The Generalized Riemann Problem
of Linear Conjugation
for Polyanalytic Functions of Order n in $W_{n,p}(D)$

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Abstract. We consider a homogeneous polyanalytic differential equation of order n in a simply-connected domain D with a smooth boundary ∂D in the complex plane $\mathbb{C}$. We pose and then prove solvability of a generalized Riemann problem of linear conjugation to the differential equation. This is done by reducing the problem into n classical Riemann problems of linear conjugation for holomorphic functions, the solution of which is available in the literature.

Keywords: *Polyanalytic functions, Riemann problem of linear conjugation*

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1. Introduction

We pose the following generalized version of the Riemann problem of linear conjugation for a complex-valued polyanalytic function Φ of order n on a simply-connected and bounded domain D with a sufficiently smooth boundary ∂D in the complex plane $\mathbb{C}$:

$$\left[\bar{z}^{q_j}\frac{\partial^{2j-1}\Phi}{\partial z^{j-1}\partial\bar{z}^j}\right]^+(t) - G_j(t)\left[\bar{z}^{q_j}\frac{\partial^{2j-1}\Phi}{\partial z^{j-1}\partial\bar{z}^j}\right]^-(t) = g_j(t) \quad \text{on } \partial D \qquad (G)$$

$$\left[\frac{\partial^{2k-2}\Phi}{\partial z^{k-1}\partial\bar{z}^{k-1}}\right]^+(t) - H_k(t)\left[\frac{\partial^{2k-2}\Phi}{\partial z^{k-1}\partial\bar{z}^{k-1}}\right]^-(t) = f_k(t) \quad \text{on } \partial D \qquad (H)$$

where $q_j \in \mathbb{Z}$ and

$$\left.\begin{array}{l} g_j, G_j \in W_{n-2j+1-\frac{1}{p},p}(\partial D) \\[2mm] f_k, H_k \in W_{n-2k+2-\frac{1}{p},p}(\partial D) \end{array}\right\}$$

for $2 < p < \infty$ and $j, k, n \in \mathbb{N}$ with $2j < n+1$ and $2k < n+2$.

   The idea behind the method for solving any boundary value problem for a polyanalytic function Φ is to reduce the problem posed to corresponding classical ones for holomorphic functions. The latter are expressed in terms of the holomorphic functions which define the polyanalytic function Φ. If the latter admit holomorphic solutions, then the original problem posed is solvable as well.

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2. Problem reduction to one for holomorphic functions

Since a polyanalytic function contains both the complex variable z and its complex conjugate $\bar{z}$ in its expression we shall assume, without restricting the general case, that our domain D is the disc of radius R centred at the origin, i.e. $D = \{x \in \mathbb{C} : |z| < R\}$.

A polyanalytic function Φ of order n can be represented in the form

$$\Phi = \Phi(z, \bar{z}) = \sum_{r=0}^{n-1} \bar{z}^r \varphi_r(z)$$

(cf [1, 4]) where φ_r are holomorphic functions in D. On substituting the boundary condition (G) in the general representation for Φ we see that

$$\bar{z}^{q_j} \frac{\partial^{2j-1}\Phi}{\partial z^{j-1}\partial\bar{z}^j} = \bar{z}^{q_j} \sum_{r=0}^{n-1} \frac{\partial^{2j-1}}{\partial z^{j-1}\partial\bar{z}^j} (\bar{z}^r \varphi_r(z))$$

$$= \sum_{r=j}^{n-1} \frac{r!\,\bar{z}^{q_j+r-j}}{(r-j)!} \frac{d^{j-1}}{dz^{j-1}} \varphi_r(z)$$

$$= \sum_{r=j}^{n-1} \frac{r!\,R^{2(q_j+r-j)}}{(r-j)!} z^{j-r-q_j} \frac{d^{j-1}}{dz^{j-1}} \varphi_r(z)$$

$$= z^{1-n-q_j} \sum_{r=j}^{n-1} \frac{r!\,R^{2(q_j+r-j)}}{(r-j)!} z^{n+j-r-1} \frac{d^{j-1}}{dz^{j-1}} \varphi_r(z)$$

on ∂D. The jump conditions (G) for the polyanalytic function Φ have thus been transformed into $\left[\frac{n}{2}\right]$ classical Riemann problems for the holomorphic functions E_j $(j \in \mathbb{N}, 2j-1 < n)$:

$$E_j^+(t) - G_j(t)E_j^-(t) = t^{q_j+n-1}g_j(t) \quad \text{on } \partial D \tag{1}$$

$$E_j(z) = \sum_{r=j}^{n-1} \frac{r!\,R^{2(q_j+r-j)}}{(r-j)!} z^{n+j-r-1} \frac{d^{j-1}}{dz^{j-1}} \varphi_r(z). \tag{2}$$

The solution of the Riemann problem of linear conjugation (1) for the piecewise holomorphic functions E_j is known (cf. [2 - 6, 10]). If $k_j = \text{index}\,(G_j; \partial D) \geq 0$, then the solution may also be expressed in the form

$$E_j(z) = z^n X_j(z) \left(\frac{1}{2\pi i} \int_{\partial D} \frac{t^{q_j-1}g_j(t)}{X_j^+(t)\,(t-z)} \, dt + P_{k_j}(z) \right) \tag{3}$$

where P_{k_j} is a polynomial of degree k_j (so that the solution is bounded at infinity) and X_j is the canonical solution of the corresponding homogeneous equation:

$$\left.\begin{aligned}
X_j(z) &= \exp\left[\frac{1}{2\pi i} \int_{\partial D} \log(t^{-k_j}G_j(t))\, \frac{dt}{t-z}\right] := \exp\Gamma_j(z) \quad \text{on } D^+ \\
X_j^+(t) &= \exp\left(\frac{1}{2}\log(t^{-k_j}G_j(t)) + \Gamma_j(t)\right) \\
X_j(z) &= z^{-k_j}\exp\Gamma_j(z) \quad \text{on } D^-.
\end{aligned}\right\}$$

Remark. We make use of the general solution $z^n X_j(z)$ of the homogeneous Riemann problem of conjugation in place of the canonical solution $X_j(z)$ when solving the corresponding non-homogeneous problem.

If $k_j \geq 0$, then the canonical function X_j has a zero of order k_j at infinity (and it is bounded if $k_j = 0$). Consequently, the general solution (3) of the non-homogeneous problem (1) which is bounded at infinity contains $k_j + 1$ arbitrary complex constants. We thus impose $k_j + 1$ point conditions on the solution E_j in order to make it uniquely determined:

$$E_j(z_s) = C_s \qquad (z_s \in D^+ \text{ and } C_s \in \mathbb{C} \text{ for } s = 1, 2, \ldots, k_j + 1). \qquad (4)$$

If any $k_j < 0$, then the corresponding canonical function X_j has a pole of order $-k_j$ at infinity. Thus, in order to obtain a solution which is bounded at infinity, we must set $P_{k_j} \equiv 0$ and demand, in addition, that the Cauchy-type integral in (3) has a zero of order not less than $-k_j$ at infinity. The substitution of the identity

$$\frac{1}{t-z} = -\frac{1}{z}\left(\frac{1}{1-\frac{t}{z}}\right) = -\frac{1}{z}\left(1 + \frac{t}{z} + \left(\frac{t}{z}\right)^2 + \ldots\right) = -\sum_{k=0}^{\infty} \frac{t^k}{z^{k+1}}$$

in a neighbourhood of $z = \infty$ in (3) (with $P_{k_j} \equiv 0$) yields

$$E_j(z) = -\frac{X_j(z)}{2\pi i} \sum_{k=0}^{\infty} z^{n-k-1} \int_{\partial D} \frac{1}{X_j^+(t)} t^{q_j-1} g_j(t)\, t^k dt.$$

The latter shows that E_j is bounded at infinity if and only if

$$\int_{\partial D} \frac{1}{X_j^+(t)} t^{q_j-1} g_j(t)\, t^{k-1} dt = 0 \qquad (k = 1, 2, \ldots, n - k_j - 1). \qquad (5)$$

These are $2(|k_j| - 1)$ real solvability conditions on the free term g_j of the boundary conditions (1).

The possibility for the satisfaction of these solvability conditions may be checked by considering the following modified Riemann problem of conjugation:

$$E_j^+(t) - G_j(t)E_j^-(t) = t^{q_j+n-1} g_j(t) - \sum_{s=1}^{n-k_j-1} \lambda_s t^{-s} \quad \text{on } \partial D \qquad (1)'$$

where λ_s are complex constants yet to be fixed appropriately. It will be demanded, moreover, that if the modified problem has a solution, then it should be representable in the form (3) as well (cf. [2, 9, 11]).

Going by the preceeding discussion, the modified problem has a (unique) solution if and only if the linear system

$$\sum_{s=1}^{n-k_j-1} \lambda_s \int_{\partial D} \frac{t^{k-s-1}}{X_j^+(t)}\, dt = \int_{\partial D} \frac{g_j(t)}{X_j^+(t)} t^{k+q_j+n-2} dt \quad (k = 1, 2, n - k_j - 1) \qquad (6)$$

in λ_s is uniquely solvable. This is indeed the case (cf. [2, 9, 11]). We deduce, for instance, from the Cauchy theorems that

$$\int_{\partial D} \frac{1}{X_j^+(t)} t^{k-s-1} dt = \begin{cases} 0 & \text{for } s \leq k-1 \\ \frac{2\pi i}{(s-k)!} \lim_{z \to 0} \frac{d^{s-k}}{dz^{s-k}}\left(\frac{1}{X_j^+(z)}\right) & \text{for } s \geq k. \end{cases}$$

As such the coefficients matrix of the linear system (6) is upper triangular with the elements $\frac{2\pi i}{X_j(0)} \neq 0$ on its principal diagonal. It is therefore non-singular and hence the constants λ_j are uniquely determinable. The solution of the original Riemann problem of linear conjugation (1) is then

$$E_j(z) = \frac{z^n X_j(z)}{2\pi i} \int_{\partial D} \left(t^{q_j-1} g_j(t) - \sum_{s=1}^{n-k_j-1} \lambda_s t^{-n-s} \right) \frac{dt}{X_j^+(t)(t-z)} \tag{3$'$}$$

(cf [2, 9, 11]).

We have assumed that $g_j, G_j \in W_{n-2j+1-\frac{1}{p},p}(\partial D)$ $(2 < p < \infty)$. It thus follows from properties of Cauchy-type integrals that $E_j \in W_{n-2j+1,p}(D)$, and estimates of the form

$$\|E_j\|_{n-2j+1,p,D} \leq C(p, D, G_j) \|g_j\|_{n-2j+1-\frac{1}{p},p,\partial D}$$

hold (cf. [6 - 8]).

We now substitute the boundary conditions (H) in the general representation of a plyanalytic function Φ of order n. We have in the first place

$$\frac{d^{2k-2}\Phi}{\partial z^{k-1}\partial \bar{z}^{k-1}} = \frac{d^{2k-2}}{\partial z^{k-1}\partial \bar{z}^{k-1}} \sum_{r=0}^{n-1} \bar{z}^r \varphi_r(z)$$

$$= \sum_{r=k-1}^{n-1} \frac{r!\,\bar{z}^{r-k+1}}{(r-k+1)!} \frac{d^{k-1}}{dz^{k-1}} \varphi_r(z)$$

$$= \sum_{r=k-1}^{n-1} \frac{r!\,R^{2(r-k+1)}}{(r-k+1)!} z^{k-r-1} \frac{d^{k-1}}{dz^{k-1}} \varphi_r(z)$$

$$= z^{1-n} \sum_{r=k-1}^{n-1} \frac{r!\,R^{2(r-k+1)}}{(r-k+1)!} z^{n+k-r-2} \frac{d^{k-1}}{dz^{k-1}} \varphi_r(z)$$

on ∂D and the junp conditions (H) are thus converted into $\left[\frac{n+1}{2}\right]$ Riemann problems of linear conjugation for the unknown holomorphic functions D_k $(k \in \mathbb{N}, 2k < n+2)$:

$$D_k^+(t) - H_k(t) D_k^-(t) = t^{n-1} f_k(t) \quad \text{on } \partial D \tag{7}$$

$$D_k(z) = \sum_{r=k-1}^{n-1} \frac{r!\,R^{2(r-k+1)}}{(r-k+1)!} z^{n+k-r-2} \frac{d^{k-1}}{dz^{k-1}} \varphi_r(z). \tag{8}$$

We obtain unique holomorphic solutions D_k in the same manner as in the case of the previously discussed holomorphic functions E_j. If $h_k = \mathrm{index}\,(H_k; \partial D) \geq 0$, then the solution may be expressed in the form

$$D_k(z) = z^n X_k(z) \left(\frac{1}{2\pi i} \int_{\partial D} \frac{t^{-1} f_k(t)}{X_k^+(t)\,(t-z)}\, dt + P_{h_k}(z) \right) \tag{9}$$

where P_{h_k} is a polynomial of degreee h_k (so that the solution is bounded at infinity) and X_k is the canonical solution of the corresponding homogeneous equation:

$$X_k(z) = \exp\left[\frac{1}{2\pi i} \int_{\partial D} \log\left(t^{-h_k} H_k(t)\right) \frac{dt}{t-z} \right] =: \exp Y_k(z) \quad \text{on } D^+$$

$$X_k^+(t) = \exp\left(\frac{1}{2} \log\left(t^{-h_k} H_k(t)\right) + Y_k(t) \right)$$

$$X_k^-(z) = z^{-h_k} \exp Y_k(z) \quad \text{on } D^-.$$

If $h_k < 0$, then

$$D_k(z) = \frac{z^n X_k(z)}{2\pi i} \int_{\partial D} \left(t^{-1} f_k(t) - \sum_{s=1}^{n-h_k-1} \lambda_s t^{-n-s} \right) \frac{dt}{X_k^+(t)\,(t-z)}. \tag{9$'$}$$

Moreover, we find that $D_k \in W_{n-2k+2,p}(D)$ $(2 < p < \infty)$, and estimates of the form

$$\|D_k\|_{n-2k+2,p,D} \leq C(p, D, H_k)\, \|f_k\|_{n-2k+2-\frac{1}{p},p,\partial D}$$

hold (cf. [6 - 8]).

3. Determination of the polyanalytic function

Suppose we have determined all the $\left[\frac{n+1}{2}\right]$ functions D_k and $\left[\frac{n}{2}\right]$ functions E_j:

$$
\left.
\begin{aligned}
& \sum_{r=0}^{n-1} R^{2r} z^{n-r-1} \varphi_r(z) = D_1(z) \\[2mm]
& \sum_{r=1}^{n-1} r R^{2(q_1+r-1)} z^{n-r} \varphi_r(z) = E_1(z) \\[2mm]
& \sum_{r=1}^{n-1} r R^{2r-2} z^{n-r} \frac{d}{dz} \varphi_r(z) = D_2(z) \\[2mm]
& \sum_{r=2}^{n-1} r(r-1) R^{2(q_2+r-2)} z^{n-r+1} \frac{d}{dz} \varphi_r(z) = E_2(z) \\[2mm]
& \qquad\qquad\qquad\qquad \vdots \\[2mm]
& \sum_{r=\left[\frac{n-1}{2}\right]}^{n-1} \frac{r!\, R^{2\left(r-\left[\frac{n+3}{2}\right]\right)}}{\left(r-\left[\frac{n+3}{2}\right]\right)!} z^{n+\left[\frac{n-3}{2}\right]-r} \frac{d^{\left[\frac{n-1}{2}\right]}}{dz^{\left[\frac{n-1}{2}\right]}} \varphi_r(z) = D_{\left[\frac{n+1}{2}\right]}(z) \\[2mm]
& \sum_{r=\left[\frac{n}{2}\right]}^{n-1} \frac{r!\, R^{2\left(q_{\left[\frac{n}{2}\right]}+r-\left[\frac{n}{2}\right]\right)}}{\left(r-\left[\frac{n}{2}\right]\right)!} z^{n+\left[\frac{n}{2}\right]-r-1} \frac{d^{\left[\frac{n}{2}\right]-1}}{dz^{\left[\frac{n}{2}\right]-1}} \varphi_r(z) = E_{\left[\frac{n}{2}\right]}(z).
\end{aligned}
\right\}
$$

We shall use them to compute the holomorphic functions φ_r $(r = 0, 1, \ldots, n - 1)$, which define the polyanalytic function Φ. For simplicity and without loss of generality, we shall assume that $R = 1$ and proceed to consider the system for various values of n.

The case $n = 1$: Then

$$\varphi_0 = D_1.$$

The case $n = 2$: Then

$$\left.\begin{aligned} z\varphi_0 + \varphi_1 &= D_1 \\ z\varphi_1 &= E_1 \end{aligned}\right\}.$$

Thus

$$\left.\begin{aligned} \varphi_0 &= D_1\frac{1}{z} - E_1\frac{1}{z^2} \\ \varphi_1 &= E_1\frac{1}{z} \end{aligned}\right\}.$$

The case $n = 3$: Then

$$\left.\begin{aligned} z^2\varphi_0 + z\varphi_1 + \varphi_2 &= D_1 \\ z^2\varphi_1 + 2z\varphi_2 &= E_1 \\ z^2\varphi_1' + 2z\varphi_2' &= D_2 \end{aligned}\right\}. \tag{10}$$

Differentiation of $(10)_2$ yields

$$z^2\varphi_1' + 2z\varphi_1 + 2z\varphi_2' + 2\varphi_2 = E_1'.$$

From this and $(10)_3$ we get

$$2z\varphi_1 + 2\varphi_2 = E' - D_2$$

which when combined with $(10)_2$ delivers

$$\left.\begin{aligned} \varphi_0(z) &= \frac{1}{z^2}\left(D_1 + D_2 - \frac{D_2 + E_1}{2}\right) \\ \varphi_1(z) &= \frac{1}{z^2}\left[(E_1' - D_2)z - E_1\right] \\ \varphi_2(z) &= \frac{1}{z}\left[E_1 - (E_1' - D_2)z\right] \end{aligned}\right\}.$$

The case $n = 4$: Then

$$\left.\begin{aligned} z^3\varphi_0 + z^2\varphi_1 + z\varphi_2 + \varphi_3 &= D_1 \\ z^3\varphi_1 + 2z^2\varphi_2 + 3z\varphi_3 &= E_1 \\ z^3\varphi_1' + 2z^2\varphi_2' + 3z\varphi_3' &= D_2 \\ 2z^3\varphi_2' + 6z^2\varphi_3' &= E_2 \end{aligned}\right\}. \tag{11}$$

We differentiate $(11)_2$ to obtain

$$z^3\varphi_1' + 3z^2\varphi_1 + 2z^2\varphi_2' + 4z\varphi_2 + 3z\varphi_3' + 3\varphi_3 = E_1'$$

which when combined with $(11)_3$ yields

$$3z^2 \varphi_1 + 4z\varphi_2 + 3\varphi_3 = E_1' - D_2.$$

This equation when combinded with $(11)_2$ deliver

$$\left.\begin{aligned}\varphi_1 &= \frac{1}{z^3}\left(z(E_1' - D_2) + 3z\varphi_3 - 2E_1\right)\\\varphi_2 &= \frac{1}{2z^2}\left(3E_1 - z(E_1' - D_2)\right)\end{aligned}\right\}.$$

If we now insert these values into $(11)_2$, we arrive at the equation

$$6z^2 \varphi_3' + 4zE_1' - z^2(E_1'' - D_2') - 6E_1 - zD_2 = E_2$$

which determines φ_3 and subsequently φ_2, φ_1 and φ_0.

   We proceed in the same way for higher values of n. Somehow we can always eliminate the higher derivatives, so that we do not have any differential equation to solve. The only problem that arises seems to be the poles for the φ_i's at the origin. In fact these functions do not have any poles there on account of the zeros of order not less than n the holomorphic functions D_k and E_j at that point (cf. equations $(3), (3)'$ and $(9), (9)'$).

   Suppose μ amongst the n classical Riemann problems of linear conjugation $(1), (7)$ have non-negative indices. Then the point conditions (4) on the functions E_j as well as the corresponding ones for the functions D_k can be formulated as point conditions on the polyanalytic function Φ. Suppose, for definiteness, the sum of the indices of the μ Riemann problems is N. Then we impose $N + \mu$ point conditions on Φ:

$$\Phi(\tau_j) = d_j \qquad (\tau_j \in D;\ j = 1, 2, \ldots, N + \mu). \tag{4$'$}$$

We next show that $\Phi \in W_{n,p}(D)\ (2 < p < \infty)$. We deduce from the defining formula for the holomorphic functions D_k that

$$\sum_{r=0}^{n-1} R^{2r} z^{n-r-1} \varphi_r(z) \in W_{n,p}(D) \qquad (2 < p < \infty),$$

i.e.

$$D_1(z) = \sum_{r=0}^{n-1} z^{n-1} \bar{z}^r \varphi_r(z)$$

and hence

$$\Phi(z) = \sum_{r=0}^{n-1} \bar{z}^r \varphi_r(z) \in W_{n,p}(D) \qquad (2 < p < \infty).$$

Moreover, we can deduce from the solutions of (7) estimates of the form

$$\left.\begin{aligned}\|\Phi\|_{p,D} &\leq C_0(p, D, H_j) \max_m \|f_m\|_{p,\partial D}\\\|\Phi\|_{k,p,D} &\leq C_k(p, D, H_j) \max_m \|f_m\|_{n+2-k-2j-\frac{1}{p},p,\partial D}\end{aligned}\right\} \tag{12}$$

where $k = 1, 2, \ldots, n$ and $2 < p < \infty$. Analogous estimates hold for the solutions of (1).

Theorem. *The generalized Riemann problem of linear conjugation* (G), (H), (4)′ *admits a uniquely defined polyanalytic function* Φ *in a simply-connected, bounded and smooth domain* D. *Moreover,* $\Phi \in W_{n,p}(D)$ $(2 < p < \infty)$ *and estimates of the form* (12) *hold.*

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