in K_1)$$

including boundaries. Repeating this procedure several times in succession, we find formulas for the sums  $S_3, S_5, \dots$  etc. So we may assume that after repeating this procedure  $2m$  times the final formula will have the form

$$S_{2m-1} = \sum_{n=1}^{\infty} \frac{\sin ny \cos nx}{n^{2m-1}} = \sum_{i=0}^{m-1} (-1)^i \frac{x^{2i}}{(2i)!} \sum_{n=1}^{\infty} \frac{\sin ny}{n^{2m-2i-1}}, \quad (7)$$

for  $(x, y) \in K_1$  including boundaries, except for  $m = 1$ .

We shall prove formula (7) using the method of mathematical induction. As (6) shows, formula (7) is true for  $m = 1$ . Further, assuming its validity for  $m = k > 1$  ( $k \in \mathbb{N}$ ), it has to be proven that it is valid for  $m = k + 1$ , too. Integration of the assumed equality

$$\sum_{n=1}^{\infty} \frac{\sin ny \cos nx}{n^{2k-1}} = \sum_{i=0}^{k-1} (-1)^i \frac{x^{2i}}{(2i)!} \sum_{n=1}^{\infty} \frac{\sin ny}{n^{2k-2i-1}}$$

gives

$$\sum_{n=1}^{\infty} \frac{\sin ny}{n^{2k-1}} \int_0^x \cos nx \, dx = \int_0^x \sum_{i=0}^{k-1} (-1)^i \frac{x^{2i}}{(2i)!} \sum_{n=1}^{\infty} \frac{\sin ny}{n^{2k-2i-1}} \, dx,$$

i.e.

$$\sum_{n=1}^{\infty} \frac{\sin ny \sin nx}{n^{2k}} = \sum_{i=0}^{k-1} (-1)^i \frac{x^{2i+1}}{(2i+1)!} \sum_{n=1}^{\infty} \frac{\sin ny}{n^{2k-2i-1}}.$$

Repeating integration,

$$\sum_{n=1}^{\infty} \frac{\sin ny}{n^{2k}} \int_0^x \sin nx \, dx = \int_0^x \sum_{i=1}^k (-1)^{i-1} \frac{x^{2i-1}}{(2i-1)!} \sum_{n=1}^{\infty} \frac{\sin ny}{n^{2k-2i+1}} \, dx$$

where the sum on the right-hand side is shifted, we obtain

$$-\sum_{n=1}^{\infty} \frac{\sin ny \cos nx}{n^{2k+1}} + \sum_{n=1}^{\infty} \frac{\sin ny}{n^{2k+1}} = -\sum_{i=1}^k (-1)^i \frac{x^{2i}}{(2i)!} \sum_{n=1}^{\infty} \frac{\sin ny}{n^{2k-2i+1}}$$

and finally

$$\sum_{n=1}^{\infty} \frac{\sin ny \cos nx}{n^{2k+1}} = \sum_{i=0}^k (-1)^i \frac{x^{2i}}{(2i)!} \sum_{n=1}^{\infty} \frac{\sin ny}{n^{2k-2i+1}},$$

and that is formula (7) for  $m = k + 1$ . Thus, (7) is proven ■

In a similar way we obtain sixteen formulas which may be represented by the general formula

$$\begin{aligned}
 S_{2m-r}(s, a, b, f(y), g(x)) &= \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)y) g((an - b)x)}{(an - b)^{2m-r}} \\
 &= \sum_{i=0}^{m-1-d\delta} (-1)^i \frac{x^{2i+\delta}}{(2i + \delta)!} \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)y)}{(an - b)^{2m-2i-\delta-r}} \\
 &\quad + (-1)^{m-\delta} (b - 1)sd \frac{x^{2m-\delta}}{2(2m - \delta)!}
 \end{aligned} \tag{8}$$

where  $m \in \mathbb{N}$ ,  $g = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}$ ,  $\delta = \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\}$ ,  $r = \left\{ \begin{smallmatrix} d \\ 1-d \end{smallmatrix} \right\}$  and, independently of that, the rest of the parameters we read from Table III. The domains  $K_i$  ( $i = 1, 2, 3, 4$ ) are closed for any  $2m - r > 1$ .

Applying (2), formula (8) takes the closed form

$$\begin{aligned}
 S_{2m-r} &= \sum_{i=0}^{m-1-d\delta} (-1)^i \frac{x^{2i+\delta}}{(2i + \delta)!} \left( (-1)^{m-i-1+d-d\delta} c \frac{\pi y^{2m-2i-\delta-r-1}}{2(2m - 2i - \delta - r - 1)!} \right. \\
 &\quad \left. + \sum_{j=0}^M (-1)^j \frac{F(2m - 2i - 2j - \delta - r - t)}{(2j + t)!} y^{2j+t} \right) \\
 &\quad + (-1)^{m-\delta} (b - 1)sd \frac{x^{2m-\delta}}{2(2m - \delta)!}
 \end{aligned} \tag{9}$$

where  $g = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}$ ,  $\delta = \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\}$ ,  $r = \left\{ \begin{smallmatrix} d \\ 1-d \end{smallmatrix} \right\}$  and, independently of that,  $f = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}$  and  $t = \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\}$ ; the rest of the parameters are given in Table III. This formula contains sixteen formulas as special cases.

Table III

It should be mentioned that by using the other formula

$$\sin ny \cos nx = \frac{1}{2} (\sin n(x + y) - \sin(n(x - y)))$$

at start of the described procedure for series (4) we obtain the formula

$$\sum_{n=1}^{\infty} \frac{\sin ny \cos nx}{n^{2m-1}} = \sum_{i=0}^{m-2} (-1)^i \frac{y^{2i+1}}{(2i + 1)!} \sum_{n=1}^{\infty} \frac{\cos nx}{n^{2m-2i-2}} - (-1)^{m-1} \frac{y^{2m-1}}{2(2m - 1)!} \tag{10}$$

which is different from formula (7). This is because (10) is valid in the domain  $K'_1$ , dual to  $K_1$ . Namely,

$$K'_1 = \left\{ (x, y) \mid -\pi \leq y \leq \pi \text{ and } |y| \leq x \leq 2\pi - |y| \right\}.$$

However, the same formula (10) holds true in  $K_1$  when we interchange the variables  $x$  and  $y$  in series (4), but this is the formula

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\cos ny \sin nx}{n^{2m-1}} &= \sum_{i=0}^{m-2} (-1)^i \frac{x^{2i+1}}{(2i+1)!} \\ &\times \left( (-1)^{m-i-1} \frac{\pi y^{2m-2i-3}}{2(2m-2i-3)!} + \sum_{j=0}^{m-i-1} (-1)^j \frac{\zeta_{ij}}{(2j)!} y^{2j} \right) \\ &- (-1)^{m-1} \frac{x^{2m-1}}{2(2m-1)!} \end{aligned}$$

where  $\zeta_{ij} = \zeta(2m - 2i - 2j - 2)$  ( $(x, y) \in K_1$ ) which is obtained from (7) for a suitable choice of the functions  $f$  and  $g$ . This consideration shows that there is no need to establish formulas for dual domains.

For  $\alpha \in \mathbb{R}^+$  series (3) can be represented as series in terms of the Riemann zeta and related functions. We explain the procedure for obtaining this result by using the series

$$S_\alpha = S_\alpha(s, a, b, \sin y, \cos x) = \sum_{n=1}^{\infty} \frac{(s)^{n-1} \sin((an - b)y) \cos((an - b)x)}{(an - b)^\alpha}. \tag{11}$$

Considering that

$$\sin((an - b)y) \cos((an - b)x) = \frac{1}{2} \left( \sin((an - b)(y - x)) + \sin((an - b)(y + x)) \right)$$

we obtain

$$S_\alpha = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(s)^{n-1} \sin((an - b)(y - x))}{(an - b)^\alpha} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(s)^{n-1} \sin((an - b)(y + x))}{(an - b)^\alpha}.$$

Applying (1) to the both series, this equality can be written as

$$\begin{aligned} S_\alpha &= c\pi \frac{(y - x)^{\alpha-1} + (y + x)^{\alpha-1}}{4\Gamma(\alpha) \sin \frac{\pi\alpha}{2}} \\ &+ \frac{1}{2} \sum_{i=0}^{\infty} (-1)^i \frac{F(\alpha - 2i - 1)}{(2i + 1)!} \left( (y - x)^{2i+1} + (y + x)^{2i+1} \right) \end{aligned} \tag{12}$$

$((x, y) \in K_i),$

where the domains  $K_i$  ( $i = 1, 2, 3, 4$ ), without boundaries, and  $c, F$  depend on the parameters  $s, a, b$ , as in Table III. Using the binomial formula, we finally obtain

$$S_\alpha = c\pi \frac{(y - x)^{\alpha-1} + (y + x)^{\alpha-1}}{4\Gamma(\alpha) \sin \frac{\pi\alpha}{2}} + \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^i F(\alpha - 2i - 1) \frac{x^{2j} y^{2i-2j+1}}{(2i - 2j + 1)! (2j)!}.$$

The general formula can be stated as

$$\begin{aligned}
 S_\alpha(s, a, b, f(y), g(x)) &= \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)y) g((an - b)x)}{(an - b)^\alpha} \\
 &= (-1)^{d\delta} c\pi \frac{(y + x)^{\alpha-1} + (-1)^\delta (y - x)^{\alpha-1}}{4\Gamma(\alpha)h(\frac{\pi\alpha}{2})} \quad (\alpha \in \mathbb{R}^+) \quad (12) \\
 &\quad + \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^i F(\alpha - 2i - d - \delta) \frac{x^{2j+\delta} y^{2i-2j+d}}{(2i - 2j + d)! (2j + \delta)!}
 \end{aligned}$$

where  $f = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}$ ,  $d = \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\}$  and, independently of that,  $g = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}$ ,  $\delta = \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\}$ ,  $h = \left\{ \begin{smallmatrix} \bar{f} \\ f \end{smallmatrix} \right\}$ , and  $c, F$  and the domains  $K_i$  we read from Table III. Note that, for  $f = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}$ ,  $\bar{f} = \left\{ \begin{smallmatrix} \cos \\ \sin \end{smallmatrix} \right\}$ .

### 3. Some series involving Bessel functions

In this section we shall illustrate the application of the obtained sum of series over the product of two trigonometric functions (9) to the summation of series over the product of Bessel and trigonometric functions.

**3.1.** Let us consider the series

$$S = \sum_{n=1}^{\infty} \frac{J_{2k}(nx)}{n^{2m}} \cos ny \quad (m \in \mathbb{N}, k \in \mathbb{N}_0) \quad (13)$$

where  $J_n$  are Bessel functions of the first kind of order  $n$  [12]. In order to obtain the sum of this series, we shall use the well known integral representation of Bessel functions

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - n\theta) d\theta \quad (n \in \mathbb{N}_0). \quad (14)$$

Substituting (14) into (13) and interchanging the orders of summation and integration we get

$$S = \frac{1}{\pi} \int_0^\pi \cos 2k\theta \sum_{n=1}^{\infty} \frac{\cos(nx \sin \theta) \cos ny}{n^{2m}} d\theta.$$

For a suitable choice of the parameters  $a, b, s, f, g$  formula (9) becomes

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{\cos ny \cos nx}{n^{2m}} &= \sum_{i=0}^{m-1} \frac{x^{2i}}{(2i)!} \left( (-1)^m \frac{\pi y^{2m-2i-1}}{2(2m - 2i - 1)!} + \sum_{j=0}^{m-i} (-1)^{i+j} \frac{\zeta_{ij}}{(2j)!} y^{2j} \right) \\
 &\quad - (-1)^m \frac{x^{2m}}{2(2m)!}
 \end{aligned}$$

where  $\zeta_{ij} = \zeta(2m - 2i - 2j)$   $((x, y) \in K_1)$ , and putting  $x \sin \theta$  in place of  $x$  we evaluate  $S$ . The further procedure leads to integrals of the type

$$\int_0^\pi \sin^\mu x f(\nu x) dx = \frac{\pi}{2^\mu} f\left(\frac{\nu\pi}{2}\right) \frac{\Gamma(\mu + 1)}{\Gamma(\frac{\mu+\nu}{2} + 1)\Gamma(\frac{\mu-\nu}{2} + 1)}$$

for  $f = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}$  and  $\text{Re } \mu > -1$  (see [2]) and finally to the result (in the domain  $K_1$ )

$$S = \sum_{i=k}^{m-1} (-1)^k \frac{\left(\frac{x}{2}\right)^{2i}}{(i+k)!(i-k)!} \left( (-1)^m \frac{\pi y^{2m-2i-1}}{2(2m-2i-1)!} + \sum_{j=0}^{m-i} (-1)^{i+j} \frac{\zeta_{ij} y^{2j}}{(2j)!} \right) - (-1)^{m+k} \frac{\left(\frac{x}{2}\right)^{2m}}{2(m+k)!(m-k)!}$$

for  $m \geq k$ , but  $S = 0$  for  $m < k$ .

**3.2** Let us consider now the series

$$S_1 = \sum_{n=1}^\infty (-1)^{n-1} \frac{J_\nu((2n-1)x)}{(2n-1)^\alpha} \sin((2n-1)y) \tag{15}$$

where  $\alpha \in \mathbb{R}^+$  and  $J_\nu$  are the Bessel functions of the first kind and order  $\nu$ . In order to obtain the sum of this series we shall use the well known integral representation of Bessel functions

$$J_\nu(x) = \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\frac{1}{2})\Gamma(\nu + \frac{1}{2})} \int_0^{\pi/2} \sin^{2\nu} \theta \cos(x \cos \theta) d\theta \quad (\text{Re } \nu > -\frac{1}{2}). \tag{16}$$

Substituting (16) into (15) and interchanging the order of summation and integration we get

$$S_1 = \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\frac{1}{2})\Gamma(\nu + \frac{1}{2})} \int_0^{\pi/2} \sin^{2\nu} \theta \sum_{n=1}^\infty (-1)^{n-1} \frac{\cos((2n-1)x \cos \theta) \sin(2n-1)y}{(2n-1)^{\alpha-\nu}} d\theta.$$

Now we use formula (12) for  $s = -1, a = 2, b = 1, f = \sin, g = \cos$  and then an integral of the type

$$\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x dx = \frac{1}{2} B\left(\frac{1}{2}\mu, \frac{1}{2}\nu\right) \quad (\text{Re } \mu > 0, \text{Re } \nu > 0).$$

Without going into details we obtain the final formula

$$S_1 = \sum_{i=0}^\infty \sum_{j=0}^i (-1)^i \frac{\beta(\alpha - \nu - 2i - 1) \left(\frac{x}{2}\right)^{\nu+2j} y^{2i-2j+1}}{\Gamma(j + \nu + 1) j! (2i - 2j + 1)!} \quad (\alpha, \nu \in \mathbb{R})$$

in the domain  $K_4$ , for  $\alpha > 0$  and  $\alpha > \nu > -\frac{1}{2}$ .



### 4. Discussion and applications

In this section we show that the obtained results include as special cases formulas in [6, 13]. Also, we show how to make use of those results to obtain closed form solutions of some boundary value problems in mathematical physics.

For certain values of  $x$  or  $y$ , the obtained formula (9) reduces to formula (2). For instance, formula (9) for  $a = 2, b = 1, s = -1, c = 0, F = \beta, d = 0, f = \cos, t = 0, g = \sin, \delta = 1, r = 0$  becomes

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos((2n-1)y) \sin((2n-1)x)}{(2n-1)^{2m}} = \sum_{i=0}^{m-1} (-1)^i \frac{x^{2i+1}}{(2i+1)!} \sum_{j=0}^{m-i-1} (-1)^j \frac{\beta_{ij}}{(2j)!} y^{2j}$$

where  $\beta_{ij} = \beta(2m - 2i - 2j - 1)$  and  $(x, y) \in K_4$ . This formula for  $y = 0$  becomes

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin((2n-1)x)}{(2n-1)^{2m}} = \sum_{i=0}^{m-1} (-1)^i \frac{x^{2i+1} \beta(2m - 2i - 1)}{(2i+1)!}$$

and that is exactly formula (2) for  $f = \sin, \alpha = 2m, s = -1, a = 2, b = 1$ .

On the other hand, for  $f = g = \cos, \alpha = 2m - 1, s = -1, c = 0, a = 2, b = 1, d = 0, r = 1, t = 0, \delta = 0$  ( $\beta_{ij}$  is the same as above) formula (9) gives

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos((2n-1)y) \cos((2n-1)x)}{(2n-1)^{2m-1}} = \sum_{i=0}^{m-1} (-1)^i \frac{x^{2i}}{(2i)!} \sum_{j=0}^{m-i-1} (-1)^j \frac{\beta_{ij}}{(2j)!} y^{2j}.$$

For  $x = y = 0$  it turns to  $\beta(2m - 1)$ .

Further, the obtained general formula (9) comprises some known results. Note that our formula (9) is valid for  $\alpha = 2m - r$  ( $r = 0$  or  $r = 1, m \in N$ ), whereas in [6, 13] there are cases only for  $\alpha = 1$  or  $2$  ( $\alpha = 3$  in one case). In addition, the domains in [6, 13] are only subsets of our domains  $K_i$  ( $i = 1, 2, 3, 4$ ) or possibly equal to them. For example, formula (8) in [6: p. 743] is

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx \cos ky}{k^3} = \frac{1}{12} x(\pi^2 - x^2 - 3y^2) \quad (|x \pm y| \leq \pi).$$

This domain is equal to our  $K_2$ , and the formula

$$\begin{aligned} &\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos ny \sin nx}{n^{2m-1}} \\ &= \sum_{i=0}^{m-2} (-1)^i \frac{x^{2i+1}}{(2i+1)!} \sum_{j=0}^{m-i-1} (-1)^j \frac{\eta_{ij}}{(2j)!} y^{2j} + (-1)^{m-1} \frac{x^{2m-1}}{2(2m-1)!} \end{aligned}$$

obtained as a particular case of (9) for the chosen parameters, where  $\eta_{ij} = \eta(2m - 2i - 2j - 2)$  and  $(x, y) \in K_2$ , for  $m = 1$  gives the same result.

As another example consider formula 4Γ3 in [13: p. 435]

$$\sum_{n=1}^{\infty} \frac{\sin((2n-1)y) \cos((2n-1)x)}{2n-1} = \begin{cases} \frac{\pi}{4} & \text{for } -y < x < y \\ 0 & \text{for } y < x < \pi - y \end{cases} \quad (0 < y \leq \frac{\pi}{2}).$$

The first domain is one half of our  $K_3$ , and the formula

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\sin((2n-1)y) \cos((2n-1)x)}{(2n-1)^{2m-1}} \\ &= \sum_{i=0}^{m-1} (-1)^i \frac{x^{2i}}{(2i)!} \left( (-1)^{m-i-1} \frac{\pi y^{2m-2i-2}}{4(2m-2i-2)!} + \sum_{j=0}^{m-i-1} (-1)^j \frac{\lambda_{ij}}{(2j+1)!} y^{2j+1} \right), \end{aligned} \tag{17}$$

where  $\lambda_{ij} = \lambda(2m-2i-2j-2)$  and  $(x, y) \in K_3$ , which is a special case of (9), for  $m = 1$  gives  $\frac{\pi}{4}$ . The second domain is half of the dual domain  $K'_3$ , and the formula

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\cos((2n-1)y) \sin((2n-1)x)}{(2n-1)^{2m-1}} \\ &= \sum_{i=0}^{m-2} (-1)^i \frac{x^{2i+1}}{(2i+1)!} \left( (-1)^{m-i-1} \frac{\pi y^{2m-2i-3}}{4(2m-2i-3)!} + \sum_{j=0}^{m-i-1} (-1)^j \frac{\lambda_{ij}}{(2j)!} y^{2j} \right), \end{aligned}$$

where  $\lambda_{ij} = \lambda(2m-2i-2j-2)$  and  $(x, y) \in K_3$ , as a special case of (9), for  $m = 1$  gives 0, too, as above.

Finally, it should be noted that some results in the cited books are not valid. Namely, formula (3) in [6: p. 743] should have the result  $-\frac{x}{2}$  instead of  $-\frac{1}{2}$ , and formula 337 in [13: p. 445] should have the result  $-\frac{\pi}{4}y$  instead of  $\frac{\pi}{4}y$ . In the formula 3j8 in [13: p. 443] the results or domains are wrong because the sum

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos(2n-1)y \sin((2n-1)x)}{(2n-1)^2}$$

equals 0 for  $x = 0$ , whilst that formula gives  $\frac{\pi^2}{8}(1-y)$  in the domain  $0 < y \leq \frac{\pi}{2}$ ,  $-y \leq x \leq \frac{\pi}{2} + y$ .

It is known that the solution of the boundary value problem

$$\left. \begin{aligned} U_{tt} &= a^2 U_{xx} \\ U(x, 0) &= \frac{4hx(L-x)}{L^2} \\ U_t(x, 0) &= 0 \end{aligned} \right\}$$

for  $0 \leq x \leq L$ ,  $t \geq 0$  is given by (see [11])

$$U(x, t) = \frac{32h}{\pi^3} \sum_{n=1}^{\infty} \frac{\cos \frac{\pi(2n-1)at}{L} \sin \frac{(2n-1)\pi x}{L}}{(2n-1)^3}.$$

Using formula (17) for  $m = 2$ , we obtain that in the domain  $0 \leq \frac{a}{L}t \leq \frac{1}{2}$ ,  $|\frac{a}{L}t| \leq \frac{1}{L}x \leq 1 - |\frac{a}{L}t|$  the solution is in the closed form

$$U(x, t) = \frac{4h}{L^2}(xL - x^2 - a^2t^2)$$

in which form it is easier to use. Similarly, the solution of the boundary value problem

$$\left. \begin{aligned} U_{tt} &= U_{xx} + x(x - L) \\ U(x, 0) &= U_t(x, 0) = 0 \\ U(0, t) &= U(L, t) = 0 \end{aligned} \right\}$$

for  $0 \leq x \leq L$ ,  $t \geq 0$  is (see [11])

$$U(x, t) = \frac{8L^4}{\pi^5} \sum_{m=1}^{\infty} \frac{\cos \frac{(2m-1)\pi t}{L} \sin \frac{(2m-1)\pi x}{L}}{(2m-1)^5} - \frac{1}{12} x(x^3 - 2x^2L + L^3).$$

Applying (17) for  $m = 3$ , this solution becomes in the closed form

$$U(x, t) = \frac{1}{2}x^2t^2 - \frac{1}{2}Lxt^2 + \frac{1}{12}t^4$$

for  $0 \leq t \leq \frac{L}{2}$ ,  $t \leq x \leq L - t$ .

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### Editorial Remark

- We recommend to formulate the Abstract in self-contained manner!
- Tables I - III we will setted in later.
- It seems to us not in any case clear enough what all constitute the argument of the functions  $\cos$  and  $\sin$ . For example, in the formula for  $S_\alpha$  after (11) we have  $\sin(an - b)(y - x)$  or  $\sin (an - b)(y - x)$  ? Analogously in other cases. Thus to clear the writing, use brackets or space in between.
- After (1), the symbol  $a = \{\frac{1}{2}\}$  means that  $a = 1$  or  $a = 2$ ? Please explain! If so, then why you write out in the same line  $s = 1$  or  $s = -1$ ?
- In alternating series, for more clarity, we recommend to separate terms like  $(-1)^i$ . Analogously, maybe it should be recommended to separate also terms like  $(s)^{(n-1)}$  because, as I understand,  $s = 1$  or  $s = -1$ ? Additionally, it is not right understandable why you clip here the exponent  $n - 1$ .
- As usual, in the boundary value problems on pp. 10 – 11 the derivatives are sufficient to write like  $U_{xx}$  instead  $U''_{xx}$ .
- [8] and [9] seem to be Proceedings. If so, then concretize the coordinates, especially quote the editors.
- It seemed to us that the English was not the best. Please check it (we allowed our-selves to make already some changes).