

On the Asymptotic Behaviour of the Integral

$$\int_0^\infty e^{itx} \left(\frac{1}{x^\alpha} - \frac{1}{[x^\alpha] + 1} \right) dx \quad (t \rightarrow 0)$$

and Rates of Convergence to α -Stable Limit Laws

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Abstract. We study the behaviour of the exponential sums $\sum_{n \geq 1} \frac{\exp\{it n^{1/\alpha}\}}{n(n+1)}$ in a small neighbourhood of $t = 0$ for $0 < \alpha \leq 1$. Our main result yields an exact estimate of the remainder term in the corresponding Tauberian theorem. In particular, we prove that

$$it \int_0^\infty e^{itx} \frac{1}{x^\alpha} - \frac{1}{[x^\alpha] + 1} dx = \begin{cases} O(|t|^{1-2\alpha}) & \text{for } 0 < \alpha < \frac{1}{4} \\ O(|t|^{\frac{1}{2}} \log \frac{1}{|t|}) & \text{for } \alpha = \frac{1}{4} \\ O(|t|^{2\alpha}) & \text{for } \frac{1}{4} < \alpha < \frac{1}{2} \end{cases}$$

as $t \rightarrow 0$. These asymptotic relations provide optimal uniform rates of convergence in limit theorems for partial sums of independent random variables with common distribution function $F_\alpha(x) = 1 - \frac{1}{[x^\alpha] + 1}$ for $x > 0$.

Keywords: *Fourier-Stieltjes transform, normal domain of attraction, α -stable distribution, exponential sums, Tauberian theorem, Fourier integrals, method of stationary phase*

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1. Introduction and main results

Let X be a random variable taking positive integer values according to the distribution $P(X = n) = \frac{1}{n(n+1)}$ ($n \in \mathbb{N}$). For any real number $\alpha \in (0, 2)$, the non-lattice distribution function $F_\alpha(x) = P(X^{\frac{1}{\alpha}} \leq x) = 1 - \frac{1}{[x^\alpha] + 1}$ ($x > 0$) of the $\frac{1}{\alpha}$ -th power of X belongs to the *normal domain of attraction* of the α -stable distribution function G_α which is defined by its Fourier-Stieltjes transform

$$\widehat{G}_\alpha(t) = \int_{-\infty}^\infty e^{itx} dG_\alpha(x) = \begin{cases} \exp \left\{ -\lambda_\alpha |t|^\alpha \left(1 - i \operatorname{sgn}(t) \tan \frac{\pi\alpha}{2} \right) \right\} & \text{for } \alpha \neq 1 \\ \exp \left\{ -\frac{\pi}{2} |t| \left(1 - i \frac{2}{\pi} \operatorname{sgn}(t) \log \frac{1}{|t|} \right) \right\} & \text{for } \alpha = 1 \end{cases} \quad (1.1)$$

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with *scale parameter*

$$\lambda_\alpha = \begin{cases} \Gamma(1 - \alpha) \cos \frac{\pi\alpha}{2} & \text{for } 0 < \alpha < 1 \\ \frac{\Gamma(2-\alpha)}{\alpha-1} \sin \frac{\pi(\alpha-1)}{2} & \text{for } 1 < \alpha < 2, \end{cases}$$

where

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx \quad (s > 0).$$

For further details and the background of limit theorems for sums of random variables with stable limit law we refer to the monographs Christoph and Wolf [2], Ibragimov and Linnik [8] or Zolotarev [12]. In other words, if X_1, X_2, \dots are independent copies of X , then there is a centering sequence $A_n(\alpha) \geq 0$ such that

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left(\frac{X_1^{\frac{1}{\alpha}} + \dots + X_n^{\frac{1}{\alpha}} - A_n(\alpha)}{n^{\frac{1}{\alpha}}} \leq x \right) - G_\alpha(x) \right| \rightarrow 0 \quad (n \rightarrow \infty) \quad (1.2)$$

where we may choose

$$A_n(\alpha) = \begin{cases} 0 & \text{for } 0 < \alpha < 1 \\ n \log n & \text{for } \alpha = 1 \\ n \mathbb{E}X^{\frac{1}{\alpha}} & \text{for } 1 < \alpha < 2. \end{cases}$$

Esseen’s basic estimate of the L^∞ -distance of two distribution functions in terms of their Fourier-Stieltjes transforms (see, e.g., [8: Chapter 1] or (3.1) below) reveals that rates of convergence in (1.2) are governed by the nearness of the functions \widehat{G}_α and \widehat{F}_α defined by

$$\widehat{F}_\alpha(t) = \int_0^\infty e^{itx} dF_\alpha(x) = \sum_{n=1}^\infty \frac{1}{n(n+1)} e^{itn^{\frac{1}{\alpha}}}$$

in some small neighbourhood of $t = 0$. More precisely, we need the exact asymptotic behaviour of the difference $\log \widehat{G}_\alpha(t) - \log \widehat{F}_\alpha(t)$ as $t \rightarrow 0$ (see also de Haan and Peng [3]). Using the Taylor expansion of the log-function, we may replace $\log \widehat{F}_\alpha(t)$ (see Section 3 for details) by

$$\widehat{F}_\alpha(t) - 1 = it \int_0^\infty e^{itx} (1 - F_\alpha(x)) dx = it \int_0^\infty \frac{e^{itx}}{[x^\alpha] + 1} dx \quad (\alpha > 0, t \in \mathbb{R}).$$

Further, taking into account the well-known integral

$$\int_0^\infty e^{itx} x^{-\alpha} dx = |t|^{\alpha-1} \Gamma(1 - \alpha) \left(\sin \frac{\pi\alpha}{2} + i \operatorname{sgn}(t) \cos \frac{\pi\alpha}{2} \right) = \frac{1}{it} \log \widehat{G}_\alpha(t)$$

for $0 < \alpha < 1$ (a similar formula holds for $1 < \alpha < 2$) we see that the rates of convergence in (1.2) are determined by behaviour of the Fourier integral

$$\Delta_\alpha(t) := it \int_0^\infty e^{itx} \left(\frac{1}{x^\alpha} - \frac{1}{[x^\alpha] + 1} \right) dx = \log \widehat{G}_\alpha(t) - (\widehat{F}_\alpha(t) - 1) \quad (1.3)$$

($0 < \alpha < 1$) as $t \rightarrow 0$ (analogously for $1 \leq \alpha < 2$).

The Tauberian theorem for Fourier integrals (see Bingham et al. [1: p. 209]) tells us that $\Delta_\alpha(t) = o(|t|^\alpha)$ as $t \rightarrow 0$ for $0 < \alpha < 1$. From this point of view, the main purpose of this paper is to find best possible bounds of the remainder term in the mentioned Tauberian theorem. This problem seems to be unsolved so far and it is indeed by no means trivial, in particular for $0 < \alpha < \frac{1}{2}$. Note that $\frac{\Delta_\alpha(t)}{it}$ expresses the error made by the approximation of the exponential sum $F_\alpha(t) - 1$ by the Fourier transform of the smooth function $x^{-\alpha}$. Several methods have been devised for estimating exponential sums in lattice point and analytic number theory to treat related question, among them the powerful *van der Corput method* (see Krätzel [9], Montgomery [11] or Drmota and Tichy [4]). Theorem 1 states bounds of $\Delta_\alpha(t)$ which are certainly unimprovable for $\frac{1}{4} < \alpha < \frac{1}{2}$, whereas, for $0 < \alpha \leq \frac{1}{4}$, there is no evidence for the optimality of these bounds which mainly depend on the growth of certain finite exponential sums.

Theorem 1. *For $0 < \alpha < \frac{1}{2}$ we have*

$$\widehat{F}_\alpha(t) - 1 + |t|^\alpha \Gamma(1 - \alpha) \exp \left\{ -i \operatorname{sgn}(t) \frac{\pi\alpha}{2} \right\} = \begin{cases} O(|t|^{\frac{\alpha}{1-2\alpha}}) & \text{for } 0 < \alpha < \frac{1}{4} \\ O(|t|^{\frac{1}{2}} \log \frac{1}{|t|}) & \text{for } \alpha = \frac{1}{4} \\ O(|t|^{2\alpha}) & \text{for } \frac{1}{4} < \alpha < \frac{1}{2} \end{cases}$$

as $t \rightarrow 0$.

The corresponding stable limit theorem based on Theorem 1 is the following

Theorem 2. *For $0 < \alpha < \frac{1}{2}$ we have uniformly in $x \in \mathbb{R}$*

$$\left| \mathbb{P} \left(\frac{X_1^{\frac{1}{\alpha}} + \dots + X_n^{\frac{1}{\alpha}}}{n^{\frac{1}{\alpha}}} \leq x \right) - G_\alpha(x) \right| = \begin{cases} O(n^{-\frac{2\alpha}{1-2\alpha}}) & \text{for } 0 < \alpha < \frac{1}{4} \\ O\left(\frac{\log n}{n}\right) & \text{for } \alpha = \frac{1}{4} \\ O\left(\frac{1}{n}\right) & \text{for } \frac{1}{4} < \alpha < \frac{1}{2} \end{cases}$$

as $n \rightarrow \infty$.

It should be mentioned that both Theorems 1 and 2 play an important role in proving rates of convergence of power sums of partial quotients of continued fraction expansions (see Heinrich [6, 7] and Lévy [10] for historical background).

The paper is organized as follows: Section 2 presents the estimation method which enables us to prove Theorem 1, and Section 3 contains the proof of Theorem 2. In Section 4 we round off the topic by considering the case $\frac{1}{2} \leq \alpha \leq 1$ which requires comparatively elementary techniques. The remaining case $1 < \alpha < 2$ is omitted because its treatment follows the classical line using pseudo-moments and difference moments (see [2] and references therein).

Throughout, let $c_1(\cdot), c_2(\cdot), \dots$ denote positive constants which may be depend on parameters indicated in parenthesis.

2. Proof of Theorem 1

Throughout we use the notations $\psi(x) = [x] - x + \frac{1}{2}$ and $\beta = \min\{1, \frac{2\alpha}{1-2\alpha}\}$ for $0 < \alpha < \frac{1}{2}$. At the beginning we get rid of those parts of $\Delta_\alpha(t)$ which can be easily estimated.

Lemma 1. For $0 < \alpha < \frac{1}{2}$ and $0 < |t| \leq 1$ we have

$$\left| \Delta_\alpha(t) - it \int_{1/|t|}^{1/|t|^{1+\beta}} e^{itx} \frac{\psi(x^\alpha)}{x^{2\alpha}} dx \right| \leq c_1(\alpha)t^{\alpha(1+\beta)}.$$

Proof. Put

$$I_\alpha(t) = it \int_{1/|t|}^{1/|t|^{1+\beta}} e^{itx} \left(\frac{1}{x^\alpha} - \frac{1}{[x^\alpha] + 1} \right) dx \quad (0 < |t| \leq 1).$$

Since $I_\alpha(-t) = \overline{I_\alpha(t)}$ and $\Delta_\alpha(-t) = \overline{\Delta_\alpha(t)}$, it suffices to consider positive t -values $0 < t \leq 1$. We first show that

$$|\Delta_\alpha(t) - I_\alpha(t)| \leq \left(4 + \frac{1}{1-2\alpha} \right) t^{\alpha(1+\beta)}. \tag{2.1}$$

For this end we split up the difference $\Delta_\alpha(t) - I_\alpha(t)$ as follows:

$$\begin{aligned} \Delta_\alpha(t) - I_\alpha(t) &= it \int_0^{1/t} e^{itx} \frac{\psi(x^\alpha) + \frac{1}{2}}{x^\alpha([x^\alpha] + 1)} dx + it \int_{1/t^{1+\beta}}^\infty \frac{e^{itx}}{x^\alpha} dx - it \int_{1/t^{1+\beta}}^\infty \frac{e^{itx}}{[x^\alpha] + 1} dx \\ &=: I_\alpha^{(1)}(t) + I_\alpha^{(2)}(t) - I_\alpha^{(3)}(t). \end{aligned}$$

Obviously, since $|\psi(x)| \leq \frac{1}{2}$,

$$|I_\alpha^{(1)}(t)| \leq t \int_0^{1/t} \frac{dx}{x^{2\alpha}} = \frac{t^{2\alpha}}{1-2\alpha}$$

and integration by parts yields

$$\begin{aligned} |I_\alpha^{(3)}(t)| &= \lim_{T \rightarrow \infty} \left| \frac{e^{iT}}{[T^\alpha] + 1} - \frac{e^{it^\beta}}{[t^{-\alpha(1+\beta)}] + 1} - \int_{1/t^{1+\beta}}^T e^{itx} d\left(\frac{1}{[x^\alpha] + 1} \right) \right| \\ &\leq \frac{2}{[t^{-\alpha(1+\beta)}] + 1}. \end{aligned}$$

In the same way we get

$$|I_\alpha^{(2)}(t)| \leq 2t^{\alpha(1+\beta)}$$

which together with the preceding estimates implies (2.1). By using the identity

$$\frac{1}{[x^\alpha] + 1} = \frac{1}{x^\alpha} - \frac{\psi(x^\alpha) + \frac{1}{2}}{x^\alpha([x^\alpha] + 1)}$$

we may rewrite $I_\alpha(t)$ for $0 < \alpha < \frac{1}{2}$ in the following way:

$$\begin{aligned} I_\alpha(t) &= it \int_{1/t}^{1/t^{1+\beta}} e^{itx} \frac{\psi(x^\alpha) + \frac{1}{2}}{x^\alpha([x^\alpha] + 1)} dx \\ &= \frac{it}{2} \int_{1/t}^{1/t^{1+\beta}} \frac{e^{itx}}{x^{2\alpha}} dx \\ &\quad - it \int_{1/t}^{1/t^{1+\beta}} e^{itx} \frac{(\psi(x^\alpha) + \frac{1}{2})^2}{x^{2\alpha}([x^\alpha] + 1)} dx \\ &\quad + it \int_{1/t}^{1/t^{1+\beta}} e^{itx} \frac{\psi(x^\alpha)}{x^{2\alpha}} dx. \end{aligned}$$

Integrating by parts yields

$$\begin{aligned} \left| \frac{it}{2} \int_{1/t}^{1/t^{1+\beta}} \frac{e^{itx}}{x^{2\alpha}} dx \right| &= \frac{t^{2\alpha}}{2} \left| \int_1^{t^{-\beta}} \frac{d(e^{ix})}{x^{2\alpha}} \right| \\ &= \frac{t^{2\alpha}}{2} \left| e^{it^{-\beta}} t^{2\alpha\beta} - e^i - \int_1^{t^{-\beta}} e^{ix} d(x^{-2\alpha}) \right| \\ &\leq t^{2\alpha} \end{aligned}$$

and, by $0 \leq \psi(x) + \frac{1}{2} \leq 1$ and $2\alpha(1 + \beta) \geq \beta$, we see that

$$\left| it \int_{1/t}^{1/t^{1+\beta}} e^{itx} \frac{(\psi(x^\alpha) + \frac{1}{2})^2}{x^{2\alpha}([x^\alpha] + 1)} dx \right| \leq t^{3\alpha} \int_1^{t^{-\beta}} \frac{dx}{x^{3\alpha}} \leq \begin{cases} \frac{t^{\alpha(1+\beta)}}{1-3\alpha} & \text{for } 0 < \alpha < \frac{1}{3} \\ t \log \frac{1}{t} & \text{for } \alpha = \frac{1}{3} \\ \frac{t^{3\alpha}}{3\alpha-1} & \text{for } \frac{1}{3} < \alpha < \frac{1}{2}. \end{cases}$$

The latter two estimates combined with (2.1) prove Lemma 1 ■

The remaining integral term in the difference $\Delta_\alpha(t) - I_\alpha(t)$ is more resistant and requires deeper considerations.

Lemma 2. For $0 < \alpha < \frac{1}{2}$ and $0 < |t| \leq 1$ we have

$$\left| it \int_{1/t}^{1/t^{1+\beta}} e^{itx} \frac{\psi(x^\alpha)}{x^{2\alpha}} dx \right| \leq c_2(\alpha) \begin{cases} |t|^{\alpha(1+\beta)} & \text{for } 0 < \alpha < \frac{1}{2}, \alpha \neq \frac{1}{4} \\ |t|^{\frac{1}{2}} \log \frac{1}{|t|} & \text{for } \alpha = \frac{1}{4}. \end{cases}$$

Proof. In view of the periodicity $\psi(x) = \psi(x + 1)$ for $x \in \mathbb{R}$ the function ψ admits the Fourier series representation

$$\psi(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n} = \frac{i}{2\pi} \sum_{n=1}^{\infty} \frac{e^{-2\pi inx} - e^{2\pi inx}}{n} \quad (x \notin \mathbb{Z}).$$

This series converges uniformly in any closed interval not containing any integer, and its value is 0 for $x \in \mathbb{Z}$. Furthermore, the partial sums $\sum_{n=1}^N \frac{\sin(2\pi nx)}{n}$ are uniformly bounded which allows term-wise integration leading to

$$it \int_{1/t}^{1/t^{1+\beta}} e^{itx} \frac{\psi(x^\alpha)}{x^{2\alpha}} dx = \sum_{n=1}^{\infty} (J_{\alpha,n}^+(t) - J_{\alpha,n}^-(t)) \quad (0 < t \leq 1) \quad (2.2)$$

where

$$J_{\alpha,n}^\pm(t) = \frac{t}{2\pi n} \int_{1/t}^{1/t^{1+\beta}} e^{itx \pm 2\pi i n x^\alpha} \frac{dx}{x^{2\alpha}} \quad (n \in \mathbb{N}, 0 < \alpha < \frac{1}{2}).$$

Substituting $x = y^a \frac{\lambda_n(t)}{t}$ with $\lambda_n(t) = (\frac{2\pi n}{t^\alpha})^{\frac{1}{1-\alpha}}$ and $a = \frac{1}{\alpha}$ we may write

$$\begin{aligned} J_{\alpha,n}^\pm(t) &= \frac{i a t^{2\alpha} (\lambda_n(t))^{1-2\alpha}}{n} \int_{(\lambda_n(t))^{-\alpha}}^{(t^\beta \lambda_n(t))^{-\alpha}} e^{i \lambda_n(t)(z^a - z)} \frac{dz}{z^{3-a}} \\ &= \frac{a t^{2\alpha}}{n (\lambda_n(t))^{2\alpha}} \int_{(\lambda_n(t))^{-\alpha}}^{(t^\beta \lambda_n(t))^{-\alpha}} \frac{d(e^{i \lambda_n(t)(z^a \pm z)})}{(a z^{a-1} \pm 1) z^{3-a}}. \end{aligned}$$

We first determine an upper bound of the series $\sum_{n=1}^{\infty} J_{\alpha,n}^+(t)$. Since the function $f(z) = (a z^{a-1} + 1) z^{3-a}$ is positive and strictly increasing on $(0, \infty)$ for $1 < a \leq 3$, it follows, after rewriting the Riemann-Stieltjes integral

$$\int_A^B \frac{1}{f(z)} d(e^{i \lambda_n(t)(z^a + z)}) \quad \text{with} \quad \begin{aligned} A &= (\lambda_n(t))^{-\alpha} \\ B &= (t^\beta \lambda_n(t))^{-\alpha} \end{aligned}$$

by partial integration that

$$\left| \int_A^B \frac{d(e^{i \lambda_n(t)(z^a + z)})}{f(z)} \right| \leq \frac{2}{f(A)} \leq 2(\lambda_n(t))^{3\alpha-1}.$$

Together with $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$,

$$\sum_{n=1}^{\infty} |J_{\alpha,n}^+(t)| \leq 2 a t^{2\alpha} \sum_{n=1}^{\infty} \frac{(\lambda_n(t))^{\alpha-1}}{n} \leq \frac{\pi a}{6} t^{3\alpha} \quad (\frac{1}{3} \leq \alpha < 1). \quad (2.3)$$

In the case of $a > 3$, the function f is decreasing on $(0, z_+)$ and increasing on (z_+, ∞) where $z_+ = (\frac{a-3}{2a})^{\frac{1}{a-1}}$. Again after integrating by parts we get the estimate

$$\begin{aligned} \left| \int_A^B \frac{d(e^{i \lambda_n(t)(z^a + z)})}{f(z)} \right| &\leq \frac{1}{f(A)} + \frac{1}{f(B)} + \int_A^B \left| d\left(\frac{1}{f(z)}\right) \right| \\ &\leq \begin{cases} \frac{2}{f(z_+)} & \text{for } A \leq z_+ \leq B \\ \frac{2}{f(B)} & \text{for } A < B < z_+. \end{cases} \end{aligned}$$

Without loss of generality, assume $0 < t \leq \left(\frac{a-3}{a}\right)^a (< 1)$ implying $(\lambda_n(t))^{-\alpha} \leq z_+$. Since $(t^\beta \lambda_n(t))^{-\alpha} < z_+$ for $n > N(t) := \left\lceil \frac{t^{\alpha(1+\beta)-\beta}}{2\pi z_+^{a-1}} \right\rceil$ and $\sum_{n>N(t)} \frac{1}{n^2} \leq \frac{2}{N(t)+1}$, we find in analogy to (2.3) that

$$\begin{aligned} \sum_{n=1}^{\infty} |J_{\alpha,n}^+(t)| &\leq \frac{2at^{2\alpha}}{f(z_+)} \sum_{n=1}^{N(t)} \frac{1}{(\lambda_n(t))^{2\alpha n}} + 2at^{2\alpha} \sum_{n>N(t)} \frac{t^{(3\alpha-1)\beta}}{(\lambda_n(t))^{1-\alpha n}} \\ &\leq \frac{2at^{2\frac{\alpha}{1-\alpha}}}{(2\pi)^{\frac{2}{a-1}} f(z_+)} \sum_{n=1}^{\infty} n^{-\frac{1+\alpha}{1-\alpha}} + 4az_+^{a-1} t^{2\alpha(1+\beta)}. \end{aligned} \tag{2.4}$$

Thus, summarizing (2.3) and (2.4) we see that

$$\sum_{n=1}^{\infty} |J_{\alpha,n}^+(t)| \leq c_3(\alpha) \max\{t^{3\alpha}, t^{2\frac{\alpha}{1-\alpha}}\} \quad (0 < \alpha < \frac{1}{2}, 0 < t \leq 1). \tag{2.5}$$

To derive a bound of the second series on the right-hand side of (2.2) we decompose the sum

$$\sum_{n=1}^{\infty} J_{\alpha,n}^-(t) = at^{2\alpha} (S_1(t) + S_2(t) + S_3(t)) \tag{2.6}$$

and treat each of the sums $S_i(t)$ ($i = 1, 2, 3$) separately for sufficiently small $t > 0$. Let $z_0 = a^{\frac{1}{1-a}} (< 1)$ denote the positive root of the equation $1 - az^{a-1} = 0$. Further, define the integers

$$\begin{aligned} N_1(t) &= \left\lceil \frac{t^{-\gamma}}{2\pi(z_0 + t^\gamma)^{a-1}} \right\rceil \\ N_2(t) &= \left\lceil \frac{t^{-\gamma}}{2\pi(z_0 - t^\gamma)^{a-1}} \right\rceil \end{aligned} \quad \text{where } \gamma = \beta - \alpha(1 + \beta) = \min\left\{1 - 2\alpha, \frac{\alpha}{1 - 2\alpha}\right\}.$$

We first treat the finite sum

$$S_1(t) = \sum_{n=N_1(t)+1}^{N_2(t)} \frac{(\lambda_n(t))^{1-2\alpha}}{n} \int_{(\lambda_n(t))^{-\alpha}}^{(t^\beta \lambda_n(t))^{-\alpha}} e^{i\lambda_n(t)(z^a - z)} \frac{dz}{z^{3-a}}$$

for $0 < t \leq \left(\frac{z_0}{2}\right)^{\frac{1}{\gamma}}$. In view of

$$(z_0 + t^\gamma)^{a-1} - (z_0 - t^\gamma)^{a-1} \leq 2(a - 1)(z_0 + t^\gamma)^{a-2} t^\gamma$$

the number of summands of $S_1(t)$ is bounded by

$$N_2(t) - N_1(t) \leq 1 + \frac{t^{-\gamma} \left((z_0 + t^\gamma)^{a-1} - (z_0 - t^\gamma)^{a-1} \right)}{2\pi(z_0^2 - t^{2\gamma})^{a-1}} \leq 1 + \frac{(a - 1)2^{a-1}}{\pi z_0^a}$$

for $0 < t \leq \left(\frac{z_0}{2}\right)^{\frac{1}{\gamma}}$. Substituting $z = (\lambda_n(t))^{-\alpha} y$ in each term of $S_1(t)$, we see that

$$|S_1(t)| \leq \sum_{n=N_1(t)+1}^{N_2(t)} \frac{1}{n} \int_1^{t^{-\alpha\beta}} \frac{dy}{y^{3-a}} \leq c_3 t^{\alpha(\beta-1)}.$$

To find an estimate of the infinite sum

$$S_2(t) = \sum_{n > N_2(t)} \frac{(\lambda_n(t))^{-2\alpha}}{n} \int_{(\lambda_n(t))^{-\alpha}}^{(t^\beta \lambda_n(t))^{-\alpha}} \frac{d(e^{i\lambda_n(t)(z^a-z)})}{(az^{a-1} - 1)z^{3-a}}$$

we rewrite the Riemann-Stieltjes integral

$$\int_A^B \frac{1}{g(z)} d(e^{i\lambda_n(t)(z^a-z)}) \quad \text{with } g(z) = (1 - az^{a-1})z^{3-a}$$

by partial integration. This yields

$$\left| \int_A^B \frac{d(e^{i\lambda_n(t)(z^a-z)})}{g(z)} \right| \leq \frac{1}{g(A)} + \frac{1}{g(B)} + \int_A^B \left| d\left(\frac{1}{g(z)}\right) \right|. \tag{2.7}$$

By the choice of $N_1(t)$ and $N_2(t)$ we have

$$(t^\beta \lambda_n(t))^{-\alpha} \begin{cases} \leq z_0 - t^\gamma & \text{for } n > N_2(t) \\ \geq z_0 + t^\gamma & \text{for } n \leq N_1(t). \end{cases}$$

The function g is positive on $[A, B]$ and strictly decreasing there for $a \geq 3$, whereas for $2 < a < 3$ the function g increases on $(0, z_-)$ and decreases on (z_-, z_0) , where $z_- = \left(\frac{3-a}{2a}\right)^{\frac{1}{a-1}}$ ($< z_0$). In either case the total variation of $\frac{1}{g(z)}$ on $[A, B]$ is less than $\frac{1}{g(A)} + \frac{1}{g(B)}$. Hence, by (2.7) and using

$$1 - a(z_0 - t^\gamma)^{a-1} = a(z_0^{a-1} - (z_0 - t^\gamma)^{a-1}) \geq t^\gamma a(a-1)(z_0 - t^\gamma)^{a-2}$$

we obtain for $n > N_2(t)$ that

$$\begin{aligned} \left| \int_{(\lambda_n(t))^{-\alpha}}^{(t^\beta \lambda_n(t))^{-\alpha}} \frac{d(e^{i\lambda_n(t)(z^a-z)})}{g(z)} \right| &\leq \frac{2}{g((\lambda_n(t))^{-\alpha})} + \frac{2}{g((t^\beta \lambda_n(t))^{-\alpha})} \\ &\leq 2(\lambda_n(t))^{3\alpha-1} \left(\frac{1}{1 - \frac{at^\alpha}{2\pi}} + \frac{t^{(3\alpha-1)\beta}}{1 - a(z_0 - t^\gamma)^{a-1}} \right) \\ &\leq (\lambda_n(t))^{3\alpha-1} \left(4 + \frac{2^{a-1}t^{2(2\alpha-1)\beta+\alpha}}{a(a-1)z_0^{a-2}} \right) \end{aligned}$$

for $0 < t \leq \min \left\{ \left(\frac{z_0}{2}\right)^{\frac{1}{\gamma}}, \left(\frac{\pi}{a}\right)^a \right\}$. Remembering that $(\lambda_n(t))^{\alpha-1} = \frac{t^\alpha}{2\pi n}$ and $\sum_{n > N} \frac{1}{n^2} \leq \frac{1}{N}$, we obtain

$$\begin{aligned} |S_2(t)| &\leq \sum_{n > N_2(t)} \frac{t^\alpha}{\pi n^2} \left(2 + \frac{2^{a-2}t^{2(2\alpha-1)\beta+\alpha}}{a(a-1)z_0^{a-2}} \right) \\ &\leq c_4 t^{(1-\alpha)\beta} + c_5 t^{3\alpha\beta+\alpha-\beta} \quad (\leq (c_4 + c_5)t^{\alpha(\beta-1)}) \end{aligned}$$

for $0 < t \leq \min\{(\frac{z_0}{2})^{\frac{1}{\gamma}}, (\frac{\pi}{a})^a\}$.

To estimate the remaining finite sum

$$S_3(t) = \sum_{n=1}^{N_1(t)} \frac{(\lambda_n(t))^{-2\alpha}}{n} \int_{(\lambda_n(t))^{-\alpha}}^{(t^\beta \lambda_n(t))^{-\alpha}} \frac{d(e^{i\lambda_n(t)(z^a-z)})}{(az^{a-1}-1)z^{3-a}}$$

we need a further integer $N_0(t) = \lceil \frac{t^{-\gamma}}{2\pi(2z_0)^{a-1}} \rceil$ being less than $N_1(t)$ for $0 < t \leq (\frac{z_0}{2})^{\frac{1}{\gamma}}$ and chosen so that $z_0 + t^\gamma \leq (t^\beta \lambda_n(t))^{-\alpha} < 2z_0$ for $n \in \{N_0(t) + 1, \dots, N_1(t)\}$. Again integrating by parts yields

$$\left| \int_{(t^\beta \lambda_n(t))^{-\alpha}}^{2z_0} \frac{d(e^{i\lambda_n(t)(z^a-z)})}{(az^{a-1}-1)z^{3-a}} \right| \leq \frac{2(t^\beta \lambda_n(t))^{3\alpha-1}}{a(z_0 + t^\gamma)^{a-1} - 1} \leq c_6(\lambda_n(t))^{3\alpha-1} t^{2(2\alpha-1)\beta+\alpha}$$

for $N_0(t) < n \leq N_1(t)$. Hence, the sum

$$S_3^*(t) = \sum_{n=N_0(t)+1}^{N_1(t)} \frac{(\lambda_n(t))^{-2\alpha}}{n} \int_{(t^\beta \lambda_n(t))^{-\alpha}}^{2z_0} \frac{d(e^{i\lambda_n(t)(z^a-z)})}{(az^{a-1}-1)z^{3-a}}$$

allows the estimate

$$|S_3^*(t)| \leq c_6 t^{2(2\alpha-1)\beta+\alpha} \sum_{n=N_0(t)+1}^{N_1(t)} \frac{(\lambda_n(t))^{\alpha-1}}{n} \leq c_7 t^{3\alpha\beta+\alpha-\beta} \leq c_7 t^{\alpha(\beta-1)}. \tag{2.8}$$

Assume in addition $0 < t \leq (\frac{2\pi(a-3)}{a})^a$ implying $(\lambda_n(t))^{-\alpha} \leq z_-$ for any $n \in \mathbb{N}$. For these t -values we may rewrite $S_3(t)$ in the following way:

$$S_3(t) = i \sum_{n=1}^{N_1(t)} \frac{(\lambda_n(t))^{1-2\alpha}}{n} (K_{\alpha,n}^-(t) + K_{\alpha,n}^+(t)) - S_3^*(t) + \sum_{n=1}^{N_1(t)} \frac{(\lambda_n(t))^{-2\alpha}}{n} \int_{(\lambda_n(t))^{-\alpha}}^{z_-} \frac{d(e^{i\lambda_n(t)(z^a-z)})}{(az^{a-1}-1)z^{3-a}} \tag{2.9}$$

$$+ \sum_{n=1}^{N_0(t)} \frac{(\lambda_n(t))^{-2\alpha}}{n} \int_{2z_0}^{(t^\beta \lambda_n(t))^{-\alpha}} \frac{d(e^{i\lambda_n(t)(z^a-z)})}{(az^{a-1}-1)z^{3-a}} \tag{2.10}$$

where

$$K_{\alpha,n}^-(t) = \int_{z_-}^{z_0} e^{i\lambda_n(t)(z^a-z)} \frac{dz}{z^{3-a}} \quad \text{and} \quad K_{\alpha,n}^+(t) = \int_{z_0}^{2z_0} e^{i\lambda_n(t)(z^a-z)} \frac{dz}{z^{3-a}}.$$

Once more employing the partial integration formula, we get

$$\left| \int_{2z_0}^{(t^\beta \lambda_n(t))^{-\alpha}} \frac{d(e^{i\lambda_n(t)(z^a-z)})}{(az^{a-1}-1)z^{3-a}} \right| \leq \frac{2}{(a(2z_0)^{a-1}-1)(2z_0)^{3-a}}$$

for $1 \leq n \leq N_0(t)$ and, by applying (2.7) with $B = z_-$,

$$\left| \int_{(\lambda_n(t))^{-\alpha}}^{z_-} \frac{d(e^{i\lambda_n(t)(z^a-z)})}{(1-az^{a-1})z^{3-a}} \right| \leq \begin{cases} \frac{2\lambda_n(t)^{3\alpha-1}}{1-\frac{at^\alpha}{2\pi}} & \text{for } 0 < \alpha \leq \frac{1}{3} \\ \frac{2z_-^{a-3}}{1-az_-^{a-1}} & \text{for } \frac{1}{3} < \alpha < \frac{1}{2} \end{cases}$$

whence it follows after a short calculation that the series (2.9) and (2.10) are bounded by $c_8 \max \{t^{\frac{2}{1-\alpha}}, t^\alpha\}$.

We now turn to estimate the integrals $K_{\alpha,n}^+(t)$ and $K_{\alpha,n}^-(t)$. For this purpose, we apply the *method of stationary phase* (see Erdélyi [5] or Krätzel [9: pp. 204 – 206]) to $K_{\alpha,n}^+(t)$. The function $h(z) = z^a - z$ possesses the unique stationary point $z = z_0$, that is $h'(z_0) = 0$. We introduce the new variable of integration $u = u(z) := \sqrt{h(z) - h(z_0)}$ which is positive and strictly increasing for $z_0 < z \leq 2z_0$. Using the inverse function $z = z(u) := h^{-1}(u^2 + h(z_0))$, we obtain

$$K_{\alpha,n}^+(t) = e^{i\lambda_n(t)h(z_0)} \int_0^{u_0} e^{i\lambda_n(t)u^2} (z(u))^{a-3} z'(u) du$$

where $u_0 = \sqrt{h(2z_0) - h(z_0)}$. Further, define the twice continuously differentiable function

$$\varphi(u) = \frac{(z(u))^{a-2}}{a-2}$$

on $[0, u_0]$. The mean-value theorem gives

$$\varphi'(u) = (z(u))^{a-3} z'(u) = \varphi'(0) + u\varphi''(\vartheta(u)) \quad \text{for some } \vartheta(u) \in (0, u)$$

with $\varphi'(0) = z_0^{a-3} z'(0)$ and $z'(0) = \sqrt{2/h''(z_0)}$. The latter is a consequence of the rule of L'Hospital applied to the function $z'(u) = \frac{2u}{h'(h^{-1}(h(z_0)+u^2))}$. Therefore,

$$\begin{aligned} K_{\alpha,n}^+(t) &= \frac{\sqrt{2}z_0^{a-3}e^{i\lambda_n(t)h(z_0)}}{\sqrt{h''(z_0)}} \int_0^{u_0} e^{i\lambda_n(t)u^2} du + e^{i\lambda_n(t)h(z_0)} \int_0^{u_0} e^{i\lambda_n(t)u^2} u\varphi''(\vartheta(u)) du \\ &= \frac{z_0^{a-3}e^{i\lambda_n(t)h(z_0)}}{\sqrt{2h''(z_0)\lambda_n(t)}} \int_0^{u_0^2\lambda_n(t)} \frac{e^{iv}}{\sqrt{v}} dv + \frac{e^{i\lambda_n(t)h(z_0)}}{2\lambda_n(t)} \int_0^{u_0} \varphi''(\vartheta(u)) d(e^{i\lambda_n(t)u^2}). \end{aligned} \tag{2.11}$$

Since the function $\varphi(u)$ has a bounded third derivative on $[0, u_0]$, we see that the integral $\int_0^{u_0} \varphi''(\vartheta(u)) d(e^{i\lambda_n(t)u^2})$ is bounded in $t \in (0, 1]$ and $n \in \mathbb{N}$. This fact and the estimate

$$\left| \int_0^{u_0^2\lambda_n(t)} \frac{e^{iv}}{\sqrt{v}} dv - \int_0^\infty \frac{e^{iv}}{\sqrt{v}} dv \right| \leq c_9 \frac{1}{\sqrt{\lambda_n(t)}}$$

which is seen from the second mean-value theorem, combined with $\int_0^\infty \frac{e^{iv}}{\sqrt{v}} dv = \sqrt{\pi} e^{i\frac{\pi}{4}}$ (see Section 1) and (2.11) yield

$$\left| K_{\alpha,n}^+(t) - \frac{z_0^{a-3}\sqrt{\pi}e^{i\lambda_n(t)h(z_0)+i\frac{\pi}{4}}}{\sqrt{2h''(z_0)\lambda_n(t)}} \right| \leq c_{10} \frac{1}{\lambda_n(t)}.$$

In quite the same manner we find the estimate

$$\left| K_{\alpha,n}^-(t) - \frac{z_0^{a-3} \sqrt{\pi} e^{i\lambda_n(t)h(z_0)+i\frac{\pi}{4}}}{\sqrt{2h''(z_0)\lambda_n(t)}} \right| \leq c_{11} \frac{1}{\lambda_n(t)}.$$

Finally, on combining the preceding two estimates, (2.8) and the above-obtained bounds of (2.9) and (2.10) we arrive at

$$\left| S_3(t) - \frac{\sqrt{2\pi} z_0^{a-3} e^{i\frac{3\pi}{4}}}{\sqrt{a(a-1)z_0^{a-2}}} \sum_{n=1}^{N_1(t)} \frac{e^{i\lambda_n(t)(z_0^a-z_0)}}{n(\lambda_n(t))^{2\alpha-\frac{1}{2}}} \right| \leq c_{12} \begin{cases} t^{\alpha(\beta-1)} & \text{for } 0 < \alpha \leq \frac{1}{4} \\ \max\{t^\alpha, t^{4\alpha-1}\} & \text{for } \frac{1}{4} < \alpha < \frac{1}{2}. \end{cases}$$

The crucial step in proving Lemma 2 is the evaluation of the finite exponential sum

$$\begin{aligned} \Sigma_\alpha(t) &:= \sum_{n=1}^{N_1(t)} \frac{e^{i\lambda_n(t)(z_0^a-z_0)}}{n(\lambda_n(t))^{\frac{4\alpha-1}{2}}} \\ &= \left(\frac{t^\alpha}{2\pi}\right)^{\frac{4\alpha-1}{2(1-\alpha)}} \sum_{n=1}^{N_1(t)} \exp\left\{i(z_0^a-z_0)\left(\frac{2\pi n}{t^\alpha}\right)^{\frac{1}{1-\alpha}}\right\} n^{-\frac{1+2\alpha}{2(1-\alpha)}}. \end{aligned}$$

Using the well-known facts that

$$\begin{aligned} \sum_{n=1}^N \frac{1}{n} &\leq 1 + \log N \\ \sum_{n=1}^N \frac{1}{n^{1-s}} &\leq \frac{N^s}{s} \end{aligned} \quad ((0 < s < 1, N \in \mathbb{N}))$$

we find the estimates

$$|\Sigma_\alpha(t)| \leq c_{13}(\alpha) \begin{cases} t^{\alpha(\beta-1)} & \text{for } 0 < \alpha < \frac{1}{4} \\ \log \frac{1}{t} & \text{for } \alpha = \frac{1}{4} \\ t^{\frac{\alpha(4\alpha-1)}{2(1-\alpha)}} & \text{for } \frac{1}{4} < \alpha < \frac{1}{2} \end{cases} \tag{2.12}$$

so that, somewhat crude for $\frac{1}{4} < \alpha < \frac{1}{2}$,

$$|S_3(t)| \leq c_{14}(\alpha) \begin{cases} t^{\alpha(\beta-1)} & \text{for } 0 < \alpha < \frac{1}{2}, \alpha \neq \frac{1}{4} \\ \log \frac{1}{t} & \text{for } \alpha = \frac{1}{4}. \end{cases}$$

Finally, inserting the obtained bounds of $S_1(t)$, $S_2(t)$ and $S_3(t)$ into the right-hand side of (2.6) and on combining this with (2.5) and (2.2) we complete the proof of Lemma 2 ■

Remark 1. The asymptotic expansions of the *Airy integral* $\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{i(\frac{x^3}{3} + zx)\} dx$ for large positive values of z (see [5]) can be used to study the behaviour of $J_{\frac{1}{3},n}^\pm(t)$ (or $K_{\frac{1}{3},n}^\pm(t)$) when $\lambda_n(t)$ becomes large.

Remark 2. Estimates (2.12) for $0 < \alpha \leq \frac{1}{4}$ could not be improved even by applying *van der Corput's method* (i.e. transforming $\Sigma_\alpha(t)$ in a corresponding Fourier integral with appropriately optimized remainder term) (see [4: Chapter 2], [9: Chapter 2] or [11: Chapter 3]). On the other hand, it is an open question whether there exists an approximation of $\Sigma_\alpha(t)$ from below having the order of the left-hand side of (2.12).

3. Proof of Theorem 2

The proof of Theorem 2 is based on Esseen’s basic inequality (see, e.g., Ibragimov and Linnik [8: Theorem 1.5.2]) which, for any sequence Z_n with α -stable limit distribution $G_\alpha(x)$ takes the form

$$\sup_{x \in \mathbb{R}} |\mathbb{P}(Z_n \leq x) - G_\alpha(x)| \leq \frac{1}{\pi} \int_{-T}^T \left| \frac{\mathbb{E}e^{itZ_n} - \widehat{G}_\alpha(t)}{t} \right| dt + \frac{24}{\pi T} \sup_x G'_\alpha(x) \quad (3.1)$$

for any $T > 0$. In our situation we take $Z_n = n^{-\frac{1}{\alpha}}(X_1^{\frac{1}{\alpha}} + \dots + X_n^{\frac{1}{\alpha}})$ and $T = \rho_0 n^{\frac{1}{\alpha}}$ with suitably chosen $\rho_0 > 0$. Theorem 1 implies $\log \widehat{F}_\alpha(t) - (\widehat{F}_\alpha(t) - 1) = O(|t|^{2\alpha})$ as $t \rightarrow 0$. Together with the obvious relation $n \log \widehat{G}_\alpha(\frac{t}{n^{1/\alpha}}) = \log \widehat{G}_\alpha(t)$, Theorem 1 reveals that there exists some sufficiently small $\rho_0 > 0$ such that, for $|t| \leq \rho_0 n^{\frac{1}{\alpha}}$,

$$|\log \mathbb{E}e^{itZ_n} - \log \widehat{G}_\alpha(t)| = n \left| \log \widehat{F}_\alpha\left(\frac{t}{n^{\frac{1}{\alpha}}}\right) - \log \widehat{G}_\alpha\left(\frac{t}{n^{\frac{1}{\alpha}}}\right) \right| \leq \frac{\lambda_\alpha}{2} |t|^\alpha \quad (3.2)$$

and

$$|\log \mathbb{E}e^{itZ_n} - \log \widehat{G}_\alpha(t)| \leq c_{15}(\alpha) \begin{cases} \left(\frac{|t|^\alpha}{n^{2\alpha}}\right)^{\frac{1}{1-2\alpha}} & \text{for } 0 < \alpha < \frac{1}{4} \\ \frac{|t|^{\frac{1}{2}}}{n} \log \frac{n^4}{|t|} & \text{for } \alpha = \frac{1}{4} \\ \frac{|t|^{2\alpha}}{n} & \text{for } \frac{1}{4} < \alpha < \frac{1}{2}. \end{cases} \quad (3.3)$$

Using the elementary inequality $|e^z - 1| \leq |z|e^{|z|}$ with $z = \log \mathbb{E}e^{itZ_n} - \log \widehat{G}_\alpha(t)$ combined with (3.2) and $|\widehat{G}_\alpha(t)| = \exp\{-\lambda_\alpha |t|^\alpha\}$ we obtain the estimate

$$\left| \frac{\mathbb{E}e^{itZ_n} - \widehat{G}_\alpha(t)}{t} \right| \leq c_{15}(\alpha) \frac{R_{\alpha,n}(t)}{|t|} \exp\left\{-\frac{\lambda_\alpha}{2} |t|^\alpha\right\} \quad (|t| \leq T),$$

where $R_{\alpha,n}(t)$ denotes the terms on the right-hand side of (3.3) (after the curled bracket). In view of (1) this last inequality and $\sup_{x \in \mathbb{R}} G'_\alpha(x) < \infty$ (see [2]) prove Theorem 2 ■

4. The case $\frac{1}{2} \leq \alpha \leq 1$

As mentioned in Section 1, the estimation of $\Delta_\alpha(t)$ for $\frac{1}{2} < \alpha < 1$ becomes considerably simpler because the Fourier integral $\frac{\Delta_\alpha(t)}{it}$ is absolutely integrable. In the case of $\alpha = \frac{1}{2}$ and $\alpha = 1$ we establish a first order asymptotic expansion involving logarithmic terms which entails optimal rates of convergence in Theorem 4.

Theorem 3. For $\frac{1}{2} \leq \alpha < 1$ we have

$$\Delta_\alpha(t) = \begin{cases} \frac{it}{2} \log \frac{1}{|t|} + O(|t|) & \text{for } \alpha = \frac{1}{2} \\ it \int_0^\infty \left(\frac{1}{x^\alpha} - \frac{1}{\lceil x^\alpha \rceil + 1}\right) dx + O(|t|^{2\alpha}) & \text{for } \frac{1}{2} < \alpha < 1 \end{cases} \quad (4.1)$$

as $t \rightarrow 0$.

Proof. Assume that $\frac{1}{2} < \alpha < 1$. Using the inequalities

$$|e^{itx} - 1| \leq \min\{2, |tx|\} \quad \text{and} \quad \left| \frac{1}{x^\alpha} - \frac{1}{[x^\alpha] + 1} \right| \leq \min\{x^{-\alpha}, x^{-2\alpha}\}$$

we see that the integral $\int_0^\infty (x^{-\alpha} - ([x^\alpha] + 1)^{-1}) dx$ exists and, moreover, that

$$\begin{aligned} \left| \int_0^\infty (e^{itx} - 1) \left(\frac{1}{x^\alpha} - \frac{1}{[x^\alpha] + 1} \right) dx \right| &\leq |t| \int_0^{1/|t|} x^{1-2\alpha} dx + 2 \int_{1/|t|}^\infty x^{-2\alpha} dx \\ &= \left(\frac{1}{2(1-\alpha)} + \frac{2}{2\alpha-1} \right) |t|^{2\alpha-1} \end{aligned}$$

for any $t \in \mathbb{R}$. But this proves the desired estimate.

The case $\alpha = \frac{1}{2}$ is a little more involved. By splitting the integral on the left-hand side of (4.1) and integrating by parts, we get

$$\begin{aligned} &\left| \left(\int_0^1 + \int_{1/t^2}^\infty \right) e^{itx} \left(\frac{1}{\sqrt{x}} - \frac{1}{[\sqrt{x}] + 1} \right) dx \right| \\ &\leq \int_0^1 \left(\frac{1}{\sqrt{x}} - 1 \right) dx + \left| \frac{1}{it} \int_{1/t^2}^\infty \frac{d(e^{itx})}{\sqrt{x}} \right| + \left| \frac{1}{it} \int_{1/t^2}^\infty \frac{d(e^{itx})}{[\sqrt{x}] + 1} \right| \\ &\leq 5. \end{aligned}$$

It is easily checked that

$$\begin{aligned} &\int_1^{1/t^2} e^{itx} \left(\frac{1}{\sqrt{x}} - \frac{1}{[\sqrt{x}] + 1} \right) dx \\ &= \frac{1}{2} \int_1^{1/t^2} \frac{e^{itx}}{x} dx + \int_1^{1/t^2} \frac{e^{itx}}{x} \psi(\sqrt{x}) dx - \int_1^{1/t^2} \frac{e^{itx} (\psi(\sqrt{x}) + \frac{1}{2})^2}{x([\sqrt{x}] + 1)} dx \end{aligned}$$

and, for $0 < t \leq 1$,

$$\int_1^{1/t^2} \frac{e^{itx}}{x} dx = \int_t^{1/t} \frac{e^{ix}}{x} dx = \log \frac{1}{t} + \int_t^1 \frac{e^{ix} - 1}{x} dx + \int_1^{1/t} \frac{e^{ix}}{x} dx$$

where the latter two integrals are bounded. It remains to show the boundedness of

$$\int_1^{1/t^2} \frac{e^{itx}}{x} \psi(\sqrt{x}) dx = 2 \int_1^{1/t} \frac{e^{itx^2}}{x} \psi(x) dx.$$

This is seen after partial integration and using the fact that $0 \leq \int_1^x \psi(z) dz \leq \frac{1}{8}$ for $x > 1$. This completes the proof of Theorem 3 ■

For $\alpha = 1$, both the real and imaginary part of $\widehat{F}_1(t)$ can be expressed by the well-known trigonometric sums

$$S(t) = \sum_{k=1}^{\infty} \frac{\sin(kt)}{k} = \frac{\pi}{2} \operatorname{sgn}(t) - \frac{t}{2} \quad \text{and} \quad C(t) = \sum_{k=1}^{\infty} \frac{\cos(kt)}{k} = -\log\left(2 \sin \frac{|t|}{2}\right)$$

which converge for $t \in (-2\pi, 2\pi) \setminus \{0\}$. A short calculation shows that

$$\begin{aligned} \widehat{F}_1(t) - 1 &= \sum_{k=1}^{\infty} \frac{\cos(kt) - 1}{k(k+1)} + i \sum_{k=1}^{\infty} \frac{\sin(kt)}{k(k+1)} \\ &= (1 - \cos t + i \sin t)(C(t) + iS(t)) \\ &= (-t)(S(t) - iC(t)) + \frac{t^2}{2} C(t) + O(t^2) \\ &= \log \widehat{G}_1(t) - \frac{t^2}{2} \log |t| + O(t^2) \end{aligned}$$

as $t \rightarrow 0$, where the last two lines follows from (1.1) and by using the Taylor expansions of $\sin x$, $\cos x$ and $\log(1+x)$. Together with

$$\begin{aligned} \log \widehat{F}_1(t) &= \widehat{F}_1(t) - 1 - \frac{1}{2}(\widehat{F}_1(t) - 1)^2 + O(t^2) \\ &= \widehat{F}_1(t) - 1 + \frac{t^2}{2} \log \frac{1}{|t|} \left(i\pi \operatorname{sgn}(t) - \log \frac{1}{|t|} \right) + O(t^2) \end{aligned}$$

we arrive at

$$\log \widehat{F}_1(t) = \log \widehat{G}_1(t) + \frac{t^2}{2} \log \frac{1}{|t|} \left(1 - i\pi \operatorname{sgn}(t) + \log \frac{1}{|t|} \right) + O(t^2) \tag{4.2}$$

as $t \rightarrow 0$.

Estimates (4.1) and (4.2) enables us to derive the following counterpart of Theorem 2.

Theorem 4. *For $\frac{1}{2} \leq \alpha \leq 1$ we have uniformly in $x \in \mathbb{R}$*

$$\left| \mathbb{P} \left(\frac{X_1^{\frac{1}{\alpha}} + \dots + X_n^{\frac{1}{\alpha}} - A_n(\alpha)}{n^{\frac{1}{\alpha}}} \leq x \right) - G_{\alpha}(x) \right| = \begin{cases} O(\frac{1}{n}) & \text{for } \frac{1}{2} \leq \alpha < 1 \\ O(\frac{\log^2 n}{n}) & \text{for } \alpha = 1 \end{cases}$$

as $n \rightarrow \infty$, where

$$A_n(\alpha) = \begin{cases} -n \log n & \text{for } \alpha = \frac{1}{2} \\ -n \int_0^{\infty} \left(\frac{1}{x^{\alpha}} - \frac{1}{[x^{\alpha}] + 1} \right) dx & \text{for } \frac{1}{2} < \alpha < 1 \\ n \log n & \text{for } \alpha = 1. \end{cases}$$

Proof. We apply Esseen's inequality (3.1) to the sequence

$$Z_n = n^{-\frac{1}{\alpha}} (X_1^{\frac{1}{\alpha}} + \dots + X_n^{\frac{1}{\alpha}} - A_n(\alpha))$$

with $T = \rho_1 n^{\frac{1}{\alpha}}$. By an obvious consequence of (1.1), (1.3) and Theorem 3,

$$\log \widehat{F}_\alpha(t) - (\widehat{F}_\alpha(t) - 1) = O(|t|^{2\alpha}) \quad \text{as } n \rightarrow \infty, \text{ for } \frac{1}{2} \leq \alpha < 1$$

whence it follows that (4.1) remains valid if the left-hand side is replaced by $\log \widehat{G}_\alpha(t) - \log \widehat{F}_\alpha(t)$ for $\frac{1}{2} \leq \alpha < 1$. This fact and relations (4.2) and $\log \widehat{G}_1(t) = n \log \widehat{G}_1(\frac{t}{n}) - it \log n$ lead to the estimate

$$\begin{aligned} |\log Ee^{itZ_n} - \log \widehat{G}_\alpha(t)| &= \left| \log \widehat{G}_\alpha(t) - n \log \widehat{F}_\alpha\left(\frac{t}{n}\right) + \frac{it}{n^\alpha} A_n(\alpha) \right| \\ &\leq c_{16}(\alpha) \begin{cases} \frac{|t|}{n} \log \frac{1}{|t|} & \text{for } \alpha = \frac{1}{2} \\ \frac{|t|^{2\alpha}}{n} & \text{for } \frac{1}{2} < \alpha < 1 \\ \frac{|t|^2}{n} (\log \frac{n}{|t|})^2 & \text{for } \alpha = 1 \end{cases} \end{aligned}$$

for $|t| \leq T = \rho_1 n^{\frac{1}{\alpha}}$. Here $\rho_1 > 0$ is chosen so that, in addition,

$$|\log Ee^{itZ_n} - \log \widehat{G}_\alpha(t)| \leq \lambda_\alpha \frac{|t|^\alpha}{2} \quad \left(\frac{1}{2} \leq \alpha \leq 1\right)$$

where $\lambda_1 = \frac{\pi}{2}$. Finally, together with $|\widehat{G}_\alpha(t)| \leq \exp\{-\lambda_\alpha |t|^\alpha\}$ (see (1.1) for $\frac{1}{2} \leq \alpha \leq 1$) we complete the proof of Theorem 4 by using the same arguments as at the end of Section 3 ■

To conclude with, we formulate a Tauberian theorem including remainder term for the trigonometric series

$$T_p(t) = \sum_{n=0}^{\infty} \frac{e^{itn}}{[n^{\frac{1}{p}}] + 1}$$

which seems to be of interest for its own.

Corollary. *For any integer $p \geq 2$ we have*

$$T_p(t) = |t|^{\frac{1}{p}-1} \Gamma\left(1 - \frac{1}{p}\right) \left(\sin \frac{\pi}{2p} + i \operatorname{sgn}(t) \cos \frac{\pi}{2p}\right) + \begin{cases} \frac{1}{2} \log \frac{1}{|t|} + O(1) & \text{for } p = 2 \\ O(|t|^{-\frac{1}{3}}) & \text{for } p = 3 \\ O(|t|^{-\frac{1}{2}} \log \frac{1}{|t|}) & \text{for } p = 4 \\ O(|t|^{-\frac{p-3}{p-2}}) & \text{for } p \geq 5 \end{cases}$$

as $t \rightarrow 0$.

Proof. By the very definition of the integer part $[\cdot]$ we may write

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{e^{itn}}{[n^{\frac{1}{p}}] + 1} &= \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=n^p}^{(n+1)^p-1} e^{itk} \\ &= \frac{1}{e^{it} - 1} \sum_{n=0}^{\infty} \frac{(e^{it(n+1)^p} - e^{itn^p})}{n+1} \\ &= \frac{\widehat{F}_{1/p}(t) - 1}{e^{it} - 1} \end{aligned}$$

for $t \notin \{2\pi k : k \in \mathbb{Z}\}$. Using the Taylor expansion $e^{it} - 1 = it + O(t^2)$ as $t \rightarrow 0$ and the estimates of $\widehat{F}_{1/p}(t) - 1$ for $p \geq 3$ from Theorem 1 and for $p = 2$ from Theorem 3, we obtain the asymptotic terms as asserted in the Corollary ■

Remark 3. In order to get the asymptotic behaviour of the trigonometric series $\tilde{T}_p(t) = \sum_{n=1}^{\infty} e^{itn} n^{-\frac{1}{p}}$ we need only to replace O -terms by $O(|t|^{\frac{2-p}{p}})$ for any integer $p \geq 2$. This follows immediately by applying the Euler-Maclaurin sum formula (see, e.g., [9]).

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