

The Generalized Riemann Problem of Linear Conjugation for Non-Homogeneous Polyanalytic Equations of Order n in $W_{n,p}(D)$

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Abstract. We consider a non-homogeneous polyanalytic partial differential equation of order n in a simply-connected domain D with smooth boundary ∂D in the complex plane \mathbb{C} . Initially we transform the given equation into an equivalent system of integro-differential equations and then find the general solution of the former in $W_{n,p}(D)$. Next we pose and prove the solvability of a generalized Riemann problem of linear conjugation to the differential equation. This is effected by first reducing the Riemann problem to a corresponding one for a polyanalytic function. The latter is solved by first transforming it into n classical Riemann problems of linear conjugation for n holomorphic functions expressed in terms of the analytic functions which define the polyanalytic function. The solution of the classical Riemann problem is available in the literature.

Keywords: *Polyanalytic functions, generalized Cauchy-Pompeiu integral operators of higher order, Riemann problem of linear conjugation*

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1. Introduction

We consider the following non-homogeneous polyanalytic partial differential equation of order n in a given simply-connected and bounded domain D in the complex plane \mathbb{C} :

$$\frac{\partial^n w}{\partial \bar{z}^n} = F\left(z, w, \left\{ \frac{\partial^{n+k} w}{\partial z^m \partial \bar{z}^k} \right\}\right) \quad \text{on } D \quad (1)$$

where $n \in \mathbb{N}$ and $k, m \in \mathbb{N}_0$ are such that $m + k \leq n$ and $(0, 0) \neq (m, k) \neq (0, n)$ and $\left\{ \frac{\partial^{m+k} w}{\partial z^m \partial \bar{z}^k} \right\}$ stands for the set of all possible partial derivatives of w , with respect to z or its complex conjugate \bar{z} , of order not exceeding n and excluding $\frac{\partial^n w}{\partial \bar{z}^n}$. Equation (1) may be viewed as a complex normal form of the following system of two partial differential

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equations of real-valued functions $u = u(x, y)$ and $v = v(x, y)$:

$$\sum_{j=0}^n \left(A_{n-j,j}^{(k)}(x, y) \frac{\partial^n u}{\partial x^{n-j} \partial y^j} + B_{n-j,j}^{(k)}(x, y) \frac{\partial^n v}{\partial x^{n-j} \partial y^j} \right) + \sum_{i+j=0}^{n-1} \left(C_{i,j}^{(k)}(x, y) \frac{\partial^{i+j} u}{\partial x^i \partial y^j} + D_{i,j}^{(k)}(x, y) \frac{\partial^{i+j} v}{\partial x^i \partial y^j} \right) = f_k(x, y)$$

($k = 1, 2$). The former is obtained from the latter through the introduction of the complex notations

$$z = x + iy, \quad w = u + iv, \quad F = f_1 + if_2$$

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

and if the assumptions of the existence theorem on implicit functions are satisfied.

We shall show that equation (1) has a unique general solution w for every prescribed in D polyanalytic function of order n , before we embark on solving a generalized version of the Riemann problem of linear conjugation.

2. The requisite integral operators

When solving elliptic equations of first order (cf. [7, 15, 24, 25, 27]) the Cauchy-Pompeiu operator T_D and the strongly singular Vekua integral operator Π_D :

$$T_D f(z) = -\frac{1}{\pi} \iint_D \frac{f(\zeta)}{\zeta - z} d\xi d\eta$$

$$\Pi_D f(z) = -\frac{1}{\pi} \iint_D \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta$$

$$\begin{pmatrix} z = x + iy \\ \zeta = \xi + i\eta \end{pmatrix}$$

and the operators related to them (cf. [8, 12, 26]) play an important role. When handling elliptic differential equations of higher order, the need arises either to reduce the given equation to one of first order or, alternatively, to generalize the integral operators so that they can be applied rationally to the higher order differential equations at hand. Begehr and Hile generalized the operators as follows (see [2, 5, 6]):

$$T_{m,n,D} f(z) = \iint_D K_{m,n}(z - \zeta) f(\zeta) d\xi d\eta \quad (m + n \geq 0) \tag{2}$$

where the kernel $K_{m,n}$ of the operator is defined as follows:

$$K_{m,n}(z) = \begin{cases} \frac{(-m)! (-1)^m}{(n-1)! \pi} z^{m-1} \bar{z}^{n-1} & \text{as } 0 \geq m \in \mathbb{Z} \\ \frac{(-n)! (-1)^n}{(m-1)! \pi} z^{m-1} \bar{z}^{n-1} & \text{as } 0 \geq n \in \mathbb{Z} \\ \frac{z^{m-1} \bar{z}^{n-1}}{(m-1)! (n-1)! \pi} \left(\log |z|^2 - \sum_{r=1}^{m-1} \frac{1}{r} - \sum_{s=1}^{n-1} \frac{1}{s} \right) & \text{as } m, n \in \mathbb{N}. \end{cases} \tag{3}$$

If $m = 1$ or $n = 1$, the corresponding sum is dropped. The case $m = n = 0$ is used to define the identity operator. Thus $T_{0,0,D}f = f$ for all $f \in L_p(D)$ ($1 < p < \infty$).

We note that the kernel $K_{m,n}$ has no singularity in \mathbb{C} , except possibly at the origin. It has at worst a pole of order two, when $m = -n$. In such a case the operator $T_{m,n,D}$ is strongly singular, and hence $T_{m,n,D}f$ exists solely in the sense of the Cauchy principal value if $f \in C_\alpha(D)$ ($0 < \alpha < 1$) or $f \in L_p(D)$ ($1 < p < \infty$). As such, $T_{m,n,D}$ can be seen as a generalization of the Vekua integral operator Π_D . It satisfies, moreover, the Zygmund-Calderon inequality (cf. [5, 6, 9, 12, 19, 21, 26]):

$$\begin{aligned} \|T_{m,n,D}f\|_p &\leq A_p \|f\|_{p,D} && \left(\begin{array}{l} m+n=0 \\ 1 < p < \infty \end{array} \right). \\ 1 = \|T_{m,n,D}\|_2 &\leq \|T_{m,n,D}\|_p \leq A_p \end{aligned}$$

If $m + n > 0$, then the operator $T_{m,n,p}$ has either no singularity in \mathbb{C} , or it has a pole of order 1 at $\zeta = z$. It can thus be viewed as a generalization of the Cauchy-Pompeiu operators T_D, \bar{T}_D, T_D^* or the potential operator P_D :

$$\begin{aligned} \bar{T}_D f(z) &= -\frac{1}{\pi} \iint_D \frac{f(\zeta)}{\zeta - \bar{z}} d\xi d\eta \\ T_D^* f(z) &= -\frac{1}{\pi} \iint_D \frac{f(\zeta)}{|\zeta - z|} d\xi d\eta \\ P_D f(z) &= \frac{2}{\pi} \iint_D f(\zeta) \log |\zeta - z| d\xi d\eta. \end{aligned}$$

Since $K_{m,n} \in L_1(D)$ if D has a finite area, we deduce from the convolution theorem of Young (see [19: p. 7], for instance) that $T_{m,n,D}f \in L_p(D)$ ($1 \leq p \leq \infty$), and an estimate of the form

$$\|T_{m,n,D}f\|_{p,D} \leq C(m, n, D) \|f\|_{p,D} \quad (1 \leq p \leq \infty; m+n > 0)$$

holds.

The integral operators $T_{m,n,D}$ with $m+n > 0$ and $m+n = 0$ demonstrate properties which are similar to those of the operators T_D and Π_D , respectively, in the Banach spaces $C_{k,\alpha}(D)$ ($0 < \alpha < 1$) and $W_{k,p}(D)$ ($1 < p < \infty$) for $k \in \mathbb{N}_0$ (see [2, 5, 6]). We write here two of such properties which will play an important role in our discussion:

1. If $f \in L_p(D)$ ($2 < p < \infty$) and $D \subset \mathbb{C}$ is a bounded domain, then $T_{-m,n,D}f$ exists, for $m, n \in \mathbb{N}_0$ with $0 \leq m < n$, in the Lebesgue sense for all $z \in \mathbb{C}$. In particular, for z with $|z| < R$ the estimates

$$\left. \begin{aligned} |T_{-m,n,D}f(z)| &\leq M_1(m, n, p, D, R) \|f\|_{p,D} \\ \|T_{-m,n,D}f\|_{p,D} &\leq M_2(m, n, p, D, R) \|f\|_{p,D} \end{aligned} \right\}$$

hold.

2. If $f \in L_p(D)$ ($2 < p < \infty$) and $D \subset \mathbb{C}$ is a bounded domain, then $T_{-m,n,D}f$ satisfies, for any $z_1, z_2 \in \mathbb{C}$ and $m, n \in \mathbb{N}_0$ with $0 \leq m < n$, the Hölder condition

$$|T_{-m,n,D}f(z_1) - T_{-m,n,D}f(z_2)| \leq M_3(m, n, p, D) \|f\|_{p,D} |z_1 - z_2|^\beta$$

where

$$\beta = \begin{cases} 1 & \text{if } n - m \geq 2 \\ 1 - \frac{2}{p} & \text{if } n - m = 1. \end{cases}$$

3. Transforming (1) into an integro-differential system

We consider the given non-homogeneous polyanalytic partial differential equation (1) in a simply-connected bounded domain $D \subset \mathbb{C}$. If $T_{m,k,D}$ denotes the generalized cauchy-Pompeiu operator (2), then $\frac{\partial}{\partial z} T_{m,k,D} f(z) = T_{m-1,k,D} f(z)$, $\frac{\partial}{\partial \bar{z}} T_{m,k,D} f(z) = T_{m,k-1,D} f(z)$ and, in general,

$$\frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} T_{m,k,D} f(z) = T_{m-i,k-j,D} f(z) \tag{4}$$

for $m, k \in \mathbb{Z}$ and $i, j \in \mathbb{N}_0$ with $m + k \geq i + j \geq 0$ in the Sobolev sense (see [5]).

Suppose w is a given solution of equation (1). We define a function Φ as follows:

$$\Phi(z) = w(z) - T_{0,n,D} F\left(\zeta, w(\zeta), \left\{ \frac{\partial^{m+k} w}{\partial \zeta^m \partial \bar{\zeta}^k} \right\}\right)(z).$$

Then, by virtue of (4), we find out immediately that

$$\begin{aligned} \frac{\partial^n \Phi}{\partial \bar{z}^n} &= \frac{\partial^n w}{\partial \bar{z}^n} - T_{0,0,D} F\left(\zeta, w(\zeta), \left\{ \frac{\partial^{m+k} w}{\partial \zeta^m \partial \bar{\zeta}^k} \right\}\right)(z) \\ &= F\left(z, w, \left\{ \frac{\partial^{m+k} w}{\partial z^m \partial \bar{z}^k} \right\}\right) - F\left(z, w, \left\{ \frac{\partial^{m+k} w}{\partial z^m \partial \bar{z}^k} \right\}\right) \quad (z \in D) \\ &= 0 \end{aligned}$$

in the Sobolev sense, and as such Φ is a polyanalytic function of order n in D (in some literature the term "n-analytic function" is used synonymously). We conclude, therefore, that any solution of equation (1) may be expressed in the form

$$w(z) = \Phi(z) + T_{0,n,D} F\left(\zeta, w(\zeta), \left\{ \frac{\partial^{m+k} w}{\partial \zeta^m \partial \bar{\zeta}^k} \right\}\right)(z) \tag{5}$$

where Φ is a polyanalytic function of order n in the domain D .

On differentiating (5) we obtain

$$h_{m,k}(z) := \frac{\partial^{m+k} w}{\partial z^m \partial \bar{z}^k} = \frac{\partial^{m+k} \Phi}{\partial z^m \partial \bar{z}^k} + T_{-m,n-k,D} F\left(\zeta, w(\zeta), \left\{ \frac{\partial^{r+s} w}{\partial \zeta^r \partial \bar{\zeta}^s} \right\}\right)(z).$$

where $n \in \mathbb{N}$ and $m, k \in \mathbb{N}_0$ are such that $m + k \leq n$ and $(0, 0) \neq (m, k) \neq (0, k)$. We are therefore in a position to formulate the following

Theorem 1. *A function w is a solution of the non-homogeneous polyanalytic partial differential equation (1) of order n if and only if, for a polyanalytic function Φ of order n in the domain $D \subset \mathbb{C}$, $\{w, h_{m,k}\}$ is a solution of the integro-differential system*

$$\left. \begin{aligned} w(z) &= \Phi(z) + T_{0,n,D} F\left(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\right)(z) \\ h_{m,k}(z) &= \frac{\partial^{m+k} \Phi}{\partial z^m \partial \bar{z}^k} + T_{-m,n-k,D} F\left(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\right)(z) \end{aligned} \right\} \tag{6}$$

where $n \in \mathbb{N}$ and $m, k \in \mathbb{N}_0$ are such that $m + k \leq n$ and $(0, 0) \neq (m, k) \neq (0, k)$.

4. The general solution

We make the following assumptions on the right-hand side of the non-homogeneous polyanalytic partial differential equation (1):

- (A1) $F(z, w, \{h_{m,k}\})$ is a continuous function of its variables $z \in D$, w and the partial derivatives of w of order not exceeding n and excluding $\frac{\partial^n w}{\partial \bar{z}^n}$.
- (A2) There exists a tuple $(w^*, \{h_{m,k}^*\})$, $w^*, h_{m,k}^* \in L_p(D)$ ($2 < p < \infty$), such that $F(z, w^*, \{h_{m,k}^*\}) \in L_p(D)$.
- (A3) $F(z, w, \{h_{m,k}\})$ satisfies a Lipschitz condition of the form

$$\begin{aligned} &|F(z, w(z), \{h_{m,k}(z)\}) - F(z, \hat{w}(z), \{\hat{h}_{m,k}(z)\})| \\ &\leq L_1 \max \left\{ |w(z) - \hat{w}(z)|, \max_{m+k < n} |h_{m,k}(z) - \hat{h}_{m,k}(z)| \right\} \\ &\quad + L_2 \max_{m+k=n} |h_{m,k}(z) - \hat{h}_{m,k}(z)| \end{aligned}$$

a.e. on D , if $w, \hat{w}, h_{m,k}, \hat{h}_{m,k} \in L_p(D)$. Whereas $0 < L_2 < 1$, the constant L_1 may take any positive value.

Remark. Assumptions (A1) - (A3) guarantee that $F(z, w, \{h_{m,k}\}) \in L_p(D)$ whenever $h_{m,k} \in L_p(D)$ ($2 < p < \infty$). Indded, we have

$$\begin{aligned} &\|F(z, w, \{h_{m,k}\})\|_{p,D} \\ &\leq \|F(z, w^*, \{h_{m,k}^*\})\|_{p,D} \\ &\quad + L_1 \max \left\{ \|w - w^*\|_{p,D} + \max_{m+k < n} \|h_{m,k} - h_{m,k}^*\|_{p,D} \right\} \\ &\quad + L_2 \max_{m+k=n} \|h_{m,k} - h_{m,k}^*\|_{p,D}. \end{aligned}$$

We next introduce the Banach space $\mathcal{L}_p(D)$ ($2 < p < \infty$):

$$\mathcal{L}_p(D) = \left\{ (w, \{h_{m,k}\}) \left| \begin{aligned} &w, h_{m,k} \in L_p(D), n \in \mathbb{N}, m, k \in \mathbb{N}_0 \text{ with } m, k \leq n, (0, 0) \neq (m, k) \neq (0, n) \\ &\|(w, \{h_{m,k}\})\|_{p,D} = \max_{\gamma > 0} \left\{ \gamma \|w\|_{p,D}, \gamma \max_{m+k < n} \|h_{m,k}\|_{p,D}, \max_{m+k=n} \|h_{m,k}\|_{p,D} \right\} \right. \end{aligned} \right\}.$$

We define a mapping P in $\mathcal{L}_p(D)$ through the right-hand-side of (5). Namely, for any tuple $(w, \{h_{m,k}\}) \in \mathcal{L}_p(D)$ we set

$$P(w, \{h_{m,k}\}) = (W, \{H_{m,k}\})$$

with

$$\left. \begin{aligned} W(z) &= \Phi(z) + T_{0,n,D}F(\zeta, w(\zeta), \{h_{m,k}(\zeta)\})(z) \\ H_{m,k}(z) &= \frac{\partial^{m+k}\Phi}{\partial z^m \partial \bar{z}^k} + T_{-m,n-k,D}F(\zeta, w(\zeta), \{h_{m,k}(\zeta)\})(z) \end{aligned} \right\} \tag{7}$$

where $\Phi \in W_{n,p}(D)$ is a polyanalytic function of order n in D , $n \in \mathbb{N}$ and $m, k \in \mathbb{N}_0$ are such that $m + k \leq n$ and $(0, 0) \neq (m, k) \neq (0, n)$. Surely, P maps the Banach space $\mathcal{L}_p(D)$ into itself.

Let $(W, \{H_{m,k}\})$ and $(\hat{W}, \{\hat{H}_{m,k}\})$ be the images of $(w, \{h_{m,k}\})$ and $(\hat{w}, \{\hat{h}_{m,k}\})$ under the mapping P , respectively. Then

$$\begin{aligned} \gamma \|W - \hat{W}\|_{p,D} &\leq \gamma \|T_{0,n,D}\|_p \|F(z, w, \{h_{m,k}\}) - F(z, \hat{w}, \{\hat{h}_{m,k}\})\|_{p,D} \\ &\leq \gamma \|T_{0,n,D}\|_p L_1 \max \left\{ \|w - \hat{w}\|_{p,D}, \max_{m+k < n} \|h_{m,k} - \hat{h}_{m,k}\|_{p,D} \right\} \\ &\quad + \gamma \|T_{0,n,D}\| L_2 \max_{m+k=n} \|h_{m,k} - \hat{h}_{m,k}\|_{p,D} \\ &\leq \|T_{0,n,D}\|_p (L_1 + \gamma L_2) \|(w, \{h_{m,k}\}) - (\hat{w}, \{\hat{h}_{m,k}\})\|_{p,D}. \end{aligned}$$

Similarly we obtain

$$\left. \begin{aligned} \gamma \|H_{m,k} - \hat{H}_{m,k}\|_{p,D} &\leq \|T_{-m,n-k,D}\|_p (L_1 + \gamma L_2) \|(w, \{h_{m,k}\}) - (\hat{w}, \{\hat{h}_{m,k}\})\|_{p,D} \\ \|H_{\alpha,\beta} - \hat{H}_{\alpha,\beta}\|_{p,D} &\leq \|T_{-\alpha,n-\beta,D}\|_p \left(\frac{1}{\gamma} L_1 + L_2\right) \|(w, \{h_{m,k}\}) - (\hat{w}, \{\hat{h}_{m,k}\})\|_{p,D} \end{aligned} \right\}$$

for $0 < m + k < n$, $\alpha + \beta = n$ and $(\alpha, \beta) \neq (0, n)$. However, we know from properties of the integral operators $T_{m,n,D}$ that

$$\|T_{-m,n-k,D}\|_p = \begin{cases} C(m, k, n, D) & \text{for } 0 < m + k < n \\ \|\Pi_D\|_p & \text{for } m + k = n \end{cases} \quad (8)$$

where Π_D is the strongly singular Vekua integral operator. We thus conclude that

$$\begin{aligned} &\|(W, \{H_{m,k}\}) - (\hat{W}, \{\hat{H}_{m,k}\})\|_{p,D} \\ &\leq \max \left\{ \gamma \|T_{0,n,D}\|_p, \gamma \max_{m+k < n} \|T_{-m,n-k,D}\|_p, \|\Pi_D\|_p \right\} \\ &\quad \times \left(\frac{1}{\gamma} L_1 + L_2\right) \|(w, \{h_{m,k}\}) - (\hat{w}, \{\hat{h}_{m,k}\})\|_{p,D} \end{aligned}$$

and P turns out to be contractive in $\mathcal{L}_p(D)$ ($2 < p < \infty$) if

$$\max \left\{ \gamma \|T_{0,n,D}\|_p, \gamma \max_{m+k < n} \|T_{-m,n-k,D}\|_p, \|\Pi_D\|_p \right\} \left(\frac{1}{\gamma} L_1 + L_2\right) < 1. \quad (9)$$

If estimate (9) is realizable, then by the Banach fixed point theorem P has a unique fixed point $(w, \{h_{m,k}\}) \in \mathcal{L}_p(D)$ ($2 < p < \infty$) satisfying (6). By Theorem 1, the corresponding w is the general solution of equation (1) corresponding to the prescribed in the domain D polyanalytic function $\Phi \in W_{n,p}(D)$ of order n .

Restriction (9) can be satisfied if the constants L_1, L_2 and γ are chosen suitably and the domain D is sufficiently small. It is well-known that

- i) $\|\Pi_D\|_p \geq \|\Pi_D\|_2 = 1$ ($1 < p < \infty$)
- ii) $\|\Pi_D\|_p$ is a logarithmic convex function of $p \in (1, \infty)$

(cf. [5, 6, 9, 12, 13, 19, 20, 26]).

Suppose that L_2 satisfies the inequality $L_2\|\Pi_D\|_p < 1$ (i.e. $0 < L_2 < 1$). We then fix the constant γ large enough so that, for given L_1 , the relation $(\frac{1}{\gamma}L_1 + L_2)\|\Pi_D\|_p < 1$ is also valid. Finally, since $\|T_{0,n,D}\|_p$ and $\|T_{-m,n-k,D}\|_p$ for $m + k < n$ increase with the area of the domain D , a proper reduction of the size of D ensures the fulfilment of estimate (9), and hence the existence of a unique fixed point of the operator P in $\mathcal{L}_p(D)$ ($2 < p < \infty$).

Theorem 2. *Assuming the validity of assumptions (A1) - (A3) and estimate (9), the non-homogeneous polyanalytic partial differential equation (1) of order n admits a unique solution $w \in W_{n,p}(D)$ ($2 < p < \infty$) given by equation (5), for every prescribed in the bounded domain D polyanalytic function $\Phi \in W_{n,p}(D)$ ($2 < p < \infty$) of order n .*

It can also be shown that the operator \mathcal{R} :

$$\mathcal{R}\left(\Phi, \left\{ \frac{\partial^{m+k}\Phi}{\partial z^m \partial \bar{z}^k} \right\}\right) = (w, \{h_{m,k}\})$$

is a topological mapping of the set H of all in the domain D polyanalytic functions $\Phi \in W_{n,p}(D)$ of order n onto the set W of all solutions $(w, \{h_{m,k}\}) \in \mathcal{L}_p(D)$ of the integro-differential system (6). In the light of Theorem 1 we can thus formulate the following

Theorem 3. *The operator \mathcal{R} is a topological mapping of the set H of all in the domain D polyanalytic functions $\Phi \in W_{n,p}(D)$ of order n onto the set W of all solutions $w \in W_{n,p}(D)$ of the complex partial differential equation (1).*

5. The generalized Riemann problem of linear conjugation for non-homogeneous polyanalytic equations

We now pose the following generalized Riemann problem of linear conjugation for the function w :

$$\frac{\partial^n w}{\partial \bar{z}^n} = F\left(z, w, \left\{ \frac{\partial^{n+k} w}{\partial z^m \partial \bar{z}^k} \right\}\right) \quad \text{on } D \tag{1}$$

$$\left[\bar{z}^{q_j} \frac{\partial^{2j-1} w}{\partial z^{j-1} \partial \bar{z}^j} \right]^+ (t) - G_j(t) \left[\bar{z}^{q_j} \frac{\partial^{2j-1} w}{\partial z^{j-1} \partial \bar{z}^j} \right]^- (t) = g_j(t) \quad \text{on } \partial D \tag{G}$$

$$\left[\frac{\partial^{2s-2} w}{\partial z^{s-1} \partial \bar{z}^{s-1}} \right]^+ (t) - H_s(t) \left[\frac{\partial^{2s-2} w}{\partial z^{s-1} \partial \bar{z}^{s-1}} \right]^- (t) = f_s(t) \quad \text{on } \partial D \tag{H}$$

where $q_j \in \mathbb{Z}$,

$$\left. \begin{aligned} g_j, G_j &\in W_{n-2j+1-\frac{1}{p},p}(\partial D) \\ f_s, H_s &\in W_{n-2s+2-\frac{1}{p},p}(\partial D) \end{aligned} \right\} \quad (2 < p < \infty)$$

and

$$j, s \in \mathbb{N}, n \in \mathbb{N}_0 \text{ with } 2j < n + 1 \text{ and } 2s < n + 2$$

$k, m, n \in \mathbb{N}_0$ with $m + k \leq n$ and $(0, 0) \neq (m, k) \neq (0, n)$.

We know that the general solution of equation (1) takes the form (5), where Φ is an arbitrary polyanalytic function of order n in the given domain D . We thus first substitute the jump conditions (G) in the general solution (5) and thus arrive at the following corresponding problem for the polyanalytic function Φ :

$$\begin{aligned} & \left[\bar{z}^{q_j} \frac{\partial^{2j-1} \Phi}{\partial z^{j-1} \partial \bar{z}^j} \right]^+ (t) - G_j(t) \left[\bar{z}^{q_j} \frac{\partial^{2j-1} \Phi}{\partial z^{j-1} \partial \bar{z}^j} \right]^- (t) \\ &= g_j(t) + G_j(t) \bar{t}^{q_j} \left[\frac{\partial^{2j-1}}{\partial z^{j-1} \partial \bar{z}^j} T_{0,n,D} F \right]^- (t) - \bar{t}^{q_j} \left[\frac{\partial^{2j-1}}{\partial z^{j-1} \partial \bar{z}^j} T_{0,n,D} F \right]^+ (t) \quad (G)' \\ &= g_j(t) + (G_j(t) - 1) \bar{t}^{q_j} T_{1-j,n-j,D} F(\cdot, w, \{h_{m,k}\})(t) \end{aligned}$$

on ∂D , since $T_{m,k,D} F$ is a continuous function on the whole complex plane whenever $F \in L_p(D)$, $2 < p < \infty$ and $m + k > 0$. Similarly, we obtain from the jump conditions (H)

$$\begin{aligned} & \left[\frac{\partial^{2s-2} \Phi}{\partial z^{s-1} \partial \bar{z}^{s-1}} \right]^+ (t) - H_s(t) \left[\frac{\partial^{2s-2} \Phi}{\partial z^{s-1} \partial \bar{z}^{s-1}} \right]^- (t) \quad (H)' \\ &= f_s(t) + (H_s(T) - 1) T_{1-s,n-s+1,D} F(\cdot, w, \{h_{m,k}\})(t) \end{aligned}$$

on ∂D . We deduce from the properties of the integral operators $T_{m,k,D}$ in $L_p(D)$ ($1 < p < \infty$) that $T_{1-j,n-j,D} F \in W_{n-2j+1,p}(D)$ and $T_{1-s,n-s+2,D} F \in W_{n-2s+2,p}(D)$, and hence, by the trace theorem (see [1, 11, 22, 23]),

$$T_{1-j,n-j,D} F \in W_{n-2j+1-\frac{1}{p},p}(\partial D) \quad \text{and} \quad T_{1-s,n-s+2,D} F \in W_{n-2s+2-\frac{1}{p},p}(\partial D).$$

We next split the polyanalytic function Φ into two such functions, $\Phi = \Phi_1 + \Phi_{(w,h)}$, so that the jump conditions (G)' and (H)' are written as generalized Riemann problems of linear conjugation for Φ_1 and $\Phi_{(w,h)}$:

$$\left. \begin{aligned} & \left[\bar{z}^{q_j} \frac{\partial^{2j-1} \Phi_1}{\partial z^{j-1} \partial \bar{z}^j} \right]^+ (t) - G_j(t) \left[\bar{z}^{q_j} \frac{\partial^{2j-1} \Phi_1}{\partial z^{j-1} \partial \bar{z}^j} \right]^- (t) = g_j(t) \\ & \left[\frac{\partial^{2s-2} \Phi_1}{\partial z^{s-1} \partial \bar{z}^{s-1}} \right]^+ (t) - H_s(t) \left[\frac{\partial^{2s-2} \Phi_1}{\partial z^{s-1} \partial \bar{z}^{s-1}} \right]^- (t) = f_s(t) \end{aligned} \right\} \quad (10)$$

and

$$\left. \begin{aligned} & \left[\bar{z}^{q_j} \frac{\partial^{2j-1} \Phi_{(w,h)}}{\partial z^{j-1} \partial \bar{z}^j} \right]^+ (t) - G_j(t) \left[\bar{z}^{q_j} \frac{\partial^{2j-1} \Phi_{(w,h)}}{\partial z^{j-1} \partial \bar{z}^j} \right]^- (t) \\ &= (G_j(t) - 1) \bar{t}^{q_j} T_{1-j,n-j,D} F(\cdot, w, \{h_{m,k}\})(t) \\ & \left[\frac{\partial^{2s-2} \Phi_{(w,h)}}{\partial z^{s-1} \partial \bar{z}^{s-1}} \right]^+ (t) - H_s(t) \left[\frac{\partial^{2s-2} \Phi_{(w,h)}}{\partial z^{s-1} \partial \bar{z}^{s-1}} \right]^- (t) \\ &= (H_s(T) - 1) T_{1-s,n-s+1,D} F(\cdot, w, \{h_{m,k}\})(t) \end{aligned} \right\} \quad (11)$$

on ∂D . Such generalized Riemann problems of linear conjugation (10) - (11) were posed and solved by the author in [17], assuming that $(w, \{h_{m,k}\})$ is given. It was shown there

that such a problem admits a uniquely defined in the given domain D polyanalytic function Φ of order n , provided that sufficient point conditions on it are imposed. The solution is an element in the Sobolev space $W_{n,p}(D)$ ($2 < p < \infty$), and it satisfies estimates of the form

$$\|\Phi\|_{p,D} \leq C_0(p, D, Q_j) \max \|f_j\|_{p,\partial D}$$

and, for $1 \leq k \leq n$,

$$\|\Phi\|_{k,p,D} \leq C_k(p, D, Q_j) \max \|f_j\|_{s,p,\partial D}$$

where

$$s = \begin{cases} n + 2 - k - 2j - \frac{1}{p} \\ \text{or} \\ n + 1 - k - 2j - \frac{1}{p}, \end{cases}$$

Q_j and f_j are the coefficients and the free terms, respectively, in the respective Riemann problem.

We next revisit the mapping P defined in Section 4, whereby the arbitrary but fixed polyanalytic function $\Phi \in W_{n,p}(D)$ is now replaced by the one constructed above. Thus, for any tuple $(w, \{h_{m,k}\}) \in \mathcal{L}_p(D)$ we set

$$P(w, \{h_{m,k}\}) = (W, \{H_{m,k}\})$$

with

$$\left. \begin{aligned} W(z) &= \Phi_1(z) + \Phi_{(w,h)}(z) + T_{0,n,D}F(\zeta, w(\zeta), \{h_{m,k}(\zeta)\})(z) \\ H_{m,k}(z) &= \frac{\partial^{m+k}}{\partial z^m \partial \bar{z}^k} (\Phi_1 + \Phi_{(w,h)})(z) + T_{-m,n-k,D}F(\zeta, w(\zeta), \{h_{m,k}(\zeta)\})(z) \end{aligned} \right\}$$

where $\Phi_1, \Phi_{(m,k)} \in W_{n,p}(D)$ are polyanalytic functions of order n in D , $n \in \mathbb{N}$ and $m, k \in \mathbb{N}_0$ are such that $m + k \leq n$ and $(0, 0) \neq (m, k) \neq (0, n)$.

Let

$$(w, \{h_{m,k}\}), (\hat{w}, \{\hat{h}_{m,k}\}) \in \mathcal{L}_p(D) \quad (2 < p < \infty)$$

and let $(W, \{H_{m,k}\}), (\hat{W}, \{\hat{H}_{m,k}\})$ be their respective images under the mapping P . We then obtain

$$\begin{aligned} &\gamma \|W - \hat{W}\|_{p,D} \\ &\leq \gamma \|T_{0,n,D}\|_p \|F(z, w, \{h_{m,k}\}) - F(z, \hat{w}, \{\hat{h}_{m,k}\})\|_{p,D} + \gamma \|\Phi_{(w,h)} - \Phi_{(\hat{w},\hat{h})}\|_{p,D} \\ &\leq \gamma (C_1(p, D) \|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p) \\ &\quad \times L_1 \max \left\{ \|w - \hat{w}\|_{p,D}, \max_{m+k < n} \|h_{m,k} - \hat{h}_{m,k}\|_{p,D} \right\} \\ &\quad + \gamma (C_1(p, D) \|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p) L_2 \max_{m+k=n} \|h_{m,k} - \hat{h}_{m,k}\|_{p,D} \\ &\leq \gamma (C_1(p, D) \|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p) (L_1 + \gamma L_2) \|(w, \{h_{m,k}\}) - (\hat{w}, \{\hat{h}_{m,k}\})\|_{p,D}. \end{aligned}$$

Similarly we arrive at

$$\begin{aligned} \gamma \|H_{m,k} - \hat{H}_{m,k}\|_{p,D} &\leq (C_{m,k}(p, D) \|T_{0,n,D}\|_{n-m-k+1,p} + \|T_{-m,n-k,D}\|_p) \\ &\quad \times (L_1 + \gamma L_2) \|(w, \{h_{m,k}\}) - (\hat{w}, \{\hat{h}_{m,k}\})\|_{p,D} \end{aligned}$$

and

$$\begin{aligned} \|H_{\alpha,\beta} - \hat{H}_{\alpha,\beta}\|_{p,D} &\leq (C_n(p, D) \|T_{0,n,D}\|_{1,p} + \|T_{-\alpha,n-\beta,D}\|_p) \\ &\quad \times \left(\frac{1}{\gamma} L_1 + L_2\right) \|(w, \{h_{m,k}\}) - (\hat{w}, \{\hat{h}_{m,k}\})\|_{p,D} \end{aligned}$$

for $0 < m + k < n$, $\alpha + \beta = n$ and $(\alpha, \beta) \neq (0, n)$. Consequently,

$$\|(W, \{H_{m,k}\}) - (\hat{W}, \{\hat{H}_{m,k}\})\| \leq \kappa \|(w, \{h_{m,k}\}) - (\hat{w}, \{\hat{h}_{m,k}\})\|$$

where

$$\kappa = \left(\frac{1}{\gamma} L_1 + L_2\right) \max \begin{cases} \gamma C_1(p, D) \|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p \\ \gamma \max_{m+k < n} \{C_{m,k}(p, D) \|T_{0,n,D}\|_{n-m-k+1,p} + \|T_{-m,n-k,D}\|_p\} \\ C_n(p, D) \|T_{0,n,D}\|_{1,p} + \|\Pi_D\|_p \end{cases}$$

The mapping P is thus contractive in $\mathcal{L}_p(D)$ if $\kappa < 1$, and in this case it has a unique fixed point $(w, \{h_{m,k}\}) \in \mathcal{L}_p(D)$ ($2 < p < \infty$), thanks to the Banach fixed point theorem. By Theorem 1 and the construction of the polyanalytic functions Φ_1 and $\Phi_{(w,h)}$, w of the fixed point $(w, \{h_{m,k}\})$ solves the generalized Riemann problem of linear conjugation (1), (G), (H).

The contractiveness of P can be arrived at in essentially the same manner as the one discussed earlier in the case of a general solution. It is noteworthy in this exercise that all constants $C_1(p, D)$, $C_{m,k}(p, D)$ and $C_n(p, D)$ increase with the size of the domain D .

Theorem 4. *If assumptions (A1) - (A3) are satisfied and a requisite number of point conditions are given, then the generalized Riemann problem of linear conjugation (1), (G), (H) admits a unique solution $w \in W_{n,p}(D)$ ($2 < p < \infty$).*

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