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On the Three "Essential" Critical Values Theorem

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Abstract. Global methods of the calculus of variations and the infinite dimensional critical point theory of Ljusternik and Schnirelmann are applied to give results on the existence of so-called critical values and essential critical values. The case of continuous, not necessarily differentiable functionals is considered and studied introducing a suitable variant of the Palais-Smale condition.

Keywords: *Essential critical values, deformation lemma, Palais-Smale condition*

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1. Introduction

The aim of this paper is to extend the main result of [14]. Let $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$ and suppose that φ is coercive, i.e., $\varphi(x) \to \infty$ as $\|x\| \to \infty$. It is well known (see [9]) that under these assumptions φ reaches a minimum at some point x_0. Let now x_1 be a critical point of φ which is not a global minimum. Krasnoselskij [11] made the following observation that if x_1 is a non-degenerate singular point of the vector field $\nabla\varphi$, i.e. $\mathrm{ind}(\nabla\varphi(x_1), 0) \neq 0$, then φ admits a third critical point. In the sequel this theorem became known as the "Three Critical Points Theorem".

The above result of Krasnoselskij was extended to the context of Banach spaces (see [1, 5, 10, 18]). Another generalization was obtained by Chang [6, 7] using the methods of Morse theory (the condition $\mathrm{ind}(\nabla\varphi(x_1), 0) \neq 0$ being replaced by the weaker assumption of non-triviality of Morse critical groups at x_1). Also, Brezis and Nirenberg [4] gave a very useful variant of the Three Critical Points Theorem for applications using the principle of local linking (see also [12]).

An interesting result has been proved by Moroz, Vignoli and Zabreiko in [14], in the case of C^1 functionals satisfying the Palais-Smale (or PS-) condition (meaning that any sequence (x_n) with $|\varphi(x_n)| < c$ and $\|\nabla\varphi(x_n)\| \to 0$ as $n \to \infty$ has a convergent subsequence), using the Ljusternik-Schnirelmann category. In this case the existence of a third critical value is established by assuming the existence of an essential critical value $c > m$, where m is the minimum of φ.

The definition of essential critical value, as we shall see, does not need differentiability of φ. So we ask ourselves if the Three Critical Points Theorem is still valid for

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continuous functionals, by replacing the concept of critical values with that of essential critical values.

Essential critical values have been considered also by Degiovanni [8] in a slightly different form, where the well known Deformation Lemma has been replaced introducing the following

Definition 1.1. Let X be a real Banach space and a and b in the extension of $\mathbb{R}$ with $a \leq b$. The pair (φ^b, φ^b) is said to be *trivial*, if for every neighborhood $[\alpha', \alpha'']$ of a and $[\beta', \beta'']$ of b with $\alpha', \alpha'', \beta', \beta'' \in \mathbb{R} \cup \{-\infty, +\infty\}$ there exist two closed subsets A and B of X such that $\varphi^{\alpha'} \subseteq A \subseteq \varphi^{\alpha''}$ and $\varphi^{\beta'} \subseteq B \subseteq \varphi^{\beta''}$ and such that A is a strong deformation retract of B.

In this way we can replace the classic Deformation Lemma with the following result.

Theorem 1.1. *Let X be a real Banach space and $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$ with $a < b$. Let us assume that φ has no essential value in (a, b). Then the pair (φ^b, φ^a) is trivial.*

This theorem is not strong enough to prove our result. To this purpose we shall introduce the notion of quasi-strong deformation retract.

2. A deformation lemma for continuous functionals

Here we introduce the notion of quasi-strong deformation retract and prove a deformation result which is stronger than Theorem 1.1.

Let X be a real Banach space. We assume that $\varphi : X \to \mathbb{R}$ is continuous and coercive, i.e. $\varphi(x) \to \infty$ as $\|x\| \to \infty$. Let us denote by

$$m = \min_X \varphi$$

the minimum of φ over X and by

$$M = \{x \in X : \varphi(x) = m\}$$

the set of minimum points of φ. Moreover, by

$$\varphi^c = \{x \in X | \, \varphi(x) \leq c\}$$

we denote the *Lebesgue set* of the functional φ for the value $c \in \mathbb{R} \cup \{\infty\}$ where it is assumed that $\varphi^\infty = X$.

Let $A \subseteq B \subseteq X$. A continuous map $D : [0, 1] \times A \to B$ such that $D(0, x) = x$ for all $x \in A$ is said to be a *deformation* of A in B. The set A is *contractible* in B if there exists a deformation D of A in B such that $D(1, A) = \{p\}$, where p is a point of B.

In the case $A = B$ we say that A is *contractible* in itself. The set A is a *deformation retract* of B if there exists a deformation D of the set B in itself such that $D(1, B) \subseteq A$ and $D(1, x) = x$ for all $x \in A$. The functional $D(1, X) : B \to A$ is called a *retraction.* The set A is called a *strong deformation retract* of B, if $D(t, x) = x$ for all $x \in A$ and $t \in [0, 1]$.

It is well known that, if A is a deformation retract of B, then A and B have the same homotopy type (the converse not holding in general). If we assume that X is reflexive and φ coercive and convex, then under these assumptions φ has a minimum on X, as a consequence of the following

Theorem 2.1. *Let X be a reflexive Banach space and $\varphi : X \to \mathbb{R}$ a convex continuous coercive functional. Then φ has a minimum on X; i.e. there exists $x_0 \in X$ such that $\varphi(x_0) = m = \min_X \varphi$.*

A proof of this result can be found in [3, 9]. Note that, under our assumptions, φ needs not be differentiable, and so we cannot refer to critical values of φ. Nevertheless, for continuous functionals the following considerations are in order.

Definition 2.1. A value $c \in \mathbb{R}$ of the functional φ is called an *essential critical value*, if for every $\delta > 0$ there exists $\varepsilon \in (0, \delta]$ such that the Lebesgue set $\varphi^{c-\varepsilon}$ is not a strong deformation retract of the Lebesgue set $\varphi^{c+\varepsilon}$.

Note that if φ is a C^1 functional satisfying the PS-condition and c is not a critical value of φ, then by the PS-condition it follows that c is not limit of critical values. By the classical Deformation Lemma, for every $\delta > 0$ there exists $\varepsilon \in (0, \delta]$ such that $\varphi^{c-\varepsilon}$ is a strong deformation retract of $\varphi^{c+\varepsilon}$. In particular, for C^1 functionals satisfying the PS-condition, an essential critical value is also a critical value.The converse does not hold in general as shown in the example $\varphi(x) = x^3$ where $c = 0$ is a critical value of φ which is not an essential critical value.

We now introduce the above mentioned notion of quasi-strong deformation retract.

Definition 2.2. The Lebesgue set φ^a is called a *quasi-strong deformation retract* of φ^b if there exists $\delta > 0$ such that, for every $\varepsilon_1, \varepsilon_2 \in (0, \delta]$ and for arbitrary small $\varepsilon_1' > 0$,

a) $\varphi^{a-\varepsilon_1}$ is a strong deformation retract of $\varphi^{b+\varepsilon_2}$

b) $\varphi^{a+\varepsilon_1'}$ is a strong deformation retract of $\varphi^{b+\varepsilon_2}$.

Let us point out that if φ^a is a quasi-strong deformation retract of φ^b, then the pair (φ^b, φ^a) is trivial (the converse not holding in general).

Lemma 2.1. *Let X be a real Banach space and $\varphi : X \to \mathbb{R}$ a real continuous functional. If we assume that the interval $[a, b] \subseteq \mathbb{R} \cup \{\infty\}$ does not contain essential critical values of φ, then φ^a is a quasi-strong deformation retract of φ^b. In the case $b = \infty$ we shall set $\varepsilon_2 = 0$ in Definition* 2.2.

Proof. We consider two cases. First, $b < \infty$, and then $b = \infty$.

1) Let $b < \infty$ and $\delta > 0$ be such that $\varphi^{a-\varepsilon}$ is a strong deformation retract of $\varphi^{a+\varepsilon}$, for every $\varepsilon \in (0, \delta]$. If $c \in (a, b)$ is fixed, we choose ε in such a way that $[c-\varepsilon, c+\varepsilon] \subseteq [a, b]$ and $\varphi^{c-\varepsilon}$ is a strong deformation retract of $\varphi^{c+\varepsilon}$. Let ε_b be such that $\varphi^{b-\varepsilon_b}$ is a strong deformation retract of $\varphi^{b+\varepsilon_b}$ and $\varepsilon_a < \delta$. We may assume that

$$[a, b] \subset \bigcup_{c_i \in (a,b)} (c_i - \varepsilon_i, c_i + \varepsilon_i) \cup (a - \varepsilon_a, a + \varepsilon_a) \cup (b - \varepsilon_b, b + \varepsilon_b) \tag{2.1}$$

and $\varphi^{c_i - \varepsilon_i}$ is a strong deformation retract of $\varphi^{c_i + \varepsilon_i}$, for every $c_i \in (a, b)$. Now in (2.1) we exhibit an open covering of the compact set $[a, b]$ so that we may extract a finite subcovering. It is not difficult to verify that if $[c_i - \varepsilon_i, c_i + \varepsilon_i] \cap [c_j - \varepsilon_j, c_j + \varepsilon_j] \neq \emptyset$, then $\varphi^{c_i - \varepsilon_i}$ is a strong deformation retract of $\varphi^{c_j + \varepsilon_j}$. Arguing by induction on a finite number of intervals it is easy to see that φ^a is a strong deformation retract of φ^b.

2) Let $b = \infty$ and $\varphi^\infty = X$. For $n \in \mathbb{N}$ let ε_n be such that $\varphi^{a+n-\varepsilon_n}$ is a strong deformation retract of $\varphi^{a+n+1-\varepsilon_{n+1}}$. Note that the existence of ε_n has been proved in the first step. Consider

$$D_1(x,t) : \varphi^{a+1-\varepsilon_1} \times [0,1] \to \varphi^{a+1-\varepsilon_1}$$

a strong deformation of $\varphi^{a+1-\varepsilon_1}$ in $\varphi^{a-\varepsilon_0}$ with $\varepsilon_0 < \delta$. Let

$$D_2 : \varphi^{a+2-\varepsilon_2} \times [0,1] \to \varphi^{a+2-\varepsilon_2}$$

be a strong deformation of $\varphi^{a+2-\varepsilon_2}$ in $\varphi^{a+1-\varepsilon_1}$. Denote by $\tilde{D}_1 : X \times [0,1] \to X$ a continuous extension of D_1 such that $D_1(x,0) = x$ for every $x \in X$. The map $\tilde{D}_1$ exists by Dugundji's Extension Theorem. Set

$$\bar{D}_2(x,t) = \begin{cases} \tilde{D}_1(D_2(x,2t),2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ D_1(D_2(x,1),2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The definition of $\bar{D}_2$ is well posed, and what we obtain is a continuous map which is a deformation of $\varphi^{a+2-\varepsilon_2}$ onto $\varphi^{a-\varepsilon_0}$.

The map $\bar{D}_2$ may assume values in all of X and not only in $\varphi^{a-\varepsilon_0}$. However, we may argue as follows. Note that $\bar{D}_2(x,t) \equiv D_1(x,t)$ on $\varphi^{a+1-\varepsilon_1} \times [0,1]$. So we may define

$$D_\infty : X \times [0,1] \to X$$

as follows: for $x \in \overline{\varphi^{a+n-\varepsilon_n} \setminus \varphi^{a+n-1-\varepsilon_{n-1}}}$ we set

$$D_\infty = \bar{D}_n(x,t) \qquad \text{for every } t \in [0,1].$$

Now, the definition of D_∞ is well posed and D_∞ is a deformation of φ^∞ on $\varphi^{a-\varepsilon_0}$ in φ^∞. Since ε_0 was arbitrary, the lemma is proved ∎

Lemma 2.1 allows us to consider the case $b = \infty$, i.e. $\varphi^b = X$. Hence, if φ has not essential critical values greater or equal to c, we can retract the whole space X on Lebesgue sets arbitrarily close to φ^c.

In what follows we show some direct consequences of Lemma 2.1.

Proposition 2.1. *Let $[a,b] \subset \mathbb{R} \cup \{\infty\}$. If φ^a is not a quasi-strong deformation retract of φ^b, then φ has an essential critical value $c \in [a,b]$.*

Therefore, if there exist $a, b \in \mathbb{R} \cup \{\infty\}$ such that φ^a is not a quasi-strong deformation retract of φ^b, then there exists at least one essential critical value in the interval $[a,b]$.

The following definition will be crucial in the proof of the main result of this paper.

Definition 2.3. A value $c \in \mathbb{R}$ is called a *strong essential critical value* of φ, if there exists $\delta > 0$ such that $\varphi^{c-\varepsilon}$ is not a strong deformation retract of $\varphi^{c+\varepsilon}$, for every $\varepsilon \in (0,\delta]$.

Remark 1. If m is the minimum of φ, then m is a strong essential critical value of φ. In the sequel we shall show that if x_0 is a strict local minimum of φ, then $\varphi(x_0)$ is a strong essential critical value of φ. This is related to the fact that, for C^1 functionals, a strict local minimum is a critical point. It is also clear that a strong essential critical value is also an essential critical value.

The following result states that an essential critical value is nearly always a strong essential critical value.

Proposition 2.2. *Let φ be a C^1 real functional satisfying the PS-condition. If c is an essential critical value which is not limit of critical values, then c is a strong essential critical value.*

Proof. Let c be an essential critical value of φ. Let $\delta > 0$ be such that $\varphi^{c-\delta}$ is not a strong deformation retract of $\varphi^{c+\delta}$ and the interval $[c-\delta, c+\delta]$ does not contain critical values. Assume that $\varepsilon \in (0, \delta)$ is such that $\varphi^{c-\varepsilon}$ is a strong deformation retract of $\varphi^{c+\varepsilon}$. Then, by the classic Deformation Lemma, $\varphi^{c+\varepsilon}$ is a strong deformation retract of $\varphi^{c+\delta}$, and then $\varphi^{c-\varepsilon}$ is a strong deformation retract of $\varphi^{c+\delta}$. Moreover, $\varphi^{c-\delta}$ is a strong deformation retract of $\varphi^{c-\varepsilon}$ and so it is also a strong deformation retract of $\varphi^{c+\delta}$. This is a contradiction ∎

3. The three essential critical values theorem

In this section, with the aid of Lemma 2.1, we prove some preliminary results and use them to prove the Three Essential Critical Values Theorem.

We shall run over the approach shown in [14] for the C^1 case, using the statements we have proved for the C^0 case. We recall (see [9]) that a metric space C is called an absolute neighbourhood retract (an ANR for short) if, for any closed subset $A \subseteq B$ of a metric space B, we have that any continuous map $f : A \to C$ has a continuous extension over some neighbourhood U_A of A in B. Any Lebesgue set corresponding to a regular value of a C^1 functional φ is an ANR (see [15]).

In what follows we prove an analogue of this result for continuous functionals.

Theorem 3.1. *Let φ be a real continuous functional on X and let c be a regular value of φ. Then there exists $\delta > 0$ such that $\varphi^{c-\varepsilon}$ is an ANR, for every $\varepsilon \in (0, \delta]$.*

Proof. Let B be a metric space and $A \subseteq B$ be a closed subset of B. Choose $\delta > 0$ such that $\varphi^{c-\varepsilon}$ is a deformation retract of $\varphi^{c+\varepsilon}$, for every $\varepsilon \in (0, \delta]$, and consider $f : A \to \varphi^{c-\varepsilon}$ continuous. Since X is an absolute retract, there exists a continuous extension $\tilde{f} : B \to X$ of f. Setting $\tilde{U}_A = \tilde{f}^{-1}(\{x \in X : \varphi(x) < c+\varepsilon\})$, by the continuity of $\tilde{f}$ we have that $\tilde{U}_A$ is an open set containing A. Now, if r is the retraction of $\varphi^{c+\varepsilon}$ onto $\varphi^{c-\varepsilon}$, then $r \circ \tilde{f}|_{\tilde{U}_A}$ satisfies

a) $r \circ \tilde{f}|_{\tilde{U}_A}$ is product of continuous functionals and so it is continuous

b) the range of $r \circ \tilde{f}|_{\tilde{U}_A}$ lays in $\varphi^{c-\varepsilon}$

c) $(r \circ \tilde{f}|_{\tilde{U}_A})|_A = id \circ \tilde{f}|_A = f$

and the theorem is proved ∎

We recall now the following important result (see [17]).

Theorem 3.2. *Let B be an ANR and $A \subseteq B$ be a closed subset of B such that A is also an ANR. Then A is a strong deformation retract of B if and only if the inclusion $i : A \hookrightarrow B$ is a homotopy equivalence.*

With the aid of Theorem 3.2 we shall study the topology of Lebesgue sets. In what follows there will be crucial the fact that $\varphi^{\infty} = X$ is contractible in itself.

Theorem 3.3. *Let X be a real Banach space, φ a continuous functional on it, and $c \in \mathbb{R}$ a strong essential critical value of φ. Then there exists $\delta > 0$ such that for every $\varepsilon \in (0, \delta]$ such that $\varphi^{c-\varepsilon}$ and $\varphi^{c+\varepsilon}$ are ANR's, at least one of the two Lebesgue sets $\varphi^{c-\varepsilon}$ and $\varphi^{c+\varepsilon}$ is not contractible in itself.*

Proof. Let $\delta > 0$ be such that the Lebesgue set $\varphi^{c-\varepsilon}$ is not a strong deformation retract of the Lebesgue set $\varphi^{c+\varepsilon}$, for every $\varepsilon \in (0, \delta]$. If $\varphi^{c-\varepsilon}$ and $\varphi^{c+\varepsilon}$ are ANR's, then they are closed ANR's and, by Theorem 3.2, the inclusion $i : \varphi^{c-\varepsilon} \hookrightarrow \varphi^{c+\varepsilon}$ is not a homotopy equivalence.

Assume that both $\varphi^{c-\varepsilon}$ and $\varphi^{c+\varepsilon}$ are contractible in itself. In this case any continuous map $f : \varphi^{c-\varepsilon} \to \varphi^{c+\varepsilon}$ and, in particular, the inclusion map is a homotopy equivalence. This is a contradiction ∎

We recall now a classic result from Ljusternik-Schnirelmann theory (see [9: p. 354]). In what follows $Cat_B(A)$ stands for the category of the set $A \subseteq B$ in B. In this case it is important to note that if a set is contractible in itself, then it has category equal to one.

Proposition 3.1. *Let B be an ANR and $A \subseteq B$. Then there exists a neighbourhood U_A of A in B such that $Cat_B(U_A) = Cat_B(A)$.*

Remark 2. Let us assume, under the assumptions of Proposition 3.1, that the set A is contractible in itself. Then as above mentioned, $Cat_B(A) = 1$ and, since subsets of B of category one are contractible in B, there exists a neighbourhood U_A of A which has category equal to one and is contractible in B.

In order to prove the main result we need a condition which replaces the PS-condition in the case of continuous functionals. To this purpose we shall introduce the PST-condition and the PSV-condition. We shall use the PST-condition in the case of continuous convex functionals. Even if this case is too restrictive, it tells us how to treat the more general case of continuous, not necessarily convex functionals.

Condition PST. We say that φ satisfies the $\mathrm{PST}_{\varphi(x_0)}$- *condition in $\varphi(x_0)$ if any sequence (x_n) such that $\varphi(x_n) \to \varphi(x_0)$ and $(x_n) \rightharpoonup x \in \varphi^{-1}(\varphi(x_0))$ has a subsequence converging to x.*

Lemma 3.1. *Let φ be continuous, coercive and convex. Let m, the minimum of φ, be an isolated essential critical value of φ and let M, the set of minimum points of φ, be contractible in itself. Then for every $\bar{\delta} > 0$ there exists $\varepsilon \in (0, \bar{\delta})$ such that the Lebesgue set $\varphi^{m+\varepsilon}$ is contractible in itself.*

Proof. Choose $\varepsilon > 0$ so that the interval $[m, m+\varepsilon]$ does not contain essential critical values of φ different from m. We may also suppose that $m+\varepsilon$ is not a limit of essential critical values. Hence, by Theorem 3.1, there exists $\varepsilon' > 0$ such that $\varphi^{m+\varepsilon+\varepsilon'}$ is an ANR, and since $M \subseteq \varphi^{m+\varepsilon+\varepsilon'}$ is contractible in itself, there exists a neighbourhood U_M of M in $\varphi^{m+\varepsilon+\varepsilon'}$ which is contractible in $\varphi^{m+\varepsilon+\varepsilon'}$. Moreover, we choose $\varepsilon' > 0$ such that the interval $[m, m+\varepsilon+\varepsilon'+\delta']$ does not contain essential critical values.

Let us show now that there exists $\delta \in (0, \varepsilon)$ such that

$$M \subseteq \varphi^{m+\delta} \subseteq U_M \subseteq \varphi^{m+\varepsilon+\varepsilon'}.$$

In fact, if this is not the case, there exists a minimizing sequence $(x_n) \subseteq \varphi^{m+\delta} \backslash U_M$. Now, since $\varphi^{m+\delta}$ is weakly compact (see [3]), there exists a subsequence $(x_{n_k}) \subset (x_n)$ such that $(x_{n_k}) \rightharpoonup x_0$ and x_0 is in M from [3: Corollary III.8]. Since we have supposed that φ satisfies the PST-condition, $(x_{n_k}) \to x_0 \in M$, and $x_0 \notin U_M$. This contradicts the fact that $M \subseteq U_M$.

Finally, by Lemma 2.1 we may suppose that $\varphi^{m+\delta} \subseteq U_M$ is a strong deformation retract of $\varphi^{m+\varepsilon+\varepsilon'}$ and U_M is contractible in $\varphi^{m+\varepsilon+\varepsilon'}$. Hence $\varphi^{m+\varepsilon+\varepsilon'}$ is contractible in itself ∎

Remark 3. By Lemma 2.1 there exists $s > 0$ such that $\varphi^{m+\varepsilon-\eta}$ is a strong deformation retract of $\varphi^{m+\varepsilon+\varepsilon'}$, for every $\eta \in (0, s)$. Since $\varphi^{m+\varepsilon+\varepsilon'}$ is contractible in itself, then $\varphi^{m+\varepsilon-\eta}$ is contractible in itself for every $\eta \in (0, s)$.

4. Proof of the main result

As above mentioned, φ^∞ is the whole space X and then it is contractible in itself. This causes that any Lebesgue set which is a strong deformation retract of φ^∞ is contractible in itself. So, if we suppose that M is not contractible in itself and that there exists an essential critical value $c > m$, we shall prove the existence of a third essential critical value greater than m.

Theorem 4.1 (Three Critical Values Theorem). *Let φ be a continuous, coercive and convex functional satisfying the PST_m-condition. If φ has a strong essential critical value $c > m$, then either φ admits three distinct essential critical values or the set M of minimum points is not contractible in itself.*

Proof. We may assume that m and c are isolated critical values of φ, otherwise the theorem is trivial. Suppose that M is contractible in itself. Since c is a strong essential critical value of φ, there exists $\delta > 0$ such that $\varphi^{c-\varepsilon}$ is not a strong deformation retract of $\varphi^{c+\varepsilon}$, for every $\varepsilon \in (0, \delta)$.

Let $c - \bar{\varepsilon}$ be a regular value of φ in the interval $(c - \varepsilon, c)$. By Theorem 3.3 there exists $\delta' > 0$ such that $\varphi^{c-\bar{\varepsilon}+\varepsilon'}$ is a closed ANR for every ε' in the interval $(0, \delta')$. By Lemma 3.1 we may also suppose that $\varphi^{c-\bar{\varepsilon}+\varepsilon'}$ is contractible in itself. Arguing the same way for the regular value $c + \bar{\varepsilon}$ and, if necessary, reducing δ' and changing $\bar{\varepsilon}$, we can suppose that $\varphi^{c+\bar{\varepsilon}-\varepsilon'}$ is a closed ANR, for every ε' in the interval $(0, \delta')$ (we use the same notations even if the intervals are changed). Now, by Theorem 3.3, $\varphi^{c+\bar{\varepsilon}-\varepsilon'}$ is not contractible in itself for every $\varepsilon' \in (0, \delta')$. Since, by Lemma 2.1, $\varphi^{c+\bar{\varepsilon}-\varepsilon_0}$ is a strong deformation retract of $\varphi^\infty = X$ for some $\varepsilon_0 \in (0, \delta')$, $\varphi^{c+\bar{\varepsilon}-\varepsilon_0}$ is contractible in itself. This is a contradiction. So either φ admits a third essential critical value or M is not contractible in itself. In particular, if M is not contractible in itself, it is not a one point set ∎

5. The main theorem for continuous functionals

In this section we prove the three essential critical values theorem for continuous, not necessarily convex functionals, replacing the PS-condition with the PSV$_m$-condition defined below. As you shall see our result extends the three critical points theorem [14] to the case of continuous functionals and, in the case of differentiable functionals, it is a stronger result.

We introduce now the above mentioned PSV$_m$-condition.

Condition PSV. A continuous functional φ satisfies the PSV$_c$-*condition*, if any sequence (x_n) such that $\varphi(x_n) \to c$ has a subsequence converging to $x_0 \in M$.

Remark 4. In particular, we shall consider $c = m$, m being the minimum of φ on X. In this case, as we shall prove below, the PSV$_m$-condition is weaker than the PS-condition. Moreover, we shall exhibit a large class of functionals satisfying the PSV$_m$-condition.

Remark 5. If $X = \mathbb{R}^n$, then any continuous coercive functional $\varphi : X \to \mathbb{R}$ satisfies the PSV$_c$-condition at any level $c \in \mathbb{R}$. In fact, in this case $B_r(0)$ is compact, and so any bounded sequence (x_n) has a subsequence (x_{n_k}) converging to x_0. Now, since φ is coercive and $\varphi(x_n) \to c$, the sequence (x_n) is bounded.

If X is an infinite dimensional space, we cannot apply the arguments of Remark 4 and say that any continuous functional satisfies the PSV$_m$-condition. However, there is a large class of functionals satisfying the PSV$_m$-condition as it is shown in the next result. Let use recall that, if $\Omega \subset X$ is a closed and bounded set, then a functional F defined on it is said to be *proper* if $F^{-1}(K)$ is compact whenever K is compact.

Proposition 5.1. *Let X be a real Banach space and let $\varphi : X \to \mathbb{R}$ be a continuous functional which is proper on closed and bounded sets of X. Then φ satisfies the PSV$_c$-condition at any level $c \in \mathbb{R}$.*

Proof. For any $c \in \mathbb{R}$ consider the interval $[c-\varepsilon, c+\varepsilon]$. We set $D = \varphi^{-1}[c-\varepsilon, c+\varepsilon]$. Since φ is coercive and continuous, then D is bounded and closed. Now, since by assumption φ is proper on closed and bounded sets, then D is compact. Let now (x_n) be such that $\varphi(x_n) \to c$. It is easy to verify that $(x_n) \subset D$ for n large enough. So (x_n) has a subsequence $(x_{n_k}) \subset (x_n)$ converging to x_0 ∎

Let us recall now an important result from [4].

Proposition 5.2. *If φ is a C^1 functional bounded from below and satisfying the PS-condition, then φ satisfies the PSV$_m$-condition.*

The converse of this result does not hold in general as shown by $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -\frac{1}{1+x^2}$. Indeed, it is not difficult to verify that f satisfies the PSV$_m$-condition, even though f does not satisfy the PS-condition. So the class of functionals satisfying the PSV$_m$-condition is in fact larger than the class of functionals satisfying the PS-condition.

Lemma 5.1. *Let m be an isolated essential critical value of φ and let M be contractible in itself. Then for every $\bar{\delta} > 0$ there exists $\varepsilon \in (0, \bar{\delta})$ such that the Lebesgue set $\varphi^{m+\varepsilon}$ is contractible in itself.*

Proof. We choose $\varepsilon > 0$ such that the interval $[m, m+\varepsilon]$ does not contain essential critical values of φ different from m. We may assume that $m+\varepsilon$ is not limit of essential critical values. Hence, by Theorem 3.1, there exists $\varepsilon' > 0$ such that $\varphi^{m+\varepsilon+\varepsilon'}$ is an ANR. Since $M \subseteq \varphi^{m+\varepsilon+\varepsilon'}$ is contractible in itself, there exists a neighbourhood U_M of M in $\varphi^{m+\varepsilon+\varepsilon'}$ which is contractible in $\varphi^{m+\varepsilon+\varepsilon'}$. Moreover, we choose $\varepsilon' > 0$ such that the interval $[m, m+\varepsilon+\varepsilon'+\delta']$ does not contain essential critical values.

Let us show now that there exists $\delta \in (0, \varepsilon)$ such that

$$M \subseteq \varphi^{m+\delta} \subseteq U_M \subseteq \varphi^{m+\varepsilon+\varepsilon'}.$$

In fact, if this is not the case, there exists a minimizing sequence $(x_n) \subseteq \varphi^{m+\delta} \backslash U_M$ with $\|x_n\| \leq c$. By the PSV$_m$-condition there exists a subsequence $(x_{n_k}) \to x_0$, and $x_0 \notin U_M$. This contradicts the fact that $M \subseteq U_M$.

Finally, by Lemma 2.1 we may suppose that $\varphi^{m+\delta} \subseteq U_M$ is a strong deformation retract of $\varphi^{m+\varepsilon+\varepsilon'}$. Since U_M is contractible in $\varphi^{m+\varepsilon+\varepsilon'}$, then $\varphi^{m+\varepsilon+\varepsilon'}$ is contractible in itself ∎

Remark 6. By Lemma 2.1 there exists $s > 0$ such that $\varphi^{m+\varepsilon-\eta}$ is a strong deformation retract of $\varphi^{m+\varepsilon+\varepsilon'}$, for every $\eta \in (0, s)$. Since $\varphi^{m+\varepsilon+\varepsilon'}$ is contractible in itself, then $\varphi^{m+\varepsilon-\eta}$ is contractible in itself for every $\eta \in (0, s)$.

Replacing Lemma 3.1 with Lemma 5.1 we may run over again the proof of Theorem 4.1 and prove our main result.

Theorem 5.1 (Three Critical Values Theorem). *Let φ be a continuous and coercive functional satisfying the PSV$_m$-condition. If φ has a strong essential critical value $c > m$, then either φ admits three distinct essential critical values or the set of minimum points M is not contractible in itself.*

We suppose now that φ satisfies the PSV-condition at any level $c = \varphi(x_0)$, where x_0 is a strict local minimum. Under these assumptions, the following proposition holds.

Proposition 5.3. *Let x_0 be a strict local minimum of φ. If φ satisfies the PSV$_c$-condition with $c = \varphi(x_0)$, then c is a strong essential critical value of φ.*

Proof. Let $B_\varrho(S_\varrho)$ be the ball (sphere) with center in x_0 and radius $\varrho > 0$. We can choose ϱ such that

$$\inf_{S_\varrho} \varphi(x) = c_\varrho > c \qquad \text{and} \qquad \inf_{B_\varrho} \varphi(x) = c.$$

In fact, if $(x_n) \subset S_\varrho$ is a minimizing sequence since $\|x_n\| \leq \varrho$, then by assumption there exists a subsequence (x_{n_k}) such that $(x_{n_k}) \to \bar{x} \in S_\varrho$. Now, since φ is continuous, then $\varphi(\bar{x}) = c$. This contradicts the fact that x_0 is a strict local minimum.

Obviously, $c_\varrho \to c$ as $\varrho \to 0$. We fix $\varepsilon = \varepsilon_\varrho \in (0, c_\varrho - c)$. Then the component of the Lebesgue set $\varphi^{c+\varepsilon}$, which contains the point x_0, does not meet the Lebesgue set $\varphi^{c-\varepsilon}$. Hence $\varphi^{c-\varepsilon}$ is not a strong deformation retract of $\varphi^{c+\varepsilon}$. Since ε is arbitrarily small, the value $c = \varphi(x_0)$ is a strong essential critical value of φ ∎

6. The case of C^1 functionals

In this section we consider again the case of C^1 coercive functionals satisfying the PS-condition. In this case we have an interesting result due to Moroz, Vignoli and Zabreiko (see [14]). Let us recall it.

Theorem 6.1. *If φ has an essential critical value $c > m$, then either φ admits at least three distinct critical values or the set of minimum points M is not contractible in itself. In particular, φ admits al least three critical points.*

We are going to show that our Theorem 5.1 allows us to give a stronger version of Theorem 6.1. In fact, the following is in order.

Theorem 6.2. *Let φ be a coercive C^1 functional which satisfies the PSV_m-condition. If φ has an essential critical value $c > m$, then either φ admits a third essential critical value or the set M of its minimum points is not contractible in itself. In particular, φ has at least three critical points.*

Proof. We start by assuming that M is contractible in itself and that φ has a finite number of critical values. From this it follows that c is not limit of critical values of φ. Then by Proposition 2.2, c is a strong essential critical value of φ. Applying Theorem 5.1 we get the result ∎

Now, for C^1 functionals satisfying the PS-condition, an essential critical value is also a critical value. Remarking that the PS-condition is stronger than the PSV_m-condition (Proposition 5.2), we get as a consequence of our result the Three Critical Points Theorem of Moroz, Vignoli and Zabreiko.

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