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Unbounded C^*-Seminorms and *-Representations of Partial *-Algebras

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Abstract. The main purpose of this paper is to construct *-representations from unbounded C^*-seminorms on partial *-algebras and to investigate their *-representations.

Keywords: *Partial* **-algebras, quasi* **-algebras, unbounded* C^**-seminorms,* (*unbounded*) **-representations*

AMS subject classification: 46K70, 46L05

1. Introduction and Preliminaries

A C^*-seminorm p on a locally convex *-algebra $\mathcal{A}$ is a seminorm enjoying the so-called C^*-*property* $p(x^*x) = p(x)^2 \quad (x \in \mathcal{A})$. They have been extensively studied in the literature (see, e.g., [9 - 13, 19]). One of the main points of the theory is that every *-representation of the completion $(\mathcal{A}, p)$ is bounded.

Generalizations of this notions have led Bhatt, Ogi and one of us [11] to consider so-called *unbounded* C^**-seminorms* on *-algebras. Their main feature is that they need not be defined on the whole $\mathcal{A}$ but only on a *-subalgebra of it. This fact allows the existence of unbounded representations of $\mathcal{A}$ (and motivates the adjective "unbounded" used to name them). But it is not only for need of mathematical generalization that it makes sense to consider unbounded C^*-seminorms but also because of it appearance in some subject of mathematical physics [1, 15, 18]. However, when considering unbounded C^*-seminorms on a locally convex *-algebra $\mathcal{A}$ whose multiplication is not jointly continuous one is naturally led to consider partial algebraic structures: in that case in fact the completion of $\mathcal{A}$ is no longer, in general, a locally convex *-algebra but only a topological quasi *-algebra [16, 17]. Quasi *-algebras are a particular case of partial *-algebras [3]. Roughly speaking, a partial *-algebra $\mathcal{A}$ is a linear space with involution and a partial multiplication defined on a subset Γ of $\mathcal{A} \times \mathcal{A}$ enjoying some of the usual properties of multiplication, with the very relevant exception of associativity. Of course, as one of the main tools in the study of *-algebras is the theory of *-representations, partial *-algebras of operators (so-called partial O^*-algebras) have been considered as the main

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instance of these *new* algebraic structures and a systematic study has been undertaken [3 - 6]. From a more abstract point of view, the possibility of introducing topologies compatible with the structure of a partial *-algebra has been investigated in [2].

The present paper is organized as follows.

In Section 2, starting from a C^*-seminorm p on a partial *-algebra $\mathcal{A}$, we prove the existence of *quasi *-representations* of $\mathcal{A}$ induced by p; they are named in this way since the usual rule for the multiplication holds in a sense that remind the multiplication in quasi *-algebras. These quasi *-representations depend essentially on a certain subspace $\mathcal{N}_p$ of the domain $\mathcal{D}(p)$ of the C^*-seminorm p. Of course, by adding assumptions on $\mathcal{N}_p$ we are led to consider a variety of situations of some interest. In this perspective, we introduce the notions of *finite* and (*weakly*-) *semifinite C^*-seminorms* and study in detail the quasi *-representations that they induce.

In Section 3 we consider the problem as to whether a *-representation of $\mathcal{A}$, in the sense of [5], does really exist or in other words if the quasi *-representation, whose existence has been proved in Section 2, is indeed a *-representation.

In Section 4, we reverse the point of view: starting from a *-representation π of a partial *-algebra, we construct an unbounded C^*-seminorm r_π on $\mathcal{A}$ which turns out to admit a *-representation $\pi^N_{r_\pi}$ called *natural*. We then investigate the relationship between $\pi^N_{r_\pi}$ and the *-representation π where we had started with.

Section 5 is devoted to the discussion of some examples.

Before going forth, we shortly give some definitions needed in the sequel.

A *partial *-algebra* is a complex vector space $\mathcal{A}$, endowed with an involution $x \mapsto x^*$ (that is, a bijection such that $x^{**} = x$) and a partial multiplication defined by a set $\Gamma \subset \mathcal{A} \times \mathcal{A}$ (a binary relation) such that:

(i) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$.

(ii) $(x, y_1), (x, y_2) \in \Gamma$ implies $(x, \lambda y_1 + \mu y_2) \in \Gamma$ for all $\lambda, \mu \in \mathbb{C}$.

(iii) For any $(x, y) \in \Gamma$, there is defined a product $x\,y \in \mathcal{A}$, which is distributive with respect to addition and satisfies the relation $(x\,y)^* = y^* x^*$.

The element e of the partial *-algebra $\mathcal{A}$ is called a *unit* if $e^* = e$, $(e, x) \in \Gamma$ for all $x \in \mathcal{A}$ and $e\,x = x\,e = x$ for all $x \in \mathcal{A}$.

Given the defining set Γ, spaces of multipliers are defined in the obvious way:

$$
\begin{aligned}
(x, y) \in \Gamma \quad &\iff \quad x \in L(y) \text{ or } x \text{ is a left multiplier of } y \\
&\iff \quad y \in R(x) \text{ or } y \text{ is a right multiplier of } x.
\end{aligned}
$$

For a subset $\mathcal{B}$ of $\mathcal{A}$, we write

$$
L(\mathcal{B}) = \cap_{x \in \mathcal{B}} L(x), \qquad R(\mathcal{B}) = \cap_{x \in \mathcal{B}} R(x).
$$

Notice that the partial multiplication is *not* required to be associative (and often it is not). The following weaker notion is therefore in use: a partial *-algebra $\mathcal{A}$ is said to be *semi-associative* if $y \in R(x)$ implies $y \cdot z \in R(x)$ for every $z \in R(\mathcal{A})$ and

$$
(x \cdot y) \cdot z = x \cdot (y \cdot z).
$$

Let $\mathcal{A}[\tau]$ be a partial *-algebra, which is a topological vector space for the locally convex topology τ. Then $\mathcal{A}[\tau]$ is called a *topological partial* **-algebra* if the following two conditions are satisfied [2]:

(i) The involution $a \mapsto a^*$ is τ-continuous.

(ii) The maps $a \mapsto xa$ and $a \mapsto ay$ are τ-continuous for all $x \in L(\mathcal{A})$ and $y \in R(\mathcal{A})$.

A *quasi* **-algebra* $(\mathcal{A}, \mathcal{A}_0)$ is a partial *-algebra where the multiplication is defined via the *-algebra $\mathcal{A}_0 \subset \mathcal{A}$ by taking Γ as

$$\Gamma = \Big\{(a,b) \in \mathcal{A} \times \mathcal{A} :\ a \in \mathcal{A}_0 \text{ or } b \in \mathcal{A}_0\Big\}.$$

If $\mathcal{A}$ is endowed with a locally convex topology which makes it into a topological partial *-algebra and $\mathcal{A}_0$ is dense in $\mathcal{A}$, then $(\mathcal{A}, \mathcal{A}_0)$ is said to be a *topological* quasi *-algebra.

We turn now to partial O^*-algebras. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all (closable) linear operators X such that $D(X) = \mathcal{D}$ and $D(X^*) \supseteq \mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a partial *-algebra with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger = X^* \upharpoonright \mathcal{D}$ and the (*weak*) partial multiplication $X_1 \square X_2 = {X_1}^{\dagger *} X_2$, defined whenever X_2 is a weak right multiplier of X_1 (equivalently, X_1 is a weak left multiplier of X_2), that is, if and only if $X_2 \mathcal{D} \subset D({X_1}^{\dagger *})$ and $X_1^* \mathcal{D} \subset D(X_2^*)$ (we write $X_2 \in R^{\mathrm{w}}(X_1)$ or $X_1 \in L^{\mathrm{w}}(X_2)$). When we regard $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ as a partial *-algebra with those operations, we denote it by $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D}, \mathcal{H})$.

A partial O^*-algebra on $\mathcal{D}$ is a (partial) *-subalgebra $\mathcal{M}$ of $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D}, \mathcal{H})$, that is, $\mathcal{M}$ is a subspace of $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D}, \mathcal{H})$ containing the identity and such that $X^\dagger \in \mathcal{M}$ whenever $X \in \mathcal{M}$ and $X_1 \square X_2 \in \mathcal{M}$ for any $X_1, X_2 \in \mathcal{M}$ such that $X_2 \in R^{\mathrm{w}}(X_1)$. Thus $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D}, \mathcal{H})$ itself is the largest partial O^*-algebra on the domain $\mathcal{D}$.

Given a †-invariant subset $\mathcal{N}$ of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, the familiar weak bounded commutant is defined as

$$\mathcal{N}'_{\mathrm{w}} = \Big\{C \in \mathcal{B}(\mathcal{H}) :\ (CX\xi|\eta) = (C\xi|X^\dagger\eta) \text{ for each } \xi, \eta \in \mathcal{D} \text{ and } X \in \mathcal{N}\Big\}.$$

The last definitions we need are related with representations.

A *-*representation* of a partial *-algebra $\mathcal{A}$ is a *-homomorphism of $\mathcal{A}$ into $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D}(\pi),$ $\mathcal{H}_\pi)$, for some pair $\mathcal{D}(\pi) \subset \mathcal{H}_\pi$, that is, a linear map $\pi : \mathcal{A} \to \mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D}(\pi), \mathcal{H}_\pi)$ such that

(i) $\pi(x^*) = \pi(x)^\dagger$ for every $x \in \mathcal{A}$.

(ii) $x \in L(y)$ in $\mathcal{A}$ implies $\pi(x) \in L^{\mathrm{w}}(\pi(y))$ and $\pi(x) \square \pi(y) = \pi(xy)$.

If π is a *-representation of the partial *-algebra $\mathcal{A}$ into $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D}(\pi), \mathcal{H}_\pi)$, we define $\widetilde{\mathcal{D}(\pi)}$ as the completion of $\mathcal{D}(\pi)$ with respect to the graph topology defined by $\pi(\mathcal{A})$. Furthermore, we put

$$\widehat{\mathcal{D}(\pi)} = \cap_{x \in \mathcal{A}} \mathcal{D}(\overline{\pi(x)})$$
$$\mathcal{D}(\pi)^* = \cap_{x \in \mathcal{A}} \mathcal{D}(\pi(x)^*).$$

We say that π is

-*closed* if $\mathcal{D}(\pi) = \widetilde{\mathcal{D}(\pi)}$

-*fully-closed* if $\mathcal{D}(\pi) = \widehat{\mathcal{D}(\pi)}$
s -*self-adjoint* if $\mathcal{D}(\pi) = \mathcal{D}(\pi)^*$.

Let π_1 and π_2 be *-representations of $\mathcal{A}$. With the notation $\pi_1 \subset \pi_2$ we mean that $\mathcal{H}_{\pi_1} \subseteq \mathcal{H}_{\pi_2}$, $\mathcal{D}(\pi_1) \subseteq \mathcal{D}(\pi_2)$ and $\pi_1(a)\xi = \pi_2(a)\xi$ for each $\xi \in \mathcal{D}(\pi_1)$.

By considering the identical *-representations, the terms *fully-closed, self-adjoint* etc. can also be referred to a partial O^*-algebra on a given domain $\mathcal{D}$ and then generalized, in obvious way, to an arbitrary †-invariant subset of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$.

2. Representations induced by unbounded C^*-seminorms

In this section we construct (quasi) *-representations of partial *-algebras from unbounded C*-seminorms. Throughout this paper we treat only with partial *-algebras whose partial multiplication satisfies the property

$$\textbf{(A)} \quad \begin{cases} y^*(ax) = (y^*a)x \\ a(xy) = (ax)y \end{cases} \qquad \text{for all } a \in \mathcal{A} \text{ and all } x, y \in R(\mathcal{A}).$$

We remark that if $\mathcal{A}$ is semi-associative, then it satisfies Property (A).

Definition 2.1. A mapping p of a (partial) *-subalgebra $\mathcal{D}(p)$ of $\mathcal{A}$ into $\mathbb{R}^+$ is said to be an *unbounded* m^*-*(semi)norm* on $\mathcal{A}$ if

(i) p is a (semi) norm on $\mathcal{D}(p)$.
(ii) $p(x^*) = p(x)$ for all $x \in \mathcal{D}(p)$.
(iii) $p(xy) \leq p(x)p(y)$ for all $x, y \in \mathcal{D}(p)$ such that $x \in L(y)$.

An unbounded m^*-(semi)norm p on $\mathcal{A}$ is said to be an unbounded C^*-*(semi)norm* if

(iv) $p(x^*x) = p(x)^2$ for all $x \in \mathcal{D}(p)$ such that $x^* \in L(x)$.

An unbounded m^*-(semi)norm or C^*-(semi)norm on $\mathcal{A}$ is said to be a m^*-*(semi)norm* or C^*-*(semi)norm*, respectively, if $\mathcal{D}(p) = \mathcal{A}$.

An (unbounded) m^*-seminorm p on $\mathcal{A}$ is said to have *Property* (B) if it satisfies the following basic density-condition:

(B) $R(\mathcal{A}) \cap \mathcal{D}(p)$ is total in $\mathcal{D}(p)$ with respect to p.

Lemma 2.2. *Let p be an m^*-seminorm on $\mathcal{A}$ having Property (B), that is, $R(\mathcal{A})$ is p-dense in $\mathcal{A}$. We denote by $\hat{\mathcal{A}}$ the set of all Cauchy sequences in $\mathcal{A}$ with respect to the seminorm p and define an equivalent relation $\sim$ in $\hat{\mathcal{A}}$ by $\{a_n\} \sim \{b_n\}$ if $\lim_{n\to\infty} p(a_n - b_n) = 0$. Then the following statements hold:*

(1) *The quotient space $\hat{\mathcal{A}}/{\sim}$ is a Banach *-algebra under the following operations, involution and norm:*

$$\begin{aligned} \{a_n\}^\sim + \{b_n\}^\sim &\equiv \{a_n + b_n\}^\sim \\ \lambda\{a_n\}^\sim &\equiv \{\lambda a_n\}^\sim \\ \{a_n\}^\sim\{b_n\}^\sim &\equiv \{x_n y_n\}^\sim \ \left(\{x_n\}^\sim, \{y_n\}^\sim \in R(\mathcal{A}), \{x_n\}^\sim \equiv \{a_n\}^\sim, \{y_n\}^\sim \equiv \{b_n\}^\sim\right) \\ \{a_n\}^{\sim *} &\equiv \{a_n^*\}^\sim \\ \|\{a_n\}^\sim\|_p &\equiv \lim_{n\to\infty} p(a_n). \end{aligned}$$

(2) *For each* $a \in \mathcal{A}$ *we put* $\tilde{a} = \{a_n\}^\sim$ $(a_n = a, n \in \mathbb{N})$ *and* $\tilde{\mathcal{A}} = \{\tilde{a} : a \in \mathcal{A}\}$. *Then* $\tilde{\mathcal{A}}$ *is a dense* **-invariant subspace of* $\hat{\mathcal{A}}/\!\sim$ *satisfying* $\tilde{a}\tilde{b} = (ab)^\sim$ *whenever* $a \in L(b)$.

(3) *Suppose* p *is a* C^**-seminorm on* $\mathcal{A}$. *Then* $\hat{\mathcal{A}}/\!\sim$ *is a* C^**-algebra.*

Proof. As in the usual construction of the completion of a normed space, it can be shown that $\hat{\mathcal{A}}/\!\sim$ is a Banach space.

We first show that $\{a_n\}^\sim\{b_n\}^\sim$ is well-defined and the relation defines a multiplication of $\hat{\mathcal{A}}/\!\sim$. Since $R(\mathcal{A})$ is p-dense in $\mathcal{A}$, for each $\{a_n\}, \{b_n\} \in \hat{\mathcal{A}}$ there exist sequences $\{x_n\}, \{y_n\} \in R(\mathcal{A})$ such that $\{a_n\}^\sim = \{x_n\}^\sim$ and $\{b_n\}^\sim = \{y_n\}^\sim$. Then it follows from the submultiplicativity of p that $\{x_n y_n\}^\sim \in \hat{\mathcal{A}}$ and $\{a_n\}^\sim\{b_n\}^\sim$ is independent of the choice of the equivalent sequences $\{x_n\}$ and $\{y_n\}$. Further, the relation $\{a_n\}^\sim\{b_n\}^\sim$ defines a multiplication of $\hat{\mathcal{A}}/\!\sim$. In fact, the associativity follows from the equalities

$$\begin{aligned}\{a_n\}^\sim(\{b_n\}^\sim\{c_n\}^\sim) &= \{x_n\}^\sim(\{y_n z_n\}^\sim) \\ &= \{x_n(y_n z_n)\}^\sim \\ &= \{(x_n y_n) z_n\}^\sim \\ &= (\{a_n\}^\sim\{b_n\}^\sim)\{c_n\}^\sim\end{aligned}$$

where $\{x_n\}, \{y_n\}, \{z_n\} \subset R(\mathcal{A})$ such that $\{x_n\}^\sim = \{a_n\}^\sim, \{y_n\}^\sim = \{b_n\}^\sim$ and $\{z_n\}^\sim = \{c_n\}^\sim$, and the other properties can be proved in a similar way. Thus $\hat{\mathcal{A}}/\!\sim$ is a usual algebra.

Similarly it is shown that $\{a_n\}^\sim \mapsto \{a_n^*\}^\sim$ is an involution of the algebra $\hat{\mathcal{A}}/\!\sim$, and

$$\|\{a_n\}^\sim\{b_n\}^\sim\|_p \le \|\{a_n\}^\sim\|_p \|\{b_n\}^\sim\|_p, \qquad \|\{a_n\}^{\sim *}\|_p = \|\{a_n\}^\sim\|_p$$

for each $\{a_n\}, \{b_n\} \in \hat{\mathcal{A}}$, which implies statement (1), i.e. that $\hat{\mathcal{A}}/\sim$ is a Banach *-algebra. Statements (2) and (3) can be proved in a similar way ∎

From now on we denote by

$\Sigma(\mathcal{A})$ the set of all unbounded C^*-seminorms on $\mathcal{A}$

$\Sigma_B(\mathcal{A})$ the subset of $\Sigma(\mathcal{A})$ consisting of those satisfying Property (B).

Let p be an unbounded C^*-seminorm on $\mathcal{A}$ having Property (B), i.e. $p \in \Sigma_B(\mathcal{A})$. By Lemma 2.2, $\mathcal{A}_p \equiv \widehat{\mathcal{D}(p)}/\!\sim$ is a C^*-algebra. We denote by $\mathrm{Rep}(\mathcal{A}_p)$ the set of all *-representations Π_p of the C^*-algebra $\mathcal{A}_p$ on Hilbert space $\mathcal{H}_{\Pi_p}$ and put

$$\mathrm{FRep}(\mathcal{A}_p) = \Big\{\Pi_p \in \mathrm{Rep}(\mathcal{A}_p) : \Pi_p \text{ is faithful}\Big\}.$$

Then we have the following

Proposition 2.3. *For any* $\Pi_p \in \mathrm{Rep}(\mathcal{A}_p)$ *we put*

$$\pi_p^\circ(x) = \Pi_p(\tilde{x}) \qquad (x \in \mathcal{D}(p)).$$

Then π_p° *is a* **-representation of* $\mathcal{D}(p)$ *on* $\mathcal{H}_{\Pi_p}$.

This proposition provides the most natural way to define a *-representation of $\mathcal{D}(p)$. However, π_p° cannot be extended to the whole $\mathcal{A}$. The construction of *-representations of $\mathcal{A}$ requires a more detailed analysis. This will be the content of the next propositions.

To begin with, we put

$$\mathcal{N}_p = \Big\{x \in \mathcal{D}(p) \cap R(\mathcal{A}) : ax \in \mathcal{D}(p) \text{ for all } a \in \mathcal{A}\Big\}.$$

Then we have the following

Lemma 2.4.

(1) *$\mathcal{N}_p$ is an algebra satisfying $(\mathcal{D}(p) \cap R(\mathcal{A}))\mathcal{N}_p \subset \mathcal{N}_p$.*

(2) *We denote by $\mathcal{T}_p$ the closure of $\widetilde{\mathcal{N}_p}$ in the C^*-algebra $\mathcal{A}_p$. Then $\mathcal{T}_p$ is a closed left ideal of $\mathcal{A}_p$.*

(3) *$\Pi_p(\widetilde{\mathcal{N}_p}^2)\mathcal{H}_{\Pi_p}$ is dense in $\Pi_p(\widetilde{\mathcal{N}_p})\mathcal{H}_{\Pi_p}$.*

Proof. Statement (1) follows from the semi-associativity (A).

Statement (2): Since $\mathcal{D}(p) \cap R(\mathcal{A})$ is p-dense in $\mathcal{D}(p)$ and the above property (1), it follows that $\mathcal{D}(p)^\sim \mathcal{N}_p^\sim \subset \mathcal{T}_p$, and so $\mathcal{D}(p)^\sim \mathcal{T}_p \subset \mathcal{T}_p$. Since $\mathcal{D}(p)^\sim$ is dense in the C^*-algebra $\mathcal{A}_p$, we have $\mathcal{A}_p\mathcal{T}_p \subset \mathcal{T}_p$.

Statement (3): It is clear that $\Pi_p(\widetilde{\mathcal{N}_p}^2)\mathcal{H}_{\Pi_p}$ is dense in $\Pi_p(\widetilde{\mathcal{N}_p}\mathcal{T}_p)\mathcal{H}_{\Pi_p}$. Since $\mathcal{T}_p$ is a closed left ideal of the C^*-algebra $\mathcal{A}_p$, there exists a direct net $\{U_\lambda\}$ in $\mathcal{T}_p$ such that $\lim_\lambda \|AU_\lambda - A\|_p = 0$ for each $A \in \mathcal{T}_p$, which implies that $\Pi_p(\widetilde{\mathcal{N}_p}\mathcal{T}_p)\mathcal{H}_{\Pi_p}$ is dense in $\Pi_p(\widetilde{\mathcal{N}_p})\mathcal{H}_{\Pi_p}$. Hence $\Pi_p(\widetilde{\mathcal{N}_p}^2)\mathcal{H}_{\Pi_p}$ is dense in $\Pi_p(\widetilde{\mathcal{N}_p})\mathcal{H}_{\Pi_p}$ ∎

Let now

$$\mathcal{D}(\pi_p) = \text{Lin}\Big\{\Pi_p((xy)^\sim)\xi : x, y \in \mathcal{N}_p \text{ and } \xi \in \mathcal{H}_{\Pi_p}\Big\}$$

and $\mathcal{H}_{\pi_p}$ be the closure of $\mathcal{D}(\pi_p)$ in $\mathcal{H}_{\Pi_p}$. We define

$$\pi_p(a)\big(\textstyle\sum_k \Pi_p((x_k y_k)^\sim)\xi_k\big) = \textstyle\sum_k \Pi_p((ax_k)^\sim \widetilde{y_k})\xi_k$$

for $a \in \mathcal{A}$ and $\sum_k \Pi_p((x_k y_k)^\sim)\xi_k \in \mathcal{D}(\pi_p)$.

Remark. By Lemma 2.4/(3) we have

$$\begin{aligned}\mathcal{H}_{\pi_p} &\equiv \overline{\text{Lin}}\Big\{\Pi_p(\tilde{x_1}\tilde{x_2})\xi : x_1, x_2 \in \mathcal{N}_p \text{ and } \xi \in \mathcal{H}_{\Pi_p}\Big\} \\ &= \overline{\text{Lin}}\Big\{\Pi_p(\tilde{x})\xi : x \in \mathcal{N}_p \text{ and } \xi \in \mathcal{H}_{\Pi_p}\Big\}.\end{aligned}$$

In general, it may happen that $\mathcal{H}_{\pi_p}$ is very 'small' compared to $\mathcal{H}_{\Pi_p}$. This point will be considered at the end of this section, where *well-behaved* representations related to unbounded C^*-seminorms will be introduced.

Now, we prove the following

Lemma 2.5. *π_p is a linear map of $\mathcal{A}$ into $\mathcal{L}^\dagger(\mathcal{D}(\pi_p), \mathcal{H}_{\pi_p})$ satisfying the following properties:*

(i) $\pi_p(a^*) = \pi_p(a)^\dagger \quad (a \in \mathcal{A})$.

(ii) $\pi_p(ax) = \pi_p(a) \,\Box\, \pi_p(x) \quad \big(a \in \mathcal{A}, x \in R(\mathcal{A})\big)$.

(iii) $\|\overline{\pi_p(x)}\| \le p(x) \quad (x \in \mathcal{D}(p))$. *Further, if $\pi_p \in \mathrm{FRep}(\mathcal{A}_p)$, then* $\|\overline{\pi_p(x)}\| = p(x)$ $(x \in \mathcal{N}_p)$.

Proof. By Lemma 2.4/(2)-(3) we have

$$\Pi_p((ax)^\sim \tilde{y})\xi \in \Pi_p(\mathcal{T}_p)\mathcal{H}_{\Pi_p} \subset \mathcal{H}_{\pi_p}$$

for each $a \in \mathcal{A}$, $x, y \in \mathcal{N}_p$ and $\xi \in \mathcal{H}_{\Pi_p}$, and further by Property (A)

$$\begin{aligned}
&\big(\Pi_p((ax_1)^\sim \tilde{y_1})\xi | \Pi_p(\tilde{x_2}\tilde{y_2})\eta\big) \\
&\quad = \big(\Pi_p(\tilde{y_1})\xi | \Pi_p(((ax_1)^* x_2)^\sim)\Pi_p(\tilde{y_2})\eta\big) \\
&\quad = \big(\Pi_p(\tilde{y_1})\xi | \Pi_p((x_1^*(a^* x_2))^\sim)\Pi_p(\tilde{y_2})\eta\big) \\
&\quad = \big(\Pi_p(\tilde{x_1}\tilde{y_1})\xi | \Pi_p((a^* x_2)^\sim \tilde{y_2})\eta\big)
\end{aligned}$$

for each $a \in \mathcal{A}$, $x_1, y_1, x_2, y_2 \in \mathcal{N}_p$ and $\xi, \eta \in \mathcal{H}_{\Pi_p}$, which implies that $\pi_p(a)$ is a well-defined linear map from $\mathcal{D}(\pi_p)$ to $\mathcal{H}_{\pi_p}$ satisfying $\pi_p(a^*) = \pi_p(a)^\dagger$. It is clear that π_p is a linear map of $\mathcal{A}$ into $\mathcal{L}^\dagger(\mathcal{D}(\pi_p), \mathcal{H}_{\pi_p})$.

We next show statement (ii). Take arbitrary $a \in \mathcal{A}$ and $x \in R(\mathcal{A})$. By Property (A) we have

$$z^*((ax)y) = (z^*(ax))y = ((z^*a)x)y$$

for each $a \in \mathcal{A}, x \in R(\mathcal{A})$ and $y, z \in \mathcal{N}_p$, and hence it follows from Lemma 2.2/(2) that

$$\begin{aligned}
&\big(\pi_p(ax)\Pi_p(\tilde{y_1}\tilde{y_2})\xi | \Pi_p(\tilde{z_1}\tilde{z_2})\eta\big) \\
&\quad = \big(\Pi_p((ax)y_1)^\sim)\Pi_p(\tilde{y_2})\xi | \Pi_p(\tilde{z_1})\Pi_p(\tilde{z_2})\eta\big) \\
&\quad = \big(\Pi_p(((z_1^* a)x)^\sim \tilde{y_1})\Pi_p(\tilde{y_2})\xi | \Pi_p(\tilde{z_2})\eta\big) \\
&\quad = \big(\Pi_p(\tilde{y_1}\tilde{y_2})\xi | \Pi_p(\tilde{x^*}(z_1^* a)^{*\sim})\Pi_p(\tilde{z_2})\eta\big) \\
&\quad = \big(\Pi_p(x)\Pi_p(\tilde{y_1}\tilde{y_2})\xi | \pi_p(a)^\dagger \Pi_p(\tilde{z_1}\tilde{z_2})\eta\big) \\
&\quad = \big(\pi_p(x)\Pi_p(\tilde{y_1}\tilde{y_2})\xi | \pi_p(a)^\dagger \Pi_p(\tilde{z_1}\tilde{z_2})\eta\big)
\end{aligned}$$

for each $y_1, y_2, z_1, z_2 \in \mathcal{N}_p$ and $\xi, \eta \in \mathcal{H}_{\Pi_p}$, which implies statement (ii).

Take an arbitrary $x \in \mathcal{D}(p)$. Since $\overline{\pi_p(x)} = \Pi_p(\tilde{x}) \upharpoonright \mathcal{H}_{\pi_p}$, it follows that $\|\overline{\pi_p(x)}\| \le \|\Pi_p(\tilde{x})\| = p(x)$. Suppose $\Pi_p \in \mathrm{FRep}(\mathcal{A}_p)$. Take an arbitrary $x \in \mathcal{N}_p$. It is sufficient to show $\|\overline{\pi_p(x)}\| \ge p(x)$. If $p(x) = 0$, then this is obvious. Suppose $p(x) \ne 0$. We put $y = \frac{x}{p(x)}$. Since

$$\|\Pi_p(\tilde{y})\xi\| \le \|\Pi_p(\tilde{y})\| \, \|\xi\| = p(y)\|\xi\| \le 1$$

for each $\xi \in \mathcal{H}_{\Pi_p}$ such that $\|\xi\| \le 1$ and $\Pi_p(\tilde{\mathcal{N}_p})\mathcal{H}_{\Pi_p}$ is total in $\mathcal{H}_{\pi_p}$ (by Lemma 2.4/(3) and Remark thereafter), it follows that

$$\begin{aligned}
\|\overline{\pi_p(y)}\| &= \|\overline{\pi_p(y^*)}\| \\
&\ge \sup\left\{\|\pi_p(y^*)\Pi_p(\tilde{y})\xi\| : \xi \in \mathcal{H}_{\Pi_p} \text{ with } \|\xi\| \le 1\right\} \\
&= \sup\left\{\|\Pi_p((y^*y)^{\sim})\xi\| : \xi \in \mathcal{H}_{\Pi_p} \text{ with } \|\xi\| \le 1\right\} \\
&= \|\Pi_p((y^*y)^{\sim})\| \\
&= p(y^*y) \\
&= p(y)^2 \\
&= 1
\end{aligned}$$

which implies that $\|\overline{\pi_p(x)}\| \ge p(x)$ ∎

Remark. If, instead of following the above procedure, we would have taken

$$\begin{aligned}
\mathcal{D}(\pi) &= \mathrm{Lin}\left\{\Pi_p(\tilde{x})\xi : x \in \mathcal{N}_p \text{ and } \xi \in \mathcal{H}_{\Pi_p}\right\} \\
\mathcal{H}_\pi &= \text{closure of } \mathcal{D}(\pi) \text{ in } \mathcal{H}_{\Pi_p}
\end{aligned}$$

and

$$\pi(a)\left(\textstyle\sum_k \Pi_p(\tilde{x_k})\xi_k\right) = \textstyle\sum_k \Pi_p((ax_k)^{\sim})\xi_k$$

for $a \in \mathcal{A}$ and $\sum_k \Pi_p(\tilde{x_k})\xi_k \in \mathcal{D}(\pi)$, then we could not conclude that $\pi(a)$ belongs to $\mathcal{L}^\dagger(\mathcal{D}(\pi), \mathcal{H}_\pi)$ for each $a \in \mathcal{A}$.

So far, we do not know whether π_p is a *-representation of $\mathcal{A}$ for the lack of semi-associativity of partial multiplication, and so we define the following notion:

Definition 2.6. A linear map π of $\mathcal{A}$ into $\mathcal{L}^\dagger(\mathcal{D}(\pi), \mathcal{H}_\pi)$ is said to be a *quasi *-representation* if

(i) $\pi(a^*) = \pi(a)^\dagger$ for all $a \in \mathcal{A}$

(ii) $\pi(ax) = \pi(a) \,\square\, \pi(x)$ for all $a \in \mathcal{A}$ and all $x \in R(\mathcal{A})$.

By Lemma 2.5, for each $p \in \Sigma_B(\mathcal{A})$, every π_p is a quasi *-representation of $\mathcal{A}$, and it is said to be a *quasi *-representation of $\mathcal{A}$ induced by p.*

We summerize in the following scheme the method of construction π_p from an unbounded C^*-seminorm p described above:

Here the arrow A - - - $\rightarrow B$ means that B is constructed from A.

We put

$$\begin{aligned}\mathrm{QRep}(\mathcal{A},p) &= \big\{\pi_p : \Pi_p \in \mathrm{Rep}(\mathcal{A}_p)\big\}\\ \mathrm{Rep}(\mathcal{A},p) &= \big\{\pi_p \in \mathrm{QRep}(\mathcal{A},p) : \pi_p \text{ is a } {}^*\text{-representation}\big\}\\ \mathrm{FQRep}(\mathcal{A},p) &= \big\{\pi_p : \Pi_p \in \mathrm{FRep}(\mathcal{A}_p)\big\}.\end{aligned}$$

Definition 2.7. Let $p \in \Sigma_B(\mathcal{A})$. We say that p is *representable* if

$$\mathrm{FRep}(\mathcal{A},p) \equiv \Big\{\pi_p \in \mathrm{FQRep}(\mathcal{A},p) : \pi_p \text{ is a } {}^*\text{-representation of } \mathcal{A}\Big\} \neq \emptyset.$$

It is natural to look for conditions for p to be representable. We shall consider this problem in Section 3.

We define the notions of semifiniteness and weak semifiniteness of unbounded C^*-seminorms, and study (quasi) *-representations induced by them.

Definition 2.8. An unbounded m^*-seminorm p on $\mathcal{A}$ is said to be

-*finite* if $\mathcal{D}(p) = \mathcal{N}_p$

-*semifinite* if $\mathcal{N}_p$ is p-dense in $\mathcal{D}(p)$.

An unbounded C^*-seminorm p on $\mathcal{A}$ having Property (B) is said to be

-*weakly semifinite* if $\mathrm{QRep}^{\mathrm{WB}}(\mathcal{A},p) \equiv \big\{\pi \in \mathrm{FQRep}(\mathcal{A},p) : \mathcal{H}_{\pi_p} = \mathcal{H}_{\Pi_p}\big\} \neq \emptyset$

and an element π_p of $\mathrm{QRep}^{\mathrm{WB}}(\mathcal{A},p)$ is said to be a

-*well-behaved* quasi *-representation of $\mathcal{A}$ in $\mathrm{QRep}(\mathcal{A},p)$.

A representable unbounded C^*-seminorm p on $\mathcal{A}$ having Property (B) is said to be

-*weakly semifinite* if $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p) \equiv \mathrm{QRep}^{\mathrm{WB}}(\mathcal{A},p) \cap \mathrm{Rep}(\mathcal{A},p) \neq \emptyset$.

We remark that semifinite unbounded m^*- or C^*-seminorms automatically satisfy Property (B).

Let π be a (quasi) *-representation of $\mathcal{A}$. We put

$$\mathcal{A}_b^\pi = \left\{x \in \mathcal{A} : \overline{\pi(x)} \in \mathcal{B}(\mathcal{H}_\pi)\right\}$$
$$\mathcal{N}_\pi \equiv \left\{x \in \mathcal{A}_b^\pi \cap R(\mathcal{A}) : ax \in \mathcal{A}_b^\pi \text{ for all } a \in \mathcal{A}\right\}.$$

Definition 2.9. If $\pi(\mathcal{A})\mathcal{D}(\pi)$ is total in $\mathcal{H}_\pi$, then π is said to be *non-degenerate.* If $\pi(\mathcal{N}_\pi)\mathcal{D}(\pi)$ is total in $\mathcal{H}_\pi$, then π is said to be *strongly non-degenerate.*

Proposition 2.10. *Let p be an unbounded C^*-seminorm on $\mathcal{A}$ having Property* (B)*. Then the following statements hold:*

(1) *We have*

$$\mathrm{QRep}^{\mathrm{WB}}(\mathcal{A},p) \subset \left\{\pi_p \in \mathrm{QRep}(\mathcal{A},p) : \Pi_p \textit{ is non-degenerate}\right\}$$
$$\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p) \subset \left\{\pi_p \in \mathrm{Rep}(\mathcal{A},p) : \Pi_p \textit{ is non-degenerate}\right\}.$$

In particular, if p is semifinite, then it is weakly semifinite and

$$\mathrm{QRep}^{\mathrm{WB}}(\mathcal{A},p) = \left\{\pi_p \in \mathrm{QRep}(\mathcal{A},p) : \Pi_p \textit{ is non-degenerate}\right\}$$
$$\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p) = \left\{\pi_p \in \mathrm{Rep}(\mathcal{A},p) : \Pi_p \textit{ is non-degenerate}\right\}.$$

(2) *Suppose $\pi_p \in \mathrm{QRep}^{\mathrm{WB}}(\mathcal{A},p)$. Then:*

(i) *$\pi_p(\mathcal{N}_p)\mathcal{D}(\pi_p)$ is total in $\mathcal{H}_{\pi_p}$, and so π_p is strongly non-degenerate.*

(ii) *$\|\overline{\pi_p(x)}\| = p(x)$ for each $x \in \mathcal{D}(p)$.*

(iii) *$\pi_p(\mathcal{A})'_{\mathrm{w}} = \overline{\pi_p(\mathcal{D}(p))}'$ and $\pi_p(\mathcal{A})'_{\mathrm{w}}\mathcal{D}(\pi_p) \subset \mathcal{D}(\pi_p)$.*

Conversely, suppose $\pi_p \in \mathrm{QRep}(\mathcal{A},p)$ or $\pi_p \in \mathrm{Rep}(\mathcal{A},p)$ satisfy conditions (i) *and* (ii) *above. Then there exists an element π_p^{WB} of $\mathrm{QRep}^{\mathrm{WB}}(\mathcal{A},p)$ or $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)$, respectively, which is a restriction of π_p.*

Proof.

Statement (1): Take an arbitrary $\pi_p \in \mathrm{QRep}^{\mathrm{WB}}(\mathcal{A},p)$. Then since

$$\mathcal{D}(\pi_p) \subset \mathrm{Lin}\,\Pi_p(\mathcal{A}_p)\mathcal{H}_{\Pi_p} \subset \mathcal{H}_{\Pi_p} = \mathcal{H}_{\pi_p}$$

it follows that Π_p is non-degenerate. Suppose p is semifinite. Let $\Pi_p \in \mathrm{Rep}(\mathcal{A}_p)$ be non-degenerate. Since p is semifinite, it follows that $\{\Pi_p(\tilde{x}) : x \in \mathcal{N}_p\}$ is uniformly dense in the C^*-algebra $\Pi_p(\mathcal{A}_p)$, which implies $\mathcal{H}_{\Pi_p} = \mathcal{H}_{\pi_p}$.

Statement (2): Let $\pi_p \in \mathrm{QRep}^{\mathrm{WB}}(\mathcal{A},p)$. Since $\mathcal{H}_{\pi_p} = \mathcal{H}_{\Pi_p}$ and $\pi_p(x) = \Pi_p(\tilde{x}) \upharpoonright \mathcal{D}(\pi_p)$ for each $x \in \mathcal{D}(p)$, it follows that $\pi_p(\mathcal{N}_p)\mathcal{D}(\pi_p)$ is total in $\Pi_p(\widetilde{\mathcal{N}_p})\mathcal{H}_{\Pi_p}$ and $\mathcal{N}_p \subset \mathcal{N}_{\pi_p}$, which implies by Lemma 2.4/(3) that statement (i) holds. Further, we have

$$\overline{\pi_p(x)} = \Pi_p(\tilde{x}) \qquad (x \in \mathcal{D}(p)) \tag{2.2}$$

and hence

$$\|\overline{\pi_p(x)}\| = \|\Pi_p(\tilde{x})\| = p(x) \qquad (x \in \mathcal{D}(p)).$$

We next show statement (iii). Take an arbitrary $C \in \overline{\pi_p(\mathcal{D}(p))}'$. By (2.1) we have

$$C\Pi_p(\tilde{x}) = C\overline{\pi_p(x)} = \overline{\pi_p(x)}C = \Pi_p(\tilde{x})C \qquad (x \in \mathcal{D}(p))$$

which implies that $C\Pi_p(\tilde{x_1}\tilde{x_2})\xi \in \mathcal{D}(\pi_p)$ for each $x_1, x_2 \in \mathcal{N}_p$ and $\xi \in \mathcal{H}_{\Pi_p}$ and

$$\begin{aligned}\pi_p(a)C\Pi_p(\tilde{x_1}\tilde{x_2})\xi &= \pi_p(a)\Pi_p(\tilde{x_1}\tilde{x_2})C\xi \\ &= \Pi_p((ax_1)^{\sim})C\Pi_p(\tilde{x_2})\xi \\ &= C\Pi_p((ax_1)^{\sim})\Pi_p(\tilde{x_2})\xi \\ &= C\pi_p(a)\Pi_p(\tilde{x_1}\tilde{x_2})\xi\end{aligned}$$

for each $a \in \mathcal{A}$, $x_1, x_2 \in \mathcal{N}_p$ and $\xi \in \mathcal{H}_{\Pi_p}$. Hence $C \in \pi_p(\mathcal{A})'_{\mathrm{w}}$ and $C\mathcal{D}(\pi_p) \subset \mathcal{D}(\pi_p)$. The converse inclusion $\pi_p(\mathcal{A})'_{\mathrm{w}} \subset \overline{\pi_p(\mathcal{D}(p))}'$ is trivial. Thus statement (iii) holds.

Conversely, suppose that $\pi_p \in \mathrm{QRep}(\mathcal{A}, p)$ satisfies conditions (i) and (ii). We put

$$\Pi_p^{\mathrm{WB}}(\tilde{x}) = \overline{\pi_p(x)} \qquad (x \in \mathcal{D}(p)).$$

Then it follows from (ii) that

$$\|\Pi_p^{\mathrm{WB}}(\tilde{x})\| = \|\overline{\pi_p(x)}\| = p(x) = \|\tilde{x}\|_p \qquad (x \in \mathcal{D}(p))$$

and hence Π_p^{WB} can be extended to a faithful *-representation of the C^*-algebra $\mathcal{A}_p$ on the Hilbert space $\mathcal{H}_{\Pi_p^{\mathrm{WB}}} = \mathcal{H}_{\pi_p}$. We denote it by the same symbol Π_p^{WB} and denote by π_p^{WB} the quasi *-representation of $\mathcal{A}$ induced by Π_p^{WB}. Then it follows from Lemma 2.4/(3) and statement (i) that

$$\mathcal{H}_{\pi_p^{\mathrm{WB}}} = \overline{\mathrm{Lin}}\Pi_p^{\mathrm{WB}}(\tilde{\mathcal{N}_p})\mathcal{H}_{\Pi_p^{\mathrm{WB}}} = \overline{\mathrm{Lin}\overline{\pi_p(\mathcal{N}_p)}}\mathcal{H}_{\pi_p} = \mathcal{H}_{\pi_p} = \mathcal{H}_{\Pi_p^{\mathrm{WB}}}$$

so that $\pi_p^{\mathrm{WB}} \in \mathrm{QRep}^{\mathrm{WB}}(\mathcal{A}, p)$. Further, since

$$\Pi_p^{\mathrm{WB}}(\tilde{x}) = \overline{\pi_p(x)} = \Pi_p(\tilde{x})\lceil\mathcal{H}_{\pi_p^{\mathrm{WB}}} \qquad (x \in \mathcal{D}(p))$$

it follows that π_p^{WB} is a restriction of π_p. Suppose $\pi_p \in \mathrm{Rep}(\mathcal{A}, p)$. Then, since π_p^{WB} is a restriction of π_p, it follows that π_p^{WB} is a *-representation of $\mathcal{A}$ ∎

The set $\Sigma_B(\mathcal{A})$ of all unbounded C^*-seminorms on $\mathcal{A}$ having Property (B) is an ordered set with respect to the order relation $\subset$ defined by

$$p \subset q \quad \Longleftrightarrow \quad \mathcal{D}(p) \subset \mathcal{D}(q) \text{ and } p(x) = q(x)\ \forall x \in \mathcal{D}(p).$$

Proposition 2.11. *Let p and q be in $\Sigma_B(\mathcal{A})$. Suppose $p \subset q$. Then, for any $\pi_p \in \mathrm{QRep}(\mathcal{A}, p)$ there exists an element π_q of $\mathrm{QRep}(\mathcal{A}, q)$ such that $\pi_p \subset \pi_q$.*

Proof. Let $\mathcal{A}_q$ be the C^*-algebra constructed applying Lemma 2.2 to $\mathcal{D}(q)$. Then it follows from $p \subset q$ that for each $x \in \mathcal{D}(p)$ we can define

$$\Phi : \tilde{x} \in \widetilde{\mathcal{D}(p)} \mapsto \tilde{x} \in \widetilde{\mathcal{D}(q)}.$$

Then Φ is an isometric *-isomorphism of the dense subspace $\widetilde{\mathcal{D}(p)}$ of the C^*-algebra $\mathcal{A}_p$ into the C^*-algebra $\mathcal{A}_q$, and so it can be extended to a *-isomorphism of the C^*-algebra $\mathcal{A}_p$ into the C^*-algebra $\mathcal{A}_q$; we denote this extension by the same symbol Φ.

Take an arbitrary $\Pi_p \in \mathrm{Rep}(\mathcal{A}_p)$. Since $\Pi_p \circ \Phi^{-1}$ is a faithful *-representation of the C^*-algebra $\Phi(\mathcal{A}_p)$ on $\mathcal{H}_{\Pi_p}$ and every C^*-algebra is stable [14: Proposition 2.10.2], it follows that $\Pi_p \circ \Phi^{-1}$ can be extended to a *-representation Π_q of the C^*-algebra $\mathcal{A}_q$ on $\mathcal{H}_{\Pi_q}$, that is, $\mathcal{H}_{\Pi_p}$ is a closed subspace of $\mathcal{H}_{\Pi_q}$ and $\Pi_q(\Phi(A))\lceil\mathcal{H}_{\Pi_p} = \Pi_p(A)$ for each $A \in \mathcal{A}_p$. Let π_q denote the element of $\mathrm{QRep}(\mathcal{A}, q)$ induced by Π_q. Then we have

$$\begin{aligned}
\pi_p(a)\Pi_p(\tilde{x_1}\tilde{x_2})\xi &= \Pi_p((ax_1)^{\sim})\Pi_p(\tilde{x_2})\xi \\
&= \Pi_q(\Phi((ax_1)^{\sim}))\Pi_q(\Phi(\tilde{x_2}))\xi \\
&= \Pi_q((ax_1)^{\sim}\tilde{x_2})\xi \\
&= \pi_q(a)\Pi_q(\tilde{x_1}\tilde{x_2})\xi \\
&= \pi_q(a)\Pi_p(\tilde{x_1}\tilde{x_2})\xi
\end{aligned}$$

for each $a \in \mathcal{A}$, $x_1, x_2 \in \mathcal{N}_p$ and $\xi \in \mathcal{H}_{\Pi_p}$, and so $\pi_p \subset \pi_q$ ∎

3. Representability of unbounded C^*-seminorms

Let $\mathcal{A}$ be a partial *-algebra and p an unbounded C^*-seminorm on $\mathcal{A}$. In this section we give some conditions under which the equality $\mathrm{Rep}(\mathcal{A}, p) = \mathrm{QRep}(\mathcal{A}, p)$ holds. The first case we consider is that of a semi-associative partial *-algebra $\mathcal{A}$.

Lemma 3.1. *Suppose $\mathcal{A}$ is a semi-associative partial *-algebra and $p \in \Sigma_B(\mathcal{A})$. Then* $\mathrm{Rep}(\mathcal{A}, p) = \mathrm{QRep}(\mathcal{A}, p)$.

Proof. Since $\mathcal{A}$ is semi-associative, it follows that

$$y^*((ab)x) = y^*(a(bx)) = (y^*a)(bx)$$

for each $a \in L(b)$ and $x, y \in \mathcal{N}_p$ which implies

$$\begin{aligned}
&\big(\pi_p(ab)\Pi_p(\tilde{x_1}\tilde{x_2})\xi|\Pi_p(\tilde{y_1}\tilde{y_2})\eta\big) \\
&\quad = \big(\Pi_p((y_1^*((ab)x_1))^{\sim})\Pi_p(\tilde{x_2})\xi|\Pi_p(\tilde{y_2})\eta\big) \\
&\quad = \big(\Pi_p((y_1^*a)^{\sim}(bx_1)^{\sim})\Pi_p(\tilde{x_2})\xi|\Pi_p(\tilde{y_2})\eta\big) \\
&\quad = \big(\pi_p(b)\Pi_p(\tilde{x_1}\tilde{x_2})\xi|\pi_p(a^*)\Pi_p(\tilde{y_1}\tilde{y_2})\eta\big)
\end{aligned}$$

for each $x_1, x_2, y_1, y_2 \in \mathcal{N}_p$ and $\xi, \eta \in \mathcal{H}_{\Pi_p}$. Hence π_p is a *-representation of $\mathcal{A}$ ∎

We next consider the case of (everywhere defined) C^*-seminorms. Semi-associativity of $\mathcal{A}$ is no more needed.

Lemma 3.2. *Let $\mathcal{A}$ be a partial *-algebra. Suppose p is a semifinite C^*-seminorm on $\mathcal{A}$. Then* $\mathrm{Rep}(\mathcal{A},p) = \mathrm{QRep}(\mathcal{A},p)$ *and every π_p in* $\mathrm{Rep}(\mathcal{A},p)$ *is bounded.*

Proof. Since p is a C^*-seminorm on $\mathcal{A}$, we have $\mathcal{D}(p) = \mathcal{A}$ and $\mathcal{N}_p = R(\mathcal{A})$. For any $a \in \mathcal{A}$ we have $\pi_p(a) = \Pi_p(\tilde{a}) \upharpoonright \mathcal{D}(\pi_p)$, and so $\pi_p(a)$ is bounded. Take arbitrary $a, b \in \mathcal{A}$ such that $a \in L(b)$. Then there exist sequences $\{x_n\}, \{y_n\} \in R(\mathcal{A})$ such that $\{x_n\}^\sim = \tilde{a}$ and $\{y_n\}^\sim = \tilde{b}$, and hence it follows from Lemma 2.2/(2) and Property (A) that

$$\begin{aligned}
&\big(\pi_p(ab)\Pi_p(\tilde{x_1}\tilde{x_2})\xi|\Pi_p(\tilde{y_1}\tilde{y_2})\eta\big)\\
&\quad= \big(\Pi_p(\{x_n y_n\}^\sim \tilde{x_1})\Pi_p(\tilde{x_2})\xi|\Pi_p(\tilde{y_1}\tilde{y_2})\eta\big)\\
&\quad= \big(\Pi_p(\{x_n\}^\sim)\Pi_p(\{y_n\}^\sim)\Pi_p(\tilde{x_1}\tilde{x_2})\xi|\Pi_p(\tilde{y_1}\tilde{y_2})\eta\big)\\
&\quad= \big(\pi_p(b)\Pi_p(\tilde{x_1}\tilde{x_2})\xi|\pi_p(a^*)\Pi_p(\tilde{y_1}\tilde{y_2})\eta\big)
\end{aligned}$$

for each $x_1, x_2, y_1, y_2 \in \mathcal{N}_p$ and $\xi, \eta \in \mathcal{H}_{\Pi_p}$. Hence π_p is a *-representation of $\mathcal{A}$ ∎

Lemma 3.3. *Let $\mathcal{A}$ be a partial *-algebra $\mathcal{A}$ and $p \in \Sigma_B(\mathcal{A})$. Assume there exists a semifinite C^*-seminorm $\hat{p}$ on $\mathcal{A}$ such that $p \subset \hat{p}$. Then* $\mathrm{Rep}(\mathcal{A},p) = \mathrm{QRep}(\mathcal{A},p)$.

Proof. Take an arbitrary $\pi_p \in \mathrm{QRep}(\mathcal{A},p)$. By Proposition 2.11 and Lemma 3.2 there exists an element $\pi_{\hat{p}}$ of $\mathrm{QRep}(\mathcal{A},\hat{p}) = \mathrm{Rep}(\mathcal{A},\hat{p})$ such that $\pi_p \subset \pi_{\hat{p}}$ which implies $\pi_p \in \mathrm{Rep}(\mathcal{A},p)$ ∎

We consider now the special case of topological partial *-algebras. The simplest situation is of course that of topological quasi *-algebras, where we start from.

Lemma 3.4. *Suppose $\mathcal{A}$ is a topological quasi *-algebra over $\mathcal{A}_0$ and p is an unbounded C^*-seminorm on $\mathcal{A}$ having Property* (B). *Then* $\mathrm{Rep}(\mathcal{A},p) = \mathrm{QRep}(\mathcal{A},p)$.

Proof. Since every topological quasi *-algebra $\mathcal{A}$ over $\mathcal{A}_0$ is semi-associative and $R(\mathcal{A}) = \mathcal{A}_0$, it follows from Lemma 3.1 that $\mathrm{Rep}(\mathcal{A},p) = \mathrm{QRep}(\mathcal{A},p)$ ∎

Let $\mathcal{A}[\tau]$ be a topological partial *-algebra and p an unbounded C^*-seminorm on $\mathcal{A}$. For any $x \in \mathcal{N}_p$ we define a seminorm p_x on $\mathcal{A}$ by

$$p_x(a) = p(ax) \qquad (a \in \mathcal{A}).$$

We denote by τ_p the locally convex topology on $\mathcal{A}$ defined by the family $\{p_x : x \in \mathcal{N}_p\}$ of seminorms. If $\tau_p \prec \tau$, then p is said to be *locally continuous.*

Lemma 3.5. *Let $\mathcal{A}[\tau]$ be a topological partial *-algebra satisfying the following condition*

(C) *For any $a \in \mathcal{A}$, the map $L_a : R(a) \to \mathcal{A}$, $x \mapsto ax$ is continuous.*

Suppose p is a locally continuous unbounded C^*-seminorm on $\mathcal{A}$ having Property (B) *and $R(\mathcal{A}) \cap \mathcal{D}(p)$ is τ-dense in $\mathcal{A}$. Then* $\mathrm{Rep}(\mathcal{A},p) = \mathrm{QRep}(\mathcal{A},p)$.

Proof. Take arbitrary $a, b \in \mathcal{A}$ such that $a \in L(b)$. Since $R(\mathcal{A}) \cap \mathcal{D}(p)$ is τ-dense in $\mathcal{A}$, there exists a net $\{y_\beta\}$ in $R(\mathcal{A}) \cap \mathcal{D}(p)$ such that $\tau - \lim_\beta y_\beta = b$. Further, since

$\mathcal{A}$ satisfies condition (C) we have $\tau - \lim_\beta ay_\beta = ab$, and since p is locally continuous, it follows that $\lim_\beta p(y_\beta x - bx) = 0$ and $\lim_\beta p((ay_\beta)x - (ab)x) = 0$ for each $x \in \mathcal{N}_p$. Hence we have

$$\begin{aligned}
&\big(\pi_p(b)\Pi_p(\tilde{x_1}\tilde{x_2})\xi|\pi_p(a^*)\Pi_p(\tilde{y_1}\tilde{y_2})\eta\big)\\
&\quad= (\Pi_p((bx_1)^{\sim}\tilde{x_2})\xi|\pi_p(a^*)\Pi_p(\tilde{y_1}\tilde{y_2})\eta\big)\\
&\quad= \lim_\beta \big(\Pi_p((y_\beta x_1)^{\sim}\tilde{x_2})\xi|\pi_p(a^*)\Pi_p(\tilde{y_1}\tilde{y_2})\eta\big)\\
&\quad= \lim_\beta \big(\Pi_p(\tilde{y_\beta}\tilde{x_1})\Pi_p(\tilde{x_2})\xi|\pi_p(a^*)\Pi_p(\tilde{y_1}\tilde{y_2})\eta\big)\\
&\quad= \lim_\beta \big(\Pi_p((ay_\beta)^{\sim}\tilde{x_1})\Pi_p(\tilde{x_2})\xi|\Pi_p(\tilde{y_1}\tilde{y_2})\eta\big)\\
&\quad= \big(\Pi_p(((ab)x_1)^{\sim})\Pi_p(\tilde{x_2})\xi|\Pi_p(\tilde{y_1}\tilde{y_2})\eta\big)\\
&\quad= \big(\pi_p(ab)\Pi_p(\tilde{x_1}\tilde{x_2})\xi|\Pi_p(\tilde{y_1}\tilde{y_2})\eta\big)
\end{aligned}$$

for each $x_1, x_2, y_1, y_2 \in \mathcal{N}_p$ and $\xi, \eta \in \mathcal{H}_{\Pi_p}$, which implies π_p is a *-representation of $\mathcal{A}$ ∎

4. Unbounded C^*-seminorms defined by *-representations

In the previous sections we constructed *-representations of a partial *-algebra $\mathcal{A}$ from a representable unbounded C^*-seminorm on $\mathcal{A}$ having Property (B). Now, starting from a *-representation π of $\mathcal{A}$, we try to construct a representable unbounded C^*-seminorm $r_\pi \in \Sigma_B(\mathcal{A})$. When this is possible, it makes sense to investigate on the relation between π and the natural *-representation $\pi^N_{r_\pi}$ of $\mathcal{A}$ induced by r_π

Let π be a *-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\pi$. We put, as above,

$$\mathcal{A}^\pi_b = \big\{x \in \mathcal{A} : \overline{\pi(x)} \in \mathcal{B}(\mathcal{H}_\pi)\big\}$$

and

$$\pi_b(x) = \overline{\pi(x)} \qquad (x \in \mathcal{A}^\pi_b).$$

Then $\mathcal{A}^\pi_b$ is a partial *-subalgebra of $\mathcal{A}$ and π_b is a bounded *-representation of $\mathcal{A}^\pi_b$ on $\mathcal{H}_\pi$. An unbounded C^*-seminorm r^L_π on $\mathcal{A}$ is defined by

$$\mathcal{D}(r^L_\pi) = \mathcal{A}^\pi_b \qquad \text{and} \qquad r^L_\pi(x) = \|\pi_b(x)\| \;\; (x \in \mathcal{D}(r^L_\pi)).$$

But r^L_π does not necessarily have Property (B). For this reason, we consider the family of all unbounded C^*-seminorms on $\mathcal{A}$ having Property (B) which are restrictions of r^L_π. We denote this family by $\Sigma_B(\pi)$ and call it the *family of unbounded C^*-seminorms induced by* π.

Definition 4.1. If $\Sigma_B(\pi) \neq \{0\}$, then π is said to have *Property* (B).

Suppose that π has Property (B) and $r_\pi \in \Sigma_B(\pi)$. We put

$$\Pi(\tilde{x}) = \pi(x) \qquad (x \in \mathcal{D}(r_\pi)).$$

Then since

$$\|\Pi(\tilde{x})\| = r_\pi(x) = \|\tilde{x}\|_{r_\pi} \qquad (x \in \mathcal{D}(r_\pi))$$

it follows that Π can be extended to a faithful *-representation $\Pi^N_{r_\pi}$ of the C^*-algebra $\mathcal{A}_{r_\pi} \equiv \widehat{\mathcal{D}(r_\pi)}/\sim$ on the Hilbert space $\mathcal{H}_\pi$, and $\Pi^N_{r_\pi}(\mathcal{A}_{r_\pi}) = \overline{\overline{\pi(\mathcal{D}(r_\pi))}}^{\|\ \|}$. We denote by $\pi^N_{r_\pi}$ the quasi *-representation of $\mathcal{A}$ constructed by $\Pi^N_{r_\pi}$. This is called the *natural representation of* $\mathcal{A}$ *induced by* π. Since $\mathcal{H}_{\Pi^N_{r_\pi}} = \mathcal{H}_\pi$, it follows that $\mathcal{H}_{\pi^N_{r_\pi}}$ is a closed subspace of $\mathcal{H}_\pi$.

Proposition 4.2. *Suppose that π is a *-representation of $\mathcal{A}$ having Property* (B) *and $r_\pi \in \Sigma_B(\pi)$. Then r_π is representable, $\pi^N_{r_\pi} \in \mathrm{Rep}(\mathcal{A}, r_\pi)$ and $\hat{\pi}\lceil\mathcal{D}(\pi^N_{r_\pi}) = \pi^N_{r_\pi}$.*

Proof. Since

$$\begin{aligned}\mathcal{D}(\pi^N_{r_\pi}) &= \mathrm{Lin}\big\{\Pi^N_{r_\pi}(\tilde{x_1}\tilde{x_2})\xi :\ x_1, x_2 \in \mathcal{N}_{r_\pi}, \xi \in \mathcal{H}_\pi\big\}\\ &= \mathrm{Lin}\big\{\overline{\pi(x_1x_2)}\xi :\ x_1, x_2 \in \mathcal{N}_{r_\pi}, \xi \in \mathcal{H}_\pi\big\}\end{aligned}$$

it follows that

$$\begin{aligned}\big(\pi(a)^*\eta|\Pi^N_{r_\pi}(\tilde{x_1}\tilde{x_2})\xi\big) &= \big(\pi(a)^*\eta|\overline{\pi(x_1x_2)}\xi\big)\\ &= \big(\overline{\pi((ax_1)^*)}\eta|\overline{\pi(x_2)}\xi\big)\\ &= \big(\eta|\overline{\pi((ax_1)x_2)}\xi\big)\\ &= \big(\eta|\pi^N_{r_\pi}(a)\Pi^N_{r_\pi}(\tilde{x_1}\tilde{x_2})\xi\big)\end{aligned}$$

for each $a \in \mathcal{A}$, $\eta \in \mathcal{D}(\pi(a)^*)$, $x_1, x_2 \in \mathcal{N}_{r_\pi}$ and $\xi \in \mathcal{H}_\pi$, which implies that

$$\begin{aligned}&\Pi^N_{r_\pi}(\tilde{x_1}\tilde{x_2})\xi \in \mathcal{D}(\overline{\pi(a)})\\ &\overline{\pi(a)}\Pi^N_{r_\pi}(\tilde{x_1}\tilde{x_2})\xi = \pi^N_{r_\pi}(a)\Pi^N_{r_\pi}(\tilde{x_1}\tilde{x_2})\xi.\end{aligned}$$

Hence, $\mathcal{D}(\pi^N_{r_\pi}) \subset \mathcal{D}(\hat{\pi})$ and $\hat{\pi}\lceil\mathcal{D}(\pi^N_{r_\pi}) = \pi^N_{r_\pi}$, which implies since $\hat{\pi}$ is a *-representation of $\mathcal{A}$ that $\pi^N_{r_\pi}$ is a *-representation of $\mathcal{A}$ and r_π is representable ∎

We summarize in the following scheme the method of construction of $\pi^N_{r_\pi}$ described above:

Proposition 4.3. *Suppose p is a representable weakly semifinite unbounded C^*-seminorm on $\mathcal{A}$ having Property* (B)*. Then every π_p of* $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ *has Property* (B)*.*

Take an arbitrary $r_{\pi_p} \in \Sigma_B(\pi_p)$ *which is an extension of* p. *Then* $\pi_p \subset \pi^N_{r_{\pi_p}}$ *and* $\hat{\pi}_p = \hat{\pi}_{r^N_{\pi_p}}$.

Proof. Since $p \in \Sigma_B(\mathcal{A})$ and $p \subset r_{\pi_p} \subset r^N_{\pi_p}$, it follows that π_p has Property (B) and $\mathcal{N}_p \subset \mathcal{N}_{r_{\pi_p}} \subset \mathcal{A}_b^{\pi_p}$, which implies

$$\begin{aligned}
\mathcal{D}(\pi_p) &= \overline{\mathrm{Lin}\{\pi_p(x_1x_2)\xi : x_1, x_2 \in \mathcal{N}_p \text{ and } \xi \in \mathcal{H}_{\pi_p}\}} \\
&\subset \overline{\mathrm{Lin}\{\pi_p(x_1x_2)\xi : x_1, x_2 \in \mathcal{N}_{r_{\pi_p}} \text{ and } \xi \in \mathcal{H}_{\pi_p}\}} \\
&= \mathcal{D}(\pi^N_{\pi_p}) \\
&\subset \mathcal{H}_{\pi_p}
\end{aligned}$$

and $\pi_p = \pi^N_{r_{\pi_p}} \lceil \mathcal{D}(\pi_p)$. On the other hand, it follows from Proposition 4.2 that $\pi^N_{r_{\pi_p}} \subset \hat{\pi}_p$. Hence it follows that $\mathcal{H}_{\pi_p} = \mathcal{H}_{\pi^N_{r_{\pi_p}}}, \pi_p \subset \pi^N_{r_{\pi_p}}$ and $\hat{\pi}_p = \hat{\pi}^N_{r_{\pi_p}}$ ∎

5. Examples

In this section we give some examples of unbounded C^*-seminorms on partial *-algebras having Property (B).

Example 5.1. Let S be a vector space of complex sequences containing l^∞. Suppose that $\{x_n\}^* \equiv \{\overline{x_n}\} \in S$ if $\{x_n\} \in S$ and S is l^∞-module. Then S is a partial *-algebra under the following partial multiplication and involution: $\{x_n\} \in L(\{y_n\})$ if and only if $\{x_ny_n\} \in S$ and $\{x_n\}^* = \{\overline{x_n}\}$, and it has Property (A) and $R(S) \supset l^\infty$. We define an unbounded C^*-norm on S having Property (B) by

$$\mathcal{D}(r_\infty) = l^\infty \qquad \text{and} \qquad r_\infty(\{x_n\}) = \|\{x_n\}\|_\infty \quad (\{x_n\} \in \mathcal{D}(r_\infty)).$$

For any $\{x_n\} \in l^\infty$ we put

$$\Pi_{r_\infty}(\{x_n\})\{y_n\} = \{x_ny_n\} \qquad (\{y_n\} \in l^2).$$

Then Π_{r_∞} is a faithful *-representation of the C^*-algebra l^∞ on the Hilbert space l^2 and since

$$\mathcal{N}_{r_\infty} \supset \Big\{\{x_n\} \in S : x_n \neq 0 \text{ for only finite numbers } n\Big\}$$

it follows that $\Pi_{r_\infty}(\mathcal{N}^2_{r_\infty})\mathcal{H}_{\Pi_{r_\infty}}$ is total in $\mathcal{H}_{\Pi_{r_\infty}}$. Hence r_∞ is weakly semifinite.

Example 5.2. Let $C(\mathbb{R})$ be a *-algebra of all continuous complex-valued functions on $\mathbb{R}$ equipped with the usual operations $f+g, \lambda f, fg$ and the involution $f^* : f^*(t) = \overline{f(t)}$ $(t \in \mathbb{R})$. Let $\mathcal{A}$ be a *-vector subspace of $C(\mathbb{R})$. Then $\mathcal{A}$ is a partial *-algebra having Property (A) under the following partial multiplication: $f \in L(g)$ if and only if $fg \in \mathcal{A}$. Concrete examples of partial *-algebras of this kind are, for instance:

(1) $\mathcal{A} = C(\mathbb{R}) \cap L^p(\mathbb{R})$ $(1 \leq p < \infty)$.

We here define an unbounded C^*-norm on $\mathcal{A}$ by

$$\mathcal{D}(r_\infty) = C_c(\mathbb{R}) \equiv \Big\{ f \in C(\mathbb{R}) : \mathrm{supp} f \text{ compact } \Big\}$$

and

$$r_\infty(f) = \sup_{t\in\mathbb{R}} |f(t)| \quad (f \in \mathcal{D}(r_\infty)).$$

Then since $\mathcal{D}(r_\infty) \subset R(\mathcal{A})$, it follows that r_∞ has Property (B). Further,

$$\mathcal{A}_{r_\infty} = C_0(\mathbb{R}) \equiv \Big\{ f \in C(\mathbb{R}) : \lim_{|t|\to\infty} |f(t)| = 0 \Big\}$$

and a faithful *-representation Π_{r_∞} of the C^*-algebra $C_0(\mathbb{R})$ on $L^2(\mathbb{R})$ is defined by

$$\Pi_{r_\infty}(f)g = fg \qquad (f \in C_0(\mathbb{R}), \quad g \in L^2(\mathbb{R})).$$

Since $\mathcal{N}_{r_\infty} \supset C_c(\mathbb{R})$, it follows that $\Pi_{r_\infty}(\mathcal{N}^2_{r_\infty})L^2(\mathbb{R})$ is total in $L^2(\mathbb{R})$, which means that r_∞ is weakly semifinite.

(2) $\mathcal{A} = \mathcal{A}_n \equiv \{ f \in C(\mathbb{R}) : \sup_{t\in\mathbb{R}} \frac{|f(t)|}{(1+t^2)^n} < \infty \}$ $(n \in \mathbb{N})$.
We define an unbounded C^*-norm r_∞ on $\mathcal{A}$ by

$$\mathcal{D}(r_\infty) = C_b(\mathbb{R}) \equiv \Big\{ f \in C(\mathbb{R}) : \ f \text{ bounded} \Big\}$$

and

$$r_\infty(f) = \sup_{t\in\mathbb{R}} |f(t)| \quad (f \in \mathcal{D}(r_\infty)).$$

Then it is proved similarly to (1) that r_∞ is a weakly semifinite unbounded C^*-norm on $\mathcal{A}$.

Similarly we have the following

Example 5.3.

(1) Let $C^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ $(1 \leq p < \infty)$ where $C^\infty(\mathbb{R})$ is a *-algebra of all infinitely differentiable complex functions on $\mathbb{R}$. We put

$$\mathcal{D}(r_\infty) = C_c^\infty(\mathbb{R}) \qquad \text{and} \qquad r_\infty(f) = \sup_{t\in\mathbb{R}} |f(t)| \quad (f \in \mathcal{D}(r_\infty)).$$

Then r_∞ is a weakly semifinite unbounded C^*-norm on $C^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ having Property (B).

(2) Let $n \in \mathbb{N}$ and $\mathcal{A}_n = \{ f \in C^\infty(\mathbb{R}) : \sup_{t\in\mathbb{R}} \frac{|f(t)|}{(1+t^2)^n} < \infty \}$. We put

$$\mathcal{D}(r_\infty) = C_b^\infty(\mathbb{R}) \qquad \text{and} \qquad r_\infty(f) = \sup_{t\in\mathbb{R}} |f(t)| \quad (f \in \mathcal{D}(r_\infty)).$$

Then $\mathcal{A}_n$ is a partial *-algebra having Property (A) and r_∞ is a weakly semifinite unbounded C^*-norm on $\mathcal{A}_n$ having Property (B).

Example 5.4. Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$ and $C(\mathcal{H})$ the C^*-algebra of all compact operators on $\mathcal{H}$. Suppose that the maximal partial O^*-algebra $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is self-adjoint. Then $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ has Property (A) and

$$R^{\mathrm{w}}(\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})) = \Big\{X\lceil\mathcal{D} : \overline{X} \in \mathcal{B}(\mathcal{H}) \text{ and } \overline{X}\mathcal{H} \subset \mathcal{D}\Big\}.$$

We now define an unbounded C^*-norm r_u on the maximal O^*-algebra $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ by

$$\mathcal{D}(r_u) = C(\mathcal{H})\lceil\mathcal{D} \qquad \text{and} \qquad r_u(X) = \|\overline{X}\| \ \ (X \in \mathcal{D}(r_u)).$$

Since

$$F(\mathcal{D},\mathcal{H}) \equiv \mathrm{Lin}\big\{\xi \otimes \overline{y} : \xi \in \mathcal{D} \text{ and } y \in \mathcal{H}\big\} \subset R^{\mathrm{w}}(\mathcal{L}^\dagger(\mathcal{D},\mathcal{H}))$$

where $(x \otimes \overline{y})z = (z|y)x$ for $x, y, z \in \mathcal{H}$ and $F(\mathcal{D},\mathcal{H})$ is uniformly dense in $C(\mathcal{H})$, it follows that r_u has Property (B). Further, since $\mathcal{N}_{r_\infty} \supset F(\mathcal{D},\mathcal{H})$, it follows that r_u is semifinite.

Example 5.5. Let $\mathcal{M}_0$ be an O^*-algebra on the Schwartz space $\mathcal{S}(\mathbb{R})$ and $N = \sum_{n=0}^{\infty}(n+1)f_n \otimes \overline{f_n}$ the number operator, where $\{f_n\} \subset \mathcal{S}(\mathbb{R})$ is an orthonormal basis in $L^2(\mathbb{R})$ consisting the normalized Hermite functions. Let $\mathcal{M}$ be a partial O^*-algebra on $\mathcal{S}(\mathbb{R})$ containing $\mathcal{M}_0$ and $\{f_n \otimes \overline{f_m} : m, n \in \mathbb{N}_0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Since $\mathcal{M}$ is self-adjoint, it follows that $\mathcal{M}$ has Property (A). We define an unbounded C^*-norm on $\mathcal{M}$ by

$$\begin{aligned} &\mathcal{D}(r_u) = \mathrm{Lin}\big\{Af_n \otimes \overline{Bf_m} : A, B \in \mathcal{M} \text{ and } n, m \in \mathbb{N}_0\big\} \\ &r_u(X) = \|\overline{X}\| \ \ (X \in \mathcal{D}(r_u)). \end{aligned}$$

Then

$$\mathrm{Lin}\big\{f_n \otimes \overline{Bf_m} : B \in \mathcal{M} \text{ and } n, m \in \mathbb{N}_0\big\}$$

is contained in $\mathcal{N}_{r_u}$ and is uniformly dense in $\mathcal{D}(r_u)$. Hence r_u has Property (B) and it is semifinite.

Example 5.6. Let $(\mathcal{A}, \mathcal{A}_0)$ be a proper CQ^*-algebra [7, 8], i.e. a topological quasi *-algebra $(\mathcal{A}[\tau], \mathcal{A}_0)$ such that:

a) $\mathcal{A}[\tau]$ is a Banach space under the norm $\|\cdot\|$.

b) The involution * of $\mathcal{A}$ is isometric, i.e. $\|X\| = \|X^*\|$ for all $X \in \mathcal{A}$.

c) $\|X\|_0 = \max\{\|X\|_R, \|X^*\|_R\}$ where $\|X\|_R = \sup\{\|AX\| : \|A\| \leq 1\}$.

Of course, the C^*-norm on $\mathcal{A}_0$ can be viewed as an unbounded C^*-norm r with domain $\mathcal{D}(r) = \mathcal{A}_0$. Since $R(\mathcal{A}) = \mathcal{A}_0$, it is obvious that r satisfies property (B). In order to apply the results of Section 2 we have to consider the set

$$\mathcal{N}_r = \big\{X \in \mathcal{A}_0 : AX \in \mathcal{A}_0 \text{ for all } A \in \mathcal{A}\big\}.$$

Also in this simple situation, $\mathcal{N}_r$ might be trivial. Let us sketch a concrete case, where this does not happen.

Let S be an unbounded selfadjoint operator in Hilbert space $\mathcal{H}$, $S \geq 1$. The norm

$$\|X\|_S = \|S^{-1}XS^{-1}\| \qquad (X \in \mathcal{B}(\mathcal{H}))$$

defines a topology stricly weaker than the one defined by the C^*-norm of $\mathcal{B}(\mathcal{H})$. Let

$$C(S) = \big\{X \in \mathcal{B}(\mathcal{H}) : \, XS^{-1} = S^{-1}X\big\}.$$

$C(S)$ is a C^*-algebra under the norm of $\mathcal{B}(\mathcal{H})$ and its $\|\cdot\|_S$-completion $\widehat{C}(S)$ is a CQ^*-algebra on $C(S)$ [8: Proposition 2.6]. Now define

$$\mathcal{D}(r) = C(S) \qquad \text{and} \qquad r(X) = \|X\| \;\; (X \in C(S)).$$

Then r is an unbounded C^*-norm on $\widehat{C}(S)$ satisfying property (B). It is easy to check that

$$\mathcal{N}_r \supset \big\{X \in C(S) : \, X \text{ is of finite rank}\big\}.$$

So, for instance, if S has the spectral decomposition

$$S = \sum_{n=1}^{\infty} \lambda_n P_n$$

where the P_n's are finite rank projections, then $\mathcal{N}_r$ is non-trivial and the construction of Section 2 applies. We remark that r need not be semifinite but, by Lemma 3.4, any element of $\mathrm{QRep}(\mathcal{A}, p)$ is indeed a *-representation.

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