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Regularity Results for Laplace Interface Problems in Two Dimensions

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Abstract. We investigate the regularity of solutions of interface problems for the Laplacian in two dimensions. Our objective are regularity results which are independent of global bounds of the data (the diffusion). Therefore we use a restriction on the data, the quasi-monotonicity condition, which we show to be sufficient and necessary to provide $H^{1+\frac{1}{4}}$-regularity. In the proof we use estimates of eigenvalues of a related Sturm-Liouville eigenvalue problem. Additionally we state regularity results depending on the data.

Keywords: *Elliptic equations, regularity of solutions, interface and transmission problems, singularities, discontinuous diffusion coefficients, Sturm-Liouville eigenvalue problems*

AMS subject classification: Primary 35B65, 35J25, secondary 35J20, 34L15

1. Introduction

We are interested in Laplace interface problems on a domain $\Omega \subset \mathbb{R}^2$. These are elliptic problems with piecewise constant data k. The data are constant on subdomains and can be interpreted as a diffusion term. The strong form of the problem is

$$\nabla \cdot k(x)\nabla u(x) = f(x) \qquad \forall\, x \in \Omega$$

where the coefficient k is bounded by

$$\delta \le k(x) \le \delta^{-1} \qquad \forall\, x \in \Omega$$

for some $\delta > 0$ and where mixed boundary conditions are imposed.

In this article we discuss piecewise H^s-regularity of interface problems for the Laplacian which holds independently of the number and shape of the subdomains on which the coefficient k is constant. Our main interest are H^s-regularity results for $s > 1$ which

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hold independently of the bound $\delta \le k \le \delta^{-1}$ of the coefficient k. Known regularity results, which hold independently of the shape of the subdomains, restrict the partition of Ω into subdomains to the case that the maximum number of subdomains which meet in a point is 2 for points on the boundary and 3 for interior points [2, 3, 10, 12, 13, 16, 20]. In our regularity results there are *no restrictions* on the maximum number of subdomains which share a point, but we will use a restriction on k – the quasi-monotonicity condition – introduced in [4]. Additionally, we give regularity results in Sobolev spaces H^s where s *explicitly* depends on the bounds of the coefficient, and we show that these results are sharp. The results are part of [17].

The interface problem will be posed in Section 2. The relation of piecewise and global regularity is discussed in Subsection 3.1. A short review of the connection of regularity, singular functions and a Sturm-Liouville eigenvalue problem is given in Subsections 4.1 and 4.2. Known regularity results are reviewed in Subsection 4.3. The quasi-monotonicity condition will be defined in Subsection 5.1. For showing regularity results we need a lower bound of the eigenvalues of a Sturm-Liouville eigenvalue problem. This bound is derived by investigating the structure of according eigenfunctions (Subsection 5.2).

The main results states that quasi-monotonicity is necessary and sufficient to yield $H^{1+\frac{1}{4}}$-regularity independently of the global bounds of k and without restrictions on the subdomains (Section 6). We prove further that this result is optimal and show that known regularity results are special cases of our approach.

In Section 7 we prove piecewise $H^{1+\frac{\delta}{2\pi}}$-regularity, where $\delta \le k \le \delta^{-1}$. Further, we give "worst case" regularity results being *sharp with respect to* δ. Sharpness is shown by giving the explicit definition of a special singular function ψ_1 defined for a checkerboard-like pattern of coefficients δ and δ^{-1}. This means that this singular function has the lowest H^s-regularity among all singular functions independent of the geometry.

We are able to establish a link between the regularity theory for the quasi-monotone case and between the theory for the "worst case" introducing additional parameters depending on the coefficient (Subsection 7.3). $W^{2,p}$-regularity results for the quasi-monotone and the general case are given in Subsection 7.4.

The bounds on the eigenvalues for the Laplace interface problem are directly applicable to Maxwell interface problems [2]. Following [2], applications of the two-dimensional results to Laplace interface problems in three dimensions is straightforward [17]. By imposing restrictions on the geometry or on the coefficient k our results can be used to ensure the validity of shift theorems in appropriate function spaces.

2. The interface problem for the Laplacian

Interface problems for the Laplacian are also known as *transmission problems* or, in the literature coming from numerical mathematics, as *problems with discontinuous diffusion coefficients*.

Let an open, bounded, polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$ be given. Its boundary $\partial\Omega$ is decomposed into parts $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \emptyset$ and $\mathrm{meas}_1(\Gamma_D) > 0$, corresponding to Dirichlet and Neumann boundary conditions. Let $f \in L^2(\Omega)$ be given.

Let us define the space $V = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$. We pose the interface problem in variational form: seek $u \in V$ satisfying

$$\int_\Omega k(x)\nabla u(x)\nabla v(x)\,dx = \int_\Omega f(x)v(x)\,dx \qquad \forall\, v \in V. \tag{2.1}$$

We make the following assumptions on the coefficient k: Ω can be partitioned in disjoint, open, polygonal Lipschitz subdomains $\Omega_i \ \ (i = 0, ..., n-1)$ on which the coefficient has constant value k_i. Additionally we impose the global bound

$$\delta \le k(x) \le \delta^{-1} \qquad \forall\, x \in \Omega \tag{2.2}$$

for a constant $\delta > 0$. Multiplying k by a constant one can assure that both bounds in (2.2) are sharp.

As $\mathrm{meas}_1(\Gamma_D) > 0$, relation (2.2) implies that $\|k^{\frac{1}{2}}\nabla v\|_{L^2(\Omega)}$ is a norm in V which is equivalent up to a factor depending on δ with $\|v\|_{H^1(\Omega)}$, and hence existence and uniqueness of the solution of problem (2.1) follow from Riesz's Theorem [5].

The maximum piecewise regularity under the condition $f \in L^2(\Omega)$ is $u \in H^2(\Omega_i)$. In general such regularity does not hold for solutions of problem (2.1).

3. Notation

We will use Sobolev spaces of fractional order $H^s \ \ (s \in \mathbb{R})$ as defined in [1, 7, 14]. Possibly merging subdomains Ω_i and $\Omega_m \ \ (i \ne m)$ which closures intersect one can assume $k_i \ne k_m$. We define the interface $\Gamma = \mathrm{closure}\,(\cup_{i=0}^{n-1}\partial\Omega_i \setminus \partial\Omega)$. We say that an inequality is *sharp* if it is optimal in the set of regarded problems and parameters. For a point $x \in \overline{\Omega}$ we define a neighborhood of x as the set $U \cap \overline{\Omega}$ where $x \in U$ and U is an open set in $\mathbb{R}^2$.

To discuss regularity we introduce so-called singular points, which will be subdivided into homogeneous and heterogeneous singular points, depending on whether the coefficient k is constant in a small neighborhood or not.

Definition 3.1. A point $x \in \partial\Omega \setminus \Gamma$ is a *homogeneous singular point* if one of the following conditions holds:

- the interior angle of Ω at x is greater then π
- the boundary conditions change in x and the interior angle of Ω at x is greater then $\frac{\pi}{2}$.

Definition 3.2. A point on the interface $x \in \Gamma$ is a *heterogeneous singular point* if

- either x is an interior point $x \in \Omega$ and in any neighborhood of x the interface is not a straight line
- or x lies on the boundary $x \in \partial\Omega$.

Definition 3.3. If x is a homogeneous or a heterogeneous singular point, we call x a *singular* point.

Interior heterogeneous singular points belonging to at least three subdomains are in the literature called *crosspoints*. In Figure 1 several singular points are depicted.

Figure 1
Partition into subdomains, Dirichlet and Neumann boundaries are shaded differently, x_i are homogeneous and heterogeneous singular points

Let x_l be a heterogeneous singular point. We introduce polar coordinates (r, φ) with respect to x_l. Let us identify the unit sphere with the interval $[0, 2\pi)$. Similarly, the interval $[\varphi_1, \varphi_2]$ denotes the cone containing all rays φ in between φ_1 and φ_2, where positive orientation is assumed. For instance, any two intervals $[\varphi_i, \varphi_j)$ and $[\varphi_j, \varphi_i)$ cover the sphere $[0, 2\pi)$ in a natural way.

Number the subdomains sharing the singular point x_l with $\Omega_{l,i}$ $\;(i = 0, ..., n_l - 1)$ and choose a radius $r_l > 0$ such that $\Omega_{l,i} \cap B_{r_l}(x_l)$ coincides with a cone $C_{l,i}$. The cones $C_{l,i}$ are given by the rays φ_i and φ_{i+1} $\;(i = 0, ..., n_l - 1)$ where $\varphi_0 < \varphi_1 < ... < \varphi_{n_l - 1}$. This notation is illustrated with the help of Figure 2.

Figure 2
Subdomains $\Omega_{l,i}$ coincide with cones $C_{l,i}$ in a neighborhood of an interior (left figure) and a boundary (right figure) heterogeneous singular point x_l

If x_l is an interior point, we see that $\varphi_{n_l} = \varphi_0$. If not, the rays φ_0 and φ_{n_l} coincide with a part of $\partial\Omega$. By the sequence $\varphi_0 < \varphi_1 < ... < \varphi_{n_l}$ we describe the *geometry* around the singular point x_l.

We denote by $k_{l,i}$ the value of k on $\Omega_{l,i} \cap B_{x_l}(r_l)$. Now let us define the *local diffusion coefficient* $k_{x_l}(\varphi)$ on the interval $[\varphi_0, \varphi_{n_l})$ that takes the value $k_{l,i}$ on the interval $[\varphi_i, \varphi_{i+1})$ $\;(i = 0, ..., n_l - 1)$. For simplification, we may drop the sub-indices l when choosing a singular point x_l. The notation is valid also for homogeneous singular points. Then $n_l = 1$.

3.1 Restriction to piecewise regularity. Observe that the normal derivatives of

u have a jump discontinuity across the interface. To see this, choose two adjacent subdomains Ω_i and Ω_j, and let n_i and n_j be the outward normals to the interface. Due to (2.1) the solution u fulfils

$$k_i \frac{\partial u}{\partial n_i}\Big|_{\partial\Omega_i \cap \partial\Omega_j} = k_j \frac{\partial u}{\partial n_j}\Big|_{\partial\Omega_i \cap \partial\Omega_j} \tag{3.1}$$

where the equality holds in the distributional sense. Since $k_i \neq k_j$, the normal derivatives are discontinuous. Therefore, $u \notin H^{\frac{3}{2}}(\Omega)$ and we restrict ourselves to piecewise regularity $u \in H^s(\Omega_i)$.

The next simple lemma establishes a connection between piecewise and global regularity.

Lemma 3.1. *Let the polygonal Lipschitz domain Ω be decomposed into disjoint polygonal Lipschitz subdomains Ω_1 and Ω_2. Let $0 \le \lambda < \frac{1}{2}$, $v \in H^{1+\lambda}(\Omega_i)$ $(i = 1, 2)$ and $v \in H^1(\Omega)$. Then $v \in H^{1+\lambda}(\Omega)$.*

The proof is standard and uses Gauss' theorem. It is given in [17].

4. Analytical background and known regularity results

4.1 The Sturm-Liouville eigenvalue problem. Choose a singular point x. We regard the self-adjoint and positive definite Sturm-Liouville eigenvalue problem given by

$$-s(\varphi)'' = \lambda^2 s(\varphi) \qquad \big(\varphi \in (\varphi_i, \varphi_{i+1}),\, i = 0, ..., n-1\big) \tag{4.1}$$

with interface conditions on lines $\varphi = \varphi_i$ that coincide with a part of the interface

$$\begin{aligned} s(\varphi_i - 0) &= s(\varphi_i + 0) \\ k_{i-1} s(\varphi_i - 0)' &= k_i s(\varphi_i + 0)' \end{aligned} \tag{4.2}$$

and, in the case $x \in \partial\Omega$, with

$$\begin{aligned} &\text{either} \;\; s(\varphi_0 + 0) = 0 \;\; \text{or} \;\; s(\varphi_0 + 0)' = 0 \\ &\text{either} \;\; s(\varphi_n - 0) = 0 \;\; \text{or} \;\; s(\varphi_n - 0)' = 0 \end{aligned} \tag{4.3}$$

if φ_0 or φ_n lies on Γ_D or Γ_N [9]. Here we denote by $s(\varphi_i - 0)$ and $s(\varphi_i + 0)$ the left- and right-hand side limit, respectively, of the function s in the point φ_i. We conclude that the eigenvalues are real and that the spectrum has no point of density. We denote by λ the positive square root of λ^2.

The above eigenvalue problem can be rewritten in a simplier form. The general solution of equation (4.1) on an interval $[\varphi_i, \varphi_{i+1}]$ has the form $b_i \cos(\lambda(\varphi - c_i))$ for some $b_i, c_i \in \mathbb{R}$. The interface condition reads for i such that the angle $\varphi = \varphi_i$ coincides with a part of the interface

$$b_i \cos(\lambda(\varphi_{i+1} - c_i)) = b_{i+1} \cos(\lambda(\varphi_{i+1} - c_{i+1})) \tag{4.4}$$

$$k_i b_i \sin(\lambda(\varphi_{i+1} - c_i)) = k_{i+1} b_{i+1} \sin(\lambda(\varphi_{i+1} - c_{i+1})). \tag{4.5}$$

4.2 A decomposition theorem and regularity. The next theorem establishes a connection between the above Sturm-Liouville eigenvalue problem and regularity.

Theorem 4.1. *For any singular point x_l denote by $\lambda_{l,j}^2$ $(j = 1, ..., m_l)$ all eigenvalues from the interval $(0,1]$ of the respective Sturm-Liouville eigenvalue problem (4.1) – (4.3) and suppose $\lambda_{l,j}^2 \neq 1$. Denote by $s_{l,j}(\varphi)$ the according eigenfunctions. Then the solution u of problem (2.1) admits a decomposition*

$$u = w + \sum_{x_l} \sum_{j=1}^{m_l} c_{l,j} \eta(r_l) r^{\lambda_{l,j}} s_{l,j}(\varphi) \tag{4.6}$$

where $w \in H^2(\Omega_i)$ $(i = 0, ..., n-1)$ and the sum is over all singular points x_l. Here $c_{l,j} \in \mathbb{R}$ and $\eta(r_l)$ is a smooth cut-off function vanishing outside a neighborhood of each singular point. We call $r^{\lambda_{l,j}} s_{l,j}(\varphi)$ singular function for the point x_l.

Proof. The proof of representation (4.6) follows from [9: Theorem 1] and [9: Section 3] with $s = 0$. The representation is also given in [15: Theorem 2.27] and [16] ∎

We see that the regularity of u is restricted by the regularity of the singular functions $r^{\lambda_{l,j}} s_{l,j}(\varphi) \notin H^{1+\lambda_{l,j}}(\Omega_i)$ $(i = 0, ..., n_l - 1)$. Furthermore, $r^{\lambda_{l,j}} s_{l,j}(\varphi) \in H^{1+\lambda_{l,j}-\varepsilon}(\Omega_i)$ $(i = 0, ..., n_l - 1)$ for any $\varepsilon > 0$. To show this one can use [7: Theorem 1.2.18]. The probably first decomposition theorem for the case of a smooth coefficient k can be found in [11].

Corollary 4.2. *Let $\gamma \in (0,1)$ be given and let $\lambda^2 > \gamma^2$ for all non-zero eigenvalues λ^2 of the Sturm-Liouville eigenvalue problem (4.1) – (4.3) for any singular point x. Then $u \in H^{1+\gamma}(\Omega_i)$ $(i = 0, ..., n-1)$.*

Proof. The corollary follows directly from Theorem 4.1 if all eigenvalues $\lambda_{l,j}^2$ are different from 1. If there is an eigenvalue $\lambda_{l,j}^2 = 1$, then one can rely on [15]. Using the notation of [15: Corollary 2.28] set $p_0 = \frac{2}{2-\gamma}$. As $p_0 < 2$, we see that $f \in L^2(\Omega) \subset L^{p_0}(\Omega)$. As $\gamma = 2 - \frac{2}{p_0} < \lambda$, the assumptions of [15: Corollary 2.28] are fulfilled. We conclude that $u \in W^{2,p_0}(\Omega_i \cap U)$, where W^{2,p_0} is the Sobolev space of functions having all their derivatives (in distributional sense) up to order 2 integrable with the power of p_0. Use of the continuous embedding $W^{2,p_0}(\Omega_i \cap U) \subset H^{1+\gamma}(\Omega_i \cap U)$ [6: Theorem 1.4.4.1] finishes the proof ∎

4.3 Known regularity results. In this subsection we want to review briefly known regularity results and to point out some open questions. We conclude from Corollary 4.2 that regularity is a local property. The following lemma is a simple conclusion of Corollary 4.2 and Lemma 3.1.

Lemma 4.3. *Let u be a solution of problem (2.1). Then there exists a number $\varepsilon(k) > 0$ depending on k such that $u \in H^{1+\varepsilon(k)}(\Omega)$.*

The dependence of ε on k will be given in Section 7. A similar result covering the case of more general subdomains can be found in [8].

4.3.1 Regularity for homogeneous singular points. For homogeneous singular points (i.e. points x_1 and x_2 in Figure 1) the following result is well known:

Lemma 4.4 (see [7: Corollary 2.4.4]). *Let $k = 1$ and let u be the solution of problem* (2.1). *Then for any neighborhood U of x, which does not contain any other singular point, u has regularity $H^{1+\frac{1}{2}}(U \cap \Omega)$, if the boundary conditions do not change in x, and $u \in H^{1+\frac{1}{4}}(U)$, if they do.*

The strongest singularity is of type $r^{\frac{1}{4}} \cos \frac{\lambda}{4}$ and occurs in a slit domain with mixed boundary conditions [7].

4.3.2 Regularity for heterogeneous singular points. We choose a heterogeneous singular point x. Let us denote by n the number of domains to whose boundary x belongs and by m the number of types of boundary conditions. This means that $m = 0$, if x is an interior point (points $x_4 - x_6$ in Figure 1). We set $m = 1$ if $x \in \partial\Omega$ and the boundary conditions do not change in x (points x_3, x_7 from Figure 1). If they change, then $m = 2$ (point x_8).

The following results are known:

Lemma 4.5. *Let u be the solution of problem* (2.1).

(i) *Let x be a heterogeneous singular point with a neighborhood U containing no other singular points. Then, if $n + m \leq 3$, u has regularity $u \in H^{1+\frac{1}{4}}(\Omega_i \cap U)$ $(i = 0, ..., n-1)$, and if $n + m > 3$, $u \in H^1(\Omega_i)$.*

(ii) *If x is an interior singular point and $n = 2$, then $u \in H^{1+\frac{1}{2}}(\Omega_i \cap U)$ $(i = 0, 1)$.*

These regularity bounds are optimal in the respective class of problems.

Proof. For the case of $n = 2, m = 0$ see [9, 10] or [19]. The cases of $n = 3, m = 0$ and $n = 2, m = 1$ has been studied in [12].

Let us discuss the case $n = 4 - m$. For any $\varepsilon > 0$ and the case of $n = 4, m = 0$ Kellogg gives an explicit solution ψ_1 with regularity $\psi_1 \notin H^{1+\varepsilon}(\Omega)$ (see [2, 10]). The singular function ψ_1 is discussed in more detail in the following subsection. For the cases $n = 3, m = 1$ and $n = 2, m = 2$ a problem can be constructed by restricting the domain of definition of ψ_1 to a sector with opening angle π or $\frac{\pi}{2}$. For the case $n+m > 4$ one can define a function which will be a slight modification of the function u_ε ∎

All of the assertions have been shown recently also in [2]. Related results are given in [3, 13, 20] and for the case of two Lipschitz subdomains where a different technique has been used in [18].

4.3.3 Examples of singular functions. We show an example where the best H^{1+s}-regularity result which holds independently of δ is $u \in H^1(\Omega)$. The interface consists in the vicinity of an interior heterogeneous singular point of two intersecting lines. Denote as before by (r, φ) the polar coordinates with respect to the singular point located at the origin. Set $\Omega = (-1, 1) \times (-1, 1)$ and define

$$\Omega_1 = \left\{(x, y) \in \Omega : \, 0 < \varphi < \theta \text{ or } \pi < \varphi < \theta + \pi\right\}$$
$$\Omega_2 = \operatorname{Int}(\Omega/\Omega_1).$$

Define $\psi(r,\varphi) = r^\lambda s(\varphi)$ where

$$s(\varphi) = \begin{cases} \cos(\lambda(\pi-\theta-c))\,\cos(\lambda(\varphi-\theta+b)) & \text{for } 0 \le \varphi \le \theta \\ \cos(\lambda b)\,\cos(\lambda(\varphi-\pi+c)) & \text{for } \theta \le \varphi \le \pi \\ \cos(\lambda c)\,\cos(\lambda(\varphi-\pi-b)) & \text{for } \pi \le \varphi \le \pi+\theta \\ \cos(\lambda(\theta-b))\,\cos(\lambda(\varphi-\theta-\pi-c)) & \text{for } \pi+\theta \le \varphi \le 2\pi \end{cases} \tag{4.7}$$

[10]. With given parameters $0 < \lambda \le 1$ and b, c the coefficients $k_1, ..., k_4$ are defined (up to a multiplicative constant) by the interface conditions (4.2).

Example 1. Take $\Omega = (-1,1) \times (-1,1)$ and $\psi_1 = r^\lambda s(\varphi)$, where s is defined in (4.7). Set $\theta = \frac{\pi}{2}$, $b = 0.5\theta, c = \frac{\pi}{2}(1+\frac{1}{\lambda}) - b$ and vary λ as a parameter within $(0,1]$. Then $k_1 = k_3 = -\tan(\lambda c) = \frac{1}{\tan(\lambda b)}$ and $k_2 = k_4 = \tan(\lambda b)$.

Figure 3: a) - b)
Coefficient k and singular functions a) ψ_1, $\lambda \approx 0.1$, b) ψ_2 with $\delta = 0.1$, $\lambda \approx 0.99$; regularity depends on the geometry

The singular function $\psi_1(r,\varphi)$ is illustrated in Figure 3/a) for $\lambda = 0.1$. Here the ratio of the maximum and the minimum value of k is $\frac{k_{\max}}{k_{\min}} \approx 100$. This function has been defined in [2], too. In the setting of Example 1 we see $\lim_{\lambda\to 0} \frac{k_2}{\lambda\frac{\pi}{4}} = 1$ and $k_2 = k_1^{-1}$.

Now we want to demonstrate that in general a large ratio $\frac{k_{\max}}{k_{\min}}$ not necessarily implies low regularity. We construct a singular function with smoothness $H^{2-\varepsilon}$ where the ratio $\frac{k_{\max}}{k_{\min}}$ increases to infinity as $\varepsilon \to 0$:

Example 2. Let $\varepsilon > 0$ be given. Take $\Omega = (-1,1) \times (-1,1)$ and $\psi_2 = r^\lambda s(\varphi)$, where $s(\varphi)$ is defined in (4.7). Chose $\varepsilon > 0$. Set $\lambda = 1-\varepsilon$, $\theta = \frac{\pi}{2}$, $b = \varepsilon$ and $c = \frac{\pi}{2}(1+\frac{1}{\lambda}) - b$.

The coefficients in Example 2 fulfil $k_2 < k_1 = k_3 < k_4$. The function ψ_2 is depicted in Figure 3 b) for $\lambda = 0.99$. In this case $\frac{k_{\max}}{k_{\min}} \approx 100$, too.

Remark 4.1. Examples 1 and 2 show that the regularity parameter λ where $u \in H^{1+s}(\Omega_i)$, $u \notin H^{1+\lambda}(\Omega_i)$ for any s fulfilling $0 < s < \lambda \le 1$ may tend to a value $\lambda_0 = 0$ or to a value $\lambda_0 = 1$ if $\frac{k_{\max}}{k_{\min}} \to \infty$.

4.4 Open questions. An open question is whether there are conditions on k such that regularity H^s for some $s > 1$ is guaranteed and s does not depend on the bounds of k or the geometry. Such conditions will be introduced in the next Subsection 5.1.

A further question is about the dependence of ε from Lemma 4.3 on k. We will give an answer to this question in Section 7.

5. The quasi-monotone case

5.1 The quasi-monotonicity condition. We define the quasi-monotonicity condition for the coefficient k. This condition has been introduced in [4] in the context of Finite Elements. Remember that we assumed $k_{l,i} \neq k_{l,i+1}$.

Roughly speaking the quasi-monotonicity condition means that the local diffusion coefficient $k_x(\varphi)$ has *only one* local maximum. If $x \in \bar{\Gamma}_D$, we demand alternatively that each local maximum touches the Dirichlet boundary Γ_D. This is illustrated in Figure 4.

Figure 4: a) - c)
Values of k_i around singular points. Distribution of diffusion coefficients is quasi-monotone for an interior singular point in Figure a) but not in Figure b). For a boundary singular point a quasi-monotone distribution is given in Figure c) but not in Figure d)

Definition 5.1. Let a heterogeneous singular point x be given. The distribution of the coefficients $k_i \ \ (i = 0, ..., n-1)$ will be called *quasi-monotone* with respect to the singular point x, if the following conditions are fulfilled:

Denote by N_i the indices of cones C_j that are neighbors of the cone C_i, that is $N_i = \{j : \text{meas}_1(\overline{C}_j \cap \overline{C}_i) > 0 \ \ (j \neq i,\, j = 0, ..., n-1)\}$. The following condition holds:

- If $x \in \overline{\Omega}/\overline{\Gamma}_D$, there is only one index i such that $k_i > \max_{j \in N_i}\{k_j\}$
- If $x \in \overline{\Gamma}_D$, for each index i such that $k_i > \max_{j \in N_i}\{k_j\}$ the measure $\text{meas}_1(\overline{C}_i \cap \Gamma_D \cap B_x(r))$ is positive.

Definition 5.2. The coefficient k is *quasi-monotone* if for all singular points x the distribution of coefficients $k_i \ \ (i = 0, ..., n-1)$ is quasi-monotone.

Remark 5.1. In the following sense quasi-monotonicity of k is necessary for $H^s(\Omega_i)$-regularity of solutions of problem (2.1) where $s > 1$ is independent of k. Let us first restrict to the case of an interior heterogeneous singular point x.

If the quasi-monotonicity condition is violated, $k_x(\varphi)$ has $m > 1$ local maxima. For any $s > 1$, there is a geometry and local diffusion coefficient having m local maxima and defining a problem with solution $u_s \notin H^s(\Omega_i)$. For $m = 2$ see Example 1. For $m > 2$ slightly perturb the coefficient given in Example 1 by enlarging it in parts of the domain, where it takes the lower value. This will change the singular function from Example 1 and its low regularity a little only.

Though the above counter-example is defined for a special geometry, it can be defined for any geometry for which the quasi-monotonicity is violated. In the case of a heterogeneous singular point on the boundary a singular function can be constructed by restricting the domain of definition of the singular function defined in Example 1.

We give conditions for the quasi-monotonicity conditions to hold without restrictions on k but with restrictions on the maximum number of subdomains that share singular points.

Remark 5.2. Choose a heterogeneous singular point x and denote as in Subsection 4.3.2 by n the number of subdomains Ω_i to whose boundary x belongs and by m the number of types of boundary conditions. If

$$n + m \leq 3, \tag{5.1}$$

then for any values of $k_i \quad (i = 0, ..., n-1)$ the distribution of the coefficients $k_i \quad (i = 0, .., n-1)$ is quasi-monotone with respect to x. Condition (5.1) is sharp.

Observe that for exactly these restrictions on the maximum number of subdomains regularity results with piecewise regularity $H^s \quad (s > 1)$, where s is independent of δ, are known (Lemma 4.5).

Thus the distribution of the coefficients $k_{l,i} \quad (i = 0, ..., n_l - 1)$ is always quasi-monotone for points x_1, x_2, x_3, x_4, x_5 from Figure 1. For points x_6, x_7, x_8 from Figure 1 quasi-monotonicity depends on k. For instance, coefficients $k_{6,0} = k_{6,2} = 1$ and $k_{6,1} = k_{6,3} = 100$ are not distributed quasi-monotonically with respect to the singular point x_6.

5.2 Quasi-monotonicity bounds eigenvalues from below. In this subsection we show that if the coefficient k is quasi-monotone, the eigenvalues of the Sturm-Liouville eigenvalue problem are bounded from below. We precede the proof of this fact by two technical lemmas.

Lemma 5.1. *Let functions* $t_i(\varphi) = b_i \cos(\varphi - c_i) \quad (i = 1, 2)$ *be given that fulfil the conditions*

$$\begin{aligned} t_1(\varphi_1) &= t_2(\varphi_1) \\ k_1 t_1'(\varphi_1) &= k_2 t_2'(\varphi_1) \end{aligned} \tag{5.2}$$

for some φ_1 *and* $k_i, b_i > 0 \quad (i = 1, 2)$. *If one of the conditions*

a) $t_1'(\varphi_1) < t_2'(\varphi_1)$

b) $k_1 < k_2$ *and* $t_1'(\varphi_1) < 0$ *or* $t_2'(\varphi_1) < 0$

is fulfilled, then $t_1(\varphi) \leq t_2(\varphi) \quad (\varphi_1 \leq \varphi \leq \varphi_1 + \pi)$ *and* $t_2(\varphi) \leq t_1(\varphi) \quad (\varphi_1 - \pi \leq \varphi \leq \varphi_1)$.

Proof. Observe that $t_2 - t_1 = b_3 \cos(\varphi - c_3)$ for some b_3 and c_3. It is not hard to see that $c_3 \in \{\varphi_1 - \frac{\pi}{2}, \varphi_1 + \frac{\pi}{2}\}$ and we choose $c_3 = \varphi_1 - \frac{\pi}{2}$. Then $b_3 = (t_2 - t_1)'(\varphi_1)$ and it remains to show that $0 < b_3 = (t_2 - t_1)'(\varphi_1)$.

a) If $t_1'(\varphi_1) < t_2'(\varphi_1)$, this is obviously true.

b) In this case we conclude from equation (5.2) $\frac{t_1'(\varphi_1)}{t_2'(\varphi_1)} = \frac{k_2}{k_1} > 1$ and that $t_1'(\varphi_1) < 0$ and $t_2'(\varphi_1) < 0$. This shows $t_1'(\varphi_1) < t_2'(\varphi_1)$ ∎

Lemma 5.2. *Let numbers* $0 = \varphi_0 < \varphi_1 < ... < \varphi_n < \frac{\pi}{2}$ *and* $k_i \quad (i = 0, ..., n-1)$ *with* $0 < k_0 \leq k_1 \leq ... \leq k_{n-1}$ *be given. Denote by* $\chi_{[\varphi_i, \varphi_{i+1})}$ *the characteristic function of the interval* $[\varphi_i, \varphi_{i+1})$*. Further, let numbers* c_i *and* b_i *be given such that the function*

$$s(\varphi) = \sum_{i=0}^{n-1} b_i \cos(\varphi - c_i) \chi_{[\varphi_i, \varphi_{i+1})} \tag{5.3}$$

is continuous and its derivatives weighted with k_i *are also continuous:*

$$b_i \cos(\varphi_{i+1} - c_i) = b_{i+1} \cos(\varphi_{i+1} - c_{i+1}) \tag{5.4}$$

$$k_i b_i \sin(\varphi_{i+1} - c_i) = k_{i+1} b_{i+1} \sin(\varphi_{i+1} - c_{i+1}) \tag{5.5}$$

for $i = 0, ..., n-2$*. Assume* $c_0 = 0$ *and* $b_0 > 0$*. Then* $s(\varphi) > 0$ *for all* $\varphi \in [0, \varphi_n]$*.*

Figure 5
The function s from equation (5.3) is the upper envelope and depicted with a continuous line in case of decreasing k_i, the functions t_i are indicated by a dashed line

Proof. Define auxiliary functions $t_i(\varphi) = b_i \cos(\varphi - c_i)$. These functions are illustrated in Figure 5. The idea is that if $k_{j+1} > k_j$, the function t_{j+1} will dominate the function t_j on an interval of length π starting from the point where t_{j+1} and t_j intersect. Multiplying the function $s(\varphi)$ by a constant we can assure $b_0 = 1$. We want to prove

$$0 < \cos(\varphi) = t_0(\varphi) \leq ... \leq t_j(\varphi) \quad (\varphi_j \leq \varphi \leq \varphi_n < \tfrac{\pi}{2}), \quad t_j'(\varphi_j) \leq 0 \tag{5.6}$$

with the help of Lemma 5.1 by induction over $j = 0, ..., n-1$. For $j = 0$ inequality (5.6) is clearly fulfilled. Suppose $i > 0$ and let inequality (5.6) be fulfilled for $j = i-1$. Observe that $t_{i-1}'(\varphi_{i-1}) \leq 0$ and $t_{i-1}(\varphi) > 0$ for $\varphi_{i-1} \leq \varphi \leq \varphi_i$ implies $t_{i-1}'(\varphi_i) < 0$. Condition (5.5) then gives $t_i'(\varphi_i) < 0$. Setting in terms of Lemma 5.1 $t_1 = t_{i-1}, t_2 = t_i$ and $\varphi_1 = \varphi_i$ we see that assumption b) from Lemma 5.1 is fulfilled and obtain $t_{i-1} \leq t_i$ for $\varphi \in [\varphi_i, \varphi_i + \pi]$. From assumption $0 \leq \varphi_i < \frac{\pi}{2}$ we see $[\varphi_i, \varphi_n] \subset [\varphi_i, \varphi_i + \pi]$. This together with $t_i'(\varphi_i) < 0$ finishes the proof of the induction step (5.6) for $j = i$ ∎

Remark 5.3. Lemma 5.2 can be sharpened to hold also for $\varphi_n \leq \frac{\pi}{2}$ if $n > 1$. To show this use $k_0 < k_1$ and show that $0 < c_1$.

As already pointed out, to discuss regularity in the neighborhood of a singular point it suffices to study the eigenvalues of the Sturm-Liouville eigenvalue problem (4.1) - (4.3). The eigenvalues are real and non-negative and we look for a lower bound of eigenvalues from $(0, 1)$. If $x \in \overline{\Omega}/\overline{\Gamma}_D$, then $\lambda = 0$ will be an eigenvalue with constant eigenfunction $s(\varphi)$. But the associated constant function $\psi(r, \varphi)$ does not influence regularity.

Theorem 5.3. *Let an interior heterogeneous singular point $x \in \Omega$ be given and let the distribution of the coefficients $k_i \quad (i = 0, ..., n-1)$ be quasi-monotone with respect to x. Then the smallest non-vanishing eigenvalue λ^2 of the associated Sturm-Liouville eigenvalue problem* (4.1) – (4.3) *is greater than $(\frac{1}{4})^2$. This bound is sharp.*

Proof. We choose an eigenfunction of the associated Sturm-Liouville eigenvalue problem with eigenvalue $\lambda^2 \neq 0$. The eigenfunction has the representation

$$s(\varphi) = \sum_{i=0}^{n-1} b_i \cos(\lambda(\varphi - c_i))\chi_{[\varphi_i, \varphi_{i+1})} \tag{5.7}$$

where b_i and c_i are real numbers. The eigenfunction $s(\varphi)$ fulfils the interface conditions (4.4) - (4.5) for $i = 0, ..., n-1$.

Figure 6

Eigenfunction $s_\lambda(\varphi)$ (black line) and function $b_j \cos(\lambda\varphi), c_j = 0$ (black dashed line) coincide at the maximum φ_{ex} of $s_\lambda(\varphi)$ and $s_\lambda(\varphi_{zero}) = 0, b_j \cos(\lambda\varphi') = 0$; since k increases on $[\varphi_{ex}, \varphi_{zero}]$, it holds $0 < \varphi' = \frac{\pi}{2\lambda} \leq \varphi_{zero} < 2\pi$

The idea of the proof is to show that there is an index $j \in \{0, ..., n-1\}$ such that $s(\varphi) \geq b_j \cos(\lambda(\varphi_{j+1} - c_j)) > 0$ on the interval $[c_j, c_j + \frac{\pi}{2\lambda})$. Here we need the quasi-monotonicity condition. Since $s(\varphi)$ vanishes in some points, the length of the interval $[c_j, c_j + \frac{\pi}{2\lambda})$ is bounded by 2π (Figure 6). This yields the bound $\lambda > \frac{1}{4}$.

Let us have a closer look onto $s(\varphi)$. This periodic function is continuous and therefore achieves a minimum at a point $\varphi_{\min}$ and a maximum at $\varphi_{\max}$. Choose j such that $\varphi_{\max} \in [\varphi_j, \varphi_{j+1})$. Possibly substituting c_j with $c_j \pm \frac{\pi}{\lambda}$ we can assume $b_j \geq 0$. The case $b_j = 0$ can be excluded since then the interface condition imply $s \equiv 0$ and hence $\lambda = 0$. If $\varphi_{\max}$ lies on $(\varphi_j, \varphi_{j+1})$, we conclude from $b_j > 0$ that $\varphi_{\max} = 2l\frac{\pi}{\lambda} + c_j$ for a number $l \in \mathbb{N}$. Possibly redefining c_j we may set $\varphi_{\max} = c_j$ and we see $s(\varphi_{\max}) > 0$.

If $\varphi_{\max} = \varphi_j$, proceed as follows. Since $\varphi_{\max}$ is a maximum, it is clear that $s(\varphi_j - 0)' \geq 0$ and $s(\varphi_j + 0)' \leq 0$. The interface condition (4.5) for $i = j$ implies on the other hand that $s(\varphi_j - 0)'$ and $s(\varphi_j + 0)'$ cannot have different signs. Hence $s(\varphi_j - 0)' = s(\varphi_j + 0)' = 0$ and in this case there holds $c_j = \varphi_{\max}$, too. From this $s(\varphi_{\max}) > 0$ follows. Similarly, $s(\varphi_{\min}) < 0$ and we conclude that there are at least two points $\varphi_{zero,1}$ and $\varphi_{zero,2}$ with $s(\varphi_{zero,1}) = s(\varphi_{zero,2}) = 0$. Without loss of generality we assume

$$\varphi_{zero,1} < \varphi_{\max} < \varphi_{zero,2} < \varphi_{\min}.$$

Now we exploit the quasi-monotonicity condition. We want to show that the following property (P) holds:

(P) There is an extremum $\varphi_{ex} \in \{\varphi_{\min}, \varphi_{\max}\}$ and a point $\varphi_{zero} \in \{\varphi_{zero,1}, \varphi_{zero,2}\}$ such that $k_x(\varphi)$ increases monotonically when going from φ_{ex} to φ_{zero}. This means that $k_x(\varphi)$ is increasing on $[\varphi_{ex}, \varphi_{zero}]$ or decreasing on $[\varphi_{zero}, \varphi_{ex}]$.

Figure 7

Local diffusion coefficient $k_x(\varphi)$ is piecewise monotone on two intervals covering the sphere $[0, 2\pi)$; possible location of points $\varphi_{zero,1}$, $\varphi_{\max}$, $\varphi_{zero,2}$, $\varphi_{\min}$; $k_x(\varphi)$ decreases on $[\varphi_{zero,1}, \varphi_{\max}]$

To show this denote by $[\varphi_{i_{\min}}, \varphi_{i_{\min}+1})$ and $[\varphi_{i_{\max}}, \varphi_{i_{\max}+1})$ the intervals where $k_x(\varphi)$ reaches the minimum and maximum. The sphere $[0, 2\pi)$ is then decomposed into two intervals $I_{decr} = [\varphi_{i_{\max}}, \varphi_{i_{\min}})$ and $I_{incr} = [\varphi_{i_{\min}}, \varphi_{i_{\max}})$ on which $k_x(\varphi)$ is monotone as depicted in Figure 7. There are two possibilities:

a) either there are three points from $\{\varphi_{zero,1}, \varphi_{\max}, \varphi_{zero,2}, \varphi_{\min}\}$ contained in I_{decr} or I_{incr}

or

b) each interval I_{decr}, I_{incr} contains two points from $\{\varphi_{zero,1}, \varphi_{\max}, \varphi_{zero,2}, \varphi_{\min}\}$.

In Figure 7 a possible distribution of the points $\varphi_{zero,1}, \varphi_{\max}, \varphi_{zero,2}, \varphi_{\min}$ in the intervals I_{decr} and I_{incr} in the case a) is shown. One notices that in the depicted distribution the local diffusion coefficient is decreasing on $[\varphi_{zero,1}, \varphi_{\max}]$ and property (P) is fulfilled. For the other (essentially three) possible distributions of the points $\varphi_{zero,1}, \varphi_{\max}, \varphi_{zero,2}, \varphi_{\min}$ it is easy to check that property (P) holds, too.

In the case b) the points from $\varphi_{zero,1}, \varphi_{\max}, \varphi_{zero,2}, \varphi_{\min}$ could be distributed like $\varphi_{zero,1}, \varphi_{\max} \in I_{decr}$ and $\varphi_{zero,2}, \varphi_{\min} \in I_{incr}$. In this special case the function $k_x(\varphi)$

is decreasing on $[\varphi_{zero,1}, \varphi_{\max}]$ and property (P) is fulfilled. Other distributions of the points in the case b) can be checked in the same way to satisfy property (P).

Multiplying by -1 in (5.7), rotating the polar coordinate system and possibly reflecting it on the x-axis we can assure

$$0 = \varphi_{ex} = \varphi_{\max} < \varphi_{zero} < 2\pi. \tag{5.8}$$

Remember that $k_x(\varphi)$ increases on $[\varphi_{ex}, \varphi_{zero}]$. If the function $s_\lambda(\varphi)$ vanishes on $[\varphi_{\max}, \varphi_{zero})$ in some point(s), choose φ'_{zero} to be the minimum of these points and redefine $\varphi_{zero} := \varphi'_{zero}$. Choose j such that $\varphi_{ex} \in [\varphi_j, \varphi_{j+1})$. The function $b_j \cos(\lambda(\varphi - c_i))$ is depicted in Figure 6 with a dashed line. We show as before $c_j = \varphi_{ex} = 0$. Renumbering the points φ_i we may assume $j = 0$.

Now we are nearly done with the proof. In the last step we restrict the function s_λ to the interval $[\varphi_{ex}, \varphi_{zero}]$ and apply a homogeneous scaling to transform the functions $b_i \cos(\lambda(\varphi - c_i))$ to functions $b_i \cos(\varphi - \lambda c_i)$ which satisfy similar interface conditions and apply Lemma 5.2 to the transformed functions. Choose the largest m such that $\varphi_{m-1} < \varphi_{zero}$. We introduce an homogeneous transformation

$$F: [0 = \varphi_{ex}, \varphi_{zero}] \to [0, \lambda\varphi_{zero}], \qquad F(\varphi) = \lambda\varphi \tag{5.9}$$

and define $s_F(F(\varphi)) = s(\varphi)$ for $\varphi \in [\varphi_{ex}, \varphi_{zero}]$. Under this transformation we define a sequence

$$\widehat{\varphi}_0 < \widehat{\varphi}_1 < \ldots < \widehat{\varphi}_m$$

where

$$\widehat{\varphi}_0 = F(\varphi_{ex}) = 0, \qquad \widehat{\varphi}_i = F(\varphi_i) \;\; (0 < i < m-1), \qquad \widehat{\varphi}_m = F(\varphi_{zero}).$$

It follows that s_F fulfils

$$s_F(\varphi) = \sum_{i=0}^{n-1} b_i \cos(\varphi - \lambda c_i)\chi_{[\widehat{\varphi}_i, \widehat{\varphi}_{i+1})}$$

and

$$\begin{aligned} b_i \cos(\widehat{\varphi}_{i+1} - \lambda c_i) &= b_{i+1} \cos(\widehat{\varphi}_{i+1} - \lambda c_{i+1}) \\ k_i b_i \sin(\widehat{\varphi}_{i+1} - \lambda c_i) &= k_{i+1} b_{i+1} \sin(\widehat{\varphi}_{i+1} - \lambda c_{i+1}) \end{aligned}$$

for $i = 0, .., m-1$. Recall $s_F(\widehat{\varphi}_m) = 0$. As $k_i \le k_{i+1}$ $(i = 0, .., , m-1)$ and $c_0 = 0$, the function $s_F(\varphi)$ fulfils the assumptions of Lemma 5.2 and we conclude that s_F does not vanish on $[0, \frac{\pi}{2})$. Hence $\widehat{\varphi}_m \ge \frac{\pi}{2}$. Observe that (5.8) implies $\widehat{\varphi}_m = \lambda\varphi_{zero} < \lambda 2\pi$. This yields the bound

$$\tfrac{\pi}{2} \le \widehat{\varphi}_m = \lambda\varphi_{zero} < \lambda 2\pi, \qquad \text{i.e. } \tfrac{1}{4} < \lambda.$$

From the above proof it is not hard to see how to construct an eigenfunction $s_3(\varphi)$ with eigenvalue λ^2 arbitrarily close to $(\frac{1}{4})^2$. We see in the following example that $\lambda \to \frac{1}{4}$ when the interior angle of a subdomain tends to 2π.

Choose $\varepsilon > 0$ and $n = 3$. The idea is to construct an eigenfunction $s_3(\varphi)$ for the following problem. The interval (φ_0, φ_1) will have length of order $2\pi - O(\varepsilon)$ and

$k_0 = 1$. The other two intervals will have length $O(\varepsilon)$ and $k_1 = O(\varepsilon^{-1})$, $k_2 = O(\varepsilon)$. The constructed eigenfunction will have the eigenvalue λ^2 where $\lambda = \frac{1}{2}\frac{\pi}{2\pi-4\varepsilon} \to \frac{1}{4}$ as $\varepsilon \to 0$.

For the interested reader we will give the details below: Define $\varphi_0 = -\varepsilon$, $\varphi_1 = 2\pi - 3\varepsilon$. The remaining parameter φ_2 will be defined below. The aim is to define a function that achieves a maximum at $\varphi = 0$ and vanishes at $\varphi = 2\pi - 4\varepsilon$ and at $\varphi = 2\pi - 2\varepsilon$. Furthermore, a minimum is attained in $\varphi \in (\varphi_1, \varphi_2)$.

To do so set $c_0 = 0$ and $b_0 = 1$. Define c_2 and b_2 in such a way that $\cos(\lambda(\varphi - c_2))$ vanishes in $\varphi = 2\pi - 2\varepsilon$ and that $b_2 \cos(\lambda(\varphi - c_2)) = \cos(\lambda\varphi)$ for $\varphi = \varphi_0$. Further, choose $\varphi_2 > \varphi_1$ such that $\cos(\lambda(\varphi_2 - c_2)) = \cos(\lambda(\varphi_1 - c_0))$. Set $c_1 = 0.5(\varphi_1 + \varphi_2)$ and $b_1 = -\frac{\cos(\lambda(\varphi_1-c_0))}{\cos(\lambda(\varphi_1-c_1))}$. The definition of the eigenfunction s_3 is finished by setting $k_0 = 1$ and choosing k_1 and k_2 in such a way that interface conditions for the derivatives are fulfilled ∎

Theorem 5.4. *Let a heterogeneous singular point $x \in \partial\Omega$ on the boundary be given and let the distribution of the coefficients k_i $(i = 0, ..., n-1)$ be quasi-monotone with respect to x. Then the smallest non-vanishing eigenvalue λ^2 of the associated Sturm-Liouville eigenvalue problem (4.1) – (4.3) fulfils $(\frac{1}{4})^2 < \lambda^2$. This bound is sharp.*

Proof. The proof runs similar to that of Theorem 5.3. The eigenfunction of the associated Sturm-Liouville eigenvalue problem with eigenvalue $\lambda^2 \neq 0$ has the representation

$$s(\varphi) = \sum_{i=0}^{n-1} b_i \cos(\lambda(\varphi - c_i))\chi_{[\varphi_i,\varphi_{i+1})}$$

where $\chi_{[\varphi_i,\varphi_{i+1})}$ denotes the characteristic function of the interval $[\varphi_i, \varphi_{i+1})$ and b_i, c_i are real numbers. The eigenfunction $s(\varphi)$ fulfils the interface conditions (4.4), (4.5) for $i = 0, ..., n-2$ and some boundary conditions that will be specified later.

Since we deal with two different boundary conditions, there are three possibilities how to combine them. We will treat each case separately. In any case $s(\varphi)$ is not a constant function. Denote by F_1 and F_2 parts of the boundary on both sides of $x \in \partial\Omega \cap B_x(r)$.

Case I: $F_1 \subset \Gamma_D$ and $F_2 \subset \Gamma_D$. We deduce that there exists a local extremum φ_{ex} of the function $s(\varphi)$, and multiplying $s(\varphi)$ by -1 we may assume that φ_{ex} is a maximum. We choose j such that $\varphi_{ex} \in [\varphi_j, \varphi_{j+1})$ and show as in the proof of Theorem 5.3 that $c_j = \varphi_{ex}$. The quasi-monotonicity condition implies now that $k_x(\varphi)$ is monotonically increasing on $[\varphi_{ex}, \varphi_n]$ or decreasing on $[\varphi_0, \varphi_{ex}]$. We may suppose without loss of generality that $k_x(\varphi)$ is increasing on $[\varphi_{ex}, \varphi_n]$, and since $s(\varphi_n) = 0$ we may define $\varphi_{zero} = \varphi_n$. By rotation of the coordinate system we can assume

$$0 = \varphi_{ex} < \varphi_{zero} \leq \theta < 2\pi \tag{5.10}$$

where θ is the interior angle of Ω at x. Possibly redefining φ_{zero} we can assure $s(\varphi) > 0$ for $\varphi \in [\varphi_{ex}, \varphi_{zero}]$. Choose m such that $\varphi_{j+m-1} < \varphi_{zero}$. We transform the sequence

$$\varphi_{ex} < \varphi_{j+1} < ... < \varphi_{j+m-1} < \varphi_{zero} < \theta$$

with the affine transformation defined in (5.9) and obtain a new sequence

$$\widehat{\varphi}_0 < \widehat{\varphi}_1 < ... < \widehat{\varphi}_m$$

where

$$\widehat{\varphi}_0 = 0, \quad \widehat{\varphi}_i = F(\varphi_{i+j}) = \lambda\varphi_{i+j} \;\; (1 < i < m-1), \quad \widehat{\varphi}_m = F(\varphi_{zero}) = \lambda\varphi_{zero}.$$

Defining $s_F(F(\varphi)) = s(\varphi)$ we obtain a scaled function which fulfils the modified interface conditions

$$b_i \cos(\widehat{\varphi}_{i+1} - \lambda c_i) = b_{i+1} \cos(\widehat{\varphi}_{i+1} - \lambda c_{i+1})$$
$$k_i b_i \sin(\widehat{\varphi}_{i+1} - \lambda c_i) = k_{i+1} b_{i+1} \sin(\widehat{\varphi}_{i+1} - \lambda c_{i+1})$$

for $i = 0, ..., m-1$. Recall $s_F(\widehat{\varphi}_m) = 0$. Further, $\widehat{c}_0 = 0$ and $\widehat{k}_i \leq \widehat{k}_{i+1}$ $(i = 0, ..., m-1)$. Hence s_F fulfils the assumptions of Lemma 5.2 and it follows that s_F does not vanish on $[0, \frac{\pi}{2})$. This shows that $\widehat{\varphi}_m \geq \frac{\pi}{2}$. Equation (5.10) implies $\widehat{\varphi}_m = \lambda\varphi_{zero} \leq \lambda 2\pi$. This gives

$$\frac{\pi}{2} \leq \widehat{\varphi}_m = \lambda\varphi_{zero} \leq \lambda 2\pi, \qquad \text{i.e. } \lambda > \tfrac{1}{4}.$$

Case II: $F_1 \subset \Gamma_N$ and $F_2 \subset \Gamma_D$. Suppose that the Dirichlet conditions are set on the angle φ_n. Define $\varphi_{ex} = \varphi_0$ and $\varphi_{zero} = \varphi_n$. The quasi-monotonicity condition implies that the local diffusion coefficient $k_x(\varphi)$ has not more than one local maximum $[\varphi_i, \varphi_{i+1}]$, and this local maximum is achieved for $i = n-1$. Hence $k_x(\varphi)$ is monotonically increasing on $[\varphi_{ex}, \varphi_{zero}]$. It follows from $k\frac{\partial u}{\partial n} = 0$ at $\varphi = \varphi_{ex}$ that $c_j = \varphi_{ex}$. Using Remark 5.3 we show as in the Case I that $\frac{1}{4} < \lambda$.

Case III: $F_1 \subset \Gamma_N$ and $F_2 \subset \Gamma_N$. Set $\varphi_{ex,1} = \varphi_0$ and $\varphi_{ex,2} = \varphi_n$. As in the Case II we conclude $c_0 = 0$ and $c_{n-1} = \varphi_n$. Denote by φ_{zero} a point where $s(\varphi)$ vanishes. The quasi-monotonicity condition implies that the local diffusion coefficient $k_x(\varphi)$ has not more than one local maximum $[\varphi_j, \varphi_{j+1}]$. Using the quasi-monotonicity property we show that there is a number $\varphi_{ex} \in \{\varphi_{ex,1}, \varphi_{ex,2}\}$ such that $k_x(\varphi)$ increases monotonically when going from φ_{ex} to φ_{zero}. If $\varphi_{zero} < \varphi_j$, then $k_x(\varphi)$ is monotonically increasing on $[\varphi_0, \varphi_{zero}]$ and $\varphi_{ex} := \varphi_0$. Otherwise $k_x(\varphi)$ is monotonically decreasing on $[\varphi_{zero}, \varphi_n]$ and $\varphi_{ex} := \varphi_n$. We may suppose that the first case holds. The remainder of the proof is similar to the Case II and we show $\lambda > \frac{1}{4}$.

To prove sharpness we use the example from the proof of Theorem 5.3. Denote by I the closure of the support $\max\{0, s_3(\varphi)\}$. We define the eigenfunction $s_4(\varphi) = s_3(\varphi)$ on I. This eigenfunction has the eigenvalue $\lambda = \frac{1}{2}\frac{\pi}{2\pi - 4\varepsilon}$ ∎

Remark 5.4. Denote by θ the interior angle of Ω at $x \in \partial\Omega$. Under the assumptions of Theorem 5.4 and using the bound $\theta < 2\pi$ in inequality (5.10) it is not hard to show the improved bound $(\frac{2\pi}{4\theta})^2 < \lambda^2$. The same estimate holds if in equation (5.8) 2π is substituted by θ, where θ is the length of the largest interval on which $k_x(\varphi)$ is monotone.

6. Regularity results in the quasi-monotone case

Here we present our main results.

Theorem 6.1. *Let the distribution of coefficients* $k_i \quad (i = 0, ..., n-1)$ *be quasi-monotone with respect to all singular points* x*. The solution of problem* (2.1) *fulfils*

$$u \in H^{1+\frac{1}{4}}(\Omega)$$

independent of δ*. This is the maximum regularity independent of* δ *and the geometry.*

Proof. The assertion follows with Corollary 4.2 from Theorems 5.3 and 5.4 and Lemmas 4.4 and 3.1 ∎

Necessity of the quasi-monotonicity for $H^{1+\frac{1}{4}}$-regularity independent of δ is discussed in Remark 5.1. Note that we get the same regularity as for $k = 1$ were $H^{1+\frac{1}{4}}$-regularity is the maximal regularity at a reentrant corner with changing boundary conditions. As special case of the quasi-monotonicity condition we slightly extend results from [2, 12].

Theorem 6.2. *Let a singular point* $x \in \overline{\Omega}$ *be given. Denote by* n *the number of subdomains that meet in* x *and let* U *be a neighborhood containing no other singular points. Denote by* θ *the maximum interior angle of subdomains* $\Omega_i \quad (i = 0, ..., n-1)$ *at* x*. If* x *is an interior singular point, let* $n \leq 3$*. If* $x \in \partial\Omega$*, then let* $n \leq 2$ *and additionally the boundary conditions do not change in* x*.*

Then the solution u *of problem* (2.1) *fulfils*

$$\begin{aligned} &u \in H^{1+\frac{1}{4}}(\Omega \cap U) \\ &u \in H^{1+\max(1, \frac{\pi}{2\theta})}(\Omega_i \cap U) \quad (i = 0, ..., n-1). \end{aligned}$$

The results are optimal with respect to θ *and* n*.*

Proof. One checks that under the above restrictions on n the coefficient k is quasi-monotone. The first part follows from Theorem 6.1. For the second part one has to show an eigenvalue bound using techniques exploited above. A proof can be found in [17]. To see that the restrictions on n and θ are sharp we refer to the proof of Lemma 4.5 and Theorem 5.3, respectively ∎

For heterogeneous singular points on the boundary with quasi-monotonically distributed coefficients k_i the results can be sharpened.

Corollary 6.3. *Let* x *be a boundary heterogeneous singular point and* U *a neighborhood containing no other singular points. Denote by* θ *the interior angle at* $x \in \partial\Omega$*. Assume that the distribution of coefficients* $k_i \quad (i = 0, ..., n-1)$ *is quasi-monotone with respect to* x*. Then the solution* u *of problem* (2.1) *has regularity*

$$u \in H^{1+\max(1, \frac{\pi}{2\theta})}(\Omega_i \cap U).$$

Proof. For a proof use Remark 5.4 ∎

The special case, where the interface consists locally of two intersecting lines, has been already studied in [10]. We state a regularity result for the quasi-monotone case.

Theorem 6.4. *Let an interior heterogeneous singular point $x \in \Omega$ be given and let U be a neighborhood of x containing no other singular points. The interface consists in a neighborhood of x of two intersecting lines. Let the distribution of coefficients k_i $(i = 0, ..., 3)$ be quasi-monotone with respect to x. Then the solution of problem* (2.1) *fulfils*

$$u \in H^{1+\frac{1}{2}}(\Omega_i \cap U) \quad (i = 0, ..., 3).$$

This bound is sharp.

Proof. The proof uses techniques described above. It can be found in [17] ∎

One checks that regularity results from [3, 10, 12, 16, 20] are special cases of Theorems 6.2 or 6.4.

Remark 6.1. One notices that Lemma 5.2 is the key ingredient for deriving lower bounds for the eigenfunctions of the Sturm-Liouville problem. It uses explicitly that the eigenfunctions of the Sturm-Liouville problem are piecewise scaled and shifted cosines. One could prove a similar result by using only concavity of the positive part of the cosine function. In such a way applications to other problems may be possible.

7. The general case

7.1 Eigenvalue bounds in the general case. We conclude from Lemma 5.1 that in the case of a non-quasi-monotone coefficient regularity may go down to H^1. This may happen if δ^{-1} becomes large. In this subsection we derive explicit bounds of the regularity depending on δ. We show that $u \in H^{1+\frac{\delta}{2\pi}}$. Moreover, we can derive slightly better results which are sharp. To our knowledge a result which gives *explicit* H^s-regularity where s depends on k is new.

The following technical lemma is the equivalent of Lemma 5.2. Before formulating the lemma we will stretch its content. We have given a piecewise constant function $k(\varphi)$ defined on $[0, 2\pi)$ fulfilling $k(0) = 1$ and $k(\varphi) \geq \delta^2$, where $0 < \delta^2 < 1$ is a given constant. The function k defines a continuous function $s(\varphi)$ which has piecewise the form $b_i \cos(\varphi - c_i)$ and whose derivatives satisfy interface conditions of the type $[ks'] = 0$. We demand $s'(0) = 0$ and $s(0) = 1$.

Let φ_n be the infimum of all roots φ_{zero} of these functions $s(\varphi)$. The question is about the dependence of φ_n on δ^2. To answer the question we look for the function k which defines the function $s(\varphi)$ that has φ_n as a root. This function k is defined by $k = 1$ on $[0, \frac{1}{2}\varphi_n)$ and $k = \delta^2$ on $[\frac{1}{2}\varphi_n, \varphi_n)$ where $\varphi_n = 2 \arctan \delta$ (see Figure 9).

Lemma 7.1. *Let a number $0 < \delta^2 < 1$ and numbers $0 = \varphi_0 < \varphi_1 < ... < \varphi_n = 2 \arctan \delta$ be given. Further, there are coefficients k_i given where $k_0 > 0$ and $\frac{k_i}{k_0} \geq \delta^2$ $(i = 0, ..., n-1)$. Denote by $\chi_{[\varphi_i, \varphi_{i+1})}$ the characteristic function of the interval $[\varphi_i, \varphi_{i+1})$. Let numbers $c_i, b_i \geq 0$ $(i = 0, ..., n-1)$ be given which define a* *function*

$$s(\varphi) = \sum_{i=0}^{n-1} b_i \cos(\varphi - c_i)\chi_{[\varphi_i,\varphi_{i+1})} \tag{7.1}$$

that is continuous and whose derivatives weighted with k_i are also continuous:

$$b_i \cos(\varphi_{i+1} - c_i) = b_{i+1} \cos(\varphi_{i+1} - c_{i+1}) \tag{7.2}$$

$$k_i b_i \sin(\varphi_{i+1} - c_i) = k_{i+1} b_{i+1} \sin(\varphi_{i+1} - c_{i+1}) \tag{7.3}$$

for $i = 0, ..., n-2$. Assume $c_0 = 0$ and $b_0 = 1$. Then $s(\varphi) > 0$ on $[0, \varphi_n)$.

Proof. We define $t_i(\varphi) = b_i \cos(\varphi - c_i)$. Dividing k_i by k_0 we may set $k_0 = 1$. But in order to make the dependence on k clear we will use in the proof the notation k_0 remembering $k_0 = 1$. We first assume $k_0 > k_1$. Otherwise regard the discussion at the end of the proof. The proof is done in three steps. Its idea is to bound function t_i from below by functions t_{j_i}. Here we write j_i to denote the dependence of j on i. Then we show that the function t_{j_i} is greater than a function u_{j_i}. In the last step we discuss the functions u_{j_i}.

First Step: In the first step our goal is to show that for $i = 1, ..., n-1$ there is an index $0 \le j \le n-1$ and a number φ_j^- fulfilling

$$t_j(\varphi_j^-) = t_0(\varphi_j^-) \quad (0 < \varphi_j^- \le \varphi_i) \quad \text{and} \quad t_j(\varphi) \le \begin{cases} t_0(\varphi) & \text{if } \varphi_j^- \le \varphi \le \varphi_n \\ t_i(\varphi) & \text{if } \varphi_i \le \varphi \le \varphi_n \end{cases} \tag{7.4}$$

The proof is somewhat technical. We show equation (7.4) by induction with respect to $i = 1, ..., n-1$.

Figure 8
Step $i = 3$ is illustrated, here $J = 1$, note $\varphi_J \le \varphi \le \varphi_i \le \varphi^+$

Initial step $i = 1$: Simply define $\varphi_{j_1}^- := \varphi_1$ and $j_1 = 1$. As $k_0 > k_1$, Lemma 5.1 implies $t_{j_1}(\varphi) \le t_0(\varphi)$ for $\varphi_{j_1}^- \le \varphi \le \varphi_n$. We showed equation (7.4) for $i = 1$.

Induction for $i > 1$: Set $J := j_{i-1}$. There are two cases. In the first case $t_J(\varphi) \le t_i(\varphi)$ for $\varphi_i \le \varphi \le \varphi_n$. We define $j_i := J$ and thus proved (7.4). In the second case we define $j_i := i$. This case is illustrated in Figure 8. There is a $\varphi^+ \in (\varphi_i, \varphi_n]$ with $t_J(\varphi^+) = t_i(\varphi^+)$. Further, due to equations (7.2) and (7.4), $t_J(\varphi_i) \le t_{i-1}(\varphi_i) = t_i(\varphi_i)$. The last equations imply $0 \le (t_J - t_i)'(\varphi^+)$. We may use Lemma 5.1 to show $t_J(\varphi) \le t_i(\varphi)$ for $0 \le \varphi \le \varphi^+$. From equation (7.4) and from $\varphi_J^- < \varphi_i < \varphi^+$ there follows

$$t_0(\varphi_J^-) = t_J(\varphi_J^-) \le t_i(\varphi_J^-).$$

We conclude that there is a number φ_i^- fulfilling $\varphi_J^- \le \varphi_i^- \le \varphi_i$ with $t_0(\varphi_i^-) = t_i(\varphi_i^-)$. It is not hard to see that $t_i(\varphi) \le t_0(\varphi)$ for $\varphi_i^- \le \varphi_i \le \varphi \le \varphi_n$ and hence we proved (7.4).

Second Step: For $i = 1, ..., n-1$ set $j = j_i$ and define u_j by

$$u_j = a_j \cos(\varphi - d_j) \tag{7.5}$$

where a_j and d_j are chosen in such a way that the interface conditions

$$\begin{aligned} t_0(\varphi_j^-) &= u_j(\varphi_j^-) \\ k_0 t_0'(\varphi_j^-) &= \delta^2 k_0 u_j(\varphi_j^-) \end{aligned} \tag{7.6}$$

are fulfilled for $\varphi_j^- \in [0, \varphi_n]$. Remember $k_0 = 0$. Since $u_j(\varphi_j^-) = t_0(\varphi_j^-) = t_j(\varphi_j^-)$ and $\delta^2 < 1$ we conclude with the help of Lemma 5.1 that $u_j(\varphi) \le t_j(\varphi)$ for $\varphi_j^- \le \varphi_j \le \varphi \le \varphi_n \le \frac{\pi}{2}$. This yields together with equation (7.4)

$$u_j(\varphi) \le t_j(\varphi) \le t_i(\varphi) \qquad (\varphi_i \le \varphi \le \varphi_n). \tag{7.7}$$

Third Step: We want to show $0 < u_j(\varphi)$ for $\varphi \in [0, \varphi_n)$ and $i = 0, ..., n-1$ by showing that $d_j \in [\varphi_n - \frac{\pi}{2}, 0)$. Therefore, we choose $\varphi = \varphi_j^-$ and $d := d_j$ and rewrite (7.6) as

$$\begin{aligned} \cos(\varphi) &= a_j \cos(\varphi - d) \\ k_0 \sin(\varphi) &= \delta^2 k_0 a_j \sin(\varphi - d). \end{aligned} \tag{7.8}$$

Now we look for the minimum value of d depending on φ. Dividing the two equations (7.8) by each other we obtain

$$d(\varphi) = \varphi - \arctan(\delta^{-2} \tan \varphi). \tag{7.9}$$

Differentiating with respect to φ reveals that minimum is attained for $\tan \varphi_{\min} = \delta$. Inserting the minimum into (7.9) we see that the minimum value of $d_j \in [-\frac{\pi}{2}, 0)$ is

$$d(\varphi_{\min}) = \arctan \delta - \arctan \delta^{-1} = 2(\arctan \delta) - \tfrac{\pi}{2} = \varphi_n - \tfrac{\pi}{2}.$$

Here we used the trigonometric relation $\arctan x + \arctan x^{-1} = \frac{\pi}{2}$. Thus we finished the proof of the third step.

Figure 9

The black functions $a' \cos(\varphi - d(\varphi')), a^\star \cos(\varphi - d(\varphi_{\min}))$ fulfil interface conditions with the same jump of coefficients at points $\varphi', \varphi_{\min} = \arctan \delta$; they vanish at $d(\varphi') + \frac{\pi}{2}, d(\varphi_{\min}) + \frac{\pi}{2}$; we show $\min_{\varphi'}(d(\varphi') + \frac{\pi}{2}) = d(\varphi_{\min}) + \frac{\pi}{2} = \varphi_n = 2\varphi_{\min}$

Now we collect the results from the previous three steps to obtain from inequality (7.7)

$$0 < u_j(\varphi) \leq t_j(\varphi) \leq t_i(\varphi) \qquad (\varphi_i \leq \varphi \leq \varphi_{i+1};\, i = 1, ..., n-1).$$

This shows the assertion for the case $l = 0$.

If there is an index $l > 0$ such that $k_0 < k_1 < ... < k_l$, we use Lemma 5.2 to prove that the functions $t_i \ \ (i = 0, ..., l)$ do not vanish on $[\varphi_0, \varphi_n]$. It remains to prove the assertion for functions $t_i \ \ (i > l)$. From the relation $k_0 < k_1 < ... < k_l$ there follows $0 = c_0 < c_1 < ... < c_l$. We shift the functions $t_i \ \ (i > l)$ to the left by $c_l > 0$ and prove the assertion for the shifted functions as in the case $l = 0$ ∎

7.2 A "worst case" regularity result. We use Lemma 7.1 to derive bounds for the eigenvalues of the associated Sturm-Liouville eigenvalue problem. Comparing these bounds with the function defined in Example 1 we can show that our bounds are sharp. The main result in this section is Theorem 7.3.

Remark 7.1. The singular function defined in Example 1 fulfils the conditions

$$\delta = \tan \frac{\lambda\pi}{4} \leq k \leq \tan(\frac{\lambda\pi}{4})^{-1} \qquad \text{for any number } 0 < \lambda < 1. \tag{7.10}$$

Recall that λ^2 is an eigenvalue of the associated Sturm-Liouville eigenvalue problem. Rewriting (7.10) we get $\lambda = \frac{4}{\pi} \arctan \delta$.

Theorem 7.2. *Let a heterogeneous singular point $x \in \overline{\Omega}$ be given and let $\delta \leq k_i \leq \delta^{-1} \ \ (i = 0, ..., n-1)$. Define θ as the interior angle of Ω at x (if $x \in \Omega$, set $\theta = 2\pi$) and*

$$m = \begin{cases} \frac{1}{2} & \textit{if } x \in \Omega \\ 1 & \textit{if } x \in \partial\Omega \textit{ and the boundary conditions do not change in } x \\ 2 & \textit{if } x \in \partial\Omega \textit{ and the boundary conditions change in } x. \end{cases}$$

For the smallest non-vanishing eigenvalue of the associated Sturm-Liouville eigenvalue problem λ^2 it holds

$$\left(\tfrac{2\delta}{m\theta}\right) < \left(\tfrac{4}{m\theta}\arctan\delta\right)^2 \leq \lambda^2. \tag{7.11}$$

The bound to the right is sharp with respect to δ, m and θ.

Proof. Dividing k by a δ^{-1} we may assume $\delta^2 \leq k_i \leq 1$. If x is an interior singular point, we conclude as in the proof of Theorem 5.3 that there are two points $\varphi_{\min}$ and $\varphi_{\max}$ where $s(\varphi)$ achieves an extremum and two zero points $\varphi_{zero,1}$ and $\varphi_{zero,2}$. It is easy to see that we can order these points like $\varphi_{\min} < \varphi_{zero,1} < \varphi_{\max} < \varphi_{zero,2}$. If x is an interior heterogeneous singular point, we are free in the choice of $\varphi_{ex} \in \{\varphi_{\min}, \varphi_{\max}\}$ and $\varphi_{zero} \in \{\varphi_{zero,1}, \varphi_{zero,2}\}$, and we may additionally assume $|\varphi_{ex} - \varphi_{zero}| \leq \frac{\pi}{2}$.

If x is a boundary heterogeneous singular point, we may assume there is an extremum and a zero point of $s(\varphi)$ such that $|\varphi_{ex} - \varphi_{zero}| \leq \frac{\theta}{2}$ if the boundary conditions do not change in x, and $|\varphi_{ex} - \varphi_{zero}| \leq \theta$ if they do. For the sake of brevity we continue the proof in the case of an interior singular point.

We choose j such that $\varphi_{ex} \in [\varphi_j, \varphi_{j+1})$ and show as in the proof of Theorem 5.3 that $c_j = \varphi_{ex}$. Further, we choose the maximum n such that $\varphi_{j+1} < ... < \varphi_n \leq \varphi_{zero}$. Changing the coordinate system we may set $\varphi_{ex} = 0 < \varphi_{zero} < \frac{\pi}{2}$.

We introduce the homogeneous scaling $F : [0, \varphi_{zero}] \to [0, \widehat{\varphi}_{zero}]$ with $F(\varphi) = \widehat{\varphi} = \lambda\varphi$. Define $s_F(F(\varphi)) = s(\varphi)$ for $\varphi \in [\varphi_{ex}, \varphi_{zero}]$. Then $\widehat{\varphi}_{zero} \leq \lambda\frac{\pi}{2}$ holds. Observe that $s_F(\widehat{\varphi})$ fulfils the assumption of Lemma 7.1 as $\frac{k_i}{k_0} \geq k_i \geq \delta^2$ for all $i = 0, ..., n-1$. Since s_F vanishes in $\widehat{\varphi}_{zero}$ we conclude from Lemma 7.1 that

$$2\arctan\delta \leq \widehat{\varphi}_{zero} \leq \lambda\frac{\pi}{2}. \tag{7.12}$$

The inequality $c < 2\arctan c$ for any $0 < c < 1$ is checked easily. This shows assertion (7.11). Sharpness follows from Remark 7.1.

The case $x \in \partial\Omega$ is proved similarly. Here one has to modify the singular function defined in Example 1 by restricting the domain of definition and applying a suitable affine transformation ∎

According to Theorem 7.2 we are now able to give a regularity result which will depend on the bounds of k.

Theorem 7.3. *Let $\delta < k(x) < \delta^{-1}$ for all $x \in \Omega$ and some number $0 < \delta < 1$. Then the solution of problem* (2.1) *has regularity*

$$u \in H^{1+\frac{\delta}{2\pi}}(\Omega).$$

Let a heterogeneous singular point $x \in \overline{\Omega}$ be given and let $c\nu \leq k_i \leq c\nu^{-1}$ $(i = 0, ..., n-1)$ for some constants $c > 0$ and $\nu > 0$. Denote by U a neighborhood containing no other singular point. With the notation of Theorem 7.2

$$u \in H^{1+\frac{4\arctan\nu}{m\theta}-\varepsilon}(U \cap \Omega_i) \subset H^{1+\max(1,\frac{2\nu}{m\theta})}(U \cap \Omega_i) \quad (i = 0, ..., n-1)$$

holds where $\varepsilon > 0$ is arbitrary. This is the maximum regularity with respect to δ, ν and θ independent of the number and interior angles of the subdomains.

Proof. The assertion follows with Corollary 4.2 from Theorem 7.2 ∎

An easy consequence of the above theorem is

Corollary 7.4. *Let $\delta > 0$ be given. The singular function ψ_1 defined in Example* 1 *with $\lambda = 4\pi^{-1}\arctan\delta$ is the function with lowest regularity among all singular functions for interior heterogeneous singular points under the restriction $\delta \le k \le \delta^{-1}$ and with no other restrictions on the geometry (that means there are no restrictions imposed on the number and interior angles of the subdomains).*

7.3 Regularity between the quasi-monotone and the "worst case". The question arises about regularity for coefficients that are not quasi-monotonically distributed but which have no checkerboard-like pattern as in the "worst case". In this context it seems naturally to expect that a slight perturbation of quasi-monotonically distributed coefficients will not result in large changes of the regularity. These questions will be answered in the next theorem.

Theorem 7.5. *Let a heterogeneous singular point $x \in \Omega$ be given. We assume that $k_x(\varphi)$ has more then one local maximum. Denote by $k_{\max,1}$ and $k_{\max,2}$ the two largest local maxima and let $k_{\max,1} \ge k_{\max,2}$. Further, let $k_{\min,1}$ and $k_{\min,2}$ be the two smallest local minima where $k_{\min,1} \le k_{\min,2}$. Denote by U a neighborhood containing no other singular points and define $\delta' = \sqrt{k_{\min,2}/k_{\max,2}}$. Then the solution of problem* (2.1) *has regularity*

$$u \in H^{1+\frac{\arctan\delta'}{\pi}}(U \cap \Omega_i) \subset H^{1+\frac{\delta'}{2\pi}}(U \cap \Omega_i) \quad (i = 0, ..., n-1).$$

If quasi-monotonicity is violated only "a little", that means in the case that $\frac{k_{\max,2}}{k_{\min,2}}$ is close to 1, regularity will not differ much from $H^{1+\frac{1}{4}}$-regularity in the quasi-monotone case. In the "worst case" Example 1 it holds $k_{\max,1} = k_{\max,2}$ and $k_{\min,1} = k_{\min,2}$. Accordingly, $\delta = \delta'$ and we note that regularity implied by Theorem 7.3 differs only by the constant 4 from the "worst case" result from Theorem 7.3. Hence, we can interpret Theorem 7.5 as a link between the theory of robust regularity results for quasi-monotonically distributed coefficients and results for the "worst case". Moreover, the theorem provides *regularity results independent of δ.* In the case of pure Dirichlet conditions Theorem 7.5 is valid with $\delta' = \sqrt{k_{\min,2}/k_{\max,1}}$, and in the case of pure Neumann conditions with $\delta' = \sqrt{k_{\min,1}/k_{\max,2}}$.

Proof of Theorem 7.5. The proof combines ideas from the proof of Theorems 7.3. and 5.3. It can be found in [17] ∎

7.4 $W^{2,p}$-regularity. Using the bounds of the eigenvalues in Theorems 5.3, 5.4 and 7.2 it is straightforward to formulate regularity results in Sobolev spaces $W^{2,p}$ for $p \in (1,2)$. Calculation shows that the singular function $r^\lambda s_\lambda(\varphi)$ belongs piecewise to $W^{2,p}$ for $p < \frac{2}{2-\lambda}$.

Corollary 7.6. *Denote by u the solution of problem* (2.1)*. Let a singular point $x \in \overline{\Omega}$ with neighborhood U containing no other singular point be given. If the distribution of coefficients k_i $(i = 0, ..., n-1)$ is quasi-monotone with respect to x, then u has regularity*

$$u \in W^{2,1+\frac{1}{7}}(\Omega_i \cap U) \quad (i = 0, ..., n-1).$$

If for a $\delta > 0$ *there holds* $\delta \leq k_i \leq \delta^{-1}$ $(i = 0, ..., n-1)$, *then* u *has regularity*

$$u \in W^{2,1+\frac{\delta}{4\pi}}(\Omega_i \cap U) \quad (i = 0, ..., n-1).$$

Proof. The result for the quasi-monotone case follows from Theorems 5.3 and 5.4 and [15: Corollary 2.28]. We check $\frac{2}{2-\frac{1}{4}} = 1 + \frac{1}{7}$. Regularity for the general case follows from Theorem 7.2 and [15: Corollary 2.28]. Here we use the inequality $\frac{2}{2-\frac{\delta}{2\pi}} > 1 + \frac{\delta}{4\pi}$ together with the embedding $W^{2,p} \subset W^{2,q}$ for $1 \leq q < p$ ∎

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