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On S-Homogenization of an Optimal Control Problem with Control and State Constraints

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Abstract. We study the limiting behavior of an optimal control problem for a linear elliptic equation subject to control and state constraints. Each constituent of the mathematical description of such an optimal control problem may depend on a small parameter ε. We study the limit of this problem when $\varepsilon \to 0$ in the framework of variational S-convergence which generalizes the concept of Γ-convergence. We also introduce the notion of G^*-convergence generalizing the concept of G-convergence to operators with constraints. We show convergence of the sequence of optimal control problems and identify its limit. We then apply the theory to an elliptic problem on a perforated domain.

Keywords: *Homogenization, S-convergence, optimal control*

AMS subject classification: 35B20, 49J27

1. Introduction

The aim of this paper is to study the homogenization of an optimal control problem with control and state constraints. Each component of the mathematical model of such an optimal control problem may depend on a small parameter ε (e.g. each component may contain rapidly oscillating coefficients).

Let Ω be a bounded open set in $\mathbb{R}^n$. We define the optimal control problem as follows:

$$-\mathrm{div}\,(A_\varepsilon \nabla y) = b_\varepsilon u + f_\varepsilon \quad \text{in } \Omega \tag{1.1}$$

$$y = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

$$y \in K_\varepsilon,\ u \in U_\varepsilon \tag{1.3}$$

$$I_\varepsilon(u,y) = \int_\Omega C_\varepsilon y^2 dx + \int_\Omega (\nabla y, N_\varepsilon \nabla y)_{\mathbb{R}^n} dx + \int_\Omega D_\varepsilon u^2 dx \to \inf . \tag{1.4}$$

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The purpose of this paper is to study the limiting behavior of problem (1.2) - (1.4) as $\varepsilon \to 0$.

A similar problem but one without state constraints has been studied by Kesavan and Saint Jean Paulin [7]. In contrast to the approch in [7], we stay with the optimal control problem in the original sense and look for its homogenized limit, for which we establish a solution then. The method of choice, therefore, is the so-called "direct approch" which is based on the concept of variational S-convergence [10 - 13, 15]. But before introducing the formal concept for the homogenization process via S-convergence we note that the optimal control problem (1.1) - (1.4) can be written in another form

$$\left\{ \left\langle \inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y) \right\rangle \right\}_{\varepsilon\in(0,\varepsilon_0]}$$

where by Ξ_ε we denote the set of all admissible pairs, i.e.

$$\Xi_\varepsilon = \left\{ (u,y) \in L^2(\Omega)\times H_0^1(\Omega) \;\middle|\; \begin{array}{l} -\operatorname{div}(A_\varepsilon \nabla y) = b_\varepsilon u + f_\varepsilon \text{ in } \Omega \\ y = 0 \text{ on } \partial\Omega \\ u \in U_\varepsilon,\ y \in K_\varepsilon \end{array} \right\}.$$

Let us remark that we shall differentiate between the notations $\inf_{x\in A} F(x)$ and $\langle \inf_{x\in A} F(x)\rangle$. In particular, $\inf_{x\in A} F(x)$ means the infimum of F on the set A. By $\langle \inf_{x\in A} F(x)\rangle$ we mean the constrained minimization problem as an object that is defined by the pair $(F;A)$.

We may now return to the main question of our paper. Our aim is to study the limiting behavior of the optimal control problem (1.1) - (1.4) as $\varepsilon \to 0$. The homogenization of (1.1) - (1.4) consists in studying the limit properties of the sequence (1.5). As follows from the concept of variational S-convergence, under some natural assumptions there exists a so-called absolute variational S-limit of the sequence (1.5) denoted by

$$\left\langle \inf_{(u,y)\in\mu-\mathrm{Lm}\,\Xi_\varepsilon} \mu\text{-}\mathrm{lm}^a(I_\varepsilon|_{\Xi_\varepsilon})(u,y) \right\rangle \tag{1.6}$$

where μ is some topology for the basic space $L^2(\Omega)\times H_0^1(\Omega)$, μ-$\mathrm{Lm}\,\Xi_\varepsilon$ is the topological limit of $\{\Xi_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ in Kuratowski's sense [19], μ-$\mathrm{lm}^a(I_\varepsilon|_{\Xi_\varepsilon}) : \tau\text{-}\mathrm{Lm}\,\Xi_\varepsilon \to \overline{\mathbb{R}}$ is the absolute S-limit of the sequence $\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon\in(0,\varepsilon_0]}$.

We emphazise that each of the functionals $I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}$ has its individual domain. This is a principal difference between the concept of S-limit and the of theory of Γ-convergence [4]. Note, however, that under some canonical assumptions S-convergence reduces to Γ-convergence [14].

Let us briefly describe the main result of this paper. In Section 2 we recall the principal results of S-convergence and variational S-convergence which will be used in the sequel.

The topological convergence of the graph restrictions (i.e. restrictions of graphs of linear continuous operators $\mathcal{A}_\varepsilon$ to some admissible sets) is discussed in Section 3. We study this problem for a wide class of control and state constraints. We have shown that the topological limit of such graph restrictions can be recovered or identified in terms of the G-limit operator only for the so-called "convenient" constraints. In order to explore this more in detail (Section 3) for the sequence of coercive operators we introduce the concept of G^*-convergence. We prove a G^*-compactness theorem and obtain the sufficient conditions under which the G^*-limit operator is invertible. It is interesting to note that, as a rule, the G^*-limit operator $\mathcal{A}_*$ does not coincide with the G-limit $\mathcal{A}_0$ and, moreover, the G^*-limit $\mathcal{A}_*$ can be constructed as the sum of the H-limit $\mathcal{A}_0$ and some additional term (called "strange term" by Murat).

Further, in Section 4 we give the application of the above mentioned concept to the homogenization of optimal control problems. We study the existence of the strong S-homogenized problem, recover its mathematical description and establish its variational properties. More precisely, let μ be the topology for the "control-state" space $L^2(\Omega) \times H_0^1(\Omega)$ that equals to the product of the weak topology $w_{L^2(\Omega)}$ for $L^2(\Omega)$ and the weak topology $w_{H_0^1(\Omega)}$ for $H_0^1(\Omega)$.

In Section 5 we obtain sufficient conditions of identifiability of the limit set $\mu\text{-Lm}\,\Xi_\varepsilon$. In particular, under some natural assumptions, the representation

$$\mu - \text{Lm}\Xi_\varepsilon = \left\{ (u,y) \in L^2(\Omega) \times H_0^1(\Omega) \;\middle|\; \begin{array}{l} \mathcal{A}_* y = \widehat{\mathcal{B}}_0 J_0^{-1} u + \widehat{f}_0 \\ u \in w_{L^2(\Omega)}\text{-Lm}\, U_\varepsilon \\ y \in w_{H_0^1(\Omega)}\text{-Lm}\, K_\varepsilon \end{array} \right\}$$

will be obtained where A_* is the G^*-limit of $\{A_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$, f_0 is the weak limit in $H^{-1}(\Omega)$ of $\{f_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$, $\mathcal{B}_\varepsilon$ is some linear continuos operator from $L^2(\Omega)$ into $H^{-1}(\Omega)$, and J_0 is some linear invertible operator from $L^2(\Omega)$ onto $L^2(\Omega)$.

In Section 6 we consider the problem of identification of the functional

$$\mu\text{-lm}^a(I_\varepsilon|_{\Xi_\varepsilon}) : \mu\text{-Lm}\,\Xi_\varepsilon \to \overline{\mathbb{R}}.$$

Under more general assumption than in [7] we show that the representation

$$\mu\text{-lm}^a(I_\varepsilon|_{\Xi_\varepsilon}) = \int_\Omega C_0 y^2 dx + \int_\Omega (\nabla y, N^\sharp \nabla y)_{\mathbb{R}^n} dx + F(u)$$

holds. Here C_0 is a weak-$*$ limit in $L^\infty(\Omega)$ of $\{C_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$, F is the S-limit of the sequence of fuctionals $\{F : U_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon\in(0,\varepsilon_0]}$, the matrix $N^\sharp \in [L^\infty(\Omega)]^{n^2}$ is defined by

$$N^\sharp \nabla y = w\text{-}\lim_{\varepsilon\to 0} [A_\varepsilon^t \nabla\psi_\varepsilon + N_\varepsilon^t \nabla y_\varepsilon] - A_0^t \nabla\psi_0$$

where w-lim is the weak limit in $[L^\infty(\Omega)]^{n^2}$, $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is any sequence μ-converging to $(u,y) \in \tau\text{-Lm}\,\Xi_\varepsilon$, and $\{\psi_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is some bounded sequence in $H_0^1(\Omega)$.

In Section 7 we describe some variational properties of the homogenized problem. In particular, the τ-convergence of the sequence of optimal pairs $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon\in(0,\varepsilon_0]}$ to a unique solution of S-homogenized problem is a direct consequence of the variational S-limits.

In the last section, Section 8, we give the application of our results to the S-homogenization of an optimal control problem on perforated domain.

In closing this section we would like to note that the concept pusued in this paper and the results obtained are different form those of Kesavan and Saint Jean Paulin [7 - 9]. Moreover, our results of homogenization of optimal control problems on perforated domains differ from their results (see Section 8). They differ in the fact that the state equation for homogenized control object by the method [9] has another form, namely

$$-\mathrm{div}\,(A_0 \nabla y) = u + \chi_0 f \qquad \text{in } \Omega,$$

whereas the S-homogenization of similar problem gives

$$-\mathrm{div}\,(A_0 \nabla y) = \chi_0^{-1} u + f \qquad \text{in } \Omega.$$

This discrepancy, in our opinion, can be explained in the fact that we stay with the optimal control problem in the original sense and look for the homogenized optimal control problem for which we finally obtain a solution. In contrast to this the approach in [7 - 9] is concerned with the homogenization of the optimality system with respect to the parameter ε and, hence, the convergence of the optimal pairs $(u^0_\varepsilon, y^0_\varepsilon)$ was obtained. It is,however, not obvious from their analysis to which optimal control problem the limit of optimal pairs (u^0, y^0) is in fact the optimal pair.

2. Definitions and axiliary results

Let us start with a brief discussion of the formalism of variational S-convergence. Let (X, τ) be a Banach space endowed with the weak topology τ, and let $\{F^\varepsilon : X_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon \in (0,\varepsilon_0]}$ be a family of functionals, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is the half-extended set of real numbers. Here $\{X_\varepsilon \subseteq X\}_{\varepsilon \in (0,\varepsilon_0]}$ is a collection of sets with $E = (0, \varepsilon_0]$ an index space and $\mathcal{H}$ is a filter on E. Its lower topological limit, also called the limit inferior, is the set

$$\tau\text{-Li}\, X_\varepsilon = \bigcap_{H \in \mathcal{H}^\sharp} \mathrm{cl}_\tau \left(\bigcup_{\varepsilon \in H} X_\varepsilon \right),$$

and its upper topological limit, also called the limit superior, is the set

$$\tau\text{-Ls}\, X_\varepsilon = \bigcap_{H \in \mathcal{H}} \mathrm{cl}_\tau \left(\bigcup_{\varepsilon \in H} X_\varepsilon \right),$$

where $\mathcal{H}^\sharp$ is the family of subsets of $E = (0, \varepsilon_0]$ that meet all sets H in $\mathcal{H}$. If $\tau\text{-Li}\, X_\varepsilon = \tau\text{-Ls}\, X_\varepsilon$, this set, denoted as $\tau\text{-Lm}\, X_\varepsilon$, is the (Painleve-Kuratowski) topological limit of the collection $\{X_\varepsilon \subseteq X\}_{\varepsilon \in (0,\varepsilon_0]}$.

It will be convenient to have at our disposal the following equivalent expressions for the lower and upper topological limit of sets:

$$\begin{aligned}\tau\text{-Li}\, X_\varepsilon &= \Big\{x \Big| \,\forall\, V \in \mathcal{N}(x),\, \exists\, H \in \mathcal{H},\, \forall\, \varepsilon \in H :\, X_\varepsilon \cap V \neq \emptyset\Big\} \\ &= \Big\{x \Big| \,\exists\, H \in \mathcal{H},\, \exists\, x_\varepsilon \in X_\varepsilon \,\, (\varepsilon \in H) \,\text{ with } x_\varepsilon \xrightarrow{H} x\Big\} \\ \tau\text{-Ls}\, X_\varepsilon &= \Big\{x \Big| \,\forall\, V \in \mathcal{N}(x),\, \exists\, H \in \mathcal{H}^\sharp,\, \forall\, \varepsilon \in H :\, X_\varepsilon \cap V \neq \emptyset\Big\} \\ &= \Big\{x \Big| \,\exists\, H \in \mathcal{H}^\sharp,\, \exists\, x_\varepsilon \in X_\varepsilon \,\, (\varepsilon \in H) \,\text{ with } x_\varepsilon \xrightarrow{H} x\Big\}\end{aligned}$$

where $\mathcal{N}(x)$ denotes a system of neighborhoods at x. Assume that $\tau\text{-Li}\, X_\varepsilon \neq \emptyset$. By $\text{epi}(F^\varepsilon | X_\varepsilon)$ denote the set

$$\text{epi}\big(F^\varepsilon | X_\varepsilon\big) = \big\{(x,\lambda) \in X_\varepsilon \times \mathbb{R} \big| \, F^\varepsilon(x) \leq \lambda\big\}.$$

Definition 2.1. The *S-lower limit* $\tau\text{-li}\,(F^\varepsilon|_{X_\varepsilon}) :\, \tau\text{-Ls}\, X_\varepsilon \to \overline{\mathbb{R}}$ and the S-upper limit $\tau\text{-ls}\,(F^\varepsilon|_{X_\varepsilon}) : \,\tau\text{-Li}\, X_\varepsilon \to \overline{\mathbb{R}}$ are defined by

$$\begin{aligned}\text{epi}\big(\tau\text{-li}\,(F^\varepsilon|_{X_\varepsilon}) |\, \tau\text{-Ls}\, X_\varepsilon\big) &= \rho\text{-Ls}\,(\text{epi}(F^\varepsilon | X_\varepsilon)) \\ \text{epi}\big(\tau\text{-ls}\,(F^\varepsilon|_{X_\varepsilon}) |\, \tau\text{-Li}\, X_\varepsilon\big) &= \rho\text{-Li}\,(\text{epi}(F^\varepsilon | X_\varepsilon))\end{aligned}$$

where ρ is the product topology of $X \times \mathbb{R}$.

If there exist a set A and a functional $F :\, A \to \overline{\mathbb{R}}$ such that

$$\begin{aligned}\tau\text{-Li}\, X_\varepsilon &= A = \tau\text{-Ls}\, X_\varepsilon \\ \tau\text{-li}\,(F^\varepsilon|_{X_\varepsilon}) &= F = \tau\text{-ls}\,(F^\varepsilon|_{X_\varepsilon}),\end{aligned}$$

then we write

$$A = \tau\text{-Lm}\, X_\varepsilon, \qquad F = \tau\text{-lm}^a(F^\varepsilon|_{X_\varepsilon})$$

and we say that the sequence $\{F^\varepsilon :\, X_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon\in(0,\varepsilon_0]}$ *absolutly S-converges* to F or that F is the *absolute S-limit* of $\{F^\varepsilon :\, X_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon\in(0,\varepsilon_0]}$.

The techniques of S-convergence and the basic topological properties of S-limits are disscussed more detail in [10, 12 - 14]. We state some results from [14] that we will use below.

Assume that (X,τ) is a separable Banach space and that the sequence of functionals $\{F^\varepsilon :\, X_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon\in(0,\varepsilon_0]}$ is τ-equicoercive, i.e. for every $t \in \mathbb{R}$ there exists a τ-compact set $K_t \subseteq X$ such that

$$\bigcup_{\varepsilon\in(0,\varepsilon_0]} \big\{x \in X_\varepsilon |\, F^\varepsilon(x) \leq t\big\} \subseteq K^t$$

where τ is the $\sigma(X,X^*)$-weak topology for X. It is easy to see that $\{F^\varepsilon :\, X_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon\in(0,\varepsilon_0]}$ is τ-equicoercive if and only if there exists a lower τ-semicontinuous and lower τ-semicompact functional $\Psi :\, X \to \overline{\mathbb{R}}$ such that

$$F^\varepsilon(x) \geq \Psi(x) \qquad \forall\, x \in X_\varepsilon,\ \forall\, \varepsilon \in (0,\varepsilon_0].$$

Theorem 2.1. *Let X be a separable Banach space, $\{F^\varepsilon : X_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon\in(0,\varepsilon_0]}$ be a τ-equicoercive sequence of functionals (or the sets $\{X_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ are uniformly bounded, i.e. $\sup_{\varepsilon\in(0,\varepsilon_0]}\sup_{x\in X_\varepsilon}\|x\|_X < +\infty$). Suppose $\tau\text{-Li}\, X_\varepsilon \neq \emptyset$. Then $\{F^\varepsilon : X_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon\in(0,\varepsilon_0]}$ absolutly S-converges to $F : X_0 \to \overline{\mathbb{R}}$ if and only if the following conditions hold:*

(i) *For every $x \in X_0$, $H \in \mathcal{H}^\sharp$, and for every sequence $\{y_\varepsilon\}_{\varepsilon\in H}$ τ-converging to x we have $y_\varepsilon \in X_\varepsilon$ for every $\varepsilon \in H$ and $F(x) \le \liminf_{H\ni\varepsilon\to 0} F^\varepsilon(y_\varepsilon)$.*

(ii) *For every $x \in X_0$ and index set $H \in \mathcal{H}$ there exists a sequence $\{\overline{y}_\varepsilon\}_{\varepsilon\in H}$ such that $\overline{y}_\varepsilon \xrightarrow{\tau} x, \overline{y}_\varepsilon \in X_\varepsilon$ for all $\varepsilon \in H$, and $F(x) \ge \limsup_{H\ni\varepsilon\to 0} F^\varepsilon(\overline{y}_\varepsilon)$.*

It is easy to see that each of the functionals $F^\varepsilon : X_\varepsilon \to \overline{\mathbb{R}}$ can be associated with some constrained minimization problem $\langle \inf_{x\in X_\varepsilon} F^\varepsilon(x)\rangle$, i.e there is a one-to-one correspondence between the set of such functionals and the elements of the following sequence

$$\left\{\left\langle \inf_{x\in X_\varepsilon} F^\varepsilon(x)\right\rangle\right\}_{\varepsilon\in(0,\varepsilon_0]}. \tag{2.1}$$

Definition 2.2 The *S-lower* and *S-upper variational limits* of sequence (2.1) are defined by

$$\left\langle \inf_{x\in\tau\text{-Ls}\, X_\varepsilon} \tau\text{-li}\,(F^\varepsilon|_{X_\varepsilon})(x)\right\rangle \qquad \text{and} \qquad \left\langle \inf_{x\in\tau\text{-Li}\, X_\varepsilon} \tau\text{-ls}\,(F^\varepsilon|_{X_\varepsilon})(x)\right\rangle,$$

respectively. If for the sequence of functionals $\{F^\varepsilon : X_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon\in(0,\varepsilon_0]}$ there exists an absolute S-limit $\tau\text{-lm}^a(F^\varepsilon|_{X_\varepsilon})$, then the constrained minimization problem

$$\left\langle \inf_{x\in\tau\text{-Lm}\, X_\varepsilon} \tau\text{-lm}^a(F^\varepsilon|_{X_\varepsilon})(x)\right\rangle \tag{2.2}$$

is called the *absolute variational S-limit* of sequence (2.1).

Note that if all the minimization problems in (2.1) correspond to a single optimal control problem, then problem (2.2) is called the *strong S- homogenized optimal control problem.*

Theorem 2.2. *Let $X = V^*$, V be a separable Banach space, τ be the $\sigma(V^*, V)$-topology for X, $\left\{F^\varepsilon : X_\varepsilon \to \overline{\mathbb{R}}\right\}_{\varepsilon\in(0,\varepsilon_0]}$ be an equicoercive sequence of functionals, $\tau\text{-Lm}\, X_\varepsilon \neq \emptyset$. Then the following statements hold:*

(i) *The sets of solutions of the S-lower and S-upper variational S-limits are non-empty and $\sigma(V^*, V)$-compact.*

(ii) *We can extract a subsequence from family (2.1) $\left\{\langle \inf_{x\in X_\varepsilon} F^\varepsilon(x)\rangle : \varepsilon \to \theta\right\}_{\varepsilon\in H\in\mathcal{H}^\sharp}$ for which there exists an absolute variational S-limit in the $\sigma(V^*, V)$-topology.*

Remark 2.1. A similar result can be proved in the case when $X = V$ is a separable Banach space (i.e. X may be a non-reflexive space), τ is the $\sigma(V, V^*)$-topology for X, and the principle of "compact embedding" holds, i.e. there exists a $\sigma(V, V^*)$-compact set $X_{\text{comp}} \subseteq X$ such that $X_\varepsilon \subseteq X_{\text{comp}}$ for every $\varepsilon \in (0, \varepsilon_0]$.

Let us denote by

$$\mathbf{M}(F^\varepsilon; X_\varepsilon) = \left\{ x_\varepsilon^0 \in X_\varepsilon \,\middle|\, F^\varepsilon(x_\varepsilon^0) = \inf_{x \in X_\varepsilon} F^\varepsilon(x) \right\}$$
$$\mathbf{M}^\alpha(F^\varepsilon; X_\varepsilon) = \left\{ x_\varepsilon \in X_\varepsilon \,\middle|\, F^\varepsilon(x_\varepsilon) \leq \sup\left(\inf_{x \in X_\varepsilon} F^\varepsilon(x) + \alpha, -\frac{1}{\alpha} \right) \right\}$$

the sets of all minimizers and α-minimizers of the constrained minimization problem $\langle \inf_{x \in X_\varepsilon} F^\varepsilon(x) \rangle$, respectively.

Theorem 2.3. *Assume that the sequence* $\left\{F^\varepsilon : X_\varepsilon \to \overline{\mathbb{R}}\right\}_{\varepsilon \in (0,\varepsilon_0]}$ *absolutly* S*-converges to a functional* $F : \tau\text{-Lm}\, X_\varepsilon \to \overline{\mathbb{R}}$, *and* $F \not\equiv +\infty$ *on* $\tau\text{-Lm}\, X_\varepsilon$. *Then the following statements hold:*

(i) $\min_{x \in \tau\text{-Lm}\, X_\varepsilon} F(x) = \lim_{\varepsilon \to 0} \inf_{x \in X_\varepsilon} F^\varepsilon(x)$
$\mathbf{M}(F; \tau\text{-Li}\, X_\varepsilon) = \cap_{\alpha > 0} \tau\text{-Li}\, \mathbf{M}^\alpha(F^\varepsilon; X_\varepsilon) = \cap_{\alpha > 0} \tau\text{-Ls}\, \mathbf{M}^\alpha(F^\varepsilon; X_\varepsilon)$.

(ii) *Let* x_ε^0 *be a minimizer of* F^ε *in* X_ε. *If the sequence* $\{x_\varepsilon^0 \in X_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ τ*-converges to some element* x^* (*or* x^* *is a* τ*-cluster point of this sequence*), *then* x^* *is a minimizer of* F *in* $\tau\text{-Lm}\, X_\varepsilon$ (*i.e.* $x^* \in \mathbf{M}(F; \tau\text{-Lm}\, X_\varepsilon)$), *and* $F(x^*) = \lim_{\varepsilon \to 0} F^\varepsilon(x_\varepsilon^0)$.

Morever, if the sequence $\{F^\varepsilon : X_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon \in (0,\varepsilon_0]}$ *is* τ*-equicoercive, then the set of minimizers* $\mathbf{M}(F; \tau\text{-Lm}\, X_\varepsilon)$ *is non-empty and* τ*-compact.*

3. Formalism of G^*-convergence of elliptic operators

Let us denote by $w_{H_0^1}$ the weak topology of $H_0^1(\Omega)$, by w_{L^2} the weak topology of $L^2(\Omega)$, and by $s_{H^{-1}}$ the strong topology of $H^{-1}(\Omega)$. Let us consider sequences of operators $\{\mathcal{A}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ and $\{\mathcal{B}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ such that:

(i) $\langle \mathcal{B}_\varepsilon u, \varphi \rangle_{H_0^1(\Omega)} = \int_\Omega b_\varepsilon u \varphi \, dx$ for all $\varphi \in H_0^1(\Omega)$, i.e. $\mathcal{B}_\varepsilon$ are linear continuous operators from $L^2(\Omega)$ to $H^{-1}(\Omega)$, for every $\varepsilon \in (0, \varepsilon_0]$.

(ii) $\langle \mathcal{A}_\varepsilon y, \varphi \rangle_{H_0^1(\Omega)} = \int_\Omega (\nabla \varphi, A_\varepsilon \nabla y)_{\mathbb{R}^n} dx$ for all $\varphi \in H_0^1(\Omega)$.

(iii) The family of linear operators $\{\mathcal{A}_\varepsilon \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))\}_{\varepsilon \in (0,\varepsilon_0]}$ is uniformly coercive and uniformly bounded, i.e. there exist two constants λ_0 and λ_1 ($0 < \lambda_0 \leq \lambda_1$) satisfying $\lambda_0 \|y\|^2_{H_0^1(\Omega)} \leq \langle \mathcal{A}_\varepsilon y, y \rangle_{H_0^1(\Omega)}$, $\|\mathcal{A}_\varepsilon\| \leq \lambda_1$.

As is well known (see [21, 22]), the family of operators

$$\{\mathcal{A}_\varepsilon : H_0^1(\Omega) \to H^{-1}(\Omega)\}_{\varepsilon \in (0,\varepsilon_0]}$$

is compact with respect to G-convergence, i.e. there exists a coercive bounded operator $\mathcal{A}_0 : H_0^1(\Omega) \to H^{-1}(\Omega)$ such that

$$(s_{H^{-1}} \times w_{H_0^1})\text{-Lm}\, \mathrm{gr}\,(\mathcal{A}_\varepsilon) = \mathrm{gr}\,(\mathcal{A}_0) \tag{3.1}$$

$$\langle \mathcal{A}_0 y, \varphi \rangle_{H_0^1(\Omega)} = \int_\Omega (\nabla \varphi, A_0 \nabla y)_{\mathbb{R}^n} dx \;\forall\, \varphi \in H_0^1(\Omega) \tag{3.2}$$

where gr $(\mathcal{A})$ is defined as the set $\{(x,y) \in H^{-1}(\Omega) \times H_0^1(\Omega) | \, x = \mathcal{A}y\}$. Here the matrix A_0 is the so-called H-limit of the sequence $\{A_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$. However, many authors (see, e.g., [21, 22]) define a sequence of operators $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ to be G-convergent to $\mathcal{A}_0$ if $\mathcal{A}_\varepsilon^{-1} f \to \mathcal{A}_0^{-1} f$ weakly in $H_0^1(\Omega)$ for any $f \in H^{-1}(\Omega)$. But it is easy to prove that the last definition of G-convergence is equivalent to that we use in (3.1).

As we study the state equation

$$\mathcal{A}_\varepsilon y = \mathcal{B}_\varepsilon u + f_\varepsilon \qquad \text{in } \mathcal{D}'(\Omega) \tag{3.3}$$

under the state- and control-constraints

$$\begin{aligned} &y \in K_\varepsilon, \\ &u \in U_\varepsilon \end{aligned} \tag{3.4}$$

instead of the graphs gr $(\mathcal{A}_\varepsilon)$ we have to consider their restrictions

$$\text{gr}\,(\mathcal{A}_\varepsilon)|_{Q_\varepsilon\times K_\varepsilon} := \text{gr}\,(\mathcal{A}_\varepsilon) \cap [Q_\varepsilon \times K_\varepsilon] \tag{3.5}$$

where by Q_ε we denote the images of the sets U_ε in $H^{-1}(\Omega)$ under the maps $H^\varepsilon : L^2(\Omega) \to H^{-1}(\Omega)$, where $H^\varepsilon u = \mathcal{B}_\varepsilon u + f_\varepsilon$, i.e.

$$Q_\varepsilon = \Big\{ g \in H^{-1}(\Omega) \Big| \, g = \mathcal{B}_\varepsilon u + f_\varepsilon \ \forall\, u \in U_\varepsilon \Big\}. \tag{3.6}$$

Therefore we would like to have sufficient conditions under which the topological limit of the restricted graphs with respect to the $\tau = s_{H^{-1}} \times w_{H_0^1}$-topology

$$\tau\text{-Lm}\,[\text{gr}\,(\mathcal{A}_\varepsilon)|_{Q_\varepsilon\times K_\varepsilon}]$$

can be recovered. However, in the general case, this turns out to be impossible because by the properties of topological limits in the Kuratowski sense we have the inclusion

$$\tau\text{-Ls}\,(\text{gr}\,(\mathcal{A}_\varepsilon) \cap [Q_\varepsilon \times K_\varepsilon]) \subseteq \tau\text{-Ls}\,[\text{gr}\,(\mathcal{A}_\varepsilon)] \cap \big[(s_{H^{-1}})\text{-Ls}\, Q_\varepsilon \times (w_{H_0^1})\text{-Ls}\, K_\varepsilon\big].$$

Therefore, if

$$(s_{H^{-1}})\text{-Ls}\, Q_\varepsilon = \emptyset \tag{3.7}$$

or if there is not a single sequence of admissible pairs $\{(u_\varepsilon, y_\varepsilon) \in U_\varepsilon \times K_\varepsilon\}$ for which $\{(\mathcal{B}_\varepsilon u_\varepsilon + f_\varepsilon, y_\varepsilon)\}$ is τ-convergent, we obtain

$$\big\{\tau\text{-Ls}\,[\text{gr}\,(\mathcal{A}_\varepsilon)|_{Q_\varepsilon\times K_\varepsilon}]\big\} = \emptyset.$$

Example 3.1. Let Ω be a bounded open set of $\mathbb{R}^n$. For each $\varepsilon \in (0,\varepsilon_0]$ T_i^ε $(1 \le i \le n(\varepsilon))$ there is some closed subset, which is called a "hole". The domain Ω_ε is defined by removing the holes T_i^ε from Ω, that is

$$\Omega_\varepsilon = \Omega \setminus \cup_{i=1}^{n(\varepsilon)} T_i^\varepsilon.$$

Let U_ε be the closure in $H^{-1}(\Omega)$ of the set of all functions $u \in C^\infty(\Omega)$ with $\operatorname{supp} u$ contained in Ω_ε such that $u > 0$ in Ω_ε. We denote by χ_ε the characteristic function of the perforated domain Ω_ε and we shall assume that the following conditions are fulfilled:

(i) Every weak-$*$ limit in $L^\infty(\Omega)$ of $\{\chi_\varepsilon\}$ is positive a.e. in Ω.

(ii) $\mathcal{B}_\varepsilon u + f_\varepsilon = u$ for every $\varepsilon \in (0, \varepsilon_0]$.

In this case (3.7) holds. Indeed, suppose the converse. Then there are a sequence $\{u_\varepsilon\}_{\varepsilon \in H}$, where $H \in \mathcal{H}^\sharp$ such that $u_\varepsilon \in U_\varepsilon$ for every $\varepsilon \in H$ and $\{u_\varepsilon\}$ converges strongly in $H^{-1}(\Omega)$ to some $u^* \in H^{-1}(\Omega)$. This, however, is impossible as, with any element $g \in L^1(\Omega)$,

$$\int_\Omega u_\varepsilon g \, dx = \int_\Omega u_\varepsilon \chi_\varepsilon g \, dx,$$

and passing to the limit (using the strong convergence of u_ε to u^* and the weak-$*$ convergence of $\chi_\varepsilon g$ to $\chi_0 g$ for the term on the right-hand side), we get

$$\int_\Omega u^* g \, dx = \int_\Omega \chi_0 u^* g \, dx.$$

Since g was arbitrarily chosen in $L^1(\Omega)$, it follows that $u^* = \chi_0 u^*$ in $H^{-1}(\Omega)$, which is not generally true (except when $u^* = 0$) if $\chi_0 \not\equiv 1$. Hence (3.7) holds.

Hence, in the general case we are not able to study the convergence of the graph restrictions $\{\operatorname{gr}(\mathcal{A}_\varepsilon)|_{Q_\varepsilon \times K_\varepsilon}\}_{\varepsilon \to 0}$ with respect to the τ-topology. Consequently, one should then work with a weaker topology on $H^{-1}(\Omega) \times H_0^1(\Omega)$. To this end we shall consider the convergence of graph restrictions $\{\operatorname{gr}(\mathcal{A}_\varepsilon)|_{Q_\varepsilon \times K_\varepsilon}\}_{\varepsilon \in (0,\varepsilon_0]}$ with respect to the τ^* - topology, which is defined as the product of the weak topology for $H^{-1}(\Omega)$ and the weak topology for $H_0^1(\Omega)$.

We introduce the following hypotheses:

(A1) There exist a subset $L^\varepsilon \subset H^{-1}(\Omega)$ such that $Q_\varepsilon \cap \mathcal{A}_\varepsilon(K_\varepsilon) \subseteq L^\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$ where $\mathcal{A}_\varepsilon(K_\varepsilon)$ is the image of the set K_ε under the operator $\mathcal{A}_\varepsilon$.

(A2) For every $\varepsilon \in (0, \varepsilon_0]$ there is a real reflexive separable Banach space Y_ε with norm $\|\cdot\|_\varepsilon$ and a continuous linear mapping P_ε of Y_ε into $H_0^1(\Omega)$ such that $\sup_{\varepsilon \in (0,\varepsilon_0]} \|P_\varepsilon\| = c_0 < \infty$.

(A3) For every $\varepsilon \in (0, \varepsilon_0]$ there exists a linear mapping R_ε^+ of Y_ε^* into $L^\varepsilon \subset H^{-1}(\Omega)$ such that if $g \in Y_\varepsilon^*$, then $P_\varepsilon^*(R_\varepsilon^+ g) = g$ for every $\varepsilon \in (0, \varepsilon_0]$.

Now we introduce the following concept.

Definition 3.1. We shall call the family of real reflexive separable Banach spaces $\{Y_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ *coordinated with the control object* (3.3) - (3.4) if hypotheses (A1) - (A3) hold true and there is a sequence of convex closed subsets $\{\widehat{Q}_\varepsilon \subseteq H^{-1}(\Omega)\}_{\varepsilon \in (0,\varepsilon_0]}$ such that $R_\varepsilon^+ P_\varepsilon^* : \widehat{Q}_\varepsilon \to \widetilde{Q}_\varepsilon$ where $\widetilde{Q}_\varepsilon$ satisfies $\widetilde{Q}_\varepsilon \cap \mathcal{A}_\varepsilon(K_\varepsilon) = Q_\varepsilon \cap \mathcal{A}_\varepsilon(K_\varepsilon)$ for every $\varepsilon \in (0, \varepsilon_0]$ and ${}_{H^{-1}}\text{-Li}\, \widehat{Q}_\varepsilon \neq \emptyset$.

Definition 3.2. For problem (3.3) - (3.4) with a coordinated collection of spaces $\{Y_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ the sets

$$\operatorname{Gr}(\mathcal{A}_\varepsilon) = \left\{ (f, y) \in H^{-1}(\Omega) \times H_0^1(\Omega) \,\middle|\, \mathcal{A}_\varepsilon y = R_\varepsilon^+ P_\varepsilon^* f \right\}$$

are called *prototypes of the operator graphs* $\mathrm{gr}\,(\mathcal{A}_\varepsilon)$.

Thus instead of the problem of topological convergence for the graph restrictions

$$\left\{\mathrm{gr}\,(\mathcal{A}_\varepsilon)|_{Q_\varepsilon\times K_\varepsilon}\right\}_{\varepsilon\in(0,\varepsilon_0]}$$

in the τ^*-topology we may consider the topological convergence of their graph prototypes in the τ topology for $H^{-1}(\Omega)\times H^1_0(\Omega)$. This fact leads to the following notation of G^*-convergence.

Definition 3.3. Let $\mathcal{A}_*\in\mathcal{L}\big(H^1_0(\Omega);H^{-1}(\Omega)\big)$ be a coercive operator. We say that the sequence of operators $\left\{\mathcal{A}_\varepsilon\in\mathcal{L}\big(H^1_0(\Omega);H^{-1}(\Omega)\big)\right\}_{\varepsilon\in(0,\varepsilon_0]}$ G^**-converges* to the operator $\mathcal{A}_*$ (in symbols, $\mathcal{A}_\varepsilon\xrightarrow{G^*}\mathcal{A}_*$) if τ-$\mathrm{Lm}\,\mathrm{Gr}(\mathcal{A}_\varepsilon)=\mathrm{gr}\,(\mathcal{A}_*)$.

Remark 3.1. We note that the G^*-limit of the operators $\mathcal{A}_\varepsilon$ is defined in terms of the τ-topology. Moreover, if we put $Y_\varepsilon=H^1_0(\Omega)$, $P_\varepsilon y=y$, $R^+_\varepsilon g=g$ for every $y\in H^1_0(\Omega)$, $g\in H^{-1}(\Omega)$ and $\varepsilon\in(0,\varepsilon_0]$, then $\widehat{Q}_\varepsilon=Q_\varepsilon$ and each of the graph prototypes $\mathrm{Gr}\,(\mathcal{A}_\varepsilon)$ coincides with the corresponding graph $\mathrm{gr}\,(\mathcal{A}_\varepsilon)$. Then Definition 3.3 reduces to the well known definition of G-convergence.

Proposition 3.1. *Suppose that the for the original constrained state equation there is a coordinated collection of Banach spaces* $\{Y_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$. *Let* $\mathcal{A}_*\in\mathcal{L}\big(H^1_0(\Omega);H^{-1}(\Omega)\big)$ *be a coercive operator,* $\left\{\mathcal{A}_\varepsilon\in\mathcal{L}\big(H^1_0(\Omega);H^{-1}(\Omega)\big)\right\}_{\varepsilon\in(0,\varepsilon_0]}$ *be a* G^**-compact set of uniformly bounded and uniformly coercive operators. Then the sequence* $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ G^**-converges to* $\mathcal{A}_*$ *if and only if* $\mathcal{A}^{-1}_\varepsilon R^+_\varepsilon P^*_\varepsilon f\longrightarrow\mathcal{A}^{-1}_* f$ *weakly in* $H^1_0(\Omega)$ *for any* $f\in H^{-1}(\Omega)$.

Proof. Assume that $\mathcal{A}_\varepsilon\xrightarrow{G^*}\mathcal{A}_*$. Then, by definition of G^*-convergence,

$$\mathcal{A}^{-1}_\varepsilon R^+_\varepsilon P^*_\varepsilon f\to\mathcal{A}^{-1}_* f\qquad\text{weakly in } H^1_0(\Omega)$$

and the "only if" part of the statement is proved.

Let us prove the "if" part. Suppose that $\mathcal{A}^{-1}_\varepsilon R^+_\varepsilon P^*_\varepsilon f\to\mathcal{A}^{-1}_* f$ weakly in $H^1_0(\Omega)$ for any $f\in H^{-1}(\Omega)$. By G^*-compactness of the set $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$, there exists an index set $H\in\mathcal{H}^\sharp$ and a subsequence $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in H}$ such that $\mathcal{A}_{\varepsilon\in H}\xrightarrow{G^*}\widehat{\mathcal{A}}_*$, where $\widehat{\mathcal{A}}_*$ is a linear bounded coercive operator from $H^1_0(\Omega)$ into $H^{-1}(\Omega)$. Consequently, for $\widehat{\mathcal{A}}_*$ there exists an invertible bounded operator $\widehat{\mathcal{A}}^{-1}_*$. The definition of G^*-convergence implies that $\widehat{\mathcal{A}}^{-1}_* f=\mathcal{A}^{-1}_* f$ for any $f\in H^{-1}(\Omega)$. Therefore $\widehat{\mathcal{A}}^{-1}_*=\mathcal{A}^{-1}_*$ and $\widehat{\mathcal{A}}_*=\mathcal{A}_*$. Thus $\mathcal{A}_{\varepsilon(k)}\xrightarrow{G^*}\mathcal{A}_*$.

Theorem 3.1. *Let the following assumptions hold:*

(i) $\left\{\mathcal{A}_\varepsilon\in\mathcal{L}\big(H^1_0(\Omega),H^{-1}(\Omega)\big)\right\}_{\varepsilon\in(0,\varepsilon_0]}$ *is a sequence of uniformly coercive and uniformly bounded operators.*

(ii) *The collection of Banach spaces* $\{Y_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ *is coordinated with the constrained state equation* (3.3) – (3.4) *in the sense of Definition* 3.1.

Then there exist an index set $H\in\mathcal{H}^\sharp$, *a subsequence* $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in H}$ *and a coercive linear operator* $\mathcal{A}_*$ *of* $H^1_0(\Omega)$ *into* $H^{-1}(\Omega)$ *such that* $\mathcal{A}_\varepsilon\xrightarrow{G^*}\mathcal{A}_*$, *i.e.* τ-$\mathrm{Lm}\,\mathrm{Gr}\,(\mathcal{A}_\varepsilon)=\mathrm{gr}\,(\mathcal{A}_*)$.

Proof. Since the space $H_0^1(\Omega)$ is separable and reflexive, there exists a metric d such that for any sequence $\{y_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ the following conditions are equivalent:

(1) $y_\varepsilon \to y$ weakly in $H_0^1(\Omega)$

(2) $\{y_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is bounded and $d(y_\varepsilon, y) \to 0$

(see, e.g., [6]). We denote by μ the topology associated with the metric d on $H_0^1(\Omega)$. This topology has a countable base.

Since the topology $s_{H^{-1}} \times \mu$ has a countable base, by Kuratowski's compactness theorem [19], there exists a subsequence $\{\mathrm{Gr}\,(\mathcal{A}_\varepsilon)\}_{\varepsilon\in H}$, with $H \in \mathcal{H}^\sharp$, convergeing to a set $C \subset H^{-1}(\Omega) \times H_0^1(\Omega)$ in the $s_{H^{-1}} \times \mu$-topology.

We proceed to prove $C = \tau\text{-Lm}\,\mathrm{Gr}\,(\mathcal{A}_\varepsilon)$. To this end we show

$$\tau\text{-Ls}\,\mathrm{Gr}\,(\mathcal{A}_\varepsilon) \subseteq C, \tag{3.8}$$

$$C \subseteq \tau\text{-Li}\,\mathrm{Gr}\,(\mathcal{A}_\varepsilon). \tag{3.9}$$

Firstly, let us verify (3.8). Suppose $(f, y) \in \tau\text{-Ls}\,\mathrm{Gr}\,(\mathcal{A}_\varepsilon)$. Then there exist an index set $H \in \mathcal{H}^\sharp$ and a sequence $\{(\widehat{f}_\varepsilon, y_\varepsilon)\}_{\varepsilon\in H}$ converging to (f, y) in the topology τ such that $(\widehat{f}_\varepsilon, y_\varepsilon) \in \mathrm{Gr}\,(\mathcal{A}_\varepsilon)$ for every $\varepsilon \in H$. Since (1) implies (2), we see that $(\widehat{f}_\varepsilon, y_\varepsilon)$ converges to (f, y) with respect to the topology $s_{H^{-1}} \times \mu$. Hence, $(f, y) \in C$.

As for (3.9), let $(f, y) \in C$. Then there exists a sequence $\{(\widehat{f}_\varepsilon, y_\varepsilon)\}$ converging to (f, y) in the topology $s_{H^{-1}} \times \mu$ such that $(\widehat{f}_\varepsilon, y_\varepsilon) \in \mathrm{Gr}\,(\mathcal{A}_\varepsilon)$ for all ε small enough. Since $\{\widehat{f}_\varepsilon\}$ is bounded in $H^{-1}(\Omega)$,

$$y_\varepsilon = \mathcal{A}_\varepsilon^{-1} R_\varepsilon^+ P_\varepsilon^* \widehat{f}_\varepsilon = P_\varepsilon \Lambda_\varepsilon^{-1} P_\varepsilon^* \widehat{f}_\varepsilon$$

is bounded in $H_0^1(\Omega)$ as well. Then the equivalence between conditions (1) and (2) yields weak convergence of $\{y_\varepsilon\}$ to y. Hence, $\{(\widehat{f}_\varepsilon, y_\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0]}$ converges to (f, y) in the τ-topology, which implies (3.9).

Finally, we prove the existence of an invertible linear bounded operator $\mathcal{A}_* : H_0^1(\Omega) \to H^{-1}(\Omega)$ with $C = \mathrm{gr}\,(\mathcal{A}_*)$. Using Proposition 3.1, we see that there exists a linear operator $C_* : H^{-1}(\Omega) \to H_0^1(\Omega)$ such that for all $f \in H^{-1}(\Omega)$

$$y_\varepsilon = \mathcal{A}_\varepsilon^{-1} R_\varepsilon^+ P_\varepsilon^* f \longrightarrow C_* f \qquad \text{weakly in } H_0^1(\Omega).$$

Then by analogy with [17] (see Proposition 1.7) it can be proved that there is a constant $\alpha > 0$ such that the inequalities

$$\|f - g\|_{H^{-1}}^2 \le \alpha\, \|C_* f - C_* g\|_{H_0^1}^2 \tag{3.10}$$

$$\langle f - g, C_* f - C_* g\rangle \ge \alpha^{-1} \|C_* f - C_* g\|_{H_0^1}^2 \tag{3.11}$$

hold for every $f, g \in H^{-1}(\Omega)$. Therefore from (3.10) - (3.11) we deduce that for any $f \in H^{-1}(\Omega)$

$$\|f\|_{H^{-1}}^2 \le \alpha\, \|C_* f\|_{H_0^1}^2, \tag{3.12}$$

$$\langle f, C_* f\rangle \ge \alpha^{-1} \|C_* f\|_{H_0^1}^2. \tag{3.13}$$

Consequently, the operator C_* is invertible, i.e. we may set $\mathcal{A}_* = C_*^{-1}$. Moreover, we obtain for the operator $\mathcal{A}_*$ the properties of boundedness and coerciveness taking arbitrary $y \in H_0^1(\Omega)$ and substituting $f = \mathcal{A}_* y$ into (3.12) - (3.13). The theorem is proved ∎

Now we are in the position to state the main result of this section.

Theorem 3.2. *Suppose that the following conditions hold true:*

(i) $\{\mathcal{A}_\varepsilon \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))\}_{\varepsilon\in(0,\varepsilon_0]}$ *is a sequence of uniformly coercive and uniformly bounded operators.*

(ii) *For the constrained state equation* (3.3) – (3.4) *there exists a coordinated collection of Banach spaces* $\{Y_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$.

(iii) $\{K_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ *is a sequence of weakly closed convex subsets of* $H_0^1(\Omega)$ *for which there exists a non-empty topological limit* $(w_{H_0^1})\text{-Lm}\, K_\varepsilon \neq \emptyset$.

(iv) *There are an index set* $H \in \mathcal{H}$ *and a* τ*-converging sequence* $\{(\widehat{f}_\varepsilon, y_\varepsilon) \in \widehat{Q}_\varepsilon \times K_\varepsilon\}_{\varepsilon\in H}$ *such that* $\mathcal{A}_\varepsilon y_\varepsilon = R_\varepsilon^+ P_\varepsilon^* \widehat{f}_\varepsilon$ *for every* $\varepsilon \in H$.

Then there exist a subsequence $\{\varepsilon\}_{\varepsilon\in H}$, *where* $H \in \mathcal{H}^\sharp$, *and a coercive bounded linear operator* $\mathcal{A}_* \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ *such that* $\mathcal{A}_\varepsilon \xrightarrow{G^*} \mathcal{A}_*$ *and*

$$\tau\text{-Lm}\left[\mathrm{Gr}\,(\mathcal{A}_\varepsilon)|_{\widehat{Q}_\varepsilon\times K_\varepsilon}\right] = \mathrm{gr}\,(\mathcal{A}_*)|_{(s_{H^{-1}})\text{-Lm}\,[\widehat{Q}_\varepsilon]\times(w_{H_0^1})\text{-Lm}\,[K_\varepsilon]}. \tag{3.14}$$

For the proof we need the following result (see [16]).

Lemma 3.1. *Let* (X, τ) *be a locally convex vector space, let* $\{W_\varepsilon\}$ *and* $\{R_\varepsilon\}$ *be sequences of* τ*-closed convex subsets of* X *for which the following conditions hold:*

(a) $W_\varepsilon \cap R_\varepsilon \neq \emptyset$ *for every* $\varepsilon \in (0, \varepsilon_0]$.

(b) *There exists topological limits* $\tau\text{-Lm}\, W_\varepsilon$ *and* $\tau\text{-Lm}\, R_\varepsilon$.

(3) $\tau\text{-Li}\,(W_\varepsilon \cap R_\varepsilon) \neq \emptyset$.

Then for the sequence of subsets $\{W_\varepsilon \cap R_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ *there exists a topological limit in the* τ*-topology such that* $\tau\text{-Lm}\,(W_\varepsilon \cap R_\varepsilon) = \tau\text{-Lm}\, W_\varepsilon \cap \tau\text{-Lm}\, R_\varepsilon$.

Proof of Theorem 3.2. In accordance with Lemma 3.1 we need to verify conditions (a) - (c) for the sets $W_\varepsilon = \mathrm{Gr}\,(\mathcal{A}_\varepsilon)$ and $R_\varepsilon = \widehat{Q}_\varepsilon \times K_\varepsilon$, where $\widehat{Q}_\varepsilon$ are defined in Definition 3.1. Condition (a) follows immediately from the uniformly regular property of the original control object, that is from supposition (iv). Since the sequence of operators $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is compact with respect to G^*-convergence and the strong topology for $H^{-1}(\Omega)$ has a countable base, by the Kuratowski compactness theorem [19] there exist an index subset $H \in \mathcal{H}^\sharp$, a set $\emptyset \neq Q \subseteq H^{-1}(\Omega)$, and a coercive bounded operator $\mathcal{A}_* \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ such that, for $\varepsilon \in H$,

$$\tau\text{-Lm}\,\mathrm{Gr}\,(\mathcal{A}_\varepsilon) = \mathrm{gr}\,(\mathcal{A}_*)$$
$$\tau\text{-Lm}\,[\widehat{Q}_\varepsilon \times K_\varepsilon] = \left[(s_{H^{-1}})\text{-Lm}\,\widehat{Q}_\varepsilon \times (w_{H_0^1})\text{-Lm}\,K_\varepsilon\right] = Q \times (w_{H_0^1})\text{-Lm}\,K_\varepsilon.$$

Therefore condition (b) of Lemma 3.1 holds. Finally, condition (c) is obvious from (iv). Hence, by Lemma 3.1 we have

$$\begin{aligned}\tau\text{-Lm}\left[\mathrm{Gr}\,(\mathcal{A}_\varepsilon)|_{\widehat{Q}_\varepsilon\times K_\varepsilon}\right] &= \tau\text{-Lm}\left(\mathrm{Gr}\,(\mathcal{A}_\varepsilon) \cap [\widehat{Q}_\varepsilon \times K_\varepsilon]\right)\\ &= \tau\text{-Lm}\,[\mathrm{Gr}\,(\mathcal{A}_\varepsilon)] \cap \left[(s_{H^{-1}})\text{-Lm}\,\widehat{Q}_\varepsilon \times (w_{H_0^1})\text{-Lm}\,K_\varepsilon\right].\end{aligned}$$

This implies (3.14) ∎

Remark 3.2. We stress that as follows from Remark 3.1 the concept of G^*-convergence may be viewed as a generalization of the well known notion of the operator G-convergence. However, even though the sequence of uniformly coercive and uniformly bounded operators $\{\mathcal{A}_\varepsilon \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))\}_{\varepsilon\in(0,\varepsilon_0]}$ is compact with respect to the G-convergence, the topological limit of the graph restrictions

$$\{\mathrm{gr}\,(\mathcal{A}_\varepsilon)|_{Q_\varepsilon\times K_\varepsilon}\}_{\varepsilon\in(0,\varepsilon_0]}$$

in the τ-topology may not be recovered in the terms of the G-limit operator. However, if we asume that for the for control object (3.3) - (3.4) there exists a sequence of admissible pairs $\{(u_\varepsilon, y_\varepsilon) \in U_\varepsilon \times K_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ such that

$$\begin{aligned} K_\varepsilon \ni y_\varepsilon \to y^* \qquad & \text{weakly in} \quad H_0^1(\Omega) \\ Q_\varepsilon \ni g_\varepsilon = \mathcal{B}_\varepsilon u_\varepsilon + f_\varepsilon \to g \qquad & \text{strongly in} \quad H^{-1}(\Omega) \\ \mathcal{A}_\varepsilon y_\varepsilon = \mathcal{B}_\varepsilon u_\varepsilon + f_\varepsilon \qquad & \text{for every} \quad \varepsilon \in (0,\varepsilon_0], \end{aligned}$$

then, indeed, we have

$$\{\tau\text{-Li}\,[\mathrm{gr}\,(\mathcal{A}_\varepsilon)|_{Q_\varepsilon\times K_\varepsilon}]\} \neq \emptyset.$$

Therefore, if we put $Y_\varepsilon = H_0^1(\Omega)$ and take the operators P_ε and R_ε^+ to be the identities, by Theorem 3.2 we obtain

$$\begin{aligned} & \mathcal{A}_\varepsilon \xrightarrow{G} \mathcal{A}_0, \quad \widehat{Q}_\varepsilon = Q_\varepsilon \ \forall\,\varepsilon \in (0,\varepsilon_0] \\ & \mathrm{Gr}\,(\mathcal{A}_\varepsilon)|_{\widehat{Q}_\varepsilon\times K_\varepsilon} = \mathrm{gr}\,(\mathcal{A}_\varepsilon)|_{Q_\varepsilon\times K_\varepsilon} \ \forall\,\varepsilon \in (0,\varepsilon_0] \\ & \tau\text{-Lm}\,[\mathrm{gr}\,(\mathcal{A}_\varepsilon)|_{Q_\varepsilon\times K_\varepsilon}] = \mathrm{gr}\,(\mathcal{A}_0)|_{(s_{H^{-1}})\text{-Lm}\,[Q_\varepsilon]\times(w_{H_0^1})\text{-Lm}\,[K_\varepsilon]}, \end{aligned}$$

i.e. the topological limit of graph restrictions $\{\mathrm{gr}\,(\mathcal{A}_\varepsilon)|_{Q_\varepsilon\times K_\varepsilon}\}$ in the $\tau = w_{H^{-1}} \times s_{H_0^1}$-topology is recovered in terms of G-limit operator $\mathcal{A}_0$.

Remark 3.3. Since in general the maps $\mathcal{F}_\varepsilon : H^{-1}(\Omega) \to H^{-1}(\Omega)$ defined by

$$f \overset{\mathcal{F}_\varepsilon}{\longmapsto} \widehat{f}, \qquad f = R_\varepsilon^+ P_\varepsilon^* \widehat{f}$$

are multi-valued, one might suspect that the topological limit of the graph prototypes may depend on the choice of sequences $\{\widehat{f}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$. This, however, does not turn out to be true. Indeed, suppose that for some sequence $\{f_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]} \subset H^{-1}(\Omega)$ there exist two sequences of prototypes $\{\widehat{f}_\varepsilon^1\}_{\varepsilon\in(0,\varepsilon_0]}$ and $\{\widehat{f}_\varepsilon^2\}_{\varepsilon\in(0,\varepsilon_0]}$ such that

(i) $\widehat{f}_\varepsilon^1 \longrightarrow \widehat{f}_*^1$ strongly in $H^{-1}(\Omega)$

(ii) $\widehat{f}_\varepsilon^2 \longrightarrow \widehat{f}_*^2$ strongly in $H^{-1}(\Omega)$

(iii) $\widehat{f}_*^1 \neq \widehat{f}_*^2$

(iv) $R_\varepsilon^+ P_\varepsilon^* \widehat{f}_\varepsilon^1 = f_\varepsilon = R_\varepsilon^+ P_\varepsilon^* \widehat{f}_\varepsilon^2$ for every $\varepsilon \in (0,\varepsilon_0]$.

Then for sequences of corresponding solutions

$$y_\varepsilon^1 = \mathcal{A}_\varepsilon^{-1} R_\varepsilon^+ P_\varepsilon^* \widehat{f}_\varepsilon^1 \qquad \text{and} \qquad y_\varepsilon^2 = \mathcal{A}_\varepsilon^{-1} R_\varepsilon^+ P_\varepsilon^* \widehat{f}_\varepsilon^2$$

we have $y^1_\varepsilon = y^2_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$. Since $\{y^i_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ are bounded in $H^1_0(\Omega)$, there exists an element $y^* \in H^1_0(\Omega)$ such that $y^1_\varepsilon \longrightarrow y^*$ and $y^2_\varepsilon \longrightarrow y^*$ weakly in $H^1_0(\Omega)$, that is $(\widehat{f}^1_*, y^*) \in \mathrm{gr}\,(\mathcal{A}_*)$ and $(\widehat{f}^2_*, y^*) \in \mathrm{gr}\,(\mathcal{A}_*)$. At the same time, by Theorem 3.1, the operator $\mathcal{A}_*$ has a single-valued inverse. This implies that $\widehat{f}^1_* = \widehat{f}^2_*$. Thus the G^*-limit of the sequence $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ does not depend on the choice of functions prototypes for $\{f\}_{\varepsilon\in(0,\varepsilon_0]}$.

The following example shows that in the genaral case the G^*-limit $\mathcal{A}_*$ of the operators $\{\mathcal{A}_\varepsilon\}$ may not coincide with the G-limit $\mathcal{A}_0$ of this sequence.

Example 3.2. Let Ω be an open bounded domain of $\mathbb{R}^n$, and let $\{\Omega_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ be a sequence of open domains of $\mathbb{R}^n$ that are contained in Ω. Let $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ be a sequence of linear uniformly coercive and uniformly bounded operators from $H^1_0(\Omega)$ into $H^{-1}(\Omega)$. For every $\varepsilon \in (0, \varepsilon_0]$ we put:

(i) L^ε be the closure in $H^{-1}(\Omega)$ of the set of all functions $f \in C^\infty(\Omega)$ with $\mathrm{supp}\, f$ contained in Ω_ε.

(ii) $Y_\varepsilon = H^1_0(\Omega_\varepsilon)$.

(iii) $P_\varepsilon : H^1_0(\Omega_\varepsilon) \to H^1_0(\Omega)$ be the extension operator defined for every $y \in H^1_0(\Omega_\varepsilon)$ by $(P_\varepsilon y)|_{\Omega_\varepsilon} = y$, $(P_\varepsilon y)|_{\Omega\setminus\Omega_\varepsilon} = 0$. Since P_ε is a linear continuous operator, the conjugate operator $P^*_\varepsilon : H^{-1}(\Omega) \to H^{-1}(\Omega_\varepsilon)$ is defined.

(iv) $R^+_\varepsilon : H^{-1}(\Omega_\varepsilon) \to (L^\varepsilon \subset H^{-1}(\Omega))$ be the extension operator defined for every $y \in H^{-1}(\Omega_\varepsilon)$ by $(R^+_\varepsilon f)|_{\Omega_\varepsilon} = f$, $(R^+_\varepsilon f)|_{\Omega\setminus\Omega_\varepsilon} = 0$.

Assume furhter that each of operators $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ has the representation

$$\mathcal{A}^{-1}_\varepsilon = P_\varepsilon \Lambda^{-1}_\varepsilon P^*_\varepsilon,$$

where $\Lambda_\varepsilon \in \mathcal{L}(Y_\varepsilon; Y^*_\varepsilon)$ are invertible operators, and if $y \in C^\infty_0(\Omega)$, then there exist a constant $\nu > 0$ and a sequence $\{y_\varepsilon \in K_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ such that $y_\varepsilon \to y$ weakly in $H^1_0(\Omega)$ and for every closed cube $S \subset \Omega$,

$$\limsup_{\varepsilon\to 0} \int_S |\nabla y_\varepsilon|^2 dx \le \nu \int_S (|\nabla y|^2 + y^2)\, dx,$$

where by K_ε we denote the closure in $H^1_0(\Omega)$ of the set of all functions $y \in C^\infty(\Omega)$ with $\mathrm{supp}\, y$ contained in Ω_ε.

Thus, $\mathcal{A}_\varepsilon \xrightarrow{G^*} \mathcal{A}_*$ if and only if

$$\mathcal{A}^{-1}_\varepsilon R^+_\varepsilon P^*_\varepsilon f \equiv P_\varepsilon \Lambda^{-1}_\varepsilon P^*_\varepsilon f \longrightarrow \mathcal{A}^{-1}_* f \qquad \text{weakly in } H^1_0(\Omega),$$

for every $f \in H^{-1}(\Omega)$. Therefore, in view of Kovalevsky's theorem (see [18]) we deduce that for the G^*-limit operator $\mathcal{A}_*$ the representation

$$\mathcal{A}_* = \mathcal{A}_0 + F_\mu$$

holds where $\mathcal{A}_0$ is the G-limit of $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ in the usual sense, and the operator $F_\mu : H^1_0(\Omega) \to H^{-1}(\Omega)$ is defined by $\langle F_\mu y, z\rangle = \int_\Omega \mu(x) yz\, dx$.

4. Setting of the homogenization problem

Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary. We define the optimal control problem as follows:

$$-\operatorname{div}(A_\varepsilon \nabla y) = b_\varepsilon u + f_\varepsilon \quad \text{in } \Omega \tag{4.1}$$

$$y = 0 \quad \text{on } \partial\Omega \tag{4.2}$$

$$u \in U_\varepsilon, \quad y \in K_\varepsilon \tag{4.3}$$

$$I_\varepsilon(u,y) = \int_\Omega C_\varepsilon y^2 dx + \int_\Omega (\nabla y, N_\varepsilon \nabla y)_{\mathbb{R}^n} dx + \int_\Omega D_\varepsilon u^2 dx \longrightarrow \inf . \tag{4.4}$$

In the sequel we impose the following asumptions:

(a) $\{K_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is a family of weakly closed convex subsets of $H_0^1(\Omega)$ such that there exists a non-empty topological limit $(w_{H_0^1})$-$\mathrm{Lm}\, K_\varepsilon$.

(b) $\{U_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is a family of weakly closed convex subsets of $L^2(\Omega)$ such that there exists a non-empty topological limit (w_{L^2})-$\mathrm{Lm}\, U_\varepsilon$.

(c) There exist linear mappings $J_\varepsilon : L^2(\Omega) \to L^2(\Omega)$ and a family of closed subsets $\{\widehat{U}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]} \subseteq L^2(\Omega)$ such that $U_\varepsilon = \{u \in L^2(\Omega) |\, u = J_\varepsilon v,\, v \in \widehat{U}_\varepsilon\}$ for every $\varepsilon \in (0,\varepsilon_0]$.

(d) There exists an invertible linear operator $J_0 : L^2(\Omega) \to L^2(\Omega)$ such that $J_\varepsilon \longrightarrow J_0$ in the weak operator topology, i.e. $\langle u, J_\varepsilon v\rangle_{L^2} \longrightarrow \langle u, J_0 v\rangle_{L^2}$ for every $u, v \in L^2(\Omega)$ and the inclusion (w_{L^2})-$\mathrm{Ls}\, \widehat{U}_\varepsilon \subseteq J_0^{-1}[(w_{L^2})$-$\mathrm{Lm}\, U_\varepsilon]$ holds.

(e) For every sequence $\{u_\varepsilon \in U_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ weakly converging in $L^2(\Omega)$ there exists a sequence of prototypes $\{v_\varepsilon \in \widehat{U}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ satisfying $u_\varepsilon = J_\varepsilon v_\varepsilon$ for every $\varepsilon \in (0,\varepsilon_0]$ and $u_\varepsilon \to u = J_0 v$ weakly in $L^2(\Omega)$ where $v \in L^2(\Omega)$ is the weak limit of $\{v_\varepsilon \in \widehat{U}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$.

(f) The sequence $\{f_\varepsilon \in H^{-1}(\Omega)\}_{\varepsilon\in(0,\varepsilon_0]}$ is compact with respect to the weak topology of $H^{-1}(\Omega)$.

(g) The sequence $\{b_\varepsilon \in L^\infty(\Omega)\}_{\varepsilon\in(0,\varepsilon_0]}$ is compact with respect to the strong topology of $L^\infty(\Omega)$.

(h) $A_\varepsilon, N_\varepsilon \in [L^\infty(\Omega)]^{n^2}$ for every $\varepsilon \in (0,\varepsilon_0]$, and there are two positive constants $0 < \beta_0 \le \beta_1$ satisfying $\beta|\xi|^2 \le (\xi, N_\varepsilon\xi)_{\mathbb{R}^n}, (\xi, A_\varepsilon\xi)_{\mathbb{R}^n} \le \beta_1|\xi|^2$ a.e. in Ω, $\beta_0 \le C_\varepsilon \le \beta_1$, for any $\xi \in \mathbb{R}^n$ and $\varepsilon \in (0,\varepsilon_0]$.

(i) $\{D_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is compact with respect to the strong topology of $L^\infty(\Omega)$.

(j) The boundary value problem (4.1) - (4.3) is uniformly regular, i.e.

$$\Xi_\varepsilon = \left\{ (u,y) \in L^2(\Omega) \times H_0^1(\Omega) \;\middle|\; \begin{array}{l} -\operatorname{div}(A_\varepsilon\nabla y) = b_\varepsilon u + f_\varepsilon \text{ in } \Omega \\ y = 0 \text{ on } \partial\Omega \\ u \in U_\varepsilon, \quad y \in K_\varepsilon \end{array} \right\} \ne \emptyset$$

for every $\varepsilon \in (0,\varepsilon_0]$.

(k) For the control object (4.1) - (4.3) hypotheses (A1) - (A3) hold true.

(l) For every $\varepsilon \in (0, \varepsilon_0]$ there exist a linear continuous operator $\widehat{\mathcal{B}}_\varepsilon$ from $L^2(\Omega)$ into $H^{-1}(\Omega)$ and an element $\widehat{f}_\varepsilon \in H^{-1}(\Omega)$ such that

$$
\begin{aligned}
& R_\varepsilon^+ P_\varepsilon^* (\widehat{\mathcal{B}}_\varepsilon v + \widehat{f}_\varepsilon) = b_\varepsilon J_\varepsilon v + f_\varepsilon \quad \text{for every } v \in \widehat{U}_\varepsilon \\
& \widehat{f}_\varepsilon \to \widehat{f}_0 \quad \text{strongly in } H^{-1}(\Omega) \\
& \widehat{\mathcal{B}}_\varepsilon \longrightarrow \widehat{\mathcal{B}}_0 \in \mathcal{L}(L^2(\Omega); H^{-1}(\Omega)) \quad \text{in the uniform operator topology} \\
& \text{i.e. } \lim_{\varepsilon \to 0} \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{\mathcal{L}(L^2(\Omega);\, H^{-1}(\Omega))} = 0.
\end{aligned}
$$

Remark 4.1. As follows from conditions (k) - (l) and Definition 3.1, there is a collection of real reflexive separable Banach spaces $\{Y_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ coordinated with the constrained state equation (4.1) - (4.3). Therefore, by virtue of Theorem 3.1, there exists a coercive linear operator $\mathcal{A}_*$ of $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ such that $\mathcal{A}_\varepsilon \xrightarrow{G^*} \mathcal{A}_*$, where $\mathcal{A}_\varepsilon \in \mathcal{L}\big(H_0^1(\Omega); H^{-1}(\Omega)\big)$ are defined as

$$
\langle \mathcal{A}_\varepsilon y, \varphi \rangle_{H_0^1(\Omega)} = \int_\Omega (\nabla \varphi, A_\varepsilon \nabla y)_{\mathbb{R}^n} \, dx \qquad \forall \, \varphi \in H_0^1(\Omega).
$$

We are now in the position to apply the procedure of S-homogenization to the optimal control problem (4.1) - (4.4). Recall that our approach is based on the following representation

$$
\left\{ \left\langle \inf_{(u,y) \in \Xi_\varepsilon} I_\varepsilon(u, y) \right\rangle \right\}_{\varepsilon \in (0,\varepsilon_0]}. \tag{4.5}
$$

of the optimal control problem.

We shall consider the homogenization of the optimal control problem (4.1) - (4.4) with respect to the μ-topology for $L^2(\Omega) \times H_0^1(\Omega)$ by passing to the limit in the sequence (4.5). Namely, the μ-topology is the most natural one for the homogenization procedure in our case. Indeed, the sequence of optimal pairs $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon \in (0,\varepsilon_0]}$ for the original problem (4.1) - (4.4) is bounded and hence we may assume that it is compact with respect to this topology. Therefore, thanks to Theorem 2.3, each of the cluster points of this sequence in the μ-topology will be a minimizer for S-homogenized problem as well.

We also note that by virtue of condition (h) the sequence of cost functionals $\{I_\varepsilon : \Xi_\varepsilon \to \overline{R}\}_{\varepsilon \in (0,\varepsilon_0]}$ is μ-equicoercive, and by property (j) there exists a non-empty lower topological limit for the sequence of sets of admissible pairs $\{\Xi_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ in the μ-topology, i.e. $\mu\text{-Li}\, \Xi_\varepsilon \neq \emptyset$. Therefore, Theorems 2.2 and 2.3 immediately give the following result.

Theorem 4.1. *Suppose that assumptions* (a) - (l) *hold true and for the sequence* $\{\Xi_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ *there exists a non-empty topological limit in the* μ*-topology. Then the following statements hold:*

(i) *We can extract a subsequence from the sequence of constrained minimization problems* (4.5) (*still indexed by* ε) *for which there exists an absolute variational* S*-limit* *in sense of Definition* 2.2, *i.e. the strongly S-homogenized optimal control problem has the representation*

$$\left\langle \inf_{(u,y)\in \mu\text{-}Lm\Xi_\varepsilon} \mu\text{-}lm^a(I_\varepsilon|_{\Xi_\varepsilon})(u,y)\right\rangle. \tag{4.6}$$

(ii) *The sequence of optimal pairs* $\{(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0]}$ *for family* (4.1) – (4.4) *is compact with respect to the μ-topology.*

(iii) *If (u, y) is a cluster point of the sequence of optimal pairs* $\{(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0]}$ *in the μ-topology, then*

$$(u,y)\in \mathbf{M}(\mu\text{-lm}^a(I_\varepsilon|_{\Xi_\varepsilon}); \mu\text{-Lm}\,\Xi_\varepsilon)$$
$$\tau\text{-lm}^a(I_\varepsilon|_{\Xi_\varepsilon})(u,y) = \lim_{\varepsilon\to 0} I_\varepsilon(u^0_\varepsilon, y^0_\varepsilon).$$

In order to recover the strong S-homogenized problem (4.5) we shall use the following result.

Lemma 4.1. *A set E is the tological limit of the sequence* $\{E_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}\subset X$ *if and only if the following conditions are satisfied:*

(i) *For every $x \in E$ there exist an index set $H \in \mathcal{H}$ and a sequence $\{x_\varepsilon\}_{\varepsilon\in H}$ converging to x in X such that $x_\varepsilon \in E_\varepsilon$ for every $\varepsilon \in H$.*

(ii) *If H is any index set of $\mathcal{H}^\sharp$, $\{x_\varepsilon\}_{\varepsilon\in H}$ is a sequence converging to x in X such that $x_\varepsilon \in E_\varepsilon$ for every $\varepsilon \in H$, then $x \in E$.*

5. The topological limit of the set of admissible solutions

We begin this subsection with the following result.

Lemma 5.1. *If assumptions* (a) - (l) *hold true, then*

$$\emptyset \neq (w_{H^{-1}})\text{-Lm}\,\widehat{Q}_\varepsilon = \left\{ g\in H^{-1}(\Omega) \,\middle|\, g = \widehat{\mathcal{B}}_0 J_0^{-1}u + \widehat{f}_0 \ \forall\, u\in (w_{L^2})\text{-Lm}\, U_\varepsilon \right\} \tag{5.1}$$

where $\widehat{f}_0$ is a limit of $\{\widehat{f}_\varepsilon\}$ in the strong topology of $H^{-1}(\Omega)$ and $\widehat{Q}_\varepsilon$ are convex closed subsets defined by

$$\widehat{Q}_\varepsilon = \left\{ g\in H^{-1}(\Omega) \,\middle|\, g = \widehat{\mathcal{B}}_\varepsilon v + \widehat{f}_\varepsilon \ \forall\, v\in \widehat{U}_\varepsilon \right\}. \tag{5.2}$$

Proof. Let $g^* = \widehat{\mathcal{B}}_0 J_0^{-1}u^* + \widehat{f}_0$ be any element of the set

$$\left\{ g\in H^{-1}(\Omega) \,\middle|\, g = \widehat{\mathcal{B}}_0 J_0^{-1}u + \widehat{f}_0 \ \forall\, u\in (w_{L^2})\text{-Lm}\, U_\varepsilon \right\}.$$

Then since $u^* \in (w_{L^2})\text{-Lm}\, U_\varepsilon$, it follows that there exist an index set $H \in \mathcal{H}$, a sequence $\{u^*_\varepsilon\}_{\varepsilon\in H}$ converging to u^* in the weak topology of $L^2(\Omega)$, and a sequence of prototypes $\{v^*_\varepsilon\}_{\varepsilon\in H}$ weakly converging to v^* in $L^2(\Omega)$ such that

$$u^*_\varepsilon = J_\varepsilon v^*_\varepsilon \in U_\varepsilon,\ v^*_\varepsilon \in \widehat{U}_\varepsilon \ \forall\, \varepsilon\in H \qquad \text{and} \qquad u^* = J_0 v^*.$$

Therefore, by property (l), $\widehat{\mathcal{B}}_\varepsilon v^*_\varepsilon + \widehat{f}_\varepsilon \in \widehat{Q}_\varepsilon$ for every $\varepsilon \in H$. At the same time we have

$$\begin{aligned}\|\widehat{\mathcal{B}}_\varepsilon v^*_\varepsilon - \widehat{\mathcal{B}}_0 v^*\| &\le \|(\widehat{\mathcal{B}}_\varepsilon - \widehat{\mathcal{B}}_0)v^*_\varepsilon\| + \|\widehat{\mathcal{B}}_0(v^*_\varepsilon - v^*)\| \\ &\le \|\widehat{\mathcal{B}}_\varepsilon - \widehat{\mathcal{B}}_0\|\,\|v^*_\varepsilon\| + \sup_{\|\phi\|_{H^1_0}=1} \langle \widehat{\mathcal{B}}^*_0\phi, v^*_\varepsilon - v^*\rangle.\end{aligned}$$

Hence,

$$\widehat{\mathcal{B}}_\varepsilon v^*_\varepsilon + \widehat{f}_\varepsilon \to \widehat{\mathcal{B}}_0 v^* + \widehat{f}_0 = \widehat{\mathcal{B}}_0 J_0^{-1} u^* + \widehat{f}_0 \qquad \text{strongly in } H^{-1}(\Omega).$$

On the other hand, if H is any index set of $\mathcal{H}^\sharp$ and $\{g_\varepsilon \in \widehat{Q}_\varepsilon\}_{\varepsilon\in H}$ is a sequence converging to g in the strong topology of $H^{-1}(\Omega)$, then there is a sequence of control prototypes $\{v_\varepsilon \in \widehat{U}_\varepsilon\}_{\varepsilon\in H}$ such that $g_\varepsilon = \widehat{\mathcal{B}}_\varepsilon v_\varepsilon + \widehat{f}_\varepsilon$ for every $\varepsilon \in H$. Since the sequence $\widehat{\mathcal{B}}_\varepsilon v_\varepsilon$ is bounded in $H^{-1}(\Omega)$ and the operators $\widehat{\mathcal{B}}_\varepsilon$ are compact with respect to the uniform operator topology, it follows the the sequence $\{v_\varepsilon\}_{\varepsilon\in H}$ is bounded as well. Hence we may assume that there is an element $v_0 \in (w_{L^2})\text{-Ls}\,\widehat{U}_\varepsilon$ such that $v_\varepsilon \to v_0$ weakly in $L^2(\Omega)$. Consequently,

$$\begin{aligned}&g_\varepsilon = \widehat{\mathcal{B}}_\varepsilon v_\varepsilon + \widehat{f}_\varepsilon \in \widehat{Q}_\varepsilon \text{ for every } \varepsilon \in H\\ &g_\varepsilon \to \widehat{\mathcal{B}}_0 v_0 + \widehat{f}_0 = g_0 \text{ strongly in } H^{-1}(\Omega).\end{aligned}$$

But property (d) there is an element u_0 in $(w_{L^2})\text{-Lm}\,U_\varepsilon$ satisfying $v_0 = J_0^{-1}u_0$. Therefore $g_0 = \widehat{\mathcal{B}}_0 J_0^{-1} u_0 + \widehat{f}_0$. Thus, with Lemma 4.1 we are done ∎

By virtue of the G^*-compactness Theorem 3.1 we obtain the following result.

Theorem 5.1. *Suppose that conditions* (a) - (l) *hold true and that there is an index set $H \in \mathcal{H}$ and some μ-converging sequence of admissible pairs $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\in H}$ for the original optimal control problem* (4.1) – (4.4)*. Then for the sequence of admissible pairs sets $\{\Xi_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ there exists a topological limit in the μ-topology satisfying*

$$\mu\text{-Lm}\,\Xi_\varepsilon = \mathbf{X} \tag{5.3}$$

where

$$\mathbf{X} = \left\{ (u,y) \in L^2(\Omega)\times H^1_0(\Omega) \;\middle|\; \begin{array}{l} \mathcal{A}_* y = \widehat{\mathcal{B}}_0 J_0^{-1} u + \widehat{f}_0 \\ u \in (w_{L^2})\text{-Lm}\,U_\varepsilon,\; y \in (w_{H^1_0})\text{-Lm}\,K_\varepsilon \end{array} \right\}$$

and $\mathcal{A}_* \in \mathcal{L}\big(H^1_0(\Omega); H^{-1}(\Omega)\big)$ is the G^*-limit of the sequence of operators $\{\mathcal{A}_\varepsilon\}$ in the sense of Definition 3.3.

Proof. First of all we note that by the standing assumptions there is some sequence of admissible pair $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\in H}$ such that $(u_\varepsilon, y_\varepsilon) \xrightarrow{\mu} (u^0, y^0)$. However, by property (e) there can be found a sequence of control prototypes $\{v_\varepsilon \in \widehat{U}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ satisfying $u_\varepsilon = J_\varepsilon v_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$ and $u_\varepsilon \to u^0 = J_0 v^0$ weakly in $L^2(\Omega)$, where $v^0 \in L^2(\Omega)$ is the weak limit of $\{v_\varepsilon \in \widehat{U}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$. Therefore, in view of condition (l) instead of the original sequence of admissible pairs we may consider the sequence of their prototypes $\{(v_\varepsilon, y_\varepsilon) \in \widehat{\Xi}_\varepsilon\}_{\varepsilon\in H}$, where the sets $\widehat{\Xi}_\varepsilon$ are defined by

$$\widehat{\Xi}_\varepsilon = \left\{ (v,y) \in L^2(\Omega) \times H_0^1(\Omega) \;\middle|\; \begin{array}{l} \mathcal{A}_\varepsilon y = R_\varepsilon^+ P_\varepsilon^* (\widehat{\mathcal{B}}_\varepsilon v + \widehat{f}_\varepsilon) \\ v \in \widehat{U}_\varepsilon,\ y \in K_\varepsilon \end{array} \right\}.$$

Consequently, by Lemma 5.1 and condition (e) we have

$$\widehat{Q}_\varepsilon \ni \widehat{\mathcal{B}}_\varepsilon v_\varepsilon + \widehat{f}_\varepsilon \to \widehat{\mathcal{B}}_0 J_0^{-1} u^0 + \widehat{f}_0 \in (s_{H^{-1}})\text{-Lm}\, \widehat{Q}_\varepsilon \quad \text{strongly in } H^{-1}(\Omega),$$

i.e. all assumptions on Theorem 3.2 hold true. Therefore, for the topological limit of prototype graph restrictions $\left[\mathrm{Gr}\,(\mathcal{A}_\varepsilon)|_{\widehat{Q}_\varepsilon \times K_\varepsilon}\right]$ representation (3.14) holds.

Let $(\widehat{u}^*, \widehat{y}^*)$ be any pair of $\mathbf{X}$. Then, by Lemma 5.1,

$$\widehat{g}^* = \widehat{\mathcal{B}}_0 J_0^{-1} \widehat{u}^* + \widehat{f}_0 \in (s_{H^{-1}})\text{-Lm}\, \widehat{Q}_\varepsilon$$

where the sets $\widehat{Q}_\varepsilon$ are defined in (5.2). Using Theorem 3.2 we deduce that

$$(\widehat{g}^*, \widehat{y}^*) \in \mathrm{gr}\,(\mathcal{A}_*) \cap \left[(s_{H^{-1}})\text{-Lm}\, \widehat{Q}_\varepsilon \times (w_{H_0^1})\text{-Lm}\, K_\varepsilon\right].$$

Here $\mathcal{A}_*$ is the G^*-limit of the operators sequence $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$. Then in accordance with Theorem 3.2 we obtain

$$\begin{aligned}(\widehat{g}^*, \widehat{y}^*) &\in \tau\text{-Lm}\,\mathrm{Gr}\,(\mathcal{A}_\varepsilon) \cap \left[(s_{H^{-1}})\text{-Lm}\, \widehat{Q}_\varepsilon \times (w_{H_0^1})\text{-Lm}\, K_\varepsilon\right] \\ &= \tau\text{-Lm}\left[\mathrm{Gr}\,(\mathcal{A}_\varepsilon)|_{\widehat{Q}_\varepsilon \times K_\varepsilon}\right].\end{aligned}$$

Therefore, by the properties of topological limits (see Lemmas 4.1 and 5.1) there exist an index set $H \in \mathcal{H}$ and sequences $\{\widehat{y}_\varepsilon\}_{\varepsilon\in H}$, $\{\widehat{u}_\varepsilon\}_{\varepsilon\in H}$, $\{\widehat{v}_\varepsilon\}_{\varepsilon\in H}$ such that

$$\begin{array}{rll} K_\varepsilon \ni \widehat{y}_\varepsilon &\longrightarrow \widehat{y}^* & \text{weakly in } H_0^1(\Omega) \\ \widehat{U}_\varepsilon \ni \widehat{v}_\varepsilon &\longrightarrow \widehat{v}^* & \text{weakly in } L^2(\Omega) \\ U_\varepsilon \ni J_\varepsilon \widehat{v}_\varepsilon = \widehat{u}_\varepsilon &\longrightarrow \widehat{u}^* = J_0 \widehat{v}^* & \text{weakly in } L^2(\Omega) \\ \widehat{Q}_\varepsilon \ni \widehat{g}_\varepsilon = \widehat{\mathcal{B}}_\varepsilon \widehat{v}_\varepsilon + \widehat{f}_\varepsilon &\longrightarrow \widehat{\mathcal{B}}_0 J_0^{-1} \widehat{u}^* + \widehat{f}_0 = \widehat{g}^* & \text{strongly in } H^{-1}(\Omega) \\ \mathcal{A}_\varepsilon \widehat{y}_\varepsilon &= R_\varepsilon^+ P_\varepsilon^* \widehat{g}_\varepsilon = b_\varepsilon \widehat{u}_\varepsilon + f_\varepsilon & \text{for every } \varepsilon \in H. \end{array}$$

Thus for the pair $(\widehat{u}^*, \widehat{y}^*)$ we have found the index set $H \in \mathcal{H}$ and constructed a sequence $\{(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon)\}_{\varepsilon\in H}$ such that

$$(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \xrightarrow{\ \mu\ } (\widehat{u}^*, \widehat{y}^*) \;\text{ and }\; (\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \in \Xi_\varepsilon \qquad \text{for every } \varepsilon \in H,$$

i.e. condition (i) of Lemma 4.4 holds.

Now we consider any index set H of $\mathcal{H}^\sharp$. Let $\{(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon)\}_{\varepsilon\in H}$ be a sequence μ-converging to some pair (u, y) such that $(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \in \Xi_\varepsilon$ for every $\varepsilon \in H$. We have to show that $(u,y) \in \mathbf{X}$. Indeed, in this case there exists a sequence of prototypes $\{\widehat{v}_\varepsilon\}_{\varepsilon\in H}$ weakly converging to v in $L^2(\Omega)$ such that

$$\widehat{u}_\varepsilon = J_\varepsilon \widehat{v}_\varepsilon \in U_\varepsilon,\ \widehat{v}_\varepsilon \in \widehat{U}_\varepsilon\ \forall\, \varepsilon \in H \qquad \text{and} \qquad u = J_0 v.$$

Consequently,

$$\begin{aligned} \widehat{g}_\varepsilon &= \widehat{\mathcal{B}}_\varepsilon \widehat{v}_\varepsilon + \widehat{f}_\varepsilon \longrightarrow \widehat{\mathcal{B}}_0 J_0^{-1} u + \widehat{f}_0 = \widehat{g}_0 && \text{strongly in } H^{-1}(\Omega)\\ \widehat{y}_\varepsilon &= \mathcal{A}_\varepsilon^{-1} R_\varepsilon^{+} P_\varepsilon^{*} \widehat{g}_\varepsilon \longrightarrow y && \text{weakly in } H_0^1(\Omega) \end{aligned}$$

and by virtue of Theorem 3.2 we have

$$(\widehat{g}_0, y) \in \mathrm{gr}\,(\mathcal{A}_*)\big|_{(s_{H^{-1}})\text{-Lm}\,\widehat{Q}_\varepsilon \times (w_{H_0^1})\text{-Lm}\,K_\varepsilon}.$$

Therefore $y = \mathcal{A}_*^{-1}\widehat{g}_0 = \mathcal{A}_*^{-1}(\widehat{\mathcal{B}}_0 J_0^{-1} u + \widehat{f}_0)$, i.e. we have the inclusion $(u,y) \in \mathbf{X}$. Thus, using Lemma 4.1, we deduce that the set $\mathbf{X}$ is the topological limit of the sequence of admissible pairs sets $\{\Xi_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$. This completes the proof ∎

Remark 5.1. It is easily shown that we are able to omit the assumptions of this theorem with respect to existence some μ-converging sequence of admissible pair for the original optimal control problem (4.1) - (4.4). Indeed, thanks to the uniformly coerciveness property of the cost functionals I_ε the sequence of optimal pairs is bounded. Hence we may assume that this sequence is compact in the μ-topology.

6. On the explicit representation of the absolute S-limit of the cost functional

In this section we shall prove that, under some reasonable assumptions, there exist a convex closed subset $U_0 \subseteq (w_{L^2})\text{-Lm}\,U_\partial^\varepsilon$, a functional $F : U_0 \to \overline{\mathbb{R}}$, and a matrix $N^\sharp \in [L^\infty(\Omega)]^{n^2}$ such that

$$\mu\text{-lm}^a(\widehat{I}_\varepsilon|_{\widehat{\Xi}_\varepsilon}) = \int_\Omega C_0 y^2\,dx + \int_\Omega (\nabla y, N^\sharp \nabla y)_{\mathbb{R}^n}\,dx + F(u).$$

To this end we inroduce the following concept.

Definition 6.1. We say that the sequence of operators $\big\{\mathcal{A}_\varepsilon \in \mathcal{L}\big(H_0^1(\Omega); H^{-1}(\Omega)\big)\big\}$ *strongly G^*-converges* to the operator $\mathcal{A}_*$ (in symbols, $\mathcal{A}_\varepsilon \overset{G^*}{\Longrightarrow} \mathcal{A}_*$) if the sequence $\{\mathcal{A}_\varepsilon\}$ G^*-converges to the operator $\mathcal{A}_*$, and there exists a matrix $A_* \in [L^\infty(\Omega)]^{n^2}$ such that for every sequence $\{\widehat{g}_\varepsilon\}$ strongly converging in $H^{-1}(\Omega)$ to $\widehat{g}$ the conditions

$$\langle \mathcal{A}_* y, \varphi\rangle_{H_0^1(\Omega)} = \int_\Omega (\nabla\varphi, A_*\nabla y)_{\mathbb{R}^n}\,dx\ \forall\,\varphi \in H_0^1(\Omega) \tag{6.1}$$

$$A_\varepsilon \nabla y_\varepsilon \longrightarrow A_* \nabla y_0 \ \ \text{weakly in } [L^\infty(\Omega)]^{n^2} \tag{6.2}$$

hold where $y_\varepsilon = \mathcal{A}_\varepsilon^{-1} R_\varepsilon^{+} P_\varepsilon^{*} \widehat{g}_\varepsilon$ and $y_0 = \mathcal{A}_* \widehat{g}$.

Now we establish the following result.

Lemma 6.1. *Let* $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0]}$ *be any sequence such that*

$$(u_\varepsilon, y_\varepsilon) \xrightarrow{\mu} (u_0, y_0), \quad (u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon \quad \forall\, \varepsilon \in (0, \varepsilon_0].$$

Then, under conditions (a) - (l) *and* $\mathcal{A}_\varepsilon \stackrel{G^*}{\Longrightarrow} \mathcal{A}_*$,

$$(\nabla y_\varepsilon, N_\varepsilon \nabla y_\varepsilon)_{\mathbb{R}^n} \longrightarrow (\nabla y_0, N^\sharp \nabla y_0)_{\mathbb{R}^n} \qquad \text{in } \mathcal{D}'(\Omega)$$

where the matrix $N^\sharp$ *depends only on* $\{N_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ *and* $\{A_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ *and* $N^\sharp$ *is given by formula* (6.9) *below.*

Proof. First of all we define the functions in $\{\psi_\varepsilon \in H_0^1(\Omega)\}_{\varepsilon\in(0,\varepsilon_0]}$ by

$$-\operatorname{div}(A_\varepsilon^t \nabla \psi_\varepsilon) = \operatorname{div}(N_\varepsilon^t \nabla y_\varepsilon) \ \text{ in } \Omega \tag{6.4}$$

$$\psi_\varepsilon = 0 \ \text{ on } \partial\Omega. \tag{6.5}$$

Under our assumptions, it is easy to see that $\|\psi_\varepsilon\|_{H_0^1(\Omega)} \le C$, where the constant C is independent of ε.

Let $\varphi \in \mathcal{D}(\Omega)$ be an arbitrary function. Here by $\mathcal{D}(\Omega)$ we denote the space of all smooth real valued functions on Ω which are compactly supported in Ω. Let $\zeta_\varepsilon \in Y_\varepsilon$ be functions such that $\varphi\psi_\varepsilon = P_\varepsilon \zeta_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$. Then by property (A3) we have

$$\begin{aligned}
\big\langle R_\varepsilon^+ P_\varepsilon^* (\widehat{\mathcal{B}}_\varepsilon \widehat{v}_\varepsilon + \widehat{f}_\varepsilon), \psi_\varepsilon\varphi \big\rangle_{(H_0^1;H^{-1})} &= \big\langle R_\varepsilon^+ P_\varepsilon^* (\widehat{\mathcal{B}}_\varepsilon \widehat{v}_\varepsilon + \widehat{f}_\varepsilon), P_\varepsilon^* \zeta_\varepsilon \varphi \big\rangle_{(H_0^1;H^{-1})} \\
&= \big\langle P_\varepsilon^* (\widehat{\mathcal{B}}_\varepsilon \widehat{v}_\varepsilon + \widehat{f}_\varepsilon), \zeta_\varepsilon \varphi \big\rangle_{(Y_\varepsilon;Y_\varepsilon^*)} \\
&= \big\langle \widehat{\mathcal{B}}_\varepsilon \widehat{v}_\varepsilon + \widehat{f}_\varepsilon, \psi_\varepsilon \varphi \big\rangle_{(H_0^1;H^{-1})},
\end{aligned}$$

where the value of $F \in H^{-1}(\Omega)$ at $\mu \in H_0^1(\Omega)$ is denoted by $\langle F, \mu\rangle_{(H_0^1;H^{-1})}$ and $\{\widehat{v}\}_{\varepsilon\in(0,\varepsilon_0]}$ are the prototypes of the sequence of original controls $\{u_\varepsilon\}_{\varepsilon\in\,(0,\varepsilon_0]}$ satisfying condition (e).

Now, equation (4.5), when multiplied by $\varphi\psi_\varepsilon$ and integrated by parts, gives

$$\begin{aligned}
0 = \langle \widehat{\mathcal{B}}_\varepsilon \widehat{v}_\varepsilon + \widehat{f}_\varepsilon, \psi_\varepsilon\varphi \rangle_{(H_0^1;H^{-1})} &- \int_\Omega (A_\varepsilon \nabla y_\varepsilon, \nabla\varphi)_{\mathbb{R}^n} \psi_\varepsilon dx \\
+ \int_\Omega (N_\varepsilon^t \nabla y_\varepsilon, \nabla \varphi)_{\mathbb{R}^n} y_\varepsilon dx &+ \int_\Omega (\nabla y_\varepsilon, N_\varepsilon \nabla y_\varepsilon)_{\mathbb{R}^n} \varphi\, dx \\
+ \int_\Omega (A_\varepsilon^t \nabla \psi_\varepsilon, \nabla\varphi)_{\mathbb{R}^n} y_\varepsilon dx.
\end{aligned} \tag{6.6}$$

We may now pass to the limit in (6.6) as $\varepsilon \to 0$, since each of the term in the right-hand side is a product of two sequences, one converging weakly and the other strongly in $L^2(\Omega)$. Thus by property (e) we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_\Omega \big(\nabla y_\varepsilon, N_\varepsilon \nabla y_\varepsilon\big)_{\mathbb{R}^n} \varphi\, dx = -\big\langle \widehat{\mathcal{B}}_0 J_0^{-1} u_0 + \widehat{f}_0, \psi_0\varphi \big\rangle_{(H_0^1;H^{-1})} \\
+ \int_\Omega \big(A_* \nabla y_0, \nabla \varphi\big)_{\mathbb{R}^n} \psi_0 dx \\
- \lim_{\varepsilon\to 0} \int_\Omega \big([N_\varepsilon^t \nabla y_\varepsilon + A_\varepsilon^t \nabla \psi_\varepsilon], \nabla\varphi\big)_{\mathbb{R}^n} y_0 dx.
\end{aligned}$$

Note also that, since

$$\widehat{\mathcal{B}}_\varepsilon u_\varepsilon + \widehat{f}_\varepsilon \longrightarrow \widehat{\mathcal{B}}_0 J_0^{-1} u_0 + \widehat{f}_0 \qquad \text{strongly in } H^{-1}(\Omega),$$

by definition of the strong G^*-limit (see [21]), we have

$$\begin{aligned} y_\varepsilon &\longrightarrow y_0 && \text{weakly in } H_0^1(\Omega) \\ A_\varepsilon \nabla y_\varepsilon &\longrightarrow A_* \nabla y_0 && \text{weakly in } [L^\infty(\Omega)]^{n^2} \\ \psi_\varepsilon &\longrightarrow \psi_0 && \text{weakly in } H_0^1(\Omega) \end{aligned}$$

and

$$-\mathrm{div}\,(A_* \nabla y_0) = \widehat{\mathcal{B}}_0 J_0^{-1} u_0 + \widehat{f}_0. \tag{6.7}$$

Hence,

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_\Omega (\nabla y_\varepsilon, N_\varepsilon \nabla y_\varepsilon)_{\mathbb{R}^n} \varphi\, dx \\ &\quad = -\left\langle \widehat{\mathcal{B}}_0 J_0^{-1} u_0 + \widehat{f}_0, \psi_0 \varphi \right\rangle_{(H_0^1; H^{-1})} \\ &\qquad - \left\langle \mathrm{div}(A_* \nabla y_0), \varphi \psi_0 \right\rangle_{(H_0^1; H^{-1})} - \int_\Omega (A_* \nabla y_0, \nabla \psi_0)_{\mathbb{R}^n} \varphi\, dx \\ &\qquad + \lim_{\varepsilon\to 0} \int_\Omega \left([N_\varepsilon^t \nabla y_\varepsilon + A_\varepsilon^t \nabla \psi_\varepsilon], \nabla y_0 \right)_{\mathbb{R}^n} \varphi\, dx \\ &\qquad + \lim_{\varepsilon\to 0} \left\langle \mathrm{div}([N_\varepsilon^t \nabla y_\varepsilon + A_\varepsilon^t \nabla \psi_\varepsilon]), \varphi y_0 \right\rangle_{(H_0^1; H^{-1})}. \end{aligned}$$

Now, using (6.4) and (6.7) we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega (\nabla y_\varepsilon, N_\varepsilon \nabla y_\varepsilon)_{\mathbb{R}^n} \varphi\, dx = \lim_{\varepsilon\to 0} \int_\Omega \left([A_\varepsilon^t \nabla \psi_\varepsilon + N_\varepsilon^t \nabla y_\varepsilon] - A_*^t \nabla \psi_0, \nabla y_0 \right)_{\mathbb{R}^n} \varphi\, dx$$

for every $\varphi \in \mathcal{D}(\Omega)$.

Since $\nabla \psi_0$ is a homogeneous function with respect to ∇y_0 it follows that we may write the previous expression as

$$\lim_{\varepsilon\to 0} \int_\Omega (\nabla y_\varepsilon, N_\varepsilon \nabla y_\varepsilon)_{\mathbb{R}^n} \varphi\, dx = \int_\Omega (\nabla y_0, N^\sharp \nabla y_0)_{\mathbb{R}^n} \varphi\, dx, \tag{6.8}$$

where

$$N^\sharp \nabla y_0 = w\text{-}\lim_{\varepsilon\to 0} [A_\varepsilon^t \nabla \psi_\varepsilon + N_\varepsilon^t \nabla y_\varepsilon] - A_0^t \nabla \psi_0. \qquad (6.89$$

Here by $w\text{-}\lim_{\varepsilon\to 0}$ we denote the weak-$*$ limit in $[L^\infty(\Omega)]^{n^2}$. Finally, note that (6.8) is true for every sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0]}$ satisfying (6.3). Consequently, the matrix $N^\sharp \in [L^\infty(\Omega)]^{n^2}$ depends only on $\{N_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ and $\{A_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$. This concludes the proof ∎

Let Π_U be the linear operator from $L^2(\Omega) \times H_0^1(\Omega)$ into $L^2(\Omega$ defined by

$$\Pi_U[\Sigma] = \left\{ u \in L^2(\Omega) \middle| \ \exists\, y \in H_0^1(\Omega) \text{ such that } (u, y) \in \Sigma \right\}$$

for every $\Sigma \subseteq L^2(\Omega) \times H_0^1(\Omega)$. We denote by $F : \Pi_U[\mu\text{-Lm}\,\Xi_\varepsilon] \to \overline{\mathbb{R}}$ the absolute S-limit of the collection of functionals

$$\left\{ F_\varepsilon(u) = \int_\Omega D_\varepsilon u^2 dx \middle| \ u \in \Pi_U[\Xi_\varepsilon] \right\}_{\varepsilon \in (0, \varepsilon_0]}$$

with respect to the weak topology for $L^2(\Omega)$. Note that, by the standing assumptions (a) - (l), the absolute S-limit $F : \Pi_U[\mu\text{-Lm}\,\Xi_\varepsilon] \to \overline{\mathbb{R}}$ exists, since the sequence of functionals $\{F : U_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon \in (0,\varepsilon_0]}$ is w_{L^2}-equicoercive. Taking into account this fact and Theorem 2.1 we obtain the following analytical representation for the absolute S-limit of the cost functional sequence $\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon \in (0,\varepsilon_0]}$.

Theorem 6.1. *Under the suppositions of Lemma* 6.1 *the following representation for the homogenized cost functional*

$$\mu\text{-}lm^a(I_\varepsilon|_{\Xi_\varepsilon}) = \int_\Omega C_0 y^2 dx + \int_\Omega (\nabla y, N^\sharp \nabla y)_{\mathbb{R}^n} dx + F(u) \tag{6.10}$$

holds for every $(u, y) \in \mu\text{-}Lm\,\Xi_\varepsilon$.

Proof. Indeed, for every μ-converging sequence of admissible pairs $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon \in (0,\varepsilon_0]}$ we have

$$\int_\Omega C_\varepsilon y_\varepsilon^2 dx + \int_\Omega (\nabla y_\varepsilon, N_\varepsilon \nabla y_\varepsilon)_{\mathbb{R}^n} dx \longrightarrow \int_\Omega C_0 y_0^2 dx + \int_\Omega (\nabla y_0, N^\sharp \nabla y_0)_{\mathbb{R}^n} dx.$$

Therefore by virtue of Lemma 6.1 and Theorem 2.1 the proof of formula (6.10) is trivial ∎

Remark 6.1. Let $K_\varepsilon = H_0^1(\Omega)$ and $U_\varepsilon = U_0$ for every $\varepsilon \in (0, \varepsilon_0]$, where U_0 is some convex closed subset of $L^2(\Omega)$. Then, by the properties of S-limits, for the functional $F : \Pi_U[\Xi_\varepsilon] \to \overline{\mathbb{R}}$ the representation $F(u) = \int_\Omega D_0 u^2 dx$ holds. However, in the general case we have only the estimate

$$F(u) \geq \int_\Omega D_0 u^2 dx \qquad \forall\, u \in \Pi_U[\mu\text{-Lm}\,\Xi_\varepsilon].$$

This from the basic properties of S-limits. Indeed, by definition of S-limit we have

$$(w_{L^2})\text{-lm}^a(F_\varepsilon|_{\Pi_U[\Xi_\varepsilon]})(u) \geq \Gamma(w_{L^2})\text{-}\lim\, F_\varepsilon(u)$$

for every $u \in \Pi_U[\mu\text{-Lm}\,\Xi_\varepsilon]$. Since

$$\Gamma(w_{L^2})\text{-}\lim\, F_\varepsilon(u) = \Gamma(w_{L^2})\text{-}\lim \int_\Omega D_\varepsilon u^2 dx = \int_\Omega D_0 u^2 dx$$

we are done.

Corollary 6.2 (The one-dimensional periodic case). *Let*

$$\Omega = (c,d) \subset \mathbb{R}^1, \quad Y_\varepsilon = H_0^1(\Omega), \quad R_\varepsilon^+ P_\varepsilon^* = 1, \quad A_\varepsilon = a\Big(\frac{x}{\varepsilon}\Big), \quad Q_\varepsilon = q\Big(\frac{x}{\varepsilon}\Big)$$

where $a(\cdot)$ and $q(\cdot)$ are periodic functions on $[0,1]$. Then, under the conditions of Theorem 6.1,

$$\mu\text{-}lm^a(I_\varepsilon|_{\Xi_\varepsilon}) = \int_c^d C_0 y^2 dx + q^\sharp \int_c^d y^2 dx + F(u)$$

where

$$q^\sharp = \Big[\int_0^1 a^{-1}(\xi)\,d\xi\Big]^{-2} \cdot \int_0^1 \frac{q(\xi)}{a^2(\xi)}\,d\xi.$$

Proof. Let $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ be any sequence such that $(u_\varepsilon, y_\varepsilon) \overset{\mu}{\longrightarrow} (u,y)$, where (u,y) is an arbitrary pair of μ-Lm Ξ_ε. Then the functions u_ε, y_ε, and ψ_ε satisfy the equations

$$\left.\begin{aligned} -\frac{d}{dx}\Big(a\Big(\frac{x}{\varepsilon}\Big)\frac{dy_\varepsilon}{dx}\Big) &= b_\varepsilon u_\varepsilon + f_\varepsilon \\ -\frac{d}{dx}\Big(a\Big(\frac{x}{\varepsilon}\Big)\frac{d\psi_\varepsilon}{dx} + q\Big(\frac{x}{\varepsilon}\Big)\frac{dy_\varepsilon}{dx}\Big) &= 0 \end{aligned}\right\}.$$

By d_ε we denote the expression

$$a\Big(\frac{x}{\varepsilon}\Big)\frac{d\psi_\varepsilon}{dx} + q\Big(\frac{x}{\varepsilon}\Big)\frac{dy_\varepsilon}{dx}.$$

Then, using (6.9), we obtain

$$\lim_{\varepsilon\to 0}\int_c^d g\Big(\frac{x}{\varepsilon}\Big)\Big(\frac{dy_\varepsilon}{dx}\Big)^2 dx = \lim_{\varepsilon\to 0}\int_c^d d_\varepsilon \frac{dy}{dx}\,dx - a_0\int_c^d \frac{dy}{dx}\psi\,dx$$

where

$$a_0 = [\textstyle\int_0^1 a^{-1}(\xi)\,d\xi]^{-1} \ \text{ is the } H\text{-limit of } \Big\{a\Big(\frac{x}{\varepsilon}\Big)\Big\}$$
$$\psi \ \text{ is a weak limit of } \{\psi_\varepsilon\} \ \text{ in } H_0^1(c,d).$$

In order to find the limit of $\{d_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ we note that there exists a constant $\gamma_0 > 0$ such that

$$\|y_\varepsilon\|_{H_0^1(c,d)}, \Big\|a\Big(\tfrac{x}{\varepsilon}\Big)\tfrac{dy_\varepsilon}{dx}\Big\|_{H^1(c,d)}, \|\psi_\varepsilon\|_{H_0^1(c,d)}, |d_\varepsilon| \ \le\ \gamma_0 \quad \text{for every} \ \ \varepsilon \in (0,\varepsilon_0].$$

Let η be the strong limit of the sequence $\big\{a\big(\frac{x}{\varepsilon}\big)\frac{dy_\varepsilon}{dx}\big\}_{\varepsilon\in(0,\varepsilon_0]}$ in $L^2(c,d)$. Then:

$$\psi_\varepsilon \longrightarrow \psi \text{ weakly in } H_0^1(c,d) \tag{6.12}$$
$$y_\varepsilon \longrightarrow y \text{ weakly in } H_0^1(c,d) \tag{6.13}$$
$$d_\varepsilon \longrightarrow d_0 \text{ as numerical sequence} \tag{6.14}$$
$$a\big(\tfrac{x}{\varepsilon}\big)\tfrac{dy_\varepsilon}{dx} \longrightarrow \eta \text{ strongly in } L^2(c,d) \tag{6.15}$$

where $\eta = a_0 \frac{dy}{dx}$, by the definition of the H-limit. Now we can pass to the limit as $\varepsilon \to 0$ in the relationship

$$0 = \frac{d_\varepsilon}{a(\frac{x}{\varepsilon})} - \frac{d\psi_\varepsilon}{r dx} - \frac{q(\frac{x}{\varepsilon})}{a^2(\frac{x}{\varepsilon})}\Big[a\Big(\frac{x}{\varepsilon}\Big)\frac{dy_\varepsilon}{dx}\Big].$$

Using (6.12) - (6.15) we get

$$\begin{aligned} 0 &= d_0 \int_0^1 a^{-1}(\xi)\,d\xi - \frac{d\psi}{dx} - \int_0^1 q(\xi)a^{-2}(\xi)\,d\xi \cdot \eta \\ &= d_0 \int_0^1 a^{-1}(\xi)\,d\xi\,\frac{d\psi}{dx} - \int_0^1 q(\xi)a^{-2}(\xi)\,d\xi \cdot a_0\frac{dy}{dx} \end{aligned}$$

which yields

$$d_0 = a_0\frac{d\psi}{dx} + a_0^2\int_0^1 q(\xi)a^{-2}(\xi)\,d\xi\,\frac{dy}{dx}.$$

Substituting d_0 into (6.11) we obtain

$$\lim_{\varepsilon\to 0}\int_c^d q\Big(\frac{x}{\varepsilon}\Big)\Big[\frac{dy_\varepsilon}{dx}\Big]^2 dx = q^\sharp\int_c^d \Big(\frac{dy}{dx}\Big)^2 dx.$$

We can further proceed as in Theorem 6.1 ∎

7. Identification of the strongly S-homogenized optimal control problem and its variational properties

We now apply the procedure of S-homogenization to the optimal control problem (1.1) - (1.4). We shall assume that the conditions (a) - (l) from Section 4 hold. Recall that our approch is based on representation (4.5) of the original optimal control problem. Note that the family of cost functionals $\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon\in(0,\varepsilon_0]}$ is equicoercive in the μ-topology. Therefore, by virtue of the compactness theorem for absolute variational S-convergence in Banach spaces (see Theorem 3.3) we obtain the following result.

Theorem 7.1. *Suppose conditions* (a) - (l) *hold. Then for the family of constrained minimization problems* (4.5) *there exist a subsequence* $\{\varepsilon \in H \in \mathcal{H}^\sharp\}$ *for which*

$$\Big\langle \inf_{(u,y)\in\mu\text{-Lm}\,\Xi_\varepsilon} \mu\text{-lm}^a(I_\varepsilon|_{\Xi_\varepsilon})(u,y)\Big\rangle \tag{7.1}$$

is the absolute variational S-limit in the μ-topology.

Remark 7.1. By properties of the S-limit we know that the functional $\mu\text{-lm}^a(I_\varepsilon|_{\Xi_\varepsilon})$ is μ-lower semicontinuous and μ-coercive. Since the topological limit (in the Kuratowski sence) $\mu\text{-Lm}\,\Xi_\varepsilon$ is a μ-closed subset of $L^2(\Omega)\times H_0^1(\Omega)$, it follows that the set of solutions of (7.1) is non-empty and μ-compact.

Now for the identification of the minimization problem (7.1) we can use Theorems 5.1 and 6.1.

Theorem 7.2. *Suppose that there is a coercive operator* $\mathcal{A}_* \in \mathcal{L}\big(H_0^1(\Omega), H^{-1}(\Omega)\big)$ *such that* $\mathcal{A}_\varepsilon \overset{G^*}{\Longrightarrow} \mathcal{A}_*$. *Then, under conditions* (a) - (l) *the constrained minimization problem* (7.1) *corresponds the optimal control problem*

$$-\mathrm{div}(A_*\nabla y) = \widehat{\mathcal{B}}_0 J_0^{-1} u + \widehat{f}_0 \quad in\ \Omega \tag{7.2}$$

$$y = 0 \quad on\ \partial\Omega \tag{7.3}$$

$$u \in w_{L^2(\Omega)}\text{-}Lm\, U_\varepsilon, \quad y \in w_{H_0^1(\Omega)}\text{-}Lm\, K_\varepsilon \tag{7.4}$$

$$I_0(u,y) = \int_\Omega C_0 y^2 dx + \int_\Omega (\nabla y, Q^\sharp \nabla y)_{\mathbb{R}^n} dx + F(u) \longrightarrow \inf. \tag{7.5}$$

Further note that, since the sets $U_\varepsilon \times K_\varepsilon$ are convex μ-closed, the optimal control problem (4.1) - (4.4) is uniformly regular (i.e. $\emptyset \neq \Xi_\varepsilon \subset U_\varepsilon \times K_\varepsilon$), the functionals I_ε are strictly convex and μ-coercive, it follows easily that [20] for every $\varepsilon \in (0, \varepsilon_0]$ there exists a unique solution $(u_\varepsilon^0, y_\varepsilon^0) \in U_\varepsilon \times K_\varepsilon$ of problem (4.1) - (4.4). Using the initial conditions (a) - (l), we see that there is a constant C_1 not depending on ε such that

$$\|u_\varepsilon^0\|_{L^2(\Omega)} + \|y_\varepsilon^0\|_{H_0^1(\Omega)} \le C_1 \qquad \text{for every } \varepsilon \in (0, \varepsilon_0].$$

Consequently, there exists a subsequence $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon \in H}$ where $H \in \mathcal{H}^\sharp$ such that

$$(u_\varepsilon^0, y_\varepsilon^0) \overset{\mu}{\longrightarrow} (u^*, y^*).$$

By properties of the topological limit μ-Lm Ξ_ε we have $(u^*, y^*) \in \mu\text{-Lm}\,\Xi_\varepsilon$. On the other hand, it is easy to see that S-homogenized problem has a unique solution (u^0, y^0). Then from Theorem 2.3 we immediatly obtain the following result.

Theorem 7.3. *Let* $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon \in H}$ *be a sequence of optimal pairs for problem* (4.1) – (4.4) *where the index set* $H \in \mathcal{H}^\sharp$ *corresponds to the choice in Theorem* 7.1. *Then under conditions* (a) - (l) *we have*

$$(u_\varepsilon^0, y_\varepsilon^0) \overset{\mu}{\longrightarrow} (u^0, y^0) \tag{7.6}$$

$$\inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y) \overset{\mu}{\longrightarrow} \inf_{(u,y)\in\mu\text{-Lm}\,\Xi_\varepsilon} I_0(u,y) \tag{7.7}$$

where (u^0, y^0) *is the optimal pair for the strongly* S*-homogenized problem* (7.2).

Remark 7.2. Suppose that the assumptions of Remark 3.2 hold true. Moreover, we shall assume that $K_\varepsilon = H_0^1(\Omega)$, $D_\varepsilon = D_0 = \text{const}$ and $C_\varepsilon = 0$ for every $\varepsilon \in (0, \varepsilon_0]$, the sets U_ε do not depend on ε, J_ε are identity operators, the sequence $\{f_\varepsilon\}$ is compact with respect to the strong topology of $H^{-1}(\Omega)$. Then the original optimal control problem (4.1) – (4.4) reduces to the problem that was considered by Kesavan and Saint Jean Paulin in [7]. Since in this case $\{R_\varepsilon^+ P_\varepsilon^*\}$ are identity operators it follows that

$$\widehat{\mathcal{B}}_\varepsilon = \mathcal{B}_\varepsilon,\ \widehat{f}_\varepsilon = f_\varepsilon,\ \widehat{U}_\varepsilon = U_\varepsilon \ \text{ for every } \varepsilon \in (0, \varepsilon_0] \qquad \text{and} \qquad \mathcal{A}_\varepsilon \overset{G^*}{\longrightarrow} \mathcal{A}_0$$

where $\mathcal{A}_0$ is the strong G-limit of the sequence $\{\mathcal{A}_\varepsilon \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))\}_{\varepsilon\in(0,\varepsilon_0]}$ and

$$\langle \mathcal{A}_0 y, \varphi\rangle_{H_0^1(\Omega)} = \int_\Omega (\nabla\varphi, A_0 \nabla y)_{\mathbb{R}^n}\, dx \ \forall\ \varphi \in H_0^1(\Omega)$$
$$A_\varepsilon \longrightarrow A_0 \ \text{ in the sense of } H\text{-convergence [21]}.$$

An additional point to emphasize is that by Remark 6.1 we have $F(u) = D_0 \int_\Omega u^2\, dx$.

Thus under assumptions (a) - (l) the S-homogenized optimal control problem has the representation

$$-\mathrm{div}(A_0 \nabla y) = b_0 u + f_0 \ \text{ in } \Omega$$
$$y = 0 \ \text{ on } \partial\Omega$$
$$u \in w_{L^2(\Omega)}\text{-Lm}\, U_\varepsilon$$
$$I_0(u, y) = \int_\Omega (\nabla y, Q^\sharp \nabla y)_{\mathbb{R}^n}\, dx + D_0 \int_\Omega u^2 dx \longrightarrow \inf.$$

In addition, by virtue of Theorem 7.3 we have the following variational properties for the sequence of optimal pairs $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon\in H}$:

$$(u_\varepsilon^0, y_\varepsilon^0) \xrightarrow{\mu} (u^0, y^0)$$
$$\inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u, y) \xrightarrow{\mu} \inf_{(u,y)\in\mu\text{-Lm}\,\Xi_\varepsilon} I_0(u, y)$$
$$u_\varepsilon^0 \longrightarrow u^0 \quad \text{strongly in } L^2(\Omega)$$

where (u^0, y^0) is the optimal pair for the above-mentioned S-homogenized problem.

8. Homogenization of an optimal control problem on a perforated domain

In this section we consider the application of the procedure of S-homogenization and the results of previous sections to an optimal control problem on a perforated domain.

Our example deals with a non-classical situation in homogenization theory of optimal control problems. Let Ω be a bounded open set of $\mathbb{R}^2$. For each value of $\varepsilon \in (0, \varepsilon_0]$, we cover $\mathbb{R}^2$ by cubes Z_i^ε of size 2ε. From each cube we remove the ball T_i^ε of radius $r_\varepsilon = \exp\left(-\frac{1}{\varepsilon^2}\right)$ centered at the very center of the cube. In this way, $\mathbb{R}^2$ is perforated by spherical identical holes. Set

$$S^\varepsilon = \mathbb{R}^2 \setminus \bigcup T_i^\varepsilon \qquad \text{and} \qquad \Omega_\varepsilon = \Omega \cap Q_\varepsilon = \Omega \setminus \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon.$$

This means that we removed from Ω small balls of radius r_ε whose centers are the nodes of a lattice in $\mathbb{R}^2$ with cell size 2ε.

Let $f \in L^2(\Omega)$ and consider the following optimal comtrol problem in Ω:

$$-\Delta y = u + \chi_\varepsilon f \quad \text{in } \mathcal{D}'(\Omega) \tag{8.1}$$

$$y \in H_0^1(\Omega) \tag{8.2}$$

$$y \in K_\varepsilon, \quad u \in U_\varepsilon \tag{8.3}$$

$$I(u, y) = \int_\Omega y^2\, dx + \int_\Omega u^2 dx \longrightarrow \inf \tag{8.4}$$

where χ_ε is the characteristic function of the perforated domain Ω_ε, K_ε is the closure in $H_0^1(\Omega)$ of the set $y \in C^\infty(\Omega)$ with $\operatorname{supp} y$ contained in Ω_ε and U_ε is the closure in $L^2(\Omega)$ of the set $u \in C^\infty(\Omega)$ with $\operatorname{supp} u$ contained in Ω_ε as well.

As follows from [20], for any $\varepsilon \in (0, \varepsilon_0]$ there exists unique optimal solution $(u_\varepsilon^0, y_\varepsilon^0) \in U_\partial^\varepsilon \times H_0^1(\Omega)$ such that y_ε^0 vanishes in the holes T_i^ε $(1 \le i \le n(\varepsilon))$, i.e. optimal control problem (8.1) - (8.4) is uniformly regular for every $\varepsilon \in (0, \varepsilon_0]$.

For $\varepsilon \in (0, \varepsilon_0]$, we set $Y_\varepsilon = H_0^1(\Omega_\varepsilon)$. We define the operators P_ε and R_ε^+ as in Example 3.2. Note that for the control constraints the epresentation

$$U_\varepsilon = \{u \in L^2(\Omega) |\, u = \chi_\varepsilon \widehat{u} \ \text{ for all } \widehat{u} \in L^2(\Omega) \mathcal{B}ig\} \qquad \text{for every } \varepsilon \in (0, \varepsilon_0]$$

holds. Thus, $J_\varepsilon = \chi_\varepsilon$ and for the prototypes of control functions $\widehat{u}$ there are not any constraints, i.e. $\widehat{U}_\varepsilon = L^2(\Omega)$. Moreover, for the sequence of sets $\{U_\varepsilon \times K_\varepsilon\}$ there exists topological limit in the μ-topology

$$\mu\text{-Lm}\,[U_\varepsilon \times K_\varepsilon] = L^2(\Omega) \times H_0^1(\Omega).$$

As for the limit of the operators $\{J_\varepsilon\}$ in the weak operator topology we have $J_\varepsilon \to J_0 = \chi_0$, where χ_0 is the weal-$*$ limit point in $L^\infty(\Omega)$ of $\{\chi_\varepsilon\}$. Besides Cioranescu and Saint Jean Paulin [3] have shown that, when Ω is perforated periodically, χ_0 will be a positive constant, i.e. J_0 is invertible operator.

Finally, since $R^+ P_\varepsilon^* \widehat{u} = J_\varepsilon \widehat{u} = \chi_\varepsilon \widehat{u}$ for every $\varepsilon \in (0, \varepsilon_0]$ and $\widehat{u} \in L^2(\Omega)$, it follows that we may rewrite the original optimal control problem (8.1) - (8.4) in another form

$$-\Delta y = R_\varepsilon^+ P_\varepsilon^* (\widehat{u} + f) \text{ in } \mathcal{D}'(\Omega) \tag{8.5}$$

$$y = 0 \text{ on } \partial\Omega \tag{8.6}$$

$$I(u, y) = \int_\Omega y^2 dx + \int_\Omega \chi_\varepsilon \widehat{u}^2 dx \longrightarrow \inf. \tag{8.7}$$

Thus, it is easy to see that all conditions (a) - (l) for problem (8.1) - (8.4) hold true. However, in order to apply Theorem 7.2 we note that in our case we may omit the assumption about the existence of the strong G^*-limit for operator $-\Delta$ in (8.5), because we shall not use the result of Lemma 6.1.

We may define the structure of the G^*-limit operator $\mathcal{A}_*$ for our control object (8.5) - (8.6) by (see Proposition 3.1)

$$-\Delta^{-1} R_\varepsilon^+ P_\varepsilon^* g \longrightarrow \mathcal{A}_*^{-1} g \qquad \text{for every } g \in H^{-1}(\Omega).$$

However, for every $\widehat{u} \in L^2(\Omega)$ and $f \in L^2(\Omega)$, by virtue of [2: Theorem 1.2] (see also [5]), the solutions $y_\varepsilon(\widehat{u})$ of (8.5) - (8.6) satisfy

$$y_\varepsilon(\widehat{u}) = -\Delta^{-1} R_\varepsilon^+ P_\varepsilon^*(\widehat{u} + f) \longrightarrow \mathcal{A}_*^{-1}(\widehat{u} + f) = y(\widehat{u}) \qquad \text{weakly in } H_0^1(\Omega),$$

where $y(\widehat{u})$ is the unique solution of

$$\left.\begin{aligned} -\Delta y + \tfrac{\pi}{2}\, y = \widehat{u} + f \qquad & \text{in } \mathcal{D}'(\Omega) \\ y = 0 \qquad & \text{on } \partial\Omega \end{aligned}\right\},$$

i.e. $\mathcal{A}_* y = -\Delta y + \frac{\pi}{2}\, y$.

Since, by property (d), for every $\widehat{u} \in L^2(\Omega)$ there is an element $u \in (w_{L^2})\text{-Lm}\, U_\varepsilon$ such that $\widehat{u} = \chi_0^{-1} u$, we obtain

$$\left.\begin{aligned} -\Delta y + \left(\tfrac{\pi}{2}\right) y = \chi_0^{-1} u + f \qquad & \text{in } \mathcal{D}'(\Omega) \\ y = 0 \qquad & \text{on } \partial\Omega \end{aligned}\right\}.$$

Finally, it is easy to see that

$$(w_{L^2})\text{-lm}^a \left(\int_\Omega u^2 dx \bigg|_{U_\varepsilon} \right) = \lim_{\varepsilon \to 0} \int_\Omega \chi_\varepsilon \widehat{u}^2 dx = \int_\Omega \chi_0 \widehat{u}^2 dx$$

where $\widehat{u}$ is some prototype of the original control u. Then we have $\widehat{u} = \chi_0^{-1} u$, i.e.

$$(w_{L^2})\text{-lm}^a \left(\int_\Omega u^2 dx \bigg|_{U_\varepsilon} \right) = \int_\Omega \chi_0^{-1} u^2 dx = I_*(u, y).$$

Thus have proved the following result (see Theorem 7.2).

Theorem 8.1. *For the optimal control problem* (8.1) – (8.4) *there exists a unique strong S-homogenized problem in the μ-topology of $L^2(\Omega) \times H_0^1(\Omega)$ which has the form*

$$\begin{aligned} & -\Delta y + \left(\frac{\pi}{2}\right) y = \chi_0^{-1} u + f \quad \text{in } \mathcal{D}'(\Omega) \\ & y = 0 \quad \text{on } \partial\Omega \\ & I_*(u, y) = \int_\Omega y^2 dx + \int_\Omega \chi_0^{-1} u^2 dx \longrightarrow \inf . \end{aligned}$$

Furthermore, the sequence of optimal pairs $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon \in H}$ μ-converges to the unique solutions (u^0, y^0) of the above homogenized problem and $I(u_\varepsilon^0, y_\varepsilon^0) \to I_*(u^0, y^0)$.

Now we consider the second example that deals with homogenization of an optimal control problem on perforated domain. Let Ω be a bounded open set in $\mathbb{R}^n$. Given a sequence of parameters $\varepsilon \in (0, \varepsilon_0]$ which tends to zero, we perforate the domain Ω be "holes paramertrized by ε". Mathematically speaking, we consider a family of closed subsets $S_\varepsilon \subset \Omega$ and set $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$, which we call the perforated domain. We denote by χ_ε the characteristic function of the domain Ω_ε.

We define the constituents of the optimal control problem (4.1) - (4.4) as follows:

(H1) $C_\varepsilon = 0$, $D_\varepsilon = 1$, $b_\varepsilon = 1$, $f_\varepsilon = \chi_\varepsilon \widehat{f}$, $\widehat{f} \in L^2(\Omega)$ for all $\varepsilon \in (0, \varepsilon_0]$.

(H2) $U_\varepsilon = \{u \in L^2(\Omega) |\, u = \chi_\varepsilon \widehat{u}$ for all $\widehat{u} \in \widehat{U}\}$, for every $\varepsilon \in (0, \varepsilon_0]$, where $\widehat{U} = \{u \in L^2(\Omega) |\, \psi_1 \le u \le \psi_2$ a.e. in $\Omega\}$, ψ_1 and ψ_2 are given functions in $L^2(\Omega)$.

(H3) $Y_\varepsilon = \{y \in H^1(\Omega_\varepsilon) |\, y = 0 \text{ on } \partial\Omega\}$.

(H4) $P_\varepsilon : Y_\varepsilon \to H_0^1(\Omega)$ is an extension operator such that, for every $y \in Y_\varepsilon$, $(P_\varepsilon y)|_{\Omega_\varepsilon} = y$ and $\|\nabla P_\varepsilon y\|_{L^2(\Omega)} \le C \|y\|_{L^2(\Omega_\varepsilon)}$. Since P_ε is a linear continuous operator, the adjoint operator $P_\varepsilon^* : H^{-1}(\Omega) \to Y_\varepsilon^*$ is defined.

(H5) $R_\varepsilon^* : Y_\varepsilon^* \to H^{-1}(\Omega)$ be a linear operator such that $P_\varepsilon^* R_\varepsilon g = g$ and $R_\varepsilon P_\varepsilon^* f = \chi_\varepsilon f$ for every $f \in H^{-1}(\Omega)$ and $g \in Y_\varepsilon^*$.

(H6) $K_\varepsilon = \{y = P_\varepsilon z \in H_0^1(\Omega) |\, z \in Y_\varepsilon,\ A_\varepsilon \nabla z \cdot n_\varepsilon = 0 \text{ on } \partial S_\varepsilon\}$, where n_ε is the unit outward normal on S_ε.

In addition to these assumptions, following Kesavan and Saint Jean Paulin [9], we assume that the following conditions hold:

(H7) Every weak-$*$ limit point in $L^\infty(\Omega)$ of $\{\chi_\varepsilon\}$ is positive a.e. in Ω.

(H8) If $\chi_\varepsilon \to \chi_0$ in $L^\infty(\Omega)$ weak-$*$, then $\chi_0^{-1} \in L^\infty(\Omega)$.

Thus we have the following optimal control problem:

$$-\mathrm{div}(A_\varepsilon \nabla y) = u \chi_\varepsilon \widehat{f} \ \text{ in } \Omega \tag{8.8}$$

$$y = 0 \ \text{ on } \partial\Omega \tag{8.9}$$

$$u \in U_\varepsilon \ \ y \in K_\varepsilon \tag{8.10}$$

$$I_\varepsilon(u, y) = \int_\Omega (\nabla y, N_\varepsilon \nabla y)_{\mathbb{R}^n} dx \int_\Omega u^2 dx \longrightarrow \inf \tag{8.11}$$

where the matrices A_ε and N_ε satisfy condition (h). Our aim is to study the limiting behaviour of this problem as $\varepsilon \to 0$. Note that U_ε and K_ε are convex closed subsets of $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. Besides, it is obvious that assumptions (b) - (d) from Section 4 hold true. Indeed, since Cioranescu and Saint Jean Paulin [3] have shown that, when Ω is perforated periodically, the hypotheses (H4), (H5), (H7) and (H8) above are satisfied and, in particular, χ_0 will be a positive constant, we have

$$J_0 = \chi_0 \ \text{ and } \ (w_{L^2})\text{-Lm}\, U_\varepsilon = \left\{u \in L^2(\Omega) \,\middle|\, \chi_0 \psi_1 \le u \le \chi_0 \psi_2 \text{ a.e. in } \Omega\right\}.$$

Further, by virtue of conditions (H2) and (H5) we obtain

$$u \chi_\varepsilon \widehat{f} = J_\varepsilon v \chi_\varepsilon \widehat{f} = \chi_\varepsilon (v \widehat{f}) = R_\varepsilon P_\varepsilon^* (\widehat{\mathcal{B}}_\varepsilon v \widehat{f})$$

for every $v \in \widehat{U}$. Hence $\widehat{\mathcal{B}}_\varepsilon = 1$, i.e. $\widehat{\mathcal{B}}_\varepsilon$ is the canonical isomorphism of $L^2(\Omega)$ into $H^{-1}(\Omega)$.

As for the existence of the topological limit for the sets $\{K_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0]}$ we have the following result.

Proposition 8.1. $(w_{H_0^1})$-$Lm\, K_\varepsilon = H_0^1(\Omega)$.

Proof. Since condition (ii) of Lemma 4.1 is obvious, we need to verify (i). Let y be any element of $H_0^1(\Omega)$ and g be the element of $H^{-1}(\Omega)$ such that

$$\mathcal{A}_* y = g \tag{8.13}$$

where $\mathcal{A}_*$ is the G^*-limit of the sequence $\{\mathcal{A}_\varepsilon\}$ with respect to the above defined operators $R_\varepsilon : Y_\varepsilon^* \to H^{-1}(\Omega)$ and $P_\varepsilon^* : H^{-1}(\Omega) \to Y_\varepsilon^*$. We consider the sequence of elements $\{z_\varepsilon \in Y_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ each of which is defined by $z_\varepsilon = \Lambda_\varepsilon^{-1} P_\varepsilon^* g$. Here the operators $\Lambda_\varepsilon : Y_\varepsilon \to Y_\varepsilon^*$ are constructed as

$$\Lambda_\varepsilon z = P_\varepsilon^* g \quad\Longleftrightarrow\quad \begin{cases} -\mathrm{div}(A_\varepsilon \nabla z) = P_\varepsilon^* g & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \\ A_\varepsilon \nabla z \cdot n_\varepsilon = 0 & \text{on } \partial S_\varepsilon. \end{cases} \tag{8.14}$$

As follows from Briane, Damlamian and Donato [1], there exists a unique solution of this problem for every $g \in H^{-1}(\Omega)$, i.e. $z_\varepsilon = \Lambda_\varepsilon^{-1} P_\varepsilon^* g$.

Now we put $y_\varepsilon = P_\varepsilon z_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$. Then the sequence $\{y_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is coordinated with the collection $\{K_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$, i.e. $y_\varepsilon \in K_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$. Since

$$y_\varepsilon = P_\varepsilon z_\varepsilon = P_\varepsilon \Lambda_\varepsilon^{-1} P_\varepsilon^* g = P_\varepsilon \Lambda_\varepsilon^{-1} (P_\varepsilon^* R_\varepsilon) P_\varepsilon^* g = (P_\varepsilon \Lambda_\varepsilon^{-1} P_\varepsilon^*) R_\varepsilon P_\varepsilon^* g,$$

we denote by T_ε the operator $P_\varepsilon \Lambda_\varepsilon^{-1} P_\varepsilon^*$. As the operators $\Lambda_\varepsilon \in \mathcal{L}(Y_\varepsilon; Y_\varepsilon^*)$ are uniformly coercive and bounded, we deduce that there is a constant $\alpha > 0$ such that

$$\|f\|^2_{H^{-1}} \le \alpha \|T_\varepsilon f\|^2_{H_0^1}, \qquad \langle f, T_\varepsilon f\rangle \ge \alpha^{-1} \|T_\varepsilon f\|^2_{H_0^1} \tag{8.15}$$

for every $f \in H^{-1}(\Omega)$. Consequently, the operators T_ε are invertible, i.e. we may set $\widehat{\mathcal{A}}_\varepsilon = T_\varepsilon^{-1}$. Moreover, we conclude that the operators $\widehat{\mathcal{A}}_\varepsilon$ are uniformly bounded and coercive. Therefore the elements $\{y_\varepsilon \in K_\varepsilon\}$ may be defined as the solutions of the equations $\widehat{\mathcal{A}}_\varepsilon y_\varepsilon = R_\varepsilon P_\varepsilon^* g$. However, by condition (H5), we obtain

$$P_\varepsilon^* \widehat{\mathcal{A}}_\varepsilon P_\varepsilon z_\varepsilon = P_\varepsilon^* \widehat{\mathcal{A}}_\varepsilon y_\varepsilon = P_\varepsilon^* R_\varepsilon P_\varepsilon^* g = P_\varepsilon^* g,$$

i.e. for the operators Λ_ε we have the representation

$$\Lambda_\varepsilon = P_\varepsilon^* \widehat{\mathcal{A}}_\varepsilon P_\varepsilon \qquad \text{for every } \varepsilon \in (0, \varepsilon_0].$$

Thus in view of (8.14) the operators $\widehat{\mathcal{A}}_\varepsilon \in \mathcal{L}(H_0^1(\Omega); H^{-1}(\Omega))$ can be defined as

$$\langle \widehat{\mathcal{A}}_\varepsilon y, \varphi\rangle_{H_0^1(\Omega)} = \int_\Omega (\nabla\varphi, A_\varepsilon \nabla y)_{\mathbb{R}^n}\, dx \qquad \forall\, y, \varphi \in H_0^1(\Omega),$$

i.e. $\widehat{\mathcal{A}}_\varepsilon = \mathcal{A}_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$, where the operators $\mathcal{A}_\varepsilon$ correspond to the control object (8.8) - (8.10). It follows that the sequence $\{y_\varepsilon \in K_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ has the representation $y_\varepsilon = \mathcal{A}_\varepsilon^{-1} R_\varepsilon P_\varepsilon^* g$ for all $\varepsilon \in (0, \varepsilon_0]$.

It has been proved by Briane, Damlamian and Donato [1] that the solution of problem (8.14) satisfies $P_\varepsilon z_\varepsilon \to z_0$ weakly in $H_0^1(\Omega)$, where

$$-\mathrm{div}(A_0 \nabla z_0) = g \quad \text{in } \Omega \tag{8.16}$$

$$z_0 = 0 \quad \text{on } \partial\Omega \tag{8.17}$$

and the matrix A_0 is the H_0-limit of the sequence $\{A_\varepsilon\}$. Since $y_\varepsilon = P_\varepsilon z_\varepsilon = \mathcal{A}_\varepsilon^{-1} R_\varepsilon P_\varepsilon^* g$, it follows that (see Proposition 3.1) $y_\varepsilon = \mathcal{A}_\varepsilon^{-1} R_\varepsilon P_\varepsilon^* g \to \mathcal{A}_*^{-1} g = z_0$. Thus, by (8.16) - (8.17), the G^*-limit operator $\mathcal{A}_*$ is defined as

$$\langle \mathcal{A}_* y, \varphi \rangle_{H_0^1(\Omega)} = \int_\Omega (\nabla\varphi, A_0 \nabla y)_{\mathbb{R}^n} \, dx \qquad \forall \varphi \in H_0^1(\Omega), \tag{8.18}$$

and by virtue of (8.13) we have $z_0 = y$. Thus for any element $y \in H_0^1(\Omega)$ it is possible to construct the sequence $\{y_\varepsilon \in K_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ such that $y_\varepsilon \to y$ weakly in $H_0^1(\Omega)$, i.e. in view of Lemma 4.1 equality (8.12) holds ∎

Corollary 8.1. *For the operator sequence* $\{\mathcal{A}_\varepsilon \in \mathcal{L}(H_0^1(\Omega); H^{-1}(\Omega))\}$ *associated with* (8.8) – (8.11) *there exists the* G^**-limit* $\mathcal{A}_*$ *such that* $\mathcal{A}_*$ *is a bounded coercive operator for which representation* (8.18) *holds, whith* A_0 *being the* H_0*-limit of the matrix sequence* $\{A_\varepsilon\}$.

As follows from the proof of Proposition 8.1, for any $\varepsilon \in (0, \varepsilon_0]$ and any admissible control $u^* = J_\varepsilon v^* \in U_\varepsilon$ there is an element $z_\varepsilon^* \in Y_\varepsilon$ such that z_ε^* is the unique solution of problem (8.14) for $g = v^* f$, i.e. $z_\varepsilon^* = \Lambda_\varepsilon^{-1} P_\varepsilon^*(v^* f)$. Therefore if we set $y_\varepsilon = P_\varepsilon z_\varepsilon^*$, then

$$y_\varepsilon \in K_\varepsilon \qquad \text{and} \qquad y_\varepsilon = \mathcal{A}_\varepsilon^{-1} R_\varepsilon P_\varepsilon^*(v^* f) = \mathcal{A}_\varepsilon^{-1}(u^* \chi_\varepsilon f),$$

that is, y_ε is the unique solution of original problem (8.8) - (8.11) corresponding to the control u^*. Thus we obtain the following conclusion.

Corollary 8.2. *The original optimal control problem* (8.8) – (8.11) *is uniformly regular, i.e. assumption* (j) *of Section* 4 *is satisfied.*

Corollary 8.3. *If* $(u_\varepsilon^*, y_\varepsilon^*)$ *is any admissible pair for the original optimal control problem* (8.8) – (8.11)*, then there are elements* $v_\varepsilon^* \in L^2(\Omega)$ *and* $z_\varepsilon^* \in Y_\varepsilon$ *such that*

$$y_\varepsilon^* = P_\varepsilon z_\varepsilon^* \qquad \textit{and} \qquad u_\varepsilon^* = \chi_\varepsilon v_\varepsilon^*$$

and, consequently, the pair $(v_\varepsilon^*, z_\varepsilon^*)$ *is admissible for the optimal control problem*

$$-\mathrm{div}(A_\varepsilon \nabla z) = P_\varepsilon^*(vf) \quad \textit{in } \Omega_\varepsilon \tag{8.19}$$

$$z = 0 \quad \textit{on } \partial\Omega_\varepsilon \tag{8.20}$$

$$A_\varepsilon \nabla z \cdot n_\varepsilon = 0 \quad \textit{on } \partial S_\varepsilon \tag{8.21}$$

$$v \in \widehat{U} \tag{8.22}$$

$$I_\varepsilon(u, y) = \int_\Omega (\nabla P_\varepsilon z, N_\varepsilon \nabla P_\varepsilon z)_{\mathbb{R}^n} \, dx \int_{\Omega_\varepsilon} v^2 \, dx \longrightarrow \inf. \tag{8.23}$$

Remark 8.1. It is obvious that we can consider the optimal control problem (8.19) - (8.23) as a prototype of Kesavan and Saint Jean Paulin's problem on a perforated domain [9]. At the same time this problem can be reduced to their original problem if $\nabla P_\varepsilon z_\varepsilon = 0$ a.e. in int (S_ε) and $(P_\varepsilon^* g)|_{\Omega_\varepsilon} = g$ for all $g \in H^{-1}(\Omega)$.

Remark 8.2. As for the strong G^*-convergence of $\{\mathcal{A}_\varepsilon\}$ to the operator $\mathcal{A}_*$ we note that this can also be proved by the results in [1] after some minor modifications. Namely, for every $g \in H^{-1}(\Omega)$, the solution $z_\varepsilon \in Y_\varepsilon$ of problem (8.14) satisfies

$$\begin{aligned} P_\varepsilon z_\varepsilon &\longrightarrow z_0 && \text{weakly in } H_0^1(\Omega) \\ A_\varepsilon \nabla[P_\varepsilon z_\varepsilon] &\longrightarrow A_0 \nabla z_0 && \text{weakly in } [L^2(\Omega)]^n \\ (\text{instead of } (A_\varepsilon \nabla z_\varepsilon)^\sim &\longrightarrow A_0 \nabla z_0 && \text{weakly in } [L^2(\Omega)]^n \text{ in [1]}) \end{aligned}$$

where z_0 is the solution of problem (8.16) - (8.17) and a matrix A_0 is H_0-limit of the sequence $\{A_\varepsilon\}$.

Since $y_\varepsilon = P_\varepsilon z_\varepsilon = \mathcal{A}_\varepsilon^{-1} R_\varepsilon P_\varepsilon^* g$, it follows that (see Proposition 3.1 and Definition 6.1) the sequence $\{\mathcal{A}_\varepsilon\}$ strongly G^*-converges to the operator $\mathcal{A}_*$ with representation (8.18).

In particular, as for every $v \in \widehat{U}$ there is an element $u \in (w_{L^2})\text{-Lm}\, U_\varepsilon$ such that $v = \chi_0^{-1} u$, we infer

$$y_\varepsilon = \mathcal{A}_\varepsilon^{-1} R_\varepsilon P_\varepsilon^*(v\widehat{f}) \longrightarrow \mathcal{A}_*(v\widehat{f}) = \mathcal{A}_*(\chi_0^{-1} u \widehat{f}) \qquad \text{weakly in } H_0^1(\Omega).$$

Finally, taking into account the above mentioned results we see that all assumptions of Theorems 7.1. - 7.3 hold true with respect to the optimal control problem (8.8) - (8.11). Moreover, by Lemma 6.1 and the properties of S-limits, we can show

$$\mu\text{-lm}^a(I_\varepsilon|_{\Xi_\varepsilon})(u, y) = \int_\Omega (\nabla y, N^\sharp \nabla y)_{\mathbb{R}^n}\, dx + \int_\Omega \chi_0^{-1} u^2 dx.$$

Thus, we are now in the positon to state the main result about the S-homogenization of the optimal control problem (8.8) - (8.11) as follows.

Theorem 8.2. *For the optimal control problem* (8.8) – (8.11) *there exists a unique strongly S-homogenized problem in the μ-topology of $L^2(\Omega) \times H_0^1(\Omega)$ with*

$$\begin{aligned} -\text{div}(A_0 \nabla z) &= \chi_0^{-1} u \widehat{f} \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \partial\Omega \\ u &\in (w_{L^2})\text{-Lm}\, U_\varepsilon = \left\{ u \in L^2(\Omega) \,|\, \chi_0 \psi_1 \le u \le \chi_0 \psi_2 \ \text{a.e. in } \Omega \right\} \\ I(u, y) &= \int_\Omega (\nabla y, N^\sharp \nabla y)_{\mathbb{R}^n}\, dx + \int_\Omega \chi_0^{-1} u^2 dx \longrightarrow \inf \end{aligned}$$

where A_0 is H_0-limit of $\{A_\varepsilon\}$ in the sense of Briane, Damlamian and Donato [1]. *Furthermore, the sequence of the optimal pairs $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon \in H}$ μ-converges to the unique solutions (u^0, y^0) of the above homogenized problem and $I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) \to I(u^0, y^0)$.*

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