Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 20 (2001), No. 3, 617–636

$C^{1,\alpha}$ Local Regularity for the Solutions of the p-Laplacian on the Heisenberg Group for $2 \leq p < 1 + \sqrt{5}$

S. Marchi

Abstract. We prove local Hölder continuity of the homogeneous gradient for weak solutions $u \in W^{1,p}_{loc}$ of the p-Laplacian on the Heisenberg group $\mathbb{H}^n$ for $2 \leq p < 1 + \sqrt{5}$.

Keywords: *Degenerate elliptic equations, weak solutions, regularity of solutions, higher differentiability*

AMS subject classification: 35D10, 35J60, 35J70

1. Introduction

The purpose of this paper is to prove local Hölder continuity of the gradient of local weak solutions $u \in W^{1,p}_{loc}(\Omega, X)$ $(2 \leq p < 1 + \sqrt{5})$ of the equation

$$\mathrm{div}_{\mathbb{H}} \vec{a}(Xu) = 0 \tag{1}$$

where

- $\mathrm{div}_{\mathbb{H}} \vec{a}(Xu) = \sum_{k=1}^{2n} X_k a^k(Xu)$
- $a^k(q) = |q|^{p-2} q_k$ $(k = 1, ..., 2n)$
- Ω is an open subset of the Heisenberg group $\mathbb{H}^n$
- X_k $(k = 1, ..., 2n)$ are vector fields generating the corresponding Lie algebra with their commutators up to the first order
- $Xu = (X_1 u, ..., X_{2n} u)$.

Let us recall the definitions of the needed functional spaces (see [7]). For any positive integer i, let us set $s = (s_1, ..., s_i)$, where $s_1, \ldots, s_i \in \{1, ..., 2n\}$, and set $|s| = i$. Let us denote by X_s the operator $X_{s_1} \cdots X_{s_i}$. For any $q \geq 1$ and any positive integer j, $W^{j,q}(\mathbb{H}^n, X)$ denotes the set of functions $f \in L^q(\mathbb{H}^n)$ such that $X_s f \in L^q(\mathbb{H}^n, X)$ for $|s| \leq j$, with norm $\|f\|_{j,q} = \|f\|_{L^q(\mathbb{H}^n)} + \sum_{|s| \leq j} \|X_s f\|_{L^q(\mathbb{H}^n)}$. Further, $W^{j,q}_{loc}(\Omega, X)$ is the set of functions f such that $\varphi f \in W^{j,q}(\mathbb{H}^n, X)$ for any $\varphi \in C^\infty_0(\Omega)$.

We say that $u \in W^{1,p}_{loc}(\Omega, X)$ is a *local weak solution* of equation (1) if

$$\int_\Omega a^k(Xu)\, X_k(\varphi)\, dx = 0 \tag{2}$$

S. Marchi: Univ. of Parma, Dept. Math., IT-43100 Parma; silvana.marchi@prmat.unipr.it
This work has been performed as a part of a National Research Project supported by MURST

ISSN 0232-2064 / $ 2.50

for all $\varphi \in W^{1,p}(\Omega, X)$ with $\operatorname{supp}\varphi \subset \Omega$.

We can now state the main results of this paper. From now on Ω' will denote an arbitrary open bounded subset of Ω such that $\Omega' \subset\subset \Omega$.

Theorem 1.1. *Let $u \in W^{1,p}_{loc}(\Omega, X)$ $(2 \le p < 1+\sqrt{5})$ be a local weak solution of equation (1). Then for any $\sigma \in (0,1)$ there exists a constant $\gamma(\sigma) > 0$ depending only on σ and the data such that, for any homogeneous ball $B(R) \subset\subset \Omega'$,*

$$\|Xu\|_{\infty, B(R-\sigma R)} \le \gamma(\sigma)\left(\tfrac{1}{|B(R)|}\int_{B(R)} |Xu|^p dx\right)^{\frac{1}{p}}. \tag{3}$$

In particular, $|Xu| \in L^\infty_{loc}(\Omega')$, and for every compact $K \subset \Omega'$ there exists a constant $C_0 > 0$ depending only on the data and $\operatorname{dist}(K, \partial\Omega')$ such that $\|Xu\|_{\infty,K} \le C_0$.

If a function w is bounded on a set E, then we will set

$$\operatorname{osc} w = \sup_E w - \inf_E w.$$

Theorem 1.2. *Let $u \in W^{1,p}_{loc}(\Omega, X)$ $(2 \le p < 1+\sqrt{5})$ be a local weak solution of equation (1). Then, any homogeneous ball $B(R) \subset\subset \Omega'$, there exists constants $\nu > 0$ and $\eta \in (0,1)$ depending only on the data and $\operatorname{dist}(B(R), \partial\Omega')$ such that*

$$\max_{i=1,\dots,2n} \operatorname{osc}_{B(\rho)} X_i u \le \nu\left(\tfrac{\rho}{R}\right)^\eta \sup_{B(\frac{R}{2})} |Xu| \tag{4}$$

for all $\rho < \frac{R}{2}$. In particular, Xu is locally Hölder continuous in Ω', i.e. for every compact $K \subset \Omega'$ there exist a constant $C_1 > 0$ and $\alpha \in (0,1)$ depending only on the data and $\operatorname{dist}(K, \partial\Omega')$ such that

$$|Xu(x) - Xu(y)| \le C_1 d(x,y)^\alpha \qquad (x, y \in K)$$

where d denotes the homogeneous distance associated to $\mathbb{H}^n$.

Theorems 1.1 and 1.2 partially extend to the Heisenberg setting a well-known property of the classical p-Laplacian and even of more general equations without any restrictions on p. Their proofs always rely on some kind of differentiation of the equation. We recall among others the papers of K. Uhlenbeck [24], N. N. Ural'tzeva [25], L. Evans [6] for $p \ge 2$ and those of P. Tolksdorf [23], E. Di Benedetto [5] and J. L. Lewis [14] for $1 < p < +\infty$.

In our context such an approach is complicated by the lack of commutativity of the vector fields, and therefore more care must be put in the relative procedure. For $p = 2$ the differentiation of equations similar to (1) in the Heisenberg group has been treated in particular by L. Capogna [1, 2] and by A. Cutrí and M. G. Garroni [4]. In [4] the authors prove a local estimate of the second order horizontal derivatives of the solutions of the equation $-\Delta_H u = f$, where Δ_H is the Heisenberg Laplacian and $f \in L^2_{loc}(\Omega)$, in order to establish $W^{2,2}$ local regularity of the solutions of certain integro-differential equations, but the existence of the second order derivatives is known and exploited. In [1] L. Capogna proves the differentiability of some nonlinear Heisenberg equations $\operatorname{div}_H \vec{a}(x, Xu) = f$, including the Heisenberg Laplacian, and even the Hölder

continuity of the homogeneous gradients of their solutions. Via a refined technique using the Baker-Campbell-Hausdorff's formula he proves the differentiability along the commutators' direction and uses the relative estimate to "differentiate" the equation. At this point he can gain even the second goal by standard methods.

If $p \neq 2$, the degeneracy becames stronger and the method of [1] runs into difficulties. In this case we think it right to rely on approximate arguments introducing the regularized equations

$$\operatorname{div}_H \vec{a}_\varepsilon(Xu_\varepsilon) = 0 \tag{5}$$

for small $\varepsilon > 0$, where $\vec{a}_\varepsilon(q) = [(\varepsilon + |q|^2)^{\frac{p-2}{2}} q]$. We apply to them a trick from [4] which enables us to commute the vector fields with double difference quotiens. Thanks to this tool we can then apply standard techniques developped in [9] together with a local L^p estimate of Tu_ε, where T is the first commutator of the vector fields (see Theorem 7.1), and conclude about the $W^{2,2}$ local regularity for u_ε (see Theorem 4.1). Let us just remark that in Theorem 4.1 the $W^{2,2}$ local integrability of u_ε is not uniform in ε, but this is enough to make everything work.

Now weak solutions are actually strong solutions and it is therefore possible to differentiate equations (5) and prove local boundedness and Hölder continuity of Xu_ε by the methods of [5]. These are the contents of Theorems 5.2 and 6.5, respectively. At this point these estimates are uniform in ε and this allows us to conclude about u in Theorems 1.1 and 1.2 by standard arguments, possibly up to subsequences [13, 14].

There is a strong limitation on the range of admissibility for p. This comes from Theorem 7.1: actually, we do not know if it holds for $p \geq 1 + \sqrt{5}$. It is not clear to us whether this is just a technical limitation linked to the method used (a different technique could improve the result). However, besides its employ in Section 4, Theorem 7.1 furnishes an estimate which, as far as we know, is new in literature for $p \neq 2$. Its proof is mainly founded on the Baker-Campbell-Hausdorff formula and some arguments from the theory of function spaces.

Finally, we observe that, as it happens in the Euclidean setting, Theorems 1.1 and 1.2 can be employed to obtain $C^{1,\alpha}$ local regularity for the solutions of obstacle problems, even for more general operators.

The plan of the work is the following: in Section 2 we recall the basics about Heisenberg group and in Section 3 we present and prove some introductory lemmas. Section 4 is concerned with the $W^{2,2}$ local regularity for u_ε. Sections 5 and 6 are devoted to prove Theorems 5.2 and 6.5. Finally, in Section 7 we prove Theorem 7.1.

2. Basic knowledge

The Heisenberg group $\mathbb{H}^n$ is the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$ with the following group law: for all $x = (x', t) = (x_1, ..., x_{2n}, t)$ and $y = (y', s) = (y_1, ..., y_{2n}, s)$,

$$x \circ y = \big(x' + y', t + s + 2[x', y']\big)$$

where $[x', y'] = \sum_{i=1}^{n} (y_i x_{i+n} - x_i y_{i+n})$. This is a homogeneous group, that is a group with dilations, defined as $\delta_\lambda(x', t) = (\lambda x', \lambda^2 t)$ where the direction t plays a particular

role (the space is non-isotropic) corresponding to the definition of the group action. A norm for $\mathbb{H}^n$ which is homogeneous of degree 1 with respect to the dilations can be given by

$$|x|^4 = |(x', t)|^4 = |x'|^4 + t^2 \qquad \big(x = (x', t) \in \mathbb{H}^n\big),$$

and

$$d(x, y) = |y^{-1} \circ x| \qquad \big(x, y \in \mathbb{H}^n, y^{-1} = -y\big)$$

is then the associated distance. $B(x, r)$ will denote the homogeneous ball centered in $x \in \mathbb{H}^n$ with radius $r > 0$.

For every function w defined on $\mathbb{H}^n$, both left and right translations are defined on $\mathbb{H}^n$ as

$$L_y w(x) = w(y \circ x)$$
$$R_y w(x) = w(x \circ y).$$

The Lebesgue measure is invariant with respect to the translations of the group, though the shape of the ball changes if one shifts its center, and it is proportional to the Q-th power of the radius, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$, that is $|B(x, r)| \simeq r^Q |B(0, 1)|$.

An operator N on $\mathbb{H}^n$ is *left-invariant* if $L_y(Nw) = N(L_y w)$, and similarly for *right invariance*. The Lie algebra $\mathcal{L}(X)$ of left-invariant vector fields corresponding to $\mathbb{H}^n$ is generated by

$$X_i = \partial_{x_i} + 2x_{i+n}\partial_t$$
$$X_{i+n} = \partial_{x_{i+n}} - 2x_i\partial_t$$
$$T = -4\partial_t$$

for $i = 1, ..., n$. Since $[X_i, X_{i+n}] = -[X_{i+n}, X_i] = T \quad (i = 1, ..., n)$ and $[X_j, X_k] = 0$ in any other case, the vector fields $X_i \quad (i = 1, ..., 2n)$ satisfy the Hörmander condition of order 1 [10], that is together with their first order commutators they span the whole Lie algebra.

The vector fields X_i do not commute with right translations. In particular, we cannot interchange them with difference quotiens operators

$$D_h w(x) = \frac{w(x \circ h) - w(x)}{|h|} \qquad \big(x \in \mathbb{H}^n, h = (h', 0)\big).$$

This is the main difficulty we meet in proving the existence of the second order "horizontal derivatives" (i.e. the ones along $X_1, ..., X_{2n}$).

3. Difference quotiens and a priori bounds

For more details on this argument see also [1, 4]. Let us set, for any $w \in C_0^\infty(\Omega)$ and for any $h = (h', 0) = (h_1, ..., h_{2n}, 0)$ with $h_1, \dots, h_{2n} \geq 0$,

$$\begin{aligned} D_h w(x) &= \frac{w(x \circ h) - w(x)}{|h|} \\ D_{-h} w(x) &= \frac{w(x \circ h^{-1}) - w(x)}{-|h|}. \end{aligned} \tag{6}$$

Remark 3.1. It is easy to show that

$$D_{-h} D_h w(x) = \frac{2w(x) - w(x \circ h) - w(x \circ h^{-1})}{-|h|^2} = D_h D_{-h} w(x).$$

Remark 3.2. For any function $w \in L^p(\Omega)$ with compact support $\omega \subset \Omega$, for any $f \in L_{loc}^{\frac{p}{p-1}}(\Omega)$ and for any h such that $|h| < d(\omega, \partial\Omega)$ we have

$$\int f D_{\pm h} w \, dx = - \int w D_{\mp h} f \, dx.$$

Lemma 3.3. *For any $w \in C_0^\infty(\Omega)$ and for any $i = 1, ..., n$,*

$$\begin{aligned} X_i(D_{-h} D_h w(x)) &= D_{-h} D_h (X_i w(x)) \\ &\quad - \frac{h_{i+n}}{2|h|^2} \big[(Tw)(x \circ h) - (Tw)(x \circ h^{-1}) \big] \qquad (7) \\ X_{i+n}(D_{-h} D_h w(x)) &= D_{-h} D_h (X_{i+n} w(x)) \\ &\quad + \frac{h_i}{2|h|^2} \big[(Tw)(x \circ h) - (Tw)(x \circ h^{-1}) \big]. \qquad (8) \end{aligned}$$

Proof. We limit ourselves to (7) since (8) is similar. We have

$$\begin{aligned} &X_i(D_{-h} D_h w(x)) \\ &= -\tfrac{1}{|h|^2} \Big\{ 2(X_i w)(x) \\ &\quad - \Big[(\partial_{x_i} w)(x \circ h) + 2x_{i+n} (\partial_t w)(x \circ h) + 2h_{i+n} (\partial_t w)(x \circ h) \Big] \\ &\quad - \Big[(\partial_{x_i} w(x \circ h^{-1}) + 2x_{i+n} (\partial_t w)(x \circ h^{-1}) - 2h_{i+n} (\partial_t w)(x \circ h^{-1}) \Big] \Big\} \\ &= -\tfrac{1}{|h|^2} \Big\{ 2(X_i w)(x) \\ &\quad - \Big[(X_i w)(x \circ h) + 2h_{i+n} (\partial_t w)(x \circ h) \Big] \\ &\quad - \Big[(X_i w)(x \circ h^{-1}) - 2h_{i+n} (\partial_t w)(x \circ h^{-1}) \Big] \Big\} \\ &= D_{-h} D_h (X_i w)(x) - \tfrac{h_{i+n}}{2|h|^2} \Big[(Tw)(x \circ h) - (Tw)(x \circ h^{-1}) \Big] \end{aligned}$$

and the proof is finished ∎

For any $i = 1, ..., 2n$, h^i will be the point of $\mathbb{H}^n$ whose j-th coordinate is h_i if $j = i$ and 0 otherwise.

Lemma 3.4. *For any $w \in C_0^\infty(\Omega)$ and for any $i = 1, ..., 2n$,*

$$\lim_{h_i \to 0} D_{\pm h^i} w = X_i w. \tag{9}$$

Proof. Let us observe that for any $x \in \Omega$

$$D_{h^i} w(x) = \tfrac{1}{h_i} \int_0^1 (Xw)(x \circ \delta_\theta h^i) \cdot h^i \, d\theta = \int_0^1 (X_i w)(x \circ \delta_\theta h^i) \, d\theta.$$

The poof is accomplished observing that $x \circ \delta_\theta h^i \to x$ when $h^i \to 0$ ∎

Lemma 3.5. *Let $u \in L^p_{loc}(\Omega)$, $i \in \{1, ..., 2n\}$ and $w \in C_0^\infty(\Omega)$ with $\omega = \operatorname{supp} w \subset\subset \Omega$. If there exists constants $\varepsilon > 0$ and $C > 0$ such that*

$$\sup_{0 < h_i < \varepsilon} \int_\omega |D_{h^i} u|^p \le C^p, \tag{10}$$

then $X_i u \in L^p(\omega)$ and $\|X_i u\|_{L^p(\omega)} \le C$. Conversely, if $X_i u \in L^p_{loc}(\Omega)$, then (10) holds for any $\omega = \operatorname{supp} w \subset\subset \Omega$, $w \in C_0^\infty(\Omega)$, and $C = 2\|X_i u\|_{L^p(\omega)}$.

Proof. It follows from [1: Proposition 2.3] ∎

4. $W^{2,2}_{loc}$-regularity for solutions of the approximate equation

As we discussed in the introduction, a crucial step in the proof of Theorems 1.1 and 1.2 is to show that the weak solutions u_ε of the approximate equations are actually strong solutions, namely that $u_\varepsilon \in W^{2,2}_{loc}(\Omega, X)$. The aim of this section is to make precise this general statement. Let Ω' be an open bounded set such that $\Omega' \subset\subset \Omega$. Then we have

Theorem 4.1. *Let $2 \le p < 1 + \sqrt{5}$ and, for any $\varepsilon \in (0, 1)$, let $u_\varepsilon \in W^{1,p}_{loc}(\Omega, X)$ be a local weak solution of equation (5). Then $u_\varepsilon \in W^{2,2}_{loc}(\Omega, X)$ and, for any $\Omega'' \subset\subset \Omega'$,*

$$\int_{\Omega''} V_\varepsilon^{p-2} |X^2 u_\varepsilon|^2 dx \le C(\Omega'', \Omega') \int_{\Omega'} (|u_\varepsilon|^p + V_\varepsilon^p) \, dx$$

where $V_\varepsilon^2 = \varepsilon + |X u_\varepsilon|^2$.

Proof. For notational simplicity we will drop the subscript ε and denote the solution of equation (5) by u. We briefly recall some piece of notation used in the previous sections; for any $\varepsilon > 0$ and for any $z \in \mathbb{R}^{2n}$ we will denote

$$\begin{aligned} V^2(z) &= \varepsilon + |z|^2 \\ W^2_{h^i}(x) &= \varepsilon + |Xu(x)|^2 + |Xu(x \circ h^i)|^2 \\ z^{h^i}(\theta) &= Xu + \theta h_i D_{h^i} Xu \\ z_k^{h^i}(\theta) &= X_k u + \theta h_i D_{h^i} X_k u. \end{aligned}$$

Let now $B(3R)$ be a homogeneous ball of radius $3R$ such that $B(3R) \subset \Omega'$. For an arbitrary $i = 1, ..., n$ let

$$\varphi = -\Big(D_{-h^i} D_{h^i} + D_{-h^{i+n}} D_{h^{i+n}} + D_{h^i} D_{-h^i} + D_{h^{i+n}} D_{-h^{i+n}}\Big) w$$

where $w = g^6 u$ and g is a cut-off function between $B(R)$ and $B(2R)$. Let us observe that the existence of cut-of functions in the Heisenberg group follows by standard methods whenever one observes that the horizontal gradient of the gauge distance has lenght less or equal than one (this is a trivial computation from the definition in Section 2). Let us recall that h_i and h_{i+n} are always assumed to be non-negative.

In Section 7 we will prove that $Tw \in L^p(\Omega')$. Thanks to this fact and Lemma 3.3 we obtain $\varphi \in W_0^{1,p}(\Omega, X)$; this makes φ a right test function for equation (5). Let us multiply equation (5) by the test function φ. On account of Remark 3.2, we obtain

$$\begin{aligned} 0 = &\sum_{k=1}^{2n} \int_\Omega D_{\pm h^i} a^k D_{\pm h^i} X_k w \, dx \\ &+ \sum_{k=1}^{2n} \int_\Omega D_{\pm h^{i+n}} a^k D_{\pm h^{i+n}} X_k w \, dx \\ &+ \int_\Omega \Big[(D_{\pm h^i}) a^{i+n} - (D_{\pm h^{i+n}}) a^i\Big] T w \, dx \\ =: &\, I_1 + I_2 + I_3 \end{aligned} \tag{11}$$

where $\pm$ in I_1, I_2, I_3 means the sum of the terms corresponding to both the signs.

Estimates of I_1 and I_2: Let us observe that, for any $i, k = 1, ..., 2n$,

$$\begin{aligned} D_{h^i} a^k &= \tfrac{1}{h_i} \int_0^1 \frac{d}{d\theta} a^k (Xu + \theta h_i D_{h^i} Xu) \, d\theta \\ &= \int_0^1 a_j^k (Xu + \theta h_i D_{h^i} Xu) D_{h^i} X_j u \, d\theta \\ &= \alpha_{h^i}^{kj} D_{h^i} X_j u \end{aligned} \tag{12}$$

where $\alpha_{h^i}^{kj} := \int_0^1 a_j^k (Xu + \theta h_i D_{h^i} Xu) \, d\theta$ and the sum over j is understood even if not explicitely written. Thanks to the previous notation we have

$$a_j^k(z^{h^i}) = (p-2) V^{p-4}(z^{h^i}) z_k^{h^i} z_j^{h^i} + V^{p-2}(z^{h^i}) \delta_{kj} \tag{13}$$

where $\delta_{kj} = 1$ if $k = 1$ and $\delta_{kj} = 0$ if $k \neq j$. An easy calculation gives

$$\sum_{k,j=1}^{2n} a_j^k(z^{h^i}) D_{h^i} X_k u \, D_{h^i} X_j u \geq V^{p-2}(z^{h^i}) |D_{h^i} Xu|^2. \tag{14}$$

In virtue of (12) and (14) we easily obtain

$$\begin{aligned}\sum_{k=1}^{2n} D_{h^i} a^k D_{h^i} X_k u &= \sum_{k,j=1}^{2n} \alpha_{h^i}^{kj} D_{h^i} X_k u D_{h^i} X_j u \\ &\geq c \int_0^1 V^{p-2}(z^{h^i})\, d\theta\, |D_{h^i} X u|^2.\end{aligned} \tag{15}$$

By [9: Lemma 8.3] we have

$$\int_0^1 V^{p-2}(z^{h^i})\, d\theta \geq c\, W_{h^i}^{p-2}. \tag{16}$$

Hence, from the two inequalities above we get

$$\sum_{k=1}^{2n} D_{h^i} a^k D_{h^i} X_k u \geq c W_{h^i}^{p-2} |D_{h^i} X u|^2. \tag{17}$$

Now let us observe that

$$\begin{aligned}D_{h^i} X_k w = {} & g^6 D_{h^i} X_k u + 6 g^5 X_k u D_{h^i} g \\ & + 6 g^5 D_{h^i} u X_k g + 30 u g^4 D_{h^i} g X_k g + 6 g^5 u D_{h^i} X_k g.\end{aligned} \tag{18}$$

Then, from (17) and (18) we obtain

$$\begin{aligned}&\sum_{k=1}^{2n} \int_\Omega D_{h^i} a^k D_{h^i} X_k w\, dx \\ &\qquad \geq c \int_{\Omega'} g^6 W_{h^i}^{p-2}\, |D_{h^i} X u|^2\, dx \\ &\qquad\quad + 6 \int_{\Omega'} g^5 D_{h^i} a^k X_k u\, D_{h^i} g\, dx \\ &\qquad\quad + 6 \int_{\Omega'} g^5 D_{h^i} a^k D_{h^i} u\, X_k g\, dx \\ &\qquad\quad + 30 \int_{\Omega'} g^4 u\, D_{h^i} a^k\, D_{h^i} g\, X_k g\, dx \\ &\qquad\quad + 6 \int_{\Omega'} g^5 u\, D_{h^i} a^k\, D_{h^i} X_k g\, dx \\ &\qquad =: J_1 + J_2 + J_3 + J_4 + J_5.\end{aligned} \tag{19}$$

Estimate of $J_2, ..., J_5$. By [9: Lemma 8.3] we have, for any $k, j = 1, ..., 2n$,

$$|\alpha_{h^i}^{kj}| \leq c W_{h^i}^{p-2}. \tag{20}$$

On account of (12), (20) and the Hölder inequality we have

$$\begin{aligned}|J_2| &= 6 \left| \int_{\Omega'} g^5 \alpha_{h^i}^{kj}\, D_{h^i} X_j u\, X_k u\, D_{h^i} g\, dx \right| \\ &\leq \delta \int_{\Omega'} g^6 W_{h^i}^{p-2} |D_{h^i} X u|^2\, dx + c\delta^{-1} \int_{\Omega'} g^4 W_{h^i}^{p} |D_{h^i} g|^2\, dx.\end{aligned} \tag{21}$$

As for $h_i < R$

$$\int_{B(2R)} W_{h^i}^p \, dx \leq \int_{B(3R)} V^p \, dx \tag{22}$$

it follow from (21) and (22)

$$|J_2| \leq \delta \int_{\Omega'} g^6 W_{h^i}^{p-2} |D_{h^i} X u|^2 \, dx + c\delta^{-1} R^{-2} \int_{\Omega'} V^p dx. \tag{23}$$

Taking into account

$$\int_{\Omega'} g^2 W_{h^i}^{p-2} |D_{h^i} u|^2 dx \leq c \int_{\Omega'} g^2 (W_{h^i}^p + |D_{h^i} u|^p) \, dx \tag{24}$$

and in virtue of Lemma 3.5, the same estimate holds for $|J_3|$:

$$|J_3| \leq \delta \int_{\Omega'} g^6 W_{h^i}^{p-2} |D_{h^i} X u|^2 \, dx + c\delta^{-1} R^{-2} \int_{\Omega'} V^p dx. \tag{24}$$

On account that

$$\int_{\Omega'} g^2 W_{h^i}^{p-2} |u|^2 \, dx \leq c \int_{\Omega'} g^2 W_{h^i}^p \, dx + c \int_{\Omega'} g^2 |u|^p \, dx \tag{25}$$

we similarly obtain

$$|J_4|, |J_5| \leq \delta \int_{\Omega'} g^6 W_{h^i}^{p-2} |D_{h^i} X u|^2 \, dx + c\delta^{-1} R^{-4} \int_{\Omega'} (V^p + |u|^p) \, dx. \tag{26}$$

An analogous result can be obtained switching between h^i and $-h^i$. The estimate of I_2 proceeds exactly in the same way; therefore, from (19), (23), (24), (26) and the analogous inequalities about I_2, and taking δ small enough we obtain, for any $i = 1, ..., n$,

$$\begin{aligned} I_1 + I_2 \geq c \int_{\Omega'} g^6 \left[W_{\pm h^i}^{p-2} |D_{\pm h^i} X u|^2 + W_{\pm h^{i+n}}^{p-2} |D_{\pm h^{i+n}} X u|^2 \right] dx \\ - cR^{-4} \int_{\Omega'} (V^p + |u|^p) \, dx. \end{aligned} \tag{27}$$

Estimate of I_3. As $u, Xu \in L^p_{loc}(\Omega')$ and

$$\int_{\Omega'} |T(g^2 u)|^p dx \leq cR^{-4p} \int_{\Omega'} (V^p + |u|^p) \, dx$$

(see Theorem 7.1), we have

$$\begin{aligned}
&\left| \int_{\Omega'} D_{h^i} a^{i+n} T w \, dx \right| \\
&\quad = \left| \int_{\Omega'} g^4 D_{h^i} a^{i+n} T(g^2 u) \, dx + \int_{\Omega'} g^2 u D_{h^i} a^{i+n} T(g^4) \, dx \right| \\
&\quad \leq \delta \int_{\Omega'} g^6 W_{h^i}^{p-2} |D_{h^i} X u|^2 dx \\
&\qquad + c\delta^{-1} \int_{\Omega'} g^2 W_{h^i}^{p-2} |T(g^2 u)|^2 dx + cR^{-4} \int_{\Omega'} g^4 W_{h^i}^{p-2} |u|^2 dx \qquad (28) \\
&\quad \leq \delta \int_{\Omega'} g^6 W_{h^i}^{p-2} |D_{h^i} X u|^2 dx \\
&\qquad + c\delta^{-1} \int_{\Omega'} g^2 (W_{h^i}^p + |T(g^2 u)|^p) \, dx + cR^{-4} \int_{\Omega'} g^4 (W_{h^i}^p + |u|^p) \, dx \\
&\quad \leq \delta \int_{\Omega'} g^6 W_{h^i}^{p-2} |D_{h^i} X u|^2 dx + cR^{-4p} \int_{\Omega'} (V^p + |u|^p) \, dx.
\end{aligned}$$

The other three terms of I_3, that is

$$\int_{\Omega'} D_{-h^i} a^{i+n} T w \, dx, \quad \int_{\Omega'} D_{h^{i+n}} a^i T w \, dx, \quad \int_{\Omega'} D_{-h^{i+n}} a^i T w \, dx$$

can be estimated in the same way. Inserting all these estimates together with (27) into (11) gives

$$\int_{\Omega'} g^6 W_{h^i}^{2-p} |D_{h^i} X u|^2 \leq cR^{-4p} \int_{\Omega'} (V^p + |u|^p) \, dx \qquad (29)$$

for any $i = 1, ..., 2n$. This inequality enable us to affirm that, for any $i = 1, ..., 2n$, $D_{h^i} X u$ is bounded in $L^2(B_R)$. By Lemma 3.5, possibly up to a subsequence, $D_{h^i} X u$ converges in $L^2_{loc}(B(R))$ to $X_i X u$ for $h^i \to 0$ and then $u \in W^{2,2}_{loc}(B(R))$. Moreover, we can extract from it a subsequence converging for a.e. $x \in B(R)$. By Lemma 3.5,

$$W_{h^i} \to (\varepsilon + 2|Xu|^2)^{\frac{1}{2}} \qquad \text{for a.e. } x \in B(R) \text{ as } h^i \to 0.$$

The proof of Theorem 4.1 is then finished, passing to the limit $h^i \to 0$ in (29) for any $i = 1, ..., 2n$ and summing up the resulting inequalities over $i = 1, ..., 2n$, on account that Ω'' can be covered by a finite number of balls $B(R)$ for R small enough ∎

Remark 4.2. We would point out that, thanks to Theorem 4.1, we can now differentiate formally equations $\int_{B(R)} a^k_\varepsilon (X u_\varepsilon) X_k \varphi \, dx = 0$ along X_i $(i = 1, ...2n)$ obtaining

$$\int_{B(R)} a^k_{\varepsilon,j} (X u_\varepsilon) \, X_i X_j u_\varepsilon X_k \varphi \, dx = 0 \qquad (30)$$

for any $\varphi \in W^{1,p}_0(B(R), X)$ where $\overline{B(R)} \subset \Omega'$.

5. Local boundedness of the gradient

We can now rely on the results of the previous sections to prove Theorems 1.1 and 1.2. Here we are concerned with Theorem 1.1 and we will use a technique due to [5]. First of all we will establish uniform local boundedness of the functions u_ε. Let Ω' be an open bounded set such that $\Omega' \subset\subset \Omega$.

The following result can be found in [3: Theorem 3.4].

Lemma 5.1. *For any compact $K \subset \Omega'$ there exists a constant $C > 0$ depending only on the structural constants and on* $\operatorname{dist}(K, \partial\Omega')$ *such that*

$$\|u_\varepsilon\|_{\infty,K} \leq C. \tag{31}$$

Let $x_0 \in \Omega'$ arbitrarily fixed and, for any $\rho > 0$, let $B(\rho)$ be the ball centered at x_0 of radius ρ. Further, let $B(R) \subset\subset \Omega'$.

Theorem 5.2. *For any $\sigma \in (0.1)$ there exists a constant $\gamma(\sigma) > 0$ depending only on the structural constants and σ such that*

$$\left\|[\varepsilon + |Xu_\varepsilon|^2]\right\|^{\frac{p}{2}}_{\infty,B(R-\sigma R)} \leq \gamma(\sigma)\tfrac{1}{|B(R)|}\int_{B(R)}[\varepsilon + |Xu_\varepsilon|^2]^{\frac{p}{2}}dx \tag{32}$$

for all $\varepsilon > 0$.

Proof. Let us recall that the coefficients a^k_ε satisfy the estimates

$$a^k_{\varepsilon j}\xi_k\xi_j \geq \gamma_0 V_\varepsilon^{p-2}|\xi|^2 \quad (\xi \in \mathbb{R}^n) \tag{33}$$

$$|a^k_{\varepsilon j}| \leq \gamma_1 V_\varepsilon^{p-2} \tag{34}$$

with constants γ_0 and γ_1 independent on ε, where $V_\varepsilon^2 = \varepsilon + |Xu_\varepsilon|^2$ and, thanks to Theorem 4.1,

$$\int_{\Omega'} V_\varepsilon^{p-2}|X^2u_\varepsilon|^2 < c(\varepsilon). \tag{35}$$

In virtue of this estimate, we can set in (30) the test function $\varphi = X_i u_\varepsilon V_\varepsilon^\alpha g^2$ with $\alpha > 0$, where g is a cut-off function between $B(R - \sigma R)$ with $\sigma \in (0, 1)$ and $B(R)$. Applying now standard methods to (30) gives on account of (33) - (34)

$$\int_{B(R)} |XV_\varepsilon^{\frac{p+\alpha}{2}}|^2 g^2 dx \leq \gamma \int_{B(R)} V_\varepsilon^{p+\alpha}|Xg|^2 dx \tag{36}$$

where γ is a structural constant independent on R, ε and α. If $H = V_\varepsilon^{\frac{p}{2}}$ and $\theta = 1 + \frac{\alpha}{p}$, this estimate can be rewritten as

$$\int_{B(R)} |XH^\theta|^2 g^2 dx \leq \int_{B(R)} H^{2\theta}|Xg|^2 dx. \tag{37}$$

On account of Lemma 5.1 we can apply here the Moser iteration technique [18] in a suitable adapted version due to [3: Lemma 3.29]. Then we obtain (31) ∎

6. Local Hölder continuity of the gradient

As before let Ω' be an arbitrary open bounded subset of Ω such that $\Omega' \subset\subset \Omega$, let $x_0 \in \Omega'$ be an arbitrary point and, for any $\rho > 0$, let $B(\rho)$ be the ball centered at x_0 of radius ρ. Let $R > 0$ such that $\overline{B(2R)} \subset \Omega'$. Let us observe that thanks to Theorem 5.1 and the results of [1] the solutions u_ε of equation (5) are now smooth. Therefore, for any $\rho \leq R$ and $\varepsilon > 0$, we can set

$$\mu_\varepsilon(\rho) = \max_i \sup_{B(\rho)} |X_i u_\varepsilon|$$
$$\omega_\varepsilon(\rho) = \max_i \operatorname{osc}_{B(\rho)} X_i u_\varepsilon.$$

Our purpose is to establish Hölder continuity of Xu_ε at x_0, uniformly in $\varepsilon > 0$. The technique is due to [5, 6], but in this particular setting we use also some results of [17]. We do not want to deal with all the proofs in depth. We will mostly refer to [5, 17], even if we will discuss all needed modifications in details.

The general idea consists in proving the existence of positive structural constants $\alpha \in (0,1)$, δ_0 and σ_0, independent on ε such that, for all small ρ, if the subset of $B(\rho)$ where Xu_ε degenerates is "small", then the equation behaves in $B(\rho)$ as a non-degenerate elliptic equation (in this case we get $\omega_\varepsilon(\frac{\rho}{2}) \leq \delta_0 \rho^\alpha$), whereas if Xu_ε degenerates in a "thick" portion of $B(\rho)$, then we have $\mu_\varepsilon(\frac{\rho}{2}) \leq \sigma_0 \mu_\varepsilon(2\rho)$. The Hölder continuity follows from both cases by a standard iteration technique [12].

The following result can be found in [3: Theorem 3.35].

Lemma 6.1 (Local Hölder continuity of u_ε). *For any compact $K \subset \Omega'$ there exist some constants $C > 0$ and $\beta \in (0,1)$ depending only on the structural constants and* $\operatorname{dist}(K, \partial\Omega')$ *such that*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C|x-y|^\beta \qquad (x, y \in K) \tag{38}$$

for all $\varepsilon > 0$.

Lemma 6.2. There exists a constant C such that, for any $v \in W^{1,1}(B(R), X)$ and for any real numbers $l > k$,

$$(l-k)|A^+_{l,R}| \leq C \frac{R|B(R)|}{|B(R) \setminus A^+_{k,R}|} \int_{A^+_{k,R} \setminus A^+_{l,R}} |Xv|\, dx \tag{39}$$

where $A^+_{s,R} = \{x \in B(R) |\, v(x) > s\}$.

Proof. By Poincaré's inequality [11: Theorem 2.1 and final remarks] there exists a constant $C > 0$ such that, for any $f \in C^\infty(B(R))$,

$$\int_{B(R)} |f - \overline{f}|\, dx \leq CR \int_{B(R)} |Xf|\, dx \tag{40}$$

where $\overline{f} = \frac{1}{|B(R)|} \int_{B(R)} f$. From here we have

$$\overline{v} \leq \frac{CR}{|N_0|} \int_{B(R)} |Xv|\, dx \tag{41}$$

for all $v \in W^{1,1}(B(R), X)$, where $N_0 = \{x \in B(R) | \, v(x) = 0\}$. From both inequalities together we get

$$\int_{B(R)} |v(x)| \, dx \leq \tfrac{CR|B(R)|}{|N_0|} \int_{B(R)} |Xv| \, dx. \tag{42}$$

Applying this to the function

$$\tilde{v} = \begin{cases} 0 & \text{if } v(x) \leq k \\ v(x) - k & \text{if } k \leq v(x) \leq l \\ l - k & \text{if } v(x) \geq l \end{cases}$$

we obtain (39). So the lemma is proved ∎

Let us now set $\varphi = \pm(X_i u_\varepsilon - k)^{\pm} \xi^2$ in (35) for $k \in \mathbb{R}$ and $i = 1, ..., 2n$, where ξ is a cut-off function with support contained in $B(R)$. We obtain

$$\int_{B(R)} V_\varepsilon^{p-2} |X(X_i u_\varepsilon - k)^{\pm}|^2 dx \leq \gamma \int_{B(R)} V_\varepsilon^{p-2} |(X_i u_\varepsilon - k)^{\pm}|^2 |X\xi|^2 dx \tag{43}$$

where $V_\varepsilon^2 = \varepsilon + |Xu_\varepsilon|^2$ and γ is a structural constant independent on ε.

Proposition 6.3. *Let $\rho < \frac{R}{2}$ and set $\lambda = \frac{\mu_\varepsilon(2\rho)}{2}$. Then there exists a constant $C_0 > 0$ depending only on the data but independent on ε, R, λ such that if for some $1 \leq i \leq 2n$*

$$\big|\{x \in B(2\rho) | \, X_i u_\varepsilon < \lambda\}\big| \leq C_0 |B(2\rho)|, \qquad \textit{then } X_i u_\varepsilon \geq \tfrac{\lambda}{4} \textit{ for all } x \in B(\rho).$$

Analogously, if

$$\big|\{x \in B(2\rho) | \, X_i u_\varepsilon > -\lambda\}\big| \leq C_0 |B(2\rho)|, \qquad \textit{then } X_i u_\varepsilon \leq -\tfrac{\lambda}{4} \textit{ for all } x \in B(\rho).$$

Proof. As in [5: Proposition 4.1] we distinguish between $\varepsilon \geq \lambda^2$ and $\varepsilon < \lambda^2$. The readers are referred there for the first simpler case but we sketch the proof for the second one owing to make some changes in it with respect to the cited paper.

Also, let $\varepsilon < \lambda^2$. In the following we will drop the subscript ε. If $v = |X_i u|^{\frac{p}{2}} \operatorname{sign} X_i u$ and $A^-_{h,r} = \{x \in B(r) | \, v(x) < h\}$, then using (43) as in [5: Proposition 4.1] we easy obtain

$$\int_{B(r-\sigma r)} |X(v-h)_-|^2 dx \leq \gamma h_0^2 (\sigma r)^{-2} |A^-_{h,r}| \tag{44}$$

for any $\sigma \in (0,1)$, $r \leq 2\rho$, $h \leq h_0 = \lambda^{\frac{p}{2}}$ and for a suitable structural constant $\gamma > 0$ independent on ε, r, σ and h. Let $H = \sup_{B(2r)} (v - h_0)_-$. Let us observe that if $H < \frac{h_0}{2}$, then $X_i u > \frac{\lambda}{4}$ for any $x \in B(2\rho)$. Therefore we may assume $H \geq \frac{h_0}{2}$. For any integer $j \geq 0$ let

$$r_j = \rho + \tfrac{\rho}{2^j}, \quad h_j = h_0 - \tfrac{H}{4}(1 - \tfrac{1}{2^j}), \quad B_j = B(r_j), \quad A_j = A^-_{h_j, r_j}. \tag{45}$$

For an arbitrary $j \geq 0$ let us set $h = h_j$, $r = r_j$, $r - \sigma r = r_{j+1}$ in (44). We obtain

$$\int_{B_{j+1}} |X(v - h_j)_-|^2 dx \leq C 2^{2j} \tfrac{h_0^2}{\rho^2} |A_j|. \tag{46}$$

Let $s \in (2, \frac{2Q}{Q-2})$. Applying Poincaré's inequality [15] to the function $(v-h_j)_-\xi$, where ξ is a cut-off function between B_{j+2} and B_{j+1}, we get

$$\begin{aligned}&\Big(\int_{A_{j+1}} |(v-h_j)_-\xi|^s dx\Big)^{\frac{1}{s}}\\ &\qquad \le c\rho\Big(\int_{A_{j+1}} |X(v-h_j)_-|^2 dx + \rho^{-2}\int_{A_{j+1}} |(v-h_j)_-|^2 dx\Big)^{\frac{1}{2}} |B(\rho)|^{\frac{1}{s}-\frac{1}{2}}.\end{aligned} \tag{47}$$

By Hölder's inequality and (46) - (47) we obtain

$$\begin{aligned}\big(\tfrac{H}{2^{j+3}}\big)^2 |A_{j+2}| &\le \int_{A_{j+2}} |(v-h_j)_-|^2 dx\\ &\le \Big(\int_{A_{j+1}} |(v-h_j)_-\xi|^s dx\Big)^{\frac{2}{s}} |A_{j+1}|^{1-\frac{2}{s}}\\ &\le c\rho^2\Big(\int_{A_{j+1}} |X(v-h_j)_-|^2 dx + H^2\rho^{-2}|A_{j+1}|\Big)|B(\rho)|^{\frac{2}{s}-1}|A_{j+1}|^{1-\frac{2}{s}}\\ &\le c2^{2j}H^2|B(\rho)|^{\frac{2}{s}-1}|A_j|^{2-\frac{2}{s}}\end{aligned} \tag{48}$$

from which we obtain for any $j \ge 0$

$$\tfrac{|A_{j+2}|}{|B(\rho)|} \le c2^{4j}\big(\tfrac{|A_j|}{|B(\rho)|}\big)^{1+\chi} \tag{49}$$

where $\chi = 1 - \frac{2}{s} > 0$. In particular, this gives for any $l \ge 1$

$$\tfrac{|A_{2l}|}{|B(\rho)|} \le c(2^8)^{(l-1)}\left(\tfrac{|A_{2(l-1)}|}{|B(\rho)|}\right)^{1+\chi}. \tag{50}$$

It follows from here and [12: p. 66/Lemma 4.7] that there exists a constant $C_0 > 0$ depending only on c and $b = 2^8$ such that, if $|A_0| \le C_0|B_0|$, then $\lim_{l\to+\infty} A_{2l} = 0$ which implies $|\{x \in B(\rho) | X_i u < \frac{\lambda}{2^{2/p}}\}| = 0$, and then $X_i u \ge \frac{\lambda}{4}$ for any $x \in B(\rho)$. So Proposition 6.3 is proved ∎

Making only few and obvious changes in [5: Proposition 4.2], using (43) and Lemma 6.2 we easily establish the following counterpart of Proposition 6.3.

Proposition 6.4. *Let $\rho < \frac{R}{2}$. If the assumptions of Proposition* 6.3 *fail, then there exists a structural constant $\sigma_0 \in (0,1)$ independent on $\varepsilon > 0$ and ρ such that, for all $\varepsilon > 0$, $\mu_\varepsilon(\frac{\rho}{2}) \le \sigma_0\mu_\varepsilon(2\rho)$.*

Theorem 6.5. *There exist constants $\gamma > 0$ and $\eta \in (0,1)$ depending only on the data and* $\operatorname{dist}(B(R), \partial\Omega')$ *such that*

$$\max_i \operatorname{osc}_{B(\rho)} X_i u_\varepsilon \le \gamma(\tfrac{\rho}{R})^\eta \sup_{B(\frac{R}{2})} |Xu_\varepsilon|$$

for all $\rho < \frac{R}{2}$ and for all $\varepsilon > 0$.

Proof. The proof is the same as that of [5: Proposition 4.3] using a result of [17]. We sketch it for sake of completeness. The subscript ε will be dropped.

Let $\rho < \frac{R}{2}$ and let the assumptions of Proposition 6.3 be verified in $B(2\rho)$. Then for some i either $X_i u > \frac{1}{8}\mu(2\rho)$ or $X_i u < -\frac{1}{8}\mu(2\rho)$ in $B(\rho)$. In both cases

$$(\tfrac{1}{8})^{p-2}\mu(2\rho)^{p-2} \leq V^{p-2} \leq (4n)^{p-2}\mu(2\rho)^{p-2} \tag{51}$$

in $B(\rho)$. Therefore, writing (43) over the balls $B(\rho)$ for all $i = 1, ..., 2n$ we have

$$\int_{B(\rho-\sigma\rho)} |X(X_i u - k)^{\pm}|^2 dx \leq \gamma(\sigma\rho)^{-2} \int_{B(\rho)} |(X_i u - k)^{\pm}|^2 dx \tag{52}$$

for a new structural constant γ. These inequalities state that, for every $i = 1, ..., 2n$, $X_i u$ belongs to some "De Giorgi classes" for all whose functions we stated the local Hölder continuity [17: Theorem 2.1]. Precisely, $\omega(\frac{\rho}{2}) \leq \delta_0 \rho^{\alpha}$ where $\delta_0 = c \sup_{B(R/2)} |Xu| R^{-\alpha}$ for suitable structural constants c and $\alpha \in (0, 1)$ independent on ρ and ε. On the other hand, if the assumptions of Proposition 6.3 fail in $B(\rho)$, then by Proposition 6.4 there exists a constant $\sigma_0 > 0$ such that

$$\mu(\tfrac{\rho}{2}) \leq \sigma_0 \mu(2\rho).$$

Let us now consider a sequence of radii $R_j = \frac{R}{2^{2j}}$ $(j \geq 1)$. If for every $j \geq 1$ the assumptions of Proposition 6.3 fail in $B(R_j)$, then Proposition 6.4 gives $\mu(\frac{R_j}{2}) \leq \sigma_0 \mu(2R_j)$. On the other hand, if for some $j_0 \geq 1$ the assumptions of Proposition 6.3 are verified in $B(R_{j_0})$, then as in (51) we can estimate V^{p-2} from above and below in terms of $\mu(2R_{j_0})$ in $B(R_{j_0})$ and hence also in $B(R_j)$ for every $j > j_0$. Therefore, for every $j \geq j_0$ the equation behaves in $B(R_j)$ like a non-degenerate equation. From (52) then we obtain $\omega(\frac{R_j}{2}) \leq \delta_0 R_j^{\alpha}$ for every $j \geq j_0$, and $\mu(\frac{R_j}{2}) \leq \sigma_0 \mu(2R_j)$ for every $1 \leq j < j_0$. The proof follows from a standard modification of [12: pp. 66 – 67/Lemma 4.8] ∎

7. Estimate of Tu_ε

In this section we prove, for any $2 \leq p < 1 + \sqrt{5}$, that if u_ε is a local weak solution of equation (5), then $Tu_\varepsilon \in L^p_{loc}(\Omega')$. This is a cornerstone for the paper. Just as before, here Ω' will denote an arbitrary open bounded subset of Ω such that $\Omega' \subset\subset \Omega$.

Theorem 7.1. *Let $2 \leq p < 1 + \sqrt{5}$ and, for any $\varepsilon \in (0, 1)$, let $u_\varepsilon \in W^{1,p}_{loc}(\Omega, X)$ be a local weak solution of equation (5). Further, let $B(3R)$ be an arbitrary homogeneous ball of radius $3R$ such that $B(3R) \subset \Omega'$ and let g be a cut-off function between $B(R)$ and $B(2R)$. Then $T(g^2 u_\varepsilon) \in L^p(\Omega')$ and*

$$\int_{\Omega'} |T(g^2 u_\varepsilon)|^p dx \leq c\, R^{-4p} \int_{\Omega'} (V_\varepsilon^p + |u_\varepsilon|^p)\, dx \tag{53}$$

where $V_\varepsilon^2 = \varepsilon + |Xu_\varepsilon|^2$.

Proof. If $p = 2$, then (53) follows from [1]. Let now $p > 2$. For any $\alpha \in (0,1)$, $s > 0$ and any $w \in C_0^\infty(\Omega)$ we will denote $h_s^* = (0, s)$ and

$$D_{h^*_{s,\alpha}} w(x) = \frac{w(x \circ h_s^*) - w(x)}{s^\alpha}$$
$$D_{-h^*_{s,\alpha}} w(x) = \frac{w(x \circ (h_s^*)^{-1}) - w(x)}{-s^\alpha}.$$

We have

$$D_{-h^*_{s,\alpha}} D_{h^*_{s,\alpha}} w = \frac{w(x \circ h_s^*) + w(x \circ (h_s^*)^{-1}) - 2w(x)}{s^{2\alpha}} = D_{h^*_{s,\alpha}} D_{-h^*_{s,\alpha}} w$$

and, for every $k = 1, ..., 2n$,

$$X_k D_{\pm h^*_{s,\alpha}} = D_{\pm h^*_{s,\alpha}} X_k. \tag{54}$$

Let us multiply equation (5) by the test function $\varphi = D_{-h^*_{s,1/2}}(g^2 D_{h^*_{s,1/2}} u_\varepsilon)$, g being a cut-off function between $B(R)$ and $B(2R)$. Let us observe that $\varphi \in W_0^{1,p}(\Omega, X)$ in virtue of (54). In the following we will drop the subscript ε for the sake of simplicity. On account of (54) we obtain

$$\int_\Omega D_{h^*_{s,1/2}} a^k g^2 X_k D_{h^*_{s,1/2}} u \, dx + 2 \int_\Omega D_{h^*_{s,1/2}} a^k D_{h^*_{s,1/2}} u \, g X_k g \, dx = 0. \tag{55}$$

For any $p > 1$ the first integral in the left-hand side here can be estimated by the same argument we applied to J_2 in Section 4: as

$$D_{h^*_{s,1/2}} a^k = \alpha^{kj}_{h_s^*} D_{h^*_{s,1/2}} X_j u \tag{56}$$

where

$$\alpha^{kj}_{h_s^*} = \int_0^1 a^k_j (Xu + \theta s^{\frac{1}{2}} D_{h^*_{s,1/2}} Xu) \, d\theta,$$

then

$$\int_\Omega D_{h^*_{s,1/2}} a^k g^2 X_k D_{h^*_{s,1/2}} u \, dx \geq c \int_{\Omega'} g^2 W_{h_s^*}^{p-2} |D_{h^*_{s,1/2}} Xu|^2 dx \tag{57}$$

where

$$W^2_{h_s^*}(x) = \varepsilon + |Xu(x)|^2 + |Xu(x \circ h_s^*)|^2.$$

To estimate the second integral in the left-hand side of (55), we may count again on (56) and the estimate $|\alpha^{kj}_{h_s^*}| \leq c\, W_{h_s^*}^{p-2}$ (see [9: Lemma 8.3]) to obtain

$$\begin{aligned} &\left| \int_{\Omega'} D_{h^*_{s,1/2}} a^k D_{h^*_{s,1/2}} u \, g X_k g \, dx \right| \\ &\quad = \left| \int_{\Omega'} \alpha^{kj}_{h_s^*} D_{h^*_{s,1/2}} X_j u \, D_{h^*_{s,1/2}} u \, g X_k g \, dx \right| \\ &\quad \leq \delta \int_{\Omega'} g^2 W_{h_s^*}^{p-2} |D_{h^*_{s,1/2}} Xu|^2 + c\delta^{-1} R^{-2} \int_{\Omega'} W_{h_s^*}^{p-2} |D_{h^*_{s,\frac{1}{2}}} u|^2 dx \end{aligned} \tag{58}$$

where

$$\int_{\Omega'} W_{h_s^*}^{p-2} |D_{h^*_{s,1/2}} u|^2 dx \le c \int_{\Omega'} (W_{h_s^*}^p + |D_{h^*_{s,1/2}} u|^p)\, dx. \tag{59}$$

On account of (57), (58) with small δ, (59) and the Baker-Campbell-Hausdorff formula (see [1: Theorem 2.6] for the application we need here) we obtain from (55)

$$\int_{\Omega'} g^2 W_{h_s^*}^{p-2} |D_{h^*_{s,1/2}} Xu|^2 dx \le cR^{-2} \int_{\Omega'} V^p dx. \tag{60}$$

As

$$\begin{aligned} \left|s^{\frac{1}{2}} D_{h^*_{s,1/2}} Xu\right|^2 &= \left|Xu(x \circ h_s^*) - Xu(x)\right|^2 \\ &\le 4\left[|Xu(x \circ h_s^*)|^2 + |Xu(x)|^2\right] \\ &\le 4 W_{h_s^*}^2, \end{aligned}$$

then (60) gives

$$\int_{\Omega'} s^{\frac{p-2}{2}} \left|D_{h^*_{s,1/2}} X(g^2 u)\right|^p \le cR^{-2p} \int_{\Omega'} (V^p + |u|^p)\, dx. \tag{61}$$

From (54), (61), Lemma 3.5 and the Baker-Campbell-Hausdorff formula we have

$$\int_{\Omega'} s^{\frac{p-2}{2}} \left|D_{-h^*_{s,1/2}} D_{h^*_{s,1/2}} (g^2 u)\right|^p \le cR^{-2p} \int_{\Omega'} (V^p + |u|^p)\, dx,$$

that is

$$\int_{\Omega'} |\Delta_s^2 (g^2 u)|^p s^{-1-\frac{p}{2}} dx \le cR^{-2p} \int_{\Omega'} (V^p + |u|^p)\, dx \tag{62}$$

where $\Delta_s^2 w = w(x \circ h_s^*) - 2w(x) + w(x \circ (h_s^*)^{-1})$, and then, for any $\alpha \in (0,1)$,

$$\int_0^1 \int_{\Omega'} |\Delta_s^2 (g^2 u)|^p s^{-1-p\beta} dx ds \le cR^{-2p} \int_{\Omega'} (V^p + |u|^p)\, dx \tag{63}$$

where $\beta = \frac{\alpha}{p} + \frac{1}{2}$.

Let us now briefly recall some known functional spaces in $\mathbb{R}$ and their inclusions. We refer the readers to [16, 21] for the details. We start with the Besov space $B_{p,p}^\theta$ ($\theta \in (0,1), p > 1$), the completion of $C_0^\infty(\mathbb{R})$ with respect to the norm

$$\|\varphi\|_{L^p} + \left(\int \|\Delta_s^2 \varphi\|_{L^p}^p |s|^{-1-p\theta} ds \right)^{\frac{1}{p}},$$

where $\Delta_s^2 \varphi(t) = \varphi(t+s) - 2\varphi(t) + \varphi(t-s)$. It results [22: p. 37/Formula (9) and p. 90/Formulas (5) and (9)] $B_{p,p}^\theta = W^{\theta,p}$, where $W^{\theta,p}$ is the fractional Sobolev space, which in turn is linked to the Bessel potential spaces

$$H^{\theta,p} = \left\{ \varphi :\ \|F^{-1}((1+|\xi|^2)^{\theta/2} F\varphi)\|_{L^p} < \infty \right\}$$

where F denotes the Fourier transform, in that $H^{\theta+\tau,p} \subset W^{\theta,p} \subset H^{\theta-\tau,p}$ for any small $\tau > 0$. Moreover, the interpolation spaces

$$(L^p, W^{1,p})_{\theta,\infty} = \left\{ \varphi \in L^p : \sup_{0<|s|<\sigma} \frac{\|\varphi(\cdot+s)-\varphi(\cdot)\|_{L^p}}{|s|^\theta} < \infty \right\}$$

for some positive constant $\sigma > 0$ satisfy $H^{\theta,p} \subset (L^p, W^{1,p})_{\theta,\infty} \subset H^{\widetilde{\theta},p}$ for any $0 < \widetilde{\theta} < \theta$ (see [21: p. 64/Theorem 1, p. 25/Formulas (1) and (4), and p. 185/Formula (11)]). Collecting the previous inclusions we obtain

$$(L^p, W^{1,p})_{\theta+2\tau,\infty} \subset H^{\theta+\tau,p} \subset W^{\theta,p} = B^\theta_{p,p} \subset H^{\theta-\tau,p} \subset (L^p, W^{1,p})_{\theta-\tau,\infty} \tag{64}$$

for any $\theta \in (0,1)$, $p > 1$ and any small $\tau > 0$. It follows, in particular,

$$\|\varphi\|_{(L^p,W^{1,p})_{\beta-\tau,\infty}} \le c\|\varphi\|_{B^\beta_{p,p}} \tag{65}$$

for any $\varphi \in C^\infty_0(\mathbb{R})$, any small $\tau > 0$ and any $\alpha \in (0,1)$, where $\beta = \frac{\alpha}{p} + \frac{1}{2}$.

From (63) and (65) we obtain

$$\sup_{s<\sigma} \int_{\Omega'} |D_{h^*_{s,\beta-\tau}}(g^2 u)|^p dx \le cR^{-2p} \int_{\Omega'} (V^p + |u|^p)\, dx. \tag{66}$$

Let us multiply equation (5) by the function $D_{-h^*_{s,\beta-\tau}}(g^2 D_{h^*_{s,\beta-\tau}} u)$. Let us observe that it is a right test function for (5) in virtue of (54). Using (66) in place of the Baker-Campbell-Hausdorff formula (possibly modifying the domain of the cut-off function g) we can repeat the argument from (55) until (61). More precisely, in place of (61) we have now

$$\int_{\Omega'} s^{(p-2)(\beta-\tau)} |D_{h^*_{s,\beta-\tau}} X(g^2 u)|^p dx \le cR^{-4p} \int_{\Omega'} (V^p + |u|^p)\, dx \tag{67}$$

from which, arguing as for (62), we obtain

$$\int_{\Omega'} |\Delta^2_s (g^2 u)|^p s^{-2(\beta-\tau)-\frac{p}{2}} dx \le cR^{-4p} \int_{\Omega'} (V^p + |u|^p)\, dx \tag{68}$$

and then

$$\int_0^1 \|\Delta^2_s (g^2 u)\|_{L^p(\Omega')} s^{-\sigma} ds \le cR^{-4} \left(\int_{\Omega'} (V^p + |u|^p)\, dx \right)^{\frac{1}{p}} \tag{69}$$

where $\sigma = \frac{1}{2} + \frac{2}{p}(\beta - \tau) + \alpha$. As $\beta = \frac{\alpha}{p} + \frac{1}{2}$, then $\sigma = \frac{(2\alpha+1)p^2 + 2(1-2\tau)p + 4\alpha}{2p^2}$.

Now we have $\sigma = 2$ if $(3-2\alpha)p^2 - 2(1-2\tau)p - 4\alpha = 0$, that is

$$p = \frac{1-2\tau+\sqrt{(1-2\tau)^2+4\alpha(3-2\alpha)}}{3-2\alpha}.$$

As $p \to 2$ when $\tau \to \frac{1}{2}$, $\alpha \to 1$, and $p \to 1+\sqrt{5}$ when $\tau \to 0$, $\alpha \to 1$, then, for any $p \in (2, 1+\sqrt{5})$ we can choose α and τ in such a way that $\sigma = 2$ in (69). As for any $p \ge 1$ we have $B^1_{p,1}(\mathbb{R}) \subset W^{1,p}(\mathbb{R})$ [22: p. 90/Formulas (5) and (10)], then, for any $2 < p < 1+\sqrt{5}$, (53) follows from (69) with $\sigma = 2$.

Acknowledgements. We would like thank professors M. Biroli and U. Gianazza for many useful comments on the subject.

References

[1] Capogna, L.: *Regularity of quasilinear equations in the Heisenberg group.* Comm. Pure Appl. Math. 50 (1997), 867 – 889.

[2] Capogna, L.: *Regularity for quasilinar equations and* 1*-quasiconformal maps in Carnot groups.* Math. Ann. 313 (1999), 263 – 295.

[3] Capogna, L., Danielli, D. and N. Garofalo: *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations.* Comm. Part. Diff. Equ. 18 (1993), 1765 – 1794.

[4] Cutrí, A. and M. G. Garroni: *Existence, uniqueness and regularity results for integro-differential Heisenberg equations.* Adv. Diff. Equ. 1 (1996), 920 – 939.

[5] DiBenedetto, E.: $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations.* Nonlin. Anal. 7 (1983), 827 – 850.

[6] Evans, C. L.: *A new proof of local* $C^{1+\alpha}$ *regularity for solutions of certain degenerate elliptic P.D.E.* J. Diff. Equ. 45 (1982), 356 – 373.

[7] Folland, G. B.: *Subelliptic estimates and function spaces on nilpotent Lie groups.* Ark. Mat. 13 (1975), 161 – 207.

[8] Folland, G. B. and E. M. Stein: *Estimates for the complex and analysis on the Heisenberg group.* Comm. Pure Appl. Math. 27 (1974), 459 – 522.

[9] Giusti, E.: *Direct Methods in the Calculus of Variations* (in Italian). Bologna: Unione Mat. Italiana (UMI) 1994.

[10] Hörmander, L.: *Hypoelliptic second order differential equations.* Acta Math. 119 (1967), 147 – 171.

[11] Jerison, D.: *The Poincaré inequality for vector fields satisfying Hörmander's condition.* Duke Math. J. 53 (1986), 503 – 523.

[12] Ladyzenskaja, O. A. and Ural'tzeva, N. N.: *Linear and Quasilinear Elliptic Equations.* New York: Acad. Press 1968.

[13] Lewis, J.: *Capacitary functions in convex rings.* Arch. Rat. Mech. Anal. 66 (1977), 201 – 224.

[14] Lewis, J.: *Regularity of the derivatives of solutions of certain degenerate elliptic equations.* Indiana Univ. Math. J. 32 (1983), 849 – 858.

[15] Lu, G.: *Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications.* Rev. Mat. Iberoamer. 8 (1992), 367 – 439.

[16] Maz'ja, V. G.: *Sobolev Spaces.* Berlin et al.: Springer-Verlag 1985.

[17] Marchi, S.: *Hölder continuity and Harnack inequality for De Giorgi classes related to Hörmander vector fields.* Ann. Mat. Pura Appl. (IV) 168 (1995), 171 – 188.

[18] Moser, J.: *On Harnack's theorem for elliptic differential equations.* Comm. Pure Appl. Math. 14 (1991), 577 – 591.

[19] Nagel, A., Stein, E. M. and S. Wainger: *Balls and metrics defined by vector fields.* Part I: *Basic properties.* Acta Math. 155 (1985), 103 – 147.

[20] Serrin, J.: *Local behaviour of solutions of quasi-linear elliptic equations.* Acta Math. 111 (1964), 247 – 302.

[21] Triebel, H.: *Interpolation Theory, Function Spaces, Differential Operators.* Amsterdam et al.: North Holland Pub. Comp. 1978.

[22] Triebel, H.: *Theory of Function Spaces.* Basel et al.: Birkhäuser Verlag 1983.

[23] Tolksdorf, P.: *Regularity for a more general class of quasilinear elliptic equations.* J. Diff. Equ. 51 (1984), 126 – 150.

[24] Uhlenbeck, K.: *Regularity for a class of nonlinear elliptic systems.* Acta Math. 138 (1977), 219 – 240.

[25] Ural'tzeva, N. N.: *Degenerate quasilinear elliptic systems.* Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov 7 (1968), 184 – 222.

Received 07.06.2000; in revised form 07.02.2001