

Partial Regularity of Weak Solutions to Nonlinear Elliptic Systems Satisfying a Dini Condition

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Abstract. This paper is concerned with systems of nonlinear partial differential equations

$$-D_\alpha a_i^\alpha(x, u, \nabla u) = b_i(x, u, \nabla u) \quad (i = 1, \dots, N)$$

where the coefficients a_i^α are assumed to satisfy the condition

$$a_i^\alpha(x, u, \xi) - a_i^\alpha(y, v, \xi) \leq \omega |x - y| + |u - v| (1 + |\xi|)$$

for all $\{x, u\}, \{y, v\} \in \Omega \times \mathbb{R}^N$ and all $\xi \in \mathbb{R}^{nN}$, and where $\int_0^1 \frac{\omega(t)}{t} dt < +\infty$ while the functions $\frac{\partial a_i^\alpha}{\partial \xi_\beta^j}$ satisfy the standard boundedness and ellipticity conditions and the function $\xi \mapsto b_i(x, u, \xi)$ may have quadratic growth. With these assumptions we prove partial Hölder continuity of bounded weak solutions u to the above system provided the usual smallness condition on $\|u\|_{L^\infty(\Omega)}$ is fulfilled.

Keywords: *Nonlinear elliptic systems, partial regularity, blow-up method*

AMS subject classification: 35B65, 35K65

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ ($2 \leq n \in \mathbb{N}$) be an open and bounded set and $N \in \mathbb{N}$. In what follows a repeated Greek or Latin index implies summation over $1, \dots, n$ or $1, \dots, N$, respectively. We consider the nonlinear elliptic system

$$-D_\alpha a_i^\alpha(x, u, \nabla u) = b_i(x, u, \nabla u) \quad \text{in } \Omega \quad (i = 1, \dots, N) \quad (1.1)$$

where

$$\begin{aligned} u &= \{u^1, \dots, u^N\} \\ D_\alpha &= \frac{\partial}{\partial x_\alpha} \quad (\alpha = 1, \dots, n) \\ \nabla u &= \{D_\alpha u^i\}. \end{aligned}$$

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The coefficients a_i^α and b_i are assumed to satisfy the following conditions:

$$|a_i^\alpha(x, u, \xi) - a_i^\alpha(y, v, \xi)| \leq \omega(|x - y| + |u - v|)(1 + |\xi|) \tag{1.2}$$

for all $\{x, u\}, \{y, v\} \in \Omega \times \mathbb{R}^N$ and all $\xi \in \mathbb{R}^{nN}$ where $\omega : [0, \infty) \rightarrow [0, \infty)$ is non-decreasing, bounded and

$$\int_0^1 \frac{\omega(t)}{t} dt < +\infty \tag{1.3}$$

(that is, ω is assumed to fulfill the *Dini condition*; notice that it implies $\lim_{t \rightarrow 0} \omega(t) = 0$),

$$\left. \begin{aligned} \xi \mapsto a_i^\alpha(x, u, \xi) \text{ is differentiable on } \mathbb{R}^{nN} \text{ for all } \{x, u\} \in \Omega \times \mathbb{R}^N \\ \{x, u, \xi\} \mapsto \frac{\partial a_i^\alpha}{\partial \xi_\beta^j}(x, u, \xi) \text{ is continuous on } \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \end{aligned} \right\} \tag{1.4}$$

$$\left| \frac{\partial a_i^\alpha}{\partial \xi_\beta^j}(x, u, \xi) \right| \leq c_0 \text{ for all } \{x, u, \xi\} \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \tag{1.5}$$

for some constant $c_0 > 0$,

$$\frac{\partial a_i^\alpha}{\partial \xi_\beta^j}(x, u, \xi) \eta_\alpha^i \eta_\beta^j \geq \nu_0 |\eta|^2 \text{ for all } \{x, u\} \in \Omega \times \mathbb{R}^N \text{ and all } \xi, \eta \in \mathbb{R}^{nN} \tag{1.6}$$

for some constant $\nu_0 > 0$ ($\alpha, \beta = 1, \dots, n; i, j = 1, \dots, N$), and for all $M > 0$ there exists $a(M) > 0$ such that

$$|b_i(x, u, \xi)| \leq a(M)|\xi|^2 + b \text{ for all } \{x, u, \xi\} \in \Omega \times [-M, M]^N \times \mathbb{R}^{nN} \tag{1.7}$$

for some constant $b \geq 0$ ($i = 1, \dots, N$).

By $W^{1,2}(\Omega)$ we denote the usual Sobolev space. Further, we define $L^p(\Omega; \mathbb{R}^N) = [L^p(\Omega)]^N$, $W^{1,p}(\Omega; \mathbb{R}^N) = [W^{1,p}(\Omega)]^N$, etc.

Definition. A vector-valued function $u \in W^{1,2} \cap L^\infty(\Omega; \mathbb{R}^N)$ is called to be a *weak solution* of system (1.1) if

$$\int_\Omega a_i^\alpha(x, u, \nabla u) D_\alpha \varphi^i dx = \int_\Omega b_i(x, u, \nabla u) \varphi^i dx \tag{1.8}$$

for all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^N)$.

The following theorem is the main result of our paper.

Theorem. *Let conditions (1.2) – (1.7) be satisfied and let $u \in W^{1,2} \cap L^\infty(\Omega; \mathbb{R}^N)$ be a weak solution to system (1.1) such that*

$$2a(\|u\|_{L^\infty(\Omega)})\|u\|_{L^\infty(\Omega)} < \nu_0. \tag{1.9}$$

Then there exists an open set $\Omega_0 \subset \Omega$ such that $\text{meas}(\Omega \setminus \Omega_0) = 0$ and

$$u|_{\Omega_0} \in C^{0,\mu}(\Omega_0; \mathbb{R}^N) \tag{1.10}$$

for all $\mu \in (0, 1)$.

Remarks. The partial Hölder continuity of bounded weak solutions to quasilinear systems (i.e. $a_i^\alpha(x, u, \xi) = a_{ij}^{\alpha\beta}(x, u)\xi_j^\beta$) of type (1.1) has been proved in [7, 8] by using the *blow-up* method (notice that the monograph [6] also contains a direct proof of this result based on higher integrability of ∇u). On the other hand, partial Hölder continuity of weak solutions to system (1.1) with b_i satisfying the usual *controlled* growth condition is studied in [2] (cf. also [5] for elliptic systems of higher order). Our theorem above thus extends these results to fully nonlinear elliptic systems (1.1). Since the functions $\frac{\partial a_i^\alpha}{\partial \xi_j^\beta}$ are supposed to be only (not necessarily uniformly) continuous our result extends also Campanato’s Theorem 3.I in [3] where the author has proved partial Hölder continuity of u if $n \leq 4$ and the results obtained in [2] for the case $n > 4$.

The paper is organized as follows. In Section 2 we prove some technical lemmas which form the base for applying the blow-up method to prove partial regularity of bounded weak solutions to system (1.1). The novelty in our approach is the use of properties of the function ω (cf. (1.3)) which we present in Lemma 2.2. The key inequality for partial regularity is stated in Lemma 3.1. Its proof relies on the blow-up method, where again properties of ω play an essential role. The proof of our theorem is then given in Section 4. An appendix is devoted to the proof of a convergence result which we have used in Section 3.

2. Preliminaries

In this section we are going to present some lemmas which we will use in the sequel of the paper. We start with a result of higher integrability which is due to Giaquinta and Modica (cf. [6]) relying on an idea of Gehring [4]. For this, in the case of $v \in L^1(B_r)$ ($0 < r < +\infty$) we define the mean value $\int_{B_r} v \, dxdt = \frac{1}{\text{meas}(B_r)} \int_{B_r} v \, dxdt$.

Lemma 2.1. *Let conditions (1.2) and (1.4) – (1.7) be fulfilled and let $u \in W^{1,2} \cap L^\infty(\Omega; \mathbb{R}^N)$ be a weak solution of system (1.1) satisfying the smallness condition (1.9). Then there exists a real number $p > 2$ such that $\nabla u \in L^p_{loc}(\Omega; \mathbb{R}^{nN})$. In addition, for any concentric balls $B_{R/2} \subset B_R \subset \Omega$ ($0 < R \leq 1$) we have*

$$\left(\int_{B_{R/2}} (1 + |\nabla u|^2)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} \leq c \int_{B_R} (1 + |\nabla u|^2) dx \tag{2.1}$$

where $c > 0$ is a constant depending on $\nu_0, c_0, a, b, \|u\|_{L^\infty(\Omega)}$ only.

In the case of controlled growth, higher integrability of weak solutions to system (1.1) is obtained in [2]. The assertion of Lemma 2.1 can be proved as in [2] after having established a suitable Caccioppoli type inequality.

Next, we are going to derive some useful properties relating to the modulus of continuity occurring in (1.3) satisfying the Dini condition (1.3).

Lemma 2.2. *Let $\omega : [0, \infty) \rightarrow [0, \infty)$ be a non-decreasing function which satisfies condition (1.3). Then:*

(i) *For any $\theta > 0$ we have the inequality*

$$\int_0^1 \frac{\omega(t^\theta)}{t} dt < +\infty. \quad (2.2)$$

(ii) *For any given numbers $R, \tau \in (0, 1)$ we have the inequality*

$$\sum_{m=0}^{\infty} \omega(\tau^m R) \leq \frac{1}{1-\tau} \int_0^R \frac{\omega(t)}{t} dt + \omega(R). \quad (2.3)$$

(iii) *There exists a non-increasing function $\gamma : (0, 1] \rightarrow (0, +\infty)$ with $\lim_{t \rightarrow 0} \gamma(t) = +\infty$ such that*

$$t \mapsto \gamma(t)\omega(t) \text{ is non-decreasing on } (0, 1] \text{ and } \int_0^1 \frac{\gamma(t)\omega(t)}{t} dt < +\infty. \quad (2.4)$$

Proof. (i) Assertion (2.2) easily follows by means of the transformation formula of the Lebesgue integral.

(ii) Let $R, \tau \in (0, 1)$ be arbitrarily fixed. Taking into account the fact that ω is non-decreasing one may estimate

$$\begin{aligned} \int_0^R \frac{\omega(t)}{t} dt &= \sum_{m=1}^{\infty} \int_{\tau^m R}^{\tau^{m-1} R} \frac{\omega(t)}{t} dt \\ &\geq \sum_{m=1}^{\infty} \frac{\tau^{m-1} - \tau^m}{\tau^{m-1}} \omega(\tau^m R) \\ &= (1-\tau) \sum_{m=1}^{\infty} \omega(\tau^m R). \end{aligned}$$

Whence (2.3).

(iii) Firstly, assume $\omega(t_0) = 0$ for some $t_0 \in (0, 1]$. This implies $\omega(t) = 0$ for all $t \in [0, t_0]$ and the function

$$\gamma(t) = \begin{cases} 1 & \text{if } t \in (t_0, 1] \\ \frac{t_0}{t} & \text{if } t \in (0, 1] \end{cases}$$

fulfils the conditions we are looking for. Secondly, assume that $\omega(t) > 0$ for all $t \in (0, 1]$. We set $I = \int_0^1 \frac{\omega(t)}{t} dt$. Then there exists a sequence $1 = t_0 > t_1 > \dots > t_m > 0$ ($m \in \mathbb{N}_0$) with $\lim_{m \rightarrow \infty} t_m = 0$ such that

$$\int_0^{t_m} \frac{\omega(t)}{t} dt \leq 4^{-m} I \quad \text{and} \quad \omega(t_{m+1}) \leq \frac{1}{2} \omega(t_m) \quad (m \in \mathbb{N}_0). \quad (2.5)$$

Next, we set

$$\gamma(t) = 2^m \min \left\{ 2, \frac{\omega(t_m)}{\omega(t)} \right\} \quad (t \in (t_{m+1}, t_m], m \in \mathbb{N}_0).$$

Obviously, $\gamma : (0, 1] \rightarrow (0, +\infty)$ is non-increasing and $\gamma(t_m) = 2^m$ for all $m \in \mathbb{N}_0$. In addition, one may easily check that $\gamma(t)\omega(t)$ is non-decreasing on each interval $(t_{m+1}, t_m]$. On the other hand, from (2.5) it follows that

$$\gamma(t_{m+1})\omega(t_{m+1}) = 2^{m+1}\omega(t_{m+1}) \leq \gamma(t_m)\omega(t_m) \quad (m \in \mathbb{N}_0).$$

Therefore, the function $\gamma \cdot \omega$ is non-decreasing on $(0, 1]$. Finally, with the help of (2.5) we find

$$\int_0^1 \frac{\gamma(t)\omega(t)}{t} dt = \sum_{m=0}^{\infty} \int_{t_{m+1}}^{t_m} \frac{\gamma(t)\omega(t)}{t} dt \leq I \sum_{m=0}^{\infty} \frac{2^{m+1}}{4^m} = 2I \sum_{m=0}^{\infty} \frac{1}{2^m} = 4I$$

which completes the proof of the lemma ■

Lemma 2.3. *Let $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be a bounded function. Then there exists some constant $c_* = c_*(n) > 0$ such that for each function $u \in W^{1,2}(\Omega; \mathbb{R}^N)$ the following estimate holds for every ball $B_R \subset \Omega$ ($0 < R \leq 1$) and every $q \geq 1$:*

$$\left(\int_{B_R} [\omega(R + |u(x) - u_{B_R}|)]^q dx \right)^{\frac{1}{q}} \leq \omega(2\sqrt{R}) + k_0 R^{\frac{1}{q}} \left(c_* \int_{B_R} |\nabla u|^2 dx \right)^{\frac{1}{q}} \quad (2.6)$$

where $k_0 = \sup_{t \geq 0} \omega(t)$.

Proof. Let $B_R \subset \Omega$ ($0 < R \leq 1$) be arbitrarily fixed. Defining $A = \{x \in B_R : |u(x) - u_{B_R}| \leq \sqrt{R}\}$ we estimate

$$\left(\int_{B_R} [\omega(R + |u(x) - u_{B_R}|)]^q dx \right)^{\frac{1}{q}} \leq \omega(2\sqrt{R}) + k_0 \left(\frac{\text{meas}(B_R \setminus A)}{\text{meas}(B_R)} \right)^{\frac{1}{q}}. \quad (2.7)$$

With the help of the Poincaré inequality we estimate

$$\frac{\text{meas}(B_R \setminus A)}{\text{meas}(B_R)} \leq \frac{1}{R} \int_{B_R} |u - u_{B_R}|^2 dx \leq c_* R \int_{B_R} |\nabla u|^2 dx.$$

Then inserting this inequality into (2.7) gives (2.6) ■

Next we prove a technical lemma which describes in an abstract manner the standard iterating process playing an essential role to obtain partial Hölder continuity of weak solutions u to system (1.1).

Lemma 2.4. *Let $(\phi_m), (\mathcal{M}_m), (s_m)$ be sequences of non-negative real numbers. Moreover, assume (s_m) to be non-increasing and $\mathcal{S} = \sum_{m=0}^{+\infty} s_m < +\infty$. Furthermore, suppose there are some positive constants $\varepsilon, \tau, \lambda, M$ ($0 < \tau < 1$) such that the following conditions are fulfilled:*

$$\text{(E1)} \quad \phi_{m+1} \leq \sqrt{\tau}(\phi_m + s_m) \quad (m \in \mathbb{N}_0) \text{ where } \phi_m + s_m \leq \varepsilon \text{ and } \mathcal{M}_m \leq M.$$

$$\text{(E2)} \quad |\mathcal{M}_{m+1} - \mathcal{M}_m| \leq \lambda\phi_m \quad (m \in \mathbb{N}_0).$$

$$\text{(E3)} \quad \phi_0 + \frac{s_0}{1-\sqrt{\tau}} \leq \varepsilon.$$

$$\text{(E4)} \quad \mathcal{M}_0 \leq \frac{1}{2}M.$$

$$\text{(E5)} \quad \frac{\lambda}{1-\sqrt{\tau}}(\phi_0 + \mathcal{S}) \leq \frac{1}{2}M.$$

Then

$$\phi_m \leq \tau^{\frac{m}{2}}\phi_0 + \sum_{k=0}^{m-1} \tau^{\frac{m-k}{2}} s_k \quad \text{and} \quad \mathcal{M}_m \leq M \quad (2.8)$$

for all $m \in \mathbb{N}$.

Proof. We will prove assertion (2.8) by induction over $m \in \mathbb{N}$. For $m = 1$ the first inequality in (2.8) immediately follows after having combined conditions (E3) and (E1), whereas the second inequality holds by condition (E4). Now, we assume (2.8) to be fulfilled for $j = 1, \dots, m$. Since (s_m) is non-increasing observing condition (E3), from (2.8) we obtain

$$\phi_m + s_m \leq \phi_0 + \frac{1}{1-\sqrt{\tau}}s_0 \leq \varepsilon.$$

Now, we are in the position to apply condition (E1) (notice that condition $\mathcal{M}_m \leq M$ is fulfilled by virtue of (2.8)). Thus

$$\phi_{m+1} \leq \sqrt{\tau}(\phi_m + s_m) \leq \tau^{\frac{m+1}{2}}\phi_0 + \sum_{k=0}^m \tau^{\frac{m+1-k}{2}} s_k.$$

Next, using the triangular inequality together with conditions (E2) and (E4) we obtain

$$\mathcal{M}_{m+1} \leq \sum_{j=0}^m |\mathcal{M}_{j+1} - \mathcal{M}_j| + \mathcal{M}_0 \leq \lambda \sum_{j=0}^m \phi_j + \frac{1}{2}M.$$

Finally, applying (2.8) for $j = 1, \dots, m$ and taking into account condition (E5), from the latter inequality we deduce by an elementary calculus

$$\mathcal{M}_{m+1} \leq \frac{\lambda}{1-\sqrt{\tau}}\phi_0 + \lambda \sum_{j=1}^m \sum_{k=0}^{j-1} \tau^{\frac{j-k}{2}} s_k + \frac{1}{2}M \leq \frac{\lambda}{1-\sqrt{\tau}}(\phi_0 + \mathcal{S}) + \frac{1}{2}M \leq M.$$

Whence, (2.8) for $m + 1$ ■

The next lemma contains a fundamental estimate for weak solutions of an elliptic system with constant coefficients which is due to Campanato (cf. [1, 5]).

Lemma 2.5. *Let $A_{ij(*)}^{\alpha\beta}$ ($\alpha, \beta = 1, \dots, n; i, j = 1, \dots, N$) be constants satisfying the condition*

$$\nu_0 |\eta|^2 \leq A_{ij(*)}^{\alpha\beta} \eta_\alpha^i \eta_\beta^j \leq c_0 |\eta|^2 \quad (\eta \in \mathbb{R}^{nN}). \tag{2.9}$$

Then there exists a constant $A = A(\nu_0, c_0, n) > 0$ such that for every weak solution $u \in W^{1,2}(B_1; \mathbb{R}^N)$ of the system

$$D_\alpha (A_{ij(*)}^{\alpha\beta} D_\beta u^j) = 0 \quad \text{in } B_1 \quad (i = 1, \dots, N) \tag{2.10}$$

and every $\tau \in (0, 1)$ we have

$$\int_{B_\tau} |v - v_{B_\tau}|^2 dx \leq A^2 \tau^{n+2} \int_{B_1} |u - u_{B_1}|^2 dx. \tag{2.11}$$

3. Blow up

The aim of this section is to obtain a fundamental estimate under additional suitable conditions. Here we shall use the so-called *indirect method*. For the sake of simplicity, we are going to introduce the following notions: Let $u \in W^{1,2}(\Omega; \mathbb{R}^N)$, let $x_0 \in \Omega$ and $0 < R < \text{dist}(x_0, \partial\Omega)$. We define

$$\begin{aligned} \Phi(u; x_0, R) &= \left(\int_{B_R(x_0)} |\nabla u - (\nabla u)_{B_R(x_0)}|^2 dx \right)^{\frac{1}{2}} \\ \mathcal{M}(u; x_0, R) &= \left(\int_{B_R(x_0)} (1 + |\nabla u|^2) dx \right)^{\frac{1}{2}} + |u_{B_R(x_0)}|. \end{aligned}$$

Let ω denote the modulus of continuity of the coefficients a_i^α ($\alpha = 1, \dots, n; i = 1, \dots, N$) satisfying the Dini condition (1.3). Then by Lemma 2.2 there exists a non-increasing function $\gamma : (0, 1] \rightarrow (0, +\infty)$ with $\lim_{t \rightarrow 0} \gamma(t) = +\infty$ such that the function

$$\omega_0(t) = \gamma(t) \omega(2\sqrt{t}) + t^\sigma \quad \left(\sigma = \frac{p-2}{4p}\right)$$

is non-decreasing on $(0, 1]$ and obeys the condition

$$\int_0^1 \frac{\omega_0(t)}{t} dt < +\infty \tag{3.1}$$

where $p > 2$ refers to the exponent of higher integrability (cf. Lemma 2.1).

With the notion introduced above we have the following

Lemma 3.1. *We assume conditions (1.2) – (1.7) to be fulfilled. Let $u \in W^{1,2} \cap L^\infty(\Omega; \mathbb{R}^N)$ be a weak solution to system (1.1) satisfying the smallness condition (1.9). Then for each $\tau \in (0, \frac{1}{2})$ and $M \geq 0$ there exists a constant $\varepsilon_0 = \varepsilon_0(\tau, M) > 0$ such that the inequality*

$$\Phi(u; x_0, \tau R) \leq 2A\tau (\Phi(u; x_0, R) + \omega_0(R)) \tag{3.2}$$

is true for each $x_0 \in \Omega$ and each $0 < R < \text{dist}(x_0, \partial\Omega)$ where

$$\Phi(u; x_0, R) \leq \varepsilon_0 \quad \text{and} \quad \mathcal{M}(u; x_0, R) \leq M \tag{3.3}$$

and $A > 0$ is the constant appearing in (2.11).

Proof. Let us assume there exist some numbers $\tau \in (0, \frac{1}{2})$ and $M \geq 0$ such that the assertion of the lemma is not true. Then there must exist

- 1) sequences $(\varepsilon_m), (R_m) \subset \mathbb{R}_+$ such that $\varepsilon_m, R_m \rightarrow 0$ as $m \rightarrow +\infty$
- 2) a sequence $(x_m) \subset \Omega$ such that for all $m \in \mathbb{N}$

$$\Phi(u; x_m, R_m) + \omega_0(R_m) = \varepsilon_m, \quad \mathcal{M}(u; x_m, R_m) \leq M \tag{3.4}$$

$$\Phi(u; x_m, \tau R_m) > 2A\tau(\Phi(u; x_m, R_m) + \omega_0(R_m)). \tag{3.5}$$

Then we define for almost all $y \in B_1(0)$

$$\left. \begin{aligned} \lambda_m &= (\nabla u)_{B_{R_m}} \\ v_m(y) &= \frac{u(x_m + R_m y) - u_{B_{R_m}} - R_m \lambda_m \cdot y}{R_m \varepsilon_m} \\ A_{ij(m)}^{\alpha\beta}(y) &= \int_0^1 \frac{\partial a_i^\alpha}{\partial \xi_\beta^j}(x_m, u_{B_{R_m}}, t\varepsilon_m \nabla v_m(y) + \lambda_m) dt \end{aligned} \right\} \quad (m \in \mathbb{N})$$

($\alpha, \beta = 1, \dots, n; i, j = 1, \dots, N$). With the help of the transformation formula of the Lebesgue integral from (3.4) - (3.5) we may verify

$$\left(\int_{B_1} (1 + |\varepsilon_m \nabla v_m + \lambda_m|^2) dy \right)^{\frac{1}{2}} \leq M, \quad |\lambda_m| \leq M, \quad |u_{B_{R_m}}| \leq M \tag{3.6}$$

$$\Phi(v_m; 0, 1) + \frac{\omega_0(R_m)}{\varepsilon_m} = 1 \tag{3.7}$$

$$\Phi(v_m; 0, \tau) > 2A\tau \left(\Phi(v_m; 0, 1) + \frac{\omega_0(R_m)}{\varepsilon_m} \right). \tag{3.8}$$

Then making use of the mean value theorem, from (1.8) we deduce that v_m belongs to the space $W^{1,2}(B_1; \mathbb{R}^N)$ satisfying the integral identity

$$\begin{aligned} & \int_{B_1} A_{ij(m)}^{\alpha\beta} D_\beta v_m^j D_\alpha \psi^i dy \\ &= \frac{1}{\varepsilon_m} \int_{B_1} \left\{ a_i^\alpha(x_m, u_{B_{R_m}}, \varepsilon_m \nabla v_m + \lambda_m) \right. \\ & \quad \left. - a_i^\alpha(x_m + R_m y, u(x_m + R_m y), \varepsilon_m \nabla v_m + \lambda_m) \right\} D_\alpha \psi^i dy \\ & \quad + \frac{R_m}{\varepsilon_m} \int_{B_1} b_i(x_m + R_m y, u(x_m + R_m y), \varepsilon_m \nabla v_m + \lambda_m) \psi^i dy \end{aligned} \tag{3.9}$$

for all $m \in \mathbb{N}$ and $\psi \in C_c^\infty(B_1; \mathbb{R}^N)$.

From the definition of v_m we deduce that $(v_m)_{B_1} = (\nabla v_m)_{B_1} = 0$. Then observing (3.7) after having applied the Poincaré inequality we obtain

$$\|v_m\|_{W^{1,2}(B_1)} \leq c_* \|\nabla v_m\|_{L^2(B_1)} = c_* \Phi(v_m; 0, 1) \leq c_*. \tag{3.10}$$

If necessary passing to a subsequence we may assume the following convergence properties to be fulfilled:

$$x_m \rightarrow x_* \text{ in } \mathbb{R}^n, \quad u_{B_{R_m}} \rightarrow u_* \text{ in } \mathbb{R}^N, \quad \lambda_m \rightarrow \lambda_* \text{ in } \mathbb{R}^{nN} \tag{3.11}$$

$$v_m \rightarrow v \text{ strongly in } L^2(B_1; \mathbb{R}^N) \tag{3.12}$$

$$D_\alpha v_m \rightarrow D_\alpha v \text{ weakly in } L^2(B_1; \mathbb{R}^N) \tag{3.13}$$

$$\varepsilon_m (D_\alpha v_m)(y) \rightarrow 0 \text{ for a.a. } y \in B_1 \tag{3.14}$$

$$A_{ij(m)}^{\alpha\beta}(y) \rightarrow A_{ij(*)}^{\alpha\beta} \text{ for a.a. } y \in B_1 \tag{3.15}$$

as $m \rightarrow +\infty$ where

$$A_{ij(*)}^{\alpha\beta} = \frac{\partial a_i^\alpha}{\partial \xi_j^\beta}(x_*, u_*, \lambda_*) = \text{const}$$

($\alpha, \beta = 1, \dots, n; i, j = 1, \dots, N$). Indeed, (3.11) follows from the boundedness of the sequences $(x_m), (u_{B_{R_m}}), (\lambda_m)$ (cf. (3.6)) whereas (3.12) - (3.13) are obtained by the compactness of the imbedding $W^{1,2}(B_1; \mathbb{R}^N) \subset L^2(B_1; \mathbb{R}^N)$ and the reflexivity of the space $L^2(B_1; \mathbb{R}^N)$. To prove (3.14) we use the fact that the sequence $(\varepsilon_m D_\alpha v_m)$ converges to zero with respect to the $L^2(B_1; \mathbb{R}^N)$ -norm which shows the existence of a subsequence (v_{m_j}) such that (3.14) is fulfilled. Finally, (3.15) is a consequence of (3.11) and (3.14).

Now we are in a position to pass to the limit in (3.9). Let $\psi \in C^\infty(B_1; \mathbb{R}^N)$ with $\text{supp } \psi \subset B_1$ be fixed.

(i) Observing (3.15) we deduce

$$A_{ij(m)}^{\alpha\beta} D_\alpha \psi \rightarrow A_{ij(*)}^{\alpha\beta} D_\alpha \psi \quad \text{in } L^2(B_1; \mathbb{R}^N) \text{ as } m \rightarrow +\infty.$$

Then taking into account (3.13) we get

$$\int_{B_1} A_{ij(m)}^{\alpha\beta} D_\beta v_m^j D_\alpha \psi^i dy \rightarrow \int_{B_1} A_{ij(*)}^{\alpha\beta} D_\beta v^j D_\alpha \psi^i dy \tag{3.16}$$

as $m \rightarrow +\infty$ ($\alpha, \beta = 1, \dots, n; i, j = 1, \dots, N$).

(ii) Next we are going to estimate the right-hand side of identity (3.9).

1. Observing (1.2), by virtue of the Hölder inequality applying Lemma 2.3 we find

$$\begin{aligned}
 |I_{1(m)}| &= \frac{1}{\varepsilon_m} \left| \int_{B_1} \left\{ a_i^\alpha(x_m, u_{B_{R_m}}, \varepsilon_m \nabla v_m + \lambda_m) \right. \right. \\
 &\quad \left. \left. - a_i^\alpha(x_m + R_m y, u(x_m + R_m y), \varepsilon_m \nabla v_m + \lambda_m) \right\} D_\alpha \psi^i dy \right| \\
 &\leq \frac{\max_{B_1} |\nabla \psi|}{\varepsilon_m} \int_{B_1} \omega(R_m + |u(x_m + R_m) - u_{B_{R_m}}|) (1 + |\varepsilon_m \nabla v_m + \lambda_m|) dy \\
 &\leq \frac{2M \text{meas}(B_1) \max_{B_1} |\nabla \psi|}{\varepsilon_m} \left(\int_{B_1} \left[\omega(R_m + |u(x_m + R_m) - u_{B_{R_m}}|) \right]^2 dy \right)^{\frac{1}{2}} \\
 &\leq \frac{c}{\varepsilon_m} \{ \omega(2\sqrt{R_m}) + \sqrt{R_m} \} \\
 &\leq c \left\{ \frac{1}{\gamma(R_m)} + R_m^\sigma \right\}
 \end{aligned}$$

where the constant $c > 0$ does not depend on $m \in \mathbb{N}$.

2. By (1.7) and (3.6) we estimate

$$\begin{aligned}
 |I_{2(m)}| &= \frac{R_m}{\varepsilon_m} \left| \int_{B_1} b_i(x_m + R_m y, u(x_m + R_m y), \varepsilon_m \nabla v_m + \lambda_m) \psi^i dy \right| \\
 &\leq \max_{B_1} |\psi| \sqrt{R_m} \int_{B_1} (a |\varepsilon_m \nabla v_m + \lambda_m|^2 + b) dy \\
 &\leq c' \sqrt{R_m}
 \end{aligned}$$

where the constant $c' > 0$ does not depend on $m \in \mathbb{N}$. Thus $I_{1(m)} + I_{2(m)} \rightarrow 0$ as $m \rightarrow +\infty$. Now, in (3.9) passing to the limit we conclude that

$$\int_{B_1} A_{ij(*)}^{\alpha\beta} D_\beta v^j D_\alpha \psi^i dy = 0 \tag{3.17}$$

for all $\psi \in C_c^\infty(B_1; \mathbb{R}^N)$. In addition, by (1.5) - (1.6) we easily verify

$$\nu_0 |\eta|^2 \leq A_{ij(*)}^{\alpha\beta} \eta_\alpha^i \eta_\beta^j \leq c_0 |\eta|^2$$

for all $\eta \in \mathbb{R}^{nN}$. Therefore from Lemma 2.5 it follows

$$\Phi(v; 0, \tau) \leq A\tau \Phi(v; 0, 1). \tag{3.18}$$

On the other hand, as it will be shown in the appendix below that we have

$$\lim_{m \rightarrow +\infty} \Phi(v_m; 0, \tau) = \Phi(v; 0, \tau) \geq 2A\tau. \tag{3.19}$$

Then taking into account the lower semicontinuity of the norm from (3.8) we estimate

$$2A\tau \Phi(v; 0, 1) \leq 2A\tau \liminf_{m \rightarrow +\infty} \Phi(v_m; 0, 1) \leq \lim_{m \rightarrow +\infty} \Phi(v_m; 0, \tau) = \Phi(v; 0, \tau)$$

what clearly contradicts to (3.18) ■

4. Proof of the Theorem

Partial Hölder continuity will be proved by a standart iteration process described in Lemma 2.4, where condition (E2) will be verified by the following lemma.

Lemma 4.1. *For each $\tau \in (0, 1)$ there exists a constant $\lambda_0 = \lambda_0(\tau, n) > 0$ such that for any function $u \in W^{1,2}(\Omega; \mathbb{R}^N)$, for each $x_0 \in \Omega$ and all $0 < R < \text{dist}(x_0, \partial\Omega)$ we have the inequality*

$$|\mathcal{M}(u; x_0, \tau R) - \mathcal{M}(u; x_0, R)| \leq \lambda_0 \Phi(u; x_0, R). \tag{4.1}$$

Proof. Let $\tau \in (0, 1)$ be arbitrarily fixed, let $x_0 \in \Omega$ and let $0 < R < \text{dist}(x_0, \partial\Omega)$. We define

$$\left. \begin{aligned} F(x, y) &= (1 + |\nabla u(x)|^2)^{\frac{1}{2}} \\ G(x, y) &= (1 + |\nabla u(y)|^2)^{\frac{1}{2}} \end{aligned} \right\} \text{ for a.a. } \{x, y\} \in B_{\tau R} \times B_R.$$

Obviously, $F, G \in L^2(B_{\tau R} \times B_R)$. Moreover, making use of the triangular inequality we estimate

$$\begin{aligned} & \left| \left(\int_{B_{\tau R}} (1 + |\nabla u(x)|^2) dx \right)^{\frac{1}{2}} - \left(\int_{B_R} (1 + |\nabla u(y)|^2) dy \right)^{\frac{1}{2}} \right| \\ &= [\text{meas}_{2n}(B_{\tau R} \times B_R)]^{\frac{1}{2}} \left| \|F\|_{L^2(B_{\tau R} \times B_R)} - \|G\|_{L^2(B_{\tau R} \times B_R)} \right| \\ &\leq [\text{meas}_{2n}(B_{\tau R} \times B_R)]^{\frac{1}{2}} \left(\int_{B_R} \int_{B_R} |\nabla u(x) - \nabla u(y)|^2 dx dy \right)^{\frac{1}{2}} \\ &\leq \frac{2}{\tau^{n/2}} \left(\int_{B_R} |\nabla u - (\nabla u)_{B_R}|^2 dx \right)^{\frac{1}{2}} \\ &= \frac{2}{\tau^{n/2}} \Phi(u; x_0, R). \end{aligned}$$

Next, by virtue of the Hölder and Poincaré inequalities we get

$$\begin{aligned} & |u_{B_{\tau R}} - u_{B_R}| \\ &= \left| (u - (\nabla u)_{B_R} \cdot (x - x_0))_{B_{\tau R}} - (u - (\nabla u)_{B_R} \cdot (y - x_0))_{B_R} \right| \\ &\leq \left(\iint_{B_{\tau R} \times B_R} \left| u(x) - (\nabla u)_{B_R} \cdot (x - x_0) - u(y) + (\nabla u)_{B_R} \cdot (y - x_0) \right|^2 dx dy \right)^{\frac{1}{2}} \\ &\leq \frac{2}{\tau^{n/2}} \left(\int_{B_R} \left| u(x) - u_{B_R} - (\nabla u)_{B_R} \cdot (x - x_0) \right|^2 dx \right)^{\frac{1}{2}} \\ &\leq \frac{2Rc_*}{\tau^{n/2}} \left(\int_{B_R} |\nabla u - (\nabla u)_{B_R}|^2 dx \right)^{\frac{1}{2}} \\ &\leq \frac{2c_*}{\tau^{n/2}} \Phi(u; x_0, R). \end{aligned}$$

Thus, assertion (4.1) is obtained after having combined the two inequalities proved above ■

Proof of the Theorem. Firstly, we define the *singular set* Σ by

$$\Sigma = \left\{ x_0 \in \Omega \mid \liminf_{R \rightarrow 0} \Phi(u; x_0, R) > 0 \text{ and } \sup_{R > 0} \mathcal{M}(u; x_0, R) = +\infty \right\}.$$

It is well known that $\text{meas}(\Sigma) = 0$ (cf. [9]). Now, let $x_0 \in \Omega \setminus \Sigma$ be arbitrarily chosen. Then we set

$$M = 2 \sup_{R > 0} \mathcal{M}(u; x_0, R) + 1.$$

Next, we choose $\tau \in (0, \frac{1}{2})$ such that

$$2A\sqrt{\tau} \leq 1. \tag{4.2}$$

In addition, there exists $0 < R_0 < \min\{\text{dist}(x_0, \partial\Omega), 1\}$ such that

$$\left. \begin{aligned} \frac{\lambda_0}{1-\sqrt{\tau}} \left(\Phi(u; x_0, R_0) + \frac{1}{1-\tau} \int_0^{R_0} \frac{\omega_0(t)}{t} dy + \omega_0(R_0) \right) &< \frac{1}{2}M \\ \Phi(u; x_0, R_0) + \frac{1}{\sqrt{\tau}}\omega_0(R_0) &< \varepsilon_0 \end{aligned} \right\}. \tag{4.3}$$

By the absolute continuity of the Lebesgue integral there exists a number $r \in (0, \text{dist}(x_0, \partial\Omega) - R_0)$ such that, for every $y \in B_r(x_0)$,

$$\left. \begin{aligned} \mathcal{M}(u; y, R_0) &\leq \frac{1}{2}M \\ \frac{\lambda_0}{1-\sqrt{\tau}} \left(\Phi(u; y, R_0) + \frac{1}{1-\tau} \int_0^{R_0} \frac{\omega_0(t)}{t} dy + \omega_0(R_0) \right) &\leq \frac{1}{2}M \\ \Phi(u; y, R_0) + \frac{1}{1-\sqrt{\tau}}\omega_0(R_0) &\leq \varepsilon_0 \end{aligned} \right\}. \tag{4.4}$$

Let $y \in B_r(x_0)$ be arbitrarily fixed. Then for each $m \in \mathbb{N}$ we define

$$\begin{aligned} \phi_m &= \phi_m(y) = \phi(u; y, \tau^m R_0) \\ \mathcal{M}_m &= \mathcal{M}_m(y) = \mathcal{M}(u; y, \tau^m R_0) \\ s_m &= s_m(y) = \omega_0(\tau^m R_0). \end{aligned}$$

Now we may verify that for the sequences $(\phi_m), (\mathcal{M}_m), (s_m)$ conditions (E1) - (E5) of Lemma 2.4 are fulfilled. Indeed, condition (E1) may be verified by Lemma 3.1. Condition (E2) is obtained from (4.1) (cf. Lemma 4.1). Conditions (E3) - (E5) are finally obtained by (4.4) having used Lemma 2.2. Therefore we are in a position to apply the technical Lemma 2.4 and receive

$$\phi_m(y) \leq \tau^{\frac{m}{2}} \phi_0 + \sum_{k=0}^{m-1} \tau^{\frac{m-k}{2}} s_k \quad (m \in \mathbb{N}). \tag{4.5}$$

Finally, using a standard argument (see, for example, in [5]) we get

$$\int_{B_\rho(y)} |\nabla u - (\nabla u)_{B_\rho(y)}|^2 dx \leq C_0 \left(\frac{\rho}{R_0}\right)^n \quad (y \in B_r(x_0), \rho \in (0, R_0]) \tag{4.6}$$

where the constant $C_0 > 0$ does not depend on ρ . From (4.6) we deduce together with Campanato's theorem (cf. [1]) $u|_{B_r(x_0)} \in C^{0,\mu}(\overline{B_r(x_0)}; \mathbb{R}^N)$ for all $\mu \in (0, 1)$. In particular, from the proof above it is readily seen that $\Omega_0 = \Omega \setminus \Sigma$ is an open set with $\text{meas}(\Omega \setminus \Omega_0) = 0$ and $u|_{\Omega_0} \in C^{0,\mu}(\Omega_0; \mathbb{R}^N)$ for all $\mu \in (0, 1)$ what concludes the proof of the theorem ■

5. Appendix: Proof of (3.19)

By $\zeta \in C^\infty(\mathbb{R}^n)$ we denote a *cut-off function* such that $\zeta \equiv 0$ on $\mathbb{R}^n \setminus B_{1/2}$ and $\zeta \equiv 1$ on B_τ . Then we set

$$\psi^i(y) = (v_m^i(y) - v^i(y))\zeta^2(y) \quad \text{for a.a. } y \in B_1 \quad (i = 1, \dots, N)$$

which is an admissible test function for both identities (3.9) and (3.17). After having inserted ψ into (3.9) and (3.17), by combining these two identities we evaluate

$$\begin{aligned} & \int_{B_1} A_{ij(m)}^{\alpha\beta} (D_\beta v_m^j - D_\beta v^j) D_\alpha \psi^i dy \\ &= \int_{B_1} \{A_{ij(*)}^{\alpha\beta} - A_{ij(m)}^{\alpha\beta}\} D_\beta v^j D_\alpha \psi^i dy \\ & \quad + \frac{1}{\varepsilon_m} \int_{B_1} \left\{ a_i^\alpha(x_m, u_{B_{R_m}}, \varepsilon_m \nabla v_m + \lambda_m) \right. \\ & \quad \left. - a_i^\alpha(x_m + R_m y, u(x_m + R_m y), \varepsilon_m \nabla v_m + \lambda_m) \right\} D_\alpha \psi^i dy \\ & \quad + \frac{R_m}{\varepsilon_m} \int_{B_1} b_i(x_m + R_m y, u(x_m + R_m y), \varepsilon_m \nabla v_m + \lambda_m) \psi^i dy. \end{aligned}$$

Applying the product and chain rule and observing condition (1.5) we find

$$\begin{aligned} & \nu_0 \int_{B_1} |\nabla v_m - \nabla v|^2 \zeta^2 dy \\ & \leq -2 \int_{B_1} A_{ij(m)}^{\alpha\beta} (D_\beta v_m^j - D_\beta v^j) D_\beta v^j (v_m^i - v^i) (D_\alpha \zeta) \zeta dy \\ & \quad + \int_{B_1} \{A_{ij(*)}^{\alpha\beta} - A_{ij(m)}^{\alpha\beta}\} D_\beta v^j (D_\alpha v_m^i - D_\alpha v^i) \zeta^2 dy \\ & \quad + 2 \int_{B_1} \{A_{ij(*)}^{\alpha\beta} - A_{ij(m)}^{\alpha\beta}\} D_\beta v^j (v_m^i - v^i) (D_\alpha \zeta) \zeta dy \\ & \quad + \frac{1}{\varepsilon_m} \int_{B_1} \left\{ a_i^\alpha(x_m, u_{B_{R_m}}, \varepsilon_m \nabla v_m + \lambda_m) \right. \\ & \quad \left. - a_i^\alpha(x_m + R_m y, u(x_m + R_m y), \varepsilon_m \nabla v_m + \lambda_m) \right\} (D_\alpha v_m^i - D_\alpha v^i) \zeta^2 dy \\ & \quad + \frac{1}{\varepsilon_m} \int_{B_1} \left\{ a_i^\alpha(x_m, u_{B_{R_m}}, \varepsilon_m \nabla v_m + \lambda_m) \right. \\ & \quad \left. - a_i^\alpha(x_m + R_m y, u(x_m + R_m y), \varepsilon_m \nabla v_m + \lambda_m) \right\} (v_m^i - v^i) (D_\alpha \zeta) \zeta dy \\ & \quad + \frac{R_m}{\varepsilon_m} \int_{B_1} b_i(x_m + R_m y, u(x_m + R_m y), \varepsilon_m \nabla v_m + \lambda_m) (v_m^i - v^i) \zeta^2 dy \\ & = I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \tag{5.1} \end{aligned}$$

(i) Observing (1.5) and taking into consideration that $v \in C^1(B_{1/2}; \mathbb{R}^N)$, with the

help of Young’s inequality we have

$$I_1 + I_2 + I_3 \leq 2\delta \int_{B_1} |\nabla v_m - \nabla v|^2 \zeta^2 dy + c \left\{ \int_{B_1} |v_m - v|^2 dy + \int_{B_1} |\mathbf{A}_m - \mathbf{A}_*|^2 dy \right\}$$

where \mathbf{A}_m and \mathbf{A}_* denote the matrices $\{A_{ij}^{\alpha\beta}(m)\}$ and $\{A_{ij}^{\alpha\beta}(\ast)\}$, respectively, $\delta = \frac{1}{6}(\nu_0 - 2a\|u\|_{L^\infty(\Omega)})$ and the constant $c > 0$ does not depend on $m \in \mathbb{N}$.

(ii) Once more using Young’s inequality, observing (1.2) and applying Lemmas 2.1 and 2.3 we deduce

$$\begin{aligned} I_4 &\leq \delta \int_{B_1} |\nabla v_m - \nabla v|^2 \zeta^2 dy \\ &\quad + \frac{c}{\varepsilon_m^2} \int_{B_{R_m}} [\omega(R_m + |u - u_{B_{R_m}}|)]^2 (1 + |\nabla u|^2) \zeta^2 dx \\ &\leq \delta \int_{B_1} |\nabla v_m - \nabla v|^2 \zeta^2 dy \\ &\quad + \frac{c}{\varepsilon_m^2} \left(\int_{B_{R_m}} [\omega(R_m + |u - u_{B_{R_m}}|)]^{2p/(p-2)} dx \right)^{\frac{p-2}{p}} \\ &\quad \times \int_{B_{R_m}} (1 + |\nabla u|^2) dx \\ &\leq \delta \int_{B_1} |\nabla v_m - \nabla v|^2 \zeta^2 dy + \frac{cM^2}{\varepsilon_m^2} \left\{ [\omega(2\sqrt{R_m})]^2 + R_m^{2\sigma} M^{\frac{2(p-2)}{p}} \right\} \\ &\leq \delta \int_{B_1} |\nabla v_m - \nabla v|^2 \zeta^2 dy + cM^4 \left[\frac{1}{\gamma(R_m)} + R_m^\sigma \right]^2 \end{aligned}$$

where the constant $c > 0$ does not depend on $m \in \mathbb{N}$.

(iii) Analogously as above we estimate

$$I_5 \leq c \int_{B_1} |v_m - v|^2 dy + cM^{\frac{3-2}{p}} \left[\frac{1}{\gamma(R_m)} + R_m^\sigma \right]^2.$$

where the constant $c > 0$ does not depend on $m \in \mathbb{N}$.

(iv) In order to estimate the integral I_6 we make use of the two estimates

$$\begin{aligned} &|\varepsilon_m \nabla v_m + \lambda_m|^2 \\ &\leq \varepsilon_m^2 |\nabla v_m - \nabla v|^2 + \varepsilon_m |\nabla v_m - \nabla v| |\varepsilon_m \nabla v + \lambda_m| + |\varepsilon_m \nabla v + \lambda_m|^2 \end{aligned} \tag{5.2}$$

and

$$\operatorname{ess\,sup}_{B_1} |v_m| \leq \frac{2\|u\|_{L^\infty(\Omega)}}{\varepsilon_m R_m} + \frac{\lambda_m}{\varepsilon_m} \tag{5.3}$$

which are verified by an elementary calculus. Then by condition (1.7) we get

$$\begin{aligned} I_6 &\leq a \frac{R_m}{\varepsilon_m} \int_{B_1} |\varepsilon_m \nabla v_m + \lambda_m|^2 |v_m| \zeta^2 dy \\ &\quad + a \frac{R_m}{\varepsilon_m} \int_{B_1} |\varepsilon_m \nabla v_m + \lambda_m|^2 |v| \zeta^2 dy \\ &\quad + b \frac{R_m}{\varepsilon_m} \int_{B_1} |v_m + v|^2 dy \\ &= I'_6 + I''_6 + I'''_6. \end{aligned}$$

To estimate the integral I'_6 we make use of (5.2) - (5.3). Hence

$$\begin{aligned} I'_6 &\leq a \varepsilon_m R_m \int_{B_1} |\nabla v_m - \nabla v|^2 |v_m| \zeta^2 dy \\ &\quad + a R_m \int_{B_1} |\nabla v_m - \nabla v|^2 |\varepsilon_m \nabla v + \lambda_m| |v_m| \zeta^2 dy \\ &\quad + a \frac{R_m}{\varepsilon_m} \int_{B_1} |\varepsilon_m \nabla v + \lambda_m|^2 |v_m| \zeta^2 dy \\ &\leq 2a \|u\|_{L^\infty(\Omega)} \int_{B_1} |\nabla v_m - \nabla v|^2 \zeta^2 dy + c \sqrt{R_m}. \end{aligned}$$

With the same manner, having the estimate $I''_6 + I'''_6 \leq c \sqrt{R_m}$ we finally obtain

$$I_6 \leq 2a \|u\|_{L^\infty(\Omega)} \int_{B_1} |\nabla v_m - \nabla v|^2 \zeta^2 dy + c \sqrt{R_m}$$

where the constant $c > 0$ does not depend on $m \in \mathbb{N}$. Then inserting the estimates of I_1, \dots, I_6 into (5.1) it follows that

$$\begin{aligned} &\frac{1}{2}(\nu_0 - 2a \|u\|_{L^\infty(\Omega)}) \int_{B_\tau} |\nabla v_m - \nabla v|^2 dy \\ &\leq c \left(\int_{B_1} |v_m - v|^2 dy + \int_{B_1} |\mathbf{A}_* - \mathbf{A}_m|^2 dy + \left[\frac{1}{\gamma(R_m)} + R_m^\sigma \right]^2 \right) \end{aligned} \tag{5.4}$$

where the constant $c > 0$ does not depend on $m \in \mathbb{N}$. Finally, together with (3.12) and (3.15) we conclude that the right-hand side of (5.4) tends to zero as $m \rightarrow \infty$, what proves the required convergence property \blacksquare

Note added in proof. After the present paper has been accepted for print the author became familiar with the work F. Duzaar and A. Gastel: *Nonlinear elliptic systems with Dini continuous coefficients*. In this work the authors prove by an entirely different method the C^1 partial regularity of weak solutions to nonlinear elliptic systems.

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Received 05.07.2000