Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 20 (2001), No. 2, 489–504

On Oscillation of Equations with Distributed Delay

L. Berezansky and E. Braverman

Abstract. For the scalar delay differential equation with a distributed delay

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\, d_s R(t,s) = f(t) \qquad (t > t_0)$$

a connection between the properties

non-oscillation
positiveness of the fundamental function
existence of a non-negative solution for a certain nonlinear integral inequality

is established. This enables to obtain comparison theorems and explicit non-oscillation and oscillation conditions being generalizations of some known results for delay equations and integro-differential equations and leads to oscillation results for equations with infinite number of delays.

Keywords: *Oscillation, non-oscillation, distributed delay, comparison theorems*

AMS subject classification: 34K15

1. Introduction

Delay differential equations occur in many models of mechanics, technology, economics, biology and medicine (see [15: Chapter 2]). Oscillation of such equations is an intensively developing field (see, for example, monographs [9, 10, 13, 16] and references therein).

We consider an equation with a distributed delay

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\, d_s R(t,s) = f(t) \qquad (t > t_0) \tag{1}$$

with the initial function

$$x(t) = \varphi(t) \qquad (t < t_0). \tag{2}$$

L. Berezansky: Ben-Gurion Univ. of the Negev, Dept. Math. & Comp. Sci., Beer-Sheva 84105, Israel. Supported by Israel Ministry of Absorption
E. Braverman: Technion – Israel Inst. Techn., Dept. Comp. Sci., Haifa 32000, Israel. Supported by Israel Ministry of Absorption

ISSN 0232-2064 / $ 2.50

Equation (1) includes the following ones as special cases:

1) The delay differential equation

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x(h_k(t)) = f(t) \tag{3}$$

if we assume

$$R_1(t,s) = \sum_{k=1}^{m} A_k(t)\chi_{[h_k(t),\infty)}(s) \tag{4}$$

where $\chi_{[a,b]}$ is the characteristic function of the segment $[a,b]$.

2) The integro-differential equation

$$\dot{x}(t) + \int_{-\infty}^{t} K(t,s)x(s)\,ds = f(t) \tag{5}$$

when

$$R_2(t,s) = \int_{-\infty}^{s} K(t,\zeta)\,d\zeta. \tag{6}$$

3) The mixed equation

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x(h_k(t)) + \int_{-\infty}^{t} K(t,s)x(s)\,ds = f(t) \tag{7}$$

where $R_3(t,s) = R_1(t,s) + R_2(t,s)$, with R_1 and R_2 defined by (4) and (6), respectively.

4) The mixed equation with infinite number of delays

$$\dot{x}(t) + \sum_{k=1}^{\infty} A_k(t)x(h_k(t)) + \int_{-\infty}^{t} K(t,s)x(s)\,ds = f(t) \tag{8}$$

which is obtained if

$$R_4(t,s) = \sum_{k=1}^{\infty} A_k(t)\chi_{[h_k(t),\infty)}(s) + \int_{-\infty}^{s} K(t,\zeta)\,d\zeta. \tag{9}$$

The latter equation is the most general among (3),(5),(7),(8). However, if $R(t,\cdot)$ contains a component which is continuous but not absolutely continuous, then equation (1) turns into an equation which differs from (8).

Thus, once the oscillation of equation (1) has been studied, relevant results for equations (3), (5), (7), (8) can be obtained. Oscillation properties of an equation which is equivalent to equation (1) were investigated in [16] (see also references therein). We generalize the results of [16] in the following directions:

1. In [16] an additional integral condition on $R(t,s)$ is imposed, which lead to continuous coefficients A_k and kernels K in (3) - (8). We deal with measurable essentially

bounded (locally integrable) functions. Many applied problems lead to equations with discontinuous coefficients. In addition, this is important for mathematical applications. For example, in [14] it was demonstrated that oscillation properties of a difference equation can be derived from oscillation properties of some delay differential equation with discontinuous delays. As was shown in [2, 3], we can study oscillation of non-impulsive delay equations with discontinuous coefficients rather than oscillation of an impulsive delay equation.

2. Explicit oscillation (non-oscillation) conditions in [16] are obtained in the case when $R(t,s)$ is non-decreasing in s for each t (this corresponds to positive coefficients in (3) - (8)). Sometimes we succeed in avoiding such a constraint.

Among numerous publications on oscillation of equations with deviating argument we mention here [11, 12] which are concerned with a distributed delay.

Our approach to oscillation problems follows one employed in [4, 5]. The main result is that under some natural assumptions the following four assertions are equivalent:

- non-oscillation of the equation and the corresponding differential inequality
- positiveness of the fundamental function and existence of a non-negative solution of a certain nonlinear integral inequality, which is constructed explicitly from the differential equation.

The paper is organized as follows. Section 2 includes relevant definitions and notations. In Section 3 the equivalence of the four abovementioned properties is established. Section 4 deals with comparison results. Comparison theorems appeared to be an efficient tool in oscillation theory (see [9, 10, 13 - 16]). In Section 5 explicit non-oscillation and oscillation conditions are presented. Here we obtain some known results from [4, 5, 16] for a more general class of equations. Section 6 contains a sufficient oscillation condition which uses existence of a slowly oscillating solution.

2. Preliminaries

We study problem (1) - (2) under the following assumptions:

(a1) $R(t,\cdot)$ is a left continuous function of bounded variation, and for each s its variation on the segment $[t_0, s]$

$$P(t,s) = var_{[t_0,s]} R(t,\cdot) \tag{10}$$

is a locally integrable function in t.

(a2) $R(t,s) = R(t,t+)$ $(t < s)$.

(a3) $f : [t_0,\infty) \to \mathbb{R}$ is a Lebesgue measurable locally essentially bounded function and $\varphi : (-\infty, t_0) \to \mathbb{R}$ is a Lebesgue measurable bounded function.

Definition. A function $x : \mathbb{R} \to \mathbb{R}$ is called a *solution* of problem (1) - (2) if it satisfies equation (1) for almost every $t \in [t_0,\infty)$ and (2) holds for $t < t_0$.

Definition. For each $s \geq t_0$ denote by $X(t,s)$ the solution of the problem

$$\left.\begin{aligned} \dot{x}(t) + \textstyle\int_{-\infty}^{t} x(s)\,d_sR(t,s) &= 0 \qquad && (t \geq s) \\ x(t) &= 0 && (t < s) \\ x(s) &= 1 && \end{aligned}\right\} \tag{11}$$

called a *fundamental function* of equation (1). We assume $X(t,s) = 0$ for $0 \leq t < s$.

Consider an initial value problem for (1) - (2), i.e a solution satisfying additionally

$$x(t_0) = x_0 \tag{12}$$

for some initial value x_0.

Lemma 1 (see [1: Theorem 5.1.1]). *Let assumptions* (a1) - (a3) *hold. Then there exists one and only one solution of problem* (1), (2), (12) *that can be presented in the form*

$$x(t) = X(t,t_0)x_0 + \int_{t_0}^{t} X(t,s)f(s)\,ds - \int_{t_0}^{t} X(t,s)\,ds \int_{-\infty}^{s} \varphi(\tau)\,d_\tau R(s,\tau) \tag{13}$$

where $\varphi(\tau) = 0$ *if* $\tau > t_0$.

3. Existence of a positive solution

We study problem (1) - (2) with a bounded aftereffect, i.e. the following hypothesis holds:

- **(a4)** For each t_1 there exists $s_1 = s(t_1) \leq t_1$ such that $R(t,s) = 0$ for $s < s_1$ and $t > t_1$ and $\lim_{t\to\infty} s(t) = \infty$.

If assumption (a4) holds, we can introduce the function

$$h(t) = \inf\{s \mid R(t,s) \neq 0\} \tag{14}$$

such that $\lim_{t\to\infty} h(t) = \infty$.

Together with equation (1) let us consider the homogeneous equation

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\,d_sR(t,s) = 0 \qquad (t \geq t_0) \tag{15}$$

and the differential inequality

$$\dot{y}(t) + \int_{-\infty}^{t} y(s)\,d_sR(t,s) \leq 0 \qquad (t \geq t_0). \tag{16}$$

Let us study the existence of an eventually positive solution of problem (15), (2).

Definition. Equation (15) has a *positive solution* x for $t \geq t_1 \geq t_0$ if there exist an initial function φ and an initial value x_0 such that x is solution of problem (15),(2),(12) which is positive for $t \geq t_1$.

Theorem 1. *Suppose* $R(t,\cdot)$ *is a non-decreasing function for each* t *and assumptions* (a1), (a2), (a4) *hold. Then the following hypotheses are equivalent:*

1) *There exists* $t_1 \geq t_0$ *such that* (16) *has a positive solution for* $t \geq t_1$.

2) *There exists* $t_2 \geq t_0$ *such that the inequality*

$$u(t) \geq \int_{h(t)}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_sR(t,s) \tag{17}$$

has a non-negative locally integrable solution u *for* $t \geq t_2$ *(in* (17) *we assume* $u(t)=0$ *for* $t < t_2$*).*

3) *There exists* $t_3 \geq 0$ *such that the fundamental function of* (1) $X(t,s) > 0$ *for* $t > s \geq t_3$.

4) *There exists* $t_4 \geq 0$ *such that* (15) *has a positive solution for* $t \geq t_4$.

Proof. Let us prove the implications

$$1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4) \Rightarrow 1).$$

1) $\Rightarrow$ **2)**. Let y be a positive solution of inequality (16) for $t \geq t_1$. Let t_2 be such a number that $R(t,s)=0$ if $s \leq t_1$ and $t \geq t_2$. Denote

$$u(t) = -\frac{d}{dt}\ln\frac{y(t)}{y(t_2)}, \qquad \text{i.e. } y(t) = y(t_2)\exp\left\{-\int_{t_2}^{t} u(s)\,ds\right\}$$

and assume $u(t)=0$ for $t<t_2$. By substituting y into (16) we obtain

$$-y(t_2)\exp\left\{-\int_{t_2}^{t} u(s)\,ds\right\}u(t) + y(t_2)\int_{-\infty}^{t}\exp\left\{-\int_{t_2}^{s} u(\tau)\,d\tau\right\} d_sR(t,s) \leq 0$$

which implies

$$y(t_2)\exp\left\{-\int_{t_2}^{t} u(s)\,ds\right\}\left[-u(t) + \int_{-\infty}^{t}\exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_sR(t,s)\right] \leq 0. \tag{18}$$

The first factor is positive since $y(t_2)>0$. Consequently, the expression in the brackets $[\ldots]$ is also positive. Hence

$$u(t) \geq \int_{-\infty}^{t}\exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_sR(t,s) \qquad (t \geq t_2). \tag{19}$$

Since $R(t,s)=0$ for $s<h(t)$, then (19) is equivalent to

$$u(t) \geq \int_{h(t)}^{t}\exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_sR(t,s) \qquad (t \geq t_2).$$

The last inequality implies (17), i.e. statement 2).

2) $\Rightarrow$ **3)**. As first step, let us prove that the fundamental function $X(s,t)$ of problem (11) is non-negative for $t \geq s \geq t_3$ where $t_3 \geq t_0$ is a certain number. Consider the initial value problem

$$\left.\begin{array}{rl} \dot{x}(t) + \int_{-\infty}^{t} x(s)\, d_s R(t,s) = f(t) & \qquad (t > t_2) \\ x(t) = 0 & \qquad (t \leq t_2) \end{array}\right\}. \tag{20}$$

Denote by z the function defined by $z(t) = \dot{x}(t) + u(t)x(t)$ where u is a non-negative solution of inequality (17) and x is the solution of problem (20). Thus

$$x(t) = \int_{t_2}^{t} \exp\left\{-\int_s^t u(\tau)\, d\tau\right\} z(s)\, ds \qquad (t \geq t_2). \tag{21}$$

After substituting x into (20) we obtain

$$\begin{aligned} z(t) - u(t) \int_{t_2}^{t} \exp\left\{-\int_s^t u(\tau)\, d\tau\right\} z(s)\, ds \\ + \int_{t_2}^{t} \left(\int_{t_2}^{s} \exp\left\{-\int_\theta^s u(\tau)\, d\tau\right\} z(\theta)\, d\theta \right) d_s R(t,s) = f(t). \end{aligned}$$

In the second integral (in s) the integrand vanishes for $s < t_2$. After changing in it the order of integration we have

$$\begin{aligned} z(t) - u(t) \int_{t_2}^{t} \exp\left\{-\int_s^t u(\tau)\, d\tau\right\} z(s)\, ds \\ + \int_{t_2}^{t} z(s)\, ds \int_s^t \exp\left\{-\int_s^\theta u(\tau)\, d\tau\right\} d_\theta R(t,\theta) = f(t). \end{aligned}$$

Thus the left-hand side is equal to

$$\begin{aligned} z(t) - u(t) \int_{t_2}^{t} \exp\left\{-\int_s^t u(\tau)\, d\tau\right\} z(s)\, ds \\ + \int_{t_2}^{t} z(s)\, ds \int_{t_2}^{t} \exp\left\{-\int_s^t u(\tau)\, d\tau\right\} \exp\left\{\int_\theta^t u(\tau)\, d\tau\right\} d_\theta R(t,\theta) \\ - \int_{t_2}^{t} z(s)\, ds \int_{t_2}^{s} \exp\left\{-\int_s^\theta u(\tau)\, d\tau\right\} d_\theta R(t,\theta) = \\ z(t) - \int_{t_2}^{t} \exp\left\{-\int_s^t u(\tau)\, d\tau\right\} z(s)\, ds \\ \times \left[u(t) - \int_{t_2}^{t} \exp\left\{\int_\theta^t u(\tau)\, d\tau\right\} d_\theta R(t,\theta) \right] \\ - \int_{t_2}^{t} z(s)\, ds \int_{t_2}^{s} \exp\left\{-\int_s^\theta u(\tau)\, d\tau\right\} d_\theta R(t,\theta) = f(t). \end{aligned}$$

Consequently, we obtain an operator equation

$$z - Hz = f \tag{22}$$

which is equivalent to problem (20) where

$$\begin{aligned}(Hz)(t) = \int_{t_2}^{t} &\exp\left\{-\int_{s}^{t} u(\tau)\,d\tau\right\} z(s)\,ds \\ &\times \left[u(t) - \int_{t_2}^{t} \exp\left\{\int_{\theta}^{t} u(\tau)\,d\tau\right\} d_\theta R(t,\theta)\right] \\ &+ \int_{t_2}^{t} z(s)\,ds \int_{t_2}^{s} \exp\left\{-\int_{s}^{\theta} u(\tau)\,d\tau\right\} d_\theta R(t,\theta).\end{aligned} \tag{23}$$

Inequality (17) yields that if $z(t) \geq 0$, then $(Hz)(t) \geq 0$, i.e. H is a positive operator ($t > t_2$). Besides, in each final interval $[t_2, b]$ H is a sum of Volterra integral operators, which are compact in the space of integrable functions. Hence [8: p. 519] its spectral radius $r(H) = 0 < 1$ and consequently, if in equation (22) the right-hand side f is non-negative, then

$$z(t) = f(t) + (Hf)(t) + (H^2 f)(t) + (H^3 f)(t) + \ldots \geq 0.$$

We recall that the solution of problem (20) has form (21), with z being a solution of equation (22).Thus if in (20) $f \geq 0$, then $x(t) \geq 0$. On the other hand, the solution of problem (20) has the representation

$$x(t) = \int_{t_2}^{t} X(t,s) f(s)\,ds.$$

As was demonstrated above, $f(t) \geq 0$ implies $x(t) \geq 0$. Hence the kernel of the integral operator is non-negative, i.e. $X(t,s) \geq 0$ for $t \geq s > t_2$.

As second step let us prove that the fundamental function $X(t,s)$ is strictly positive: $X(t,s) > 0$. To this end consider the function

$$\left.\begin{aligned} &x(t) = X(t,t_2) - \exp\{-\textstyle\int_{t_2}^{t} u(s)\,ds\} \\ &x(t) = 0 \quad (t < t_2) \end{aligned}\right\}$$

and substitute x into the left-hand side of (20):

$$\begin{aligned} &X_t'(t,t_2) + u(t)\exp\left\{-\int_{t_2}^{t} u(s)\,ds\right\} \\ &\quad + \int_{t_2}^{t} X(s,t_2)\,d_s R(t,s) - \int_{t_2}^{t} \exp\left\{-\int_{t_2}^{s} u(\tau)\,d\tau\right\} d_s R(t,s) \\ &= 0 + \exp\left\{-\int_{t_2}^{t} u(s)\,ds\right\}\left[u(t) - \int_{t_2}^{t} \exp\left\{\int_{s}^{t} u(s)\,ds\right\} d_s R(t,s)\right] \\ &\geq 0. \end{aligned}$$

Therefore x is a solution of problem (20) with non-negative right-hand side. As shown above, $x(t) \geq 0$, consequently,

$$X(t, t_2) \geq \exp\left\{ -\int_{t_2}^{t} u(s)\, ds \right\} > 0.$$

For any $s > t_2$ the inequality $X(t,s) > 0$ can be verified in a similar way.

3) $\Rightarrow$ **4)**. The function $x(t) = X(t, t_2)$ is a positive solution of equation (1) for $t \geq t_2$. The implication **4)** $\Rightarrow$ **1)** is obvious ∎

Remark. The lower bound of the integrals in the proof is constant (t_2); however, due to the boundedness of the delay, in many cases it can be changed by $h(t)$, where t is an upper bound, for t being large enough.

Corollary 1.1. *Suppose $A_k \geq 0$ $(k = 1, \ldots, m)$, $K(t,s) \geq 0$ and the following conditions are satisfied:*

(b1) *$K(t,s)$ is Lebesgue integrable over each finite square $[t_0, b] \times [t_0, b]$,*

$$a(t) = \sum_{k=1}^{\infty} A_k(t) \chi_{[h_k(t), \infty)}(t)$$

is a locally essentially bounded function (for a finite number of delays h_k each A_k is assumed to be locally essentially bounded).

(b2) *$K(t,s) = 0$ $(s < t)$ and $h_k(t)$ are Lebesgue measurable functions, $h_k(t) \leq t$ $(k \in \mathbb{N})$.*

(b3) *$\lim_{t\to\infty} \inf_k h_k(t) = \infty$, and there exists a function h_0 with $\lim_{t\to\infty} h_0(t) = \infty$ such that $K(t,s) = 0$ if $s < h_0(t)$.*

Then the following hypotheses are equivalent:

1) *There exists $t_1 \geq t_0$ such that the inequality*

$$\dot{x}(t) + \sum_{k=1}^{\infty} A_k(t) x(h_k(t)) + \int_{h_0(t)}^{t} K(t,s) x(s)\, ds \leq 0 \tag{24}$$

has a positive solution for $t \geq t_1$.

2) *There exists $t_2 \geq t_0$ such that the inequality*

$$u(t) \geq \sum_{k=1}^{\infty} A_k(t) \exp\left\{ \int_{h_k(t)}^{t} u(s)\, ds \right\} + \int_{h_0(t)}^{t} K(t,s) \exp\left\{ \int_{s}^{t} u(\tau)\, d\tau \right\} ds \tag{25}$$

has a non-negative locally integrable solution u for $t \geq t_2$ (in inequality (25) it is assumed that $u(t) = 0$ if $t < t_2$).

3) *There exists $t_3 \geq t_0$ such that the fundamental function of equation (8) is positive for $t > s \geq t_3$.*

4) *There exists $t_4 \geq t_0$ such that equation (8) with $f \equiv 0$ has a positive solution for $t \geq t_4$.*

Remark. Theorem 1 in [5] is a partial case of Theorem 1 for equation (7).

4. Comparison Theorems

Consider together with equation (15) the following one:

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\, d_s T(t,s) = 0 \qquad (t \geq t_0). \tag{26}$$

We compare the properties of equations (15) and (26) concerning the existence of a non-negative solution.

Theorem 2. *Suppose R and T satisfy assumptions* (a1), (a2), (a4), *and the functions $R(t,\cdot)$ and the differences $R(t,\cdot)-T(t,\cdot)$ are non-decreasing for each t. If inequality* (17) *has a positive solution for $t \geq t_2$, then equation* (26) *has a positive solution for $t \geq t_2$ and its fundamental function $Y(t,s)$ is positive for $t \geq s \geq t_2$.*

Proof. Consider the initial value problem

$$\left.\begin{array}{lr} \dot{x}(t) + \int_{-\infty}^{t} x(s)\, d_s T(t,s) = f(t) & (t \geq t_2) \\ x(t) = 0 & (t \leq t_2) \end{array}\right\}. \tag{27}$$

Let us demonstrate that $f(t) \geq 0$ implies $x(t) \geq 0$ where x is a solution of problem (27). To this end rewrite (27) as

$$\left.\begin{array}{lr} \dot{x}(t) + \int_{-\infty}^{t} x(s)\, d_s R(t,s) + \int_{-\infty}^{t} x(s)\, d_s [T(t,s) - R(t,s)] = f(t) & (t \geq t_2) \\ x(t) = 0 & (t \leq t_2) \end{array}\right\}. \tag{28}$$

After substituting $x(t) = \int_{t_2}^{t} X(t,s) z(s)\, ds$ into (28) where $X(t,s)$ is a fundamental function of equation (1), we obtain the equation

$$z(t) - \int_{t_2}^{t} \left[\int_{t_2}^{s} X(s,\tau) z(\tau)\, d\tau \right] d_s [R(t,s) - T(t,s)] = f(t)$$

which after changing the order of integration becomes

$$z(t) - \int_{t_2}^{t} z(s)\, ds \int_{s}^{t} X(\tau,s)\, d_\tau [R(t,\tau) - T(t,\tau)] = f(t).$$

Thus problem (28) can be rewritten in the form

$$z - Hz = f \tag{29}$$

where

$$(Hz)(t) = \int_{t_2}^{t} z(s)\, ds \int_{s}^{t} X(\tau,s)\, d_\tau [R(t,\tau) - T(t,\tau)]. \tag{30}$$

Since $X(t,s)$ is a fundamental function of equation (1), it satisfies for any $b \geq t_1$ the estimate

$$|X(t,s)| \leq \exp\left\{ \int_s^b P(\tau,\tau)\, d\tau \right\} \qquad (t_2 \leq s \leq t < b) \tag{31}$$

(see [17]) where

$$P(t,s) = \mathrm{var}_{[t_1,s]} R(t,\cdot). \tag{32}$$

Consequently, the Volterra integral operator H is a compact one in the space of functions integrable in $[t_2,b]$. Then for its spectral radius $r(H)$ we have $r(H) = 0 < 1$. By Theorem 1 $X(t,s) > 0 \ \ (t \geq s \geq t_1)$, hence the operator H is positive. Then the solution of equation (29) can be presented as

$$z(t) = f(t) + (Hf)(t) + (H^2 f)(t) + \ldots \geq 0$$

if $f(t) \geq 0$. Then, as in the proof of Theorem 1, we conclude $Y(t,s) > 0 \ \ (t \geq s \geq t_1)$ and, consequently, $x(t) = Y(t,t_2)$ is a positive solution of equation (26), which completes the proof ∎

Corollary 2.1. *Let R and T satisfy assumptions* (a1), (a2), (a4). *Then:*

1. *If the function $R(t,\cdot)$ and the difference $R(t,\cdot) - T(t,\cdot)$ are non-decreasing for each t and equation* (15) *has a non-oscillatory solution, then equation* (26) *also has a non-oscillatory solution.*

2. *If the function $R(t,\cdot)$ and the difference $T(t,\cdot) - R(t,\cdot)$ are non-decreasing for each t and all the solutions of equation* (26) *are oscillatory, then all the solutions of equation* (15) *are also oscillatory.*

Corollary 2.2. *Consider the integro-differential equation*

$$\dot{x}(t) + \sum_{k=1}^{\infty} B_k(t) x(g_k(t)) + \int_{-\infty}^{t} M(t,s) x(s)\, ds = 0. \tag{33}$$

Let assumptions (b1) - (b3) *be satisfied for equations* (8), (33) *and $K(t,s) \geq 0$, $K(t,s) \geq M(t,s)$, $A_k \geq B_k$, $h_k(t) \leq g_k(t)$. Then:*

1. *If the equation*

$$\dot{x}(t) + \sum_{k=1}^{\infty} A_k(t) x(h_k(t)) + \int_{-\infty}^{t} K(t,s) x(s)\, ds = 0 \tag{34}$$

has an eventually non-oscillatory solution, then equation (33) *also has an eventually non-oscillatory solution.*

2. *If all the solutions of equation* (33) *are oscillatory, then all the solutions of equation* (34) *are also oscillatory.*

Any function of bounded variation $R(t,s)$ can be presented as a difference of two increasing functions (in s, for each t):

$$R(t,s) = P(t,s) - Q(t,s). \tag{35}$$

Corollary 2.3. *Suppose assumptions* (a1), (a2), (a4) *hold. Then:*

a) *If the inequality*

$$\dot{x}(t) + \int_{h(t)}^{t} x(s)\, d_s P(t,s) \le 0 \tag{36}$$

has an eventually positive solution, then equation (15) *has an eventually non-oscillatory solution.*

b) *If the inequality*

$$u(t) \ge \int_{h(t)}^{t} \exp\left\{ \int_{s}^{t} u(\tau)\, d\tau \right\} d_s P(t,s) \tag{37}$$

has a non-negative locally integrable solution u for all $t \ge t_2$, then equation (15) *has a positive solution for $t \ge t_2$ and for its fundamental function we have $X(t,s) > 0$ $(t \ge s \ge t_2)$.*

Proof. Let us compare the solutions x of equation (15) and the equation

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\, d_s P(t,s) = 0 \qquad (t \ge t_0).$$

Theorem 2 yields that there exists a positive solution of (15) since $Q(t,s) = P(t,s) - R(t,s)$ is non-decreasing in s for each t, and all the hypotheses of Theorem 1 are satisfied for equation (15) as well ∎

Theorem 2 generalizes comparison Theorems 2 and 3 in [4, 5]. It compares oscillation properties of two equations.

Now let us compare solutions of two problems

$$\left.\begin{aligned} \dot{x}(t) + \textstyle\int_{-\infty}^{t} x(s)\, d_s R(t,s) &= f(t) \\ x(t) &= \varphi(t) \quad (t \le t_1) \\ x(t_1) &= x_0 \end{aligned}\right\} \tag{38}$$

and

$$\left.\begin{aligned} \dot{y}(t) + \textstyle\int_{-\infty}^{t} y(s)\, d_s T(t,s) &= g(t) \\ y(t) &= \psi(t) \quad (t \le t_1) \\ y(t_1) &= y_0 \end{aligned}\right\}. \tag{39}$$

Denote by $Y(t,s)$ a fundamental function of (39) (we recall that $X(t,s)$ is a fundamental function of (38)).

Theorem 3. *Let the parameters of equations* (38) – (39) *satisfy assumptions* (a1) - (a4). *Suppose there exists a non-negative solution of inequality* (17) *for $t \ge t_2$, the functions $R(t,\cdot), T(t,\cdot)$ and the difference $R(t,\cdot) - T(t,\cdot)$ are non-decreasing for each $t \ge t_1$, and*

$$\left.\begin{aligned} g(t) &\ge f(t) \\ \varphi(t) &\ge \psi(t) \end{aligned}\right\} \ (t \le t_2) \qquad \textit{and} \qquad y_0 \ge x_0. \tag{40}$$

Then $y(t) \ge x(t) > 0$ where x and y are solutions of problems (38) *and* (39), *respectively.*

Proof. Since the difference $R(t,\cdot) - T(t,\cdot)$ is non-decreasing in s for each t, then

$$\begin{aligned}
u(t) &\ge \int_{t_2}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s R(t,s) \\
&= \int_{t_2}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s\big(R(t,s) - T(t,s) + T(t,s)\big) \\
&= \int_{t_2}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s\big(R(t,s) - T(t,s)\big) + \int_{t_2}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s T(t,s) \\
&\ge \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s T(t,s).
\end{aligned} \tag{41}$$

Thus by Theorem 1, $X(t,s) > 0$ and $Y(t,s) > 0$. Equation (38) can be rewritten as

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\,d_s T(t,s) = \int_{-\infty}^{t} x(s)\,d_s\big(T(t,s) - R(t,s)\big) + f(t) \qquad (t \ge t_2),$$

consequently

$$\begin{aligned}
x(t) &= Y(t,t_1)x_0 \\
&\quad - \int_{t_1}^{t} Y(t,s)\,ds \int_{-\infty}^{s} \varphi(\tau)\,d_\tau T(s,\tau) + \int_{t_2}^{t} Y(t,s) f(s)\,ds \qquad (t \ge t_2). \\
&\quad - \int_{t_1}^{t} Y(t,s)\,ds \int_{-\infty}^{s} x(\tau)\,d_\tau\big(R(s,\tau) - T(s,\tau)\big)
\end{aligned} \tag{42}$$

If equality (42) is compared with

$$y(t) = Y(t,t_1)y_0 - \int_{t_1}^{t} Y(t,s)\,ds \int_{-\infty}^{s} \psi(\tau)\,d_\tau T(s,\tau) + \int_{t_2}^{t} Y(t,s) g(s)\,ds,$$

one can observe $y(t) \ge x(t) \ge 0$ since (40) holds ∎

5. Explicit non-oscillation and oscillation conditions

In this section we use Theorems 1 and 2 for obtaining non-oscillation and osciallation results for equations (15) and (34).

Theorem 4. *If assumptions* (a1), (a2), (a4) *hold and*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} var_{s\in[h(\tau),\tau]} P(t,s)\,d\tau < \frac{1}{e} \tag{43}$$

where $h(t)$ is defined by (14) *and $P(t,s)$ by* (35), *then equation* (15) *has a non-oscillatory solution.*

Proof. Let $t_1 > t_0$ be such that

$$\int_{h(t)}^{t} \mathrm{var}_{s\in[h(\tau),\tau]} P(t,s)\, d\tau < \frac{1}{e} \qquad (t > t_1).$$

Let us choose

$$u(t) = \begin{cases} e\,\mathrm{var}_{s\in[h(t),t]} P(t,s) & \text{if } t > t_1 \\ 0 & \text{if } t \le t_1. \end{cases} \tag{44}$$

Suppose $t_2 > t_1$ is such that $R(t,s) = 0$ if $s \le t_1$ and $t > t_2$. Then for $t > t_2$

$$\begin{aligned} &\int_{t_2}^{t} \exp\left\{\int_s^t u(\tau)\, d\tau\right\} d_s P(t,s) \\ &\quad = \int_{h(t)}^{t} \exp\left\{e \int_s^t \mathrm{var}_{s\in[h(\tau),\tau]} P(\tau,s)\, d\tau\right\} d_s P(t,s) \\ &\quad \le \int_{h(t)}^{t} \exp\left\{e\frac{1}{e}\right\} d_s P(t,s) \\ &\quad \le e\,\mathrm{var}_{s\in[h(\tau),\tau]} P(t,s) \\ &\quad = u(t). \end{aligned}$$

By Corollary 2.3 equation (15) has an eventually positive solution ∎

Let

$$f^+(t) = \max\{f(t), 0\}$$
$$\tilde{h}(t) = \inf_{k\in\mathbb{N}} h_k(t)$$

where h_0 was defined in assumption (b3).

Corollary 4.1. *If assumptions* (b1) - (b3) *hold and*

$$\limsup_{t\to\infty} \int_{\tilde{h}(t)}^{t} \left[\sum_{k=1}^{\infty} A_k^+(\tau) + \int_{h_0(\tau)}^{\tau} K^+(\tau,s)\, ds\right] d\tau < \frac{1}{e}, \tag{45}$$

then equation (34) *has a non-oscillatory solution.*

Theorem 5. *If assumptions* (a1), (a2), (a4) *hold,* $R(t,s)$ *is non-decreasing in* s *for each* t *and*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} \mathrm{var}_{s\in[h(\tau),\tau]} R(t,s)\, d\tau > \frac{1}{e}, \tag{46}$$

then all the solutions of equation (15) *are oscillatory.*

Proof. Suppose there exists a non-oscillatory solution of equation (15). By Theorem 1 there exists a positive locally integrable function u that satisfies

$$u(t) \ge \int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)\, d\tau\right\} d_s R(t,s) \qquad (t > t_1). \tag{47}$$

Since $e^t \geq e\,t$ for positive t and $R(t,s)$ is non-decreasing in s, then

$$\begin{aligned} u(t) &\geq \int_{h(t)}^{t} e\left(\int_s^t u(\tau)\,d\tau\right) d_s R(t,s) \\ &= e\int_{h(t)}^{t} u(\tau)\,d\tau \left(\int_{h(\tau)}^{\tau} d_s R(t,s)\right) \\ &= e\int_{h(t)}^{t} u(\tau)\mathrm{var}_{s\in[h(\tau),\tau]} R(t,s)\,d\tau. \end{aligned}$$

Therefore

$$\begin{aligned} &\int_{h(t)}^{t} u(\tau)\mathrm{var}_{s\in[h(\tau),\tau]} R(t,s)\,d\tau \\ &\qquad \geq e\int_{h(t)}^{t} \mathrm{var}_{s\in[h(\tau),\tau]} R(t,s)\,d\tau \int_{h(\tau)}^{\tau} u(\eta)\mathrm{var}_{s\in[h(\eta),\eta]} R(\tau,s)\,d\eta. \end{aligned}$$

Consequently,

$$\begin{aligned} &\liminf_{t\to\infty}\left[\int_{h(t)}^{t} u(\tau)\mathrm{var}_{s\in[h(\tau),\tau]} R(t,s)\,d\tau\right] \\ &\quad \geq e\liminf_{t\to\infty}\left[\int_{h(t)}^{t} \mathrm{var}_{s\in[h(\tau),\tau]} R(t,s)\,d\tau \int_{h(\tau)}^{\tau} u(\eta)\mathrm{var}_{s\in[h(\eta),\eta]} R(\tau,s)\,d\eta\right] \\ &\quad \geq e\liminf_{t\to\infty}\left[\int_{h(t)}^{t} \mathrm{var}_{s\in[h(\tau),\tau]} R(t,s)\,d\tau\right] \\ &\qquad \times \liminf_{t\to\infty}\left[\int_{h(t)}^{t} u(\tau)\mathrm{var}_{s\in[h(\tau),\tau]} R(t,s)\,d\tau\right]. \end{aligned}$$

By comparing the left-hand and right-hand sides of the inequality we obtain

$$\liminf_{t\to\infty}\left[\int_{h(t)}^{t} \mathrm{var}_{s\in[h(\tau),\tau]} R(t,s)\,d\tau\right] \leq \frac{1}{e}$$

which contradicts to inequality (46) in the statement of the theorem ∎

Corollary 5.1. *If assumptions* (b1) - (b3) *hold and*

$$\liminf_{t\to\infty}\left[\sum_{k=1}^{\infty} A_k(\tau)(t-\bar{h}(t)) + \int_{\bar{h}(t)}^{t} K(t,s)(t-s)\,ds\right] > \frac{1}{e} \tag{48}$$

where $\bar{h}(t) = \sup_{k\in\mathbb{N}} h_k(t)$, *then all solutions of equation* (34) *are oscillatory.*

The results of this section generalize Theorems 2.9.1 - 2.9.2 in [16].

6. Slowly oscillating solutions

For ordinary linear differential equations of second order it is known that if an equation has an oscillation solution, then all its solutions are oscillating. As is well known, for delay differential equations this is not true. Y. Domshlak demonstrated that if an *associated* equation has a *slowly oscillating solution*, then every solution of equation (3) is oscillating. In [6, 7] several new explicit sufficient conditions of oscillation are obtained by an explicit construction of such slowly oscillating solutions. We present here a similar oscillation result for equation (15), only the existence of a slowly oscillating solution is assumed for equation (15) and not for the associated one.

For equation (1) with finite memory, i.e. satisfying assumption (a4), the following definition can be introduced.

Definition. A solution x of equation (15) is *slowly oscillating* if for each $t_1 \geq t_0$ there exist $t_3 > t_2 > t_1$ such that $R(t,s) = 0$ if $t > t_3$ and $s < t_2$, $x(t_2) = x(t_3) = 0$ and $x(t) > 0$ for $t \in (t_2, t_3)$.

For equation (8) this means that for every $t_1 \geq t_0$ there exist such t_2 and t_3 that $t_3 > t_2 > t_1$, $K(t,s) = 0$ for $t > t_3$ and $s < t_2$, $h_k(t) \geq t_2$ for $t > t_3$, $x(t_2) = x(t_3) = 0$ and $x(t) > 0$ for $t \in (t_2, t_3)$.

The following theorem is a more general case of the results obtained in [4] for equations of type (3).

Theorem 6. *Suppose $R(t,\cdot)$ is non-decreasing for each t. If there exists a slowly oscillating solution of equation* (15) *or inequality* (16)*, then all the solutions of* (15) *or* (16) *are oscillatory.*

Proof. Let x be a slowly oscillating solution of equation (15). Suppose there exists a non-oscillatory solution of the same equation. Then by Theorem 1 such t_1 can be found that $X(t,s) > 0$ for $t \geq s > t_1$. By definition of a slowly oscillating solution there exist t_2 and t_3 exceeding t_1 and satisfying

$$R(t,s) = 0 \text{ if } t > t_3 \text{ and } s < t_2,\ x(t_2) = x(t_3) = 0,\ x(t) > 0 \text{ for } t \in (t_2, t_3). \tag{49}$$

By Lemma 1 the solution x can be presented in the form (we treat t_3 as an initial point)

$$x(t) = -\int_{t_3}^{t} X(t,s)\, ds \int_{-\infty}^{s} x(\tau)\, d_\tau R(s,\tau) \tag{50}$$

where $x(\tau)$ under the second integral is assumed to be zero if $\tau > t_3$. Since in addition $R(t,s) = 0$ for $t > t_3$ and $s < t_2$, then the expression under the integral in the right-hand side of (50) can differ from zero only for $s \in (t_2, t_3)$. By (49) $x(t) \leq 0$ for each $t \geq t_3$ (since the right-hand side in (50) is negative for $t > t_3$). Thus x is an eventually non-oscillating solution, which contradicts the assumption that it is oscillatory. The proof for inequality (16) is similar ∎

Remark. The statement of Theorem 6 implies that if equation (15) with non-decreasing $R(t,\cdot)$ for each t has a non-oscillatory solution, then equation (15) has no slowly oscillating solutions.

Corollary 6.1. *If assumptions* (b1) - (b3) *hold,* $K(t, s)$ *and* $A_k(t)$ *are non-negative functions and there exists a slowly oscillating solution of equation* (34)*, then all solutions of equation* (34) *are oscillatory.*

References

[1] Azbelev, N. V., Maksimov, V. P. and L. F. Rakhmatullina: *Introduction to the Theory of Linear Functional Differential Equations.* Atlanta: World Fed. Publ. Comp. 1996.

[2] Berezansky, L. and E. Braverman: *Oscillation of a linear delay impulsive differential equation.* Comm. Appl. Nonl. Anal. 3 (1996), 61 – 77.

[3] Berezansky, L. and E. Braverman: *On oscillation of a second order impulsive linear delay differential equation.* J. Math. Anal. Appl. 233 (1999), 276 – 300.

[4] Berezansky, L. and E. Braverman: *On non-oscillation of a scalar delay differential equation.* Dyn. Syst. Appl. 6 (1997), 567 – 580.

[5] Berezansky, L., Braverman, E. and H. Akça: *On oscillation of a linear delay integro-differential equation.* Dyn. Syst. Appl. 8 (1999), 219 – 234.

[6] Domshlak, Y.: *Properties of delay differential equations with oscillating coefficients.* Funct. Diff. Equ., Israel Sem. 2 (1994), 59 – 68.

[7] Domshlak, Y. and I. P. Stavroulakis: *Oscillations of first-order delay differential equations in a critical state.* Appl. Anal. 61 (1996), 359 – 371.

[8] Dunford, N. and J. T. Schwartz: *Linear Operators.* Part 1: *General Theory.* New York: Intersci. Publ. Inc. 1958.

[9] Erbe, L. N., Kong, Q. and B. G. Zhang: *Oscillation Theory for Functional Differential Equations.* New York - Basel: Marcel Dekker 1995.

[10] Gopalsamy, K.: *Stability and Oscillation in Delay Differential Equations of Popular Dynamics.* Dordrecht - Boston - London: Kluwer Acad. Publ. 1992.

[11] Gusarenko, S. A. and A. I. Domoshnitskii: *Asymptotic and oscillation properties of the first order linear scalar functional differential equations.* Diff. Equ. 25 (1989), 2090 – 2103.

[12] Györi, I.: *Oscillation of retarted differential equations of the neutral and the mixed types.* J. Math. Anal. Appl. 141 (1989), 1 – 20.

[13] Györi, I. and G. Ladas: *Oscillation Theory of Delay Differential Equations.* Oxford: Clarendon Press 1991.

[14] Györi, I. and M. Pituk: *Comparison theorems and asymptotic equilibrium for delay differential and difference equations.* Dynam. Syst. Appl. 5 (1996), 277 – 303.

[15] Kolmanovskii, V. and A. Myshkis: *Introduction to the Theory and Applications of Functional Differential Equations.* Dordrecht: Kluwer Acad. Publ. 1999.

[16] Ladde, G. S., Lakshmikantham, V. and B. G. Zhang: *Oscillation Theory of Differential Equations with Deviating Argument.* New York - Basel: Marcel Dekker 1987.

[17] Tyshkevich, V. A.: *Some Questions of the Theory of Stability of Functional-Differential equations* (in Russian). Kiev: Naukova Dumka 1981.

Received 09.10.2000