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Semilinear Hyperbolic Systems with Singular Non-Local Boundary Conditions: Reflection of Singularities and Delta Waves

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Abstract. In this paper we study initial-boundary value problems for first-order semilinear hyperbolic systems where the boundary conditions are non-local. We focus on situations involving strong singularities, of the Dirac delta type, in the initial data as well as in the boundary conditions. In such cases we prove an existence and uniqueness result in an algebra of generalized functions. Furthermore, we investigate the existence and structure of delta waves, i.e., distributional limits of solutions to the regularized systems. Due to the additional singularities in the boundary data the search for delta waves requires a delicate splitting of the solution into a linearly evolving singular part and a regular part satisfying a nonlinear equation. A new feature in the splitting procedure used here, compared to delta waves in pure initial value problems, is the dependence of the singular part also on part of the regular part due to singularities enetering from the boundary. Finally, we include simple examples where the existence of delta waves breaks down.

Keywords: *Semilinear hyperbolic equations, Colombeau algebras, non-local boundary conditions, delta waves*

AMS subject classification: 46F30

1. Introduction

We study propagation and reflection of strong singularities in non-local boundary value problems for first-order semilinear hyperbolic systems. We investigate existence and uniqueness of generalized solutions to $(n \times n)$-systems in two variables with smooth coefficients in diagonal form. In the domain

$$\Pi = \left\{ (x,t) \Big| -L < x < L \text{ and } t > 0 \right\}$$

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we consider the following initial-boundary value problem for the generalized function U:

$$(\partial_t + \lambda(x,t)\partial_x)U = f(x,t,U) \qquad ((x,t) \in \Pi) \tag{1}$$

$$U|_{t=0} = A(x) \qquad (x \in (-L,L)) \tag{2}$$

$$B(t)U|_{x=-L} + C(t)U|_{x=L} + \int_{-L}^{L} D(x,t)U\,dx = H(t) \qquad (t \in (0,\infty)). \tag{3}$$

Here, $U = (U_1,\dots,U_n)$, $f = (f_1,\dots,f_n)$, $H = (H_1,\dots,H_n)$, B, C and D are real $(n\times n)$-matrices, $\lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$ is a diagonal matrix with $\lambda_1 \le \dots \le \lambda_k < 0 < \lambda_{k+1} \le \dots \le \lambda_n$ (k is fixed and $1 \le k \le n$).

Problems like (1) - (3) generalize previously considered mathematical models for age-structured populations in biology and demography (c.f. [2, 17, 18]). A particular example is the following model discussed in [18]:

$$\begin{aligned}
&(\partial_t + \partial_x)u = mu(x,t)u + f(x,t) && (x \in (0,\infty), t \in (0,\infty))\\
&u|_{t=0} = a(x) && (x \in (0,\infty))\\
&u|_{x=0} = \beta(t)\int_{x_1}^{x_2} h(x,t)K(x,t)u\,dx && (t \in (0,\infty))
\end{aligned}$$

where $u(x,t)$ is the population density of age x at time t, $f(x,t)$ is the migrant density, $\mu(x,t)$ is the death rate, $\beta(t)$, $h(x,t)$, $K(x,t)$ are the standard demographic indices.

In present paper we will study generalized solutions U to problem (1) - (3) within the Colombeau algebra $\mathcal{G}(\overline{\Pi})$ when λ and f are smooth functions and the initial and boundary data are allowed to be generalized functions. For example, the initial data can model discontinuously distributed populations by arbitrary measures and the interactions at the boundary may be shock-like or stochastic.

To be precise, we will assume that entries of A are generalized functions in the Colombeau algebra $\mathcal{G}[-L,L]$, entries of B, C, and H are generalized functions in $\mathcal{G}(\mathbb{R}_+)$, and entries of D are from $\mathcal{G}(\overline{\Pi})$. Throughout this paper $\mathcal{G}(\Omega)$ and $\mathcal{G}(\overline{\Omega})$ will always denote the full version of the Colombeau algebra over an open subset Ω of $\mathbb{R}^n$ and its closure $\overline{\Omega}$, respectively (as defined in [1, 3, 11]). We recall in particular the basic definitions of the algebra $\mathcal{G}(\overline{\Omega})$ which will contain the generalized solution U.

As a preliminary step we introduce the mollifier spaces used to parameterize the regularizing sequences of generalized functions. For $q \in \mathbb{N}_0$ denote

$$\mathcal{A}_q(\mathbb{R}) = \left\{\varphi \in \mathcal{D}(\mathbb{R}) \,\middle|\, \int \varphi(x)\,dx = 1, \int x^k\varphi(x)\,dx = 0 \text{ for } 1 \le k \le q\right\}$$

$$\mathcal{A}_q(\mathbb{R}^n) = \left\{\varphi(x_1,\dots,x_n) = \prod_{i=1}^{n}\varphi_0(x_i) \,\middle|\, \varphi_0 \in \mathcal{A}_q(\mathbb{R})\right\}.$$

For $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$ define $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$ $(\varepsilon > 0)$. Let $\Omega \subset \mathbb{R}^n$ and introduce the algebra of moderate elements $\mathcal{E}(\overline{\Omega})$ in the following way. We recall that $C^\infty(\overline{\Omega})$ is the space of smooth functions in Ω, all whose derivatives are continuously extendable up to the boundary of Ω. Define

$$\mathcal{E}(\overline{\Omega}) = \Big\{ u : \mathcal{A}_0 \times \overline{\Omega} \to \mathbb{R} \Big| \; u(\varphi, \cdot) \in C^\infty(\overline{\Omega}) \text{ for all } \varphi \in \mathcal{A}_0(\mathbb{R}) \Big\}.$$

Thus, the subalgebra $\mathcal{E}_M(\overline{\Omega})$ is defined by the elements $u \in \mathcal{E}(\overline{\Omega})$ with the property:

$$\begin{aligned} &\forall K \subset \overline{\Omega} \text{ compact and } \forall \alpha \in \mathbb{N}_0^n,\ \exists N \in \mathbb{N} \text{ such that } \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n) \\ &\exists C > 0 \text{ and } \exists \eta > 0 \text{ with } \sup_{x \in K} |\partial^\alpha u(\varphi_\varepsilon, x)| \le C\varepsilon^{-N} \quad (0 < \varepsilon < \eta). \end{aligned}$$

The ideal $\mathcal{N}(\overline{\Omega})$ consists of all $u \in \mathcal{E}(\overline{\Omega})$ with the following property:

$$\begin{aligned} &\forall K \subset \overline{\Omega} \text{ compact and } \forall \alpha \in \mathbb{N}_0^n,\ \exists N \in \mathbb{N} \text{ such that } \forall q \ge N \text{ and } \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n), \\ &\exists C > 0 \text{ and } \exists \eta > 0 \text{ with } \sup_{x \in K} |\partial^\alpha u(\varphi_\varepsilon, x)| \le C\varepsilon^{q-N} \quad (0 < \varepsilon < \eta). \end{aligned}$$

Finally,

$$\mathcal{G}(\overline{\Omega}) = \mathcal{E}_M(\overline{\Omega})/\mathcal{N}(\overline{\Omega})$$

is an associative, commutative differential algebra. The algebra $\mathcal{G}(\Omega)$ on open set is constructed in the same manner (with Ω in place of $\overline{\Omega}$ in the definition above). Note that $\mathcal{G}(\Omega)$ admits a canonical embedding of $\mathcal{D}'(\Omega)$.

The use of algebras of generalized functions, in particular Colombeau algebras, in this situation is motivated by several facts. First, when we are going to transform the boundary conditions (3) into a more convenient form, this involves division and multiplication by discontinuous functions and measures. Although this is in general impossible within the setting of distribution theory it still leads to an equivalent problem within $\mathcal{G}(\overline{\Pi})$. Second, as will be indicated by the examples in Section 3, problem (1) - (3) with distributional initial and boundary data can not be expected to admit distributional solutions in general.

In Section 2 we will prove a general existence and uniqueness result for problem (1) - (3) within the Colombeau algebra $\mathcal{G}(\overline{\Pi})$. In Section 3 we focus on the question of existence of *delta waves* in the case $D = 0$ (no integral in the boundary terms). This means that we model initial data a which are distributions of discrete support by convolution with a scaled mollifier $\varphi_\varepsilon(x)$ as the net $a^\varepsilon = a * \varphi_\varepsilon$. Assume that $b^\varepsilon, c^\varepsilon$ and h^ε are representatives of the generalized boundary data and let u^ε be the unique smooth (regularized) solution to the problem

$$(\partial_t + \lambda(x,t)\partial_x)u^\varepsilon = f(x,t,u^\varepsilon) \qquad ((x,t) \in \Pi) \tag{4}$$

$$u^\varepsilon|_{t=0} = a^\varepsilon(x) \qquad (x \in (-L, L)) \tag{5}$$

$$b^\varepsilon(t)u^\varepsilon|_{x=-L} + c^\varepsilon(t)u^\varepsilon|_{x=L} = h^\varepsilon(t) \qquad (t \in (0,\infty)). \tag{6}$$

In the case the net $(u^\varepsilon)_{\varepsilon>0}$ of smooth functions (which we also call a sequential solution) has a limit $v \in \mathcal{D}'$ as $\varepsilon \to 0$ this limit is called a *delta wave*. Note that $(u^\varepsilon)_\varepsilon$ also defines the (class of) the corresponding Colombeau solution $U = \mathrm{cl}[(u^\varepsilon)_\varepsilon]$ which is then said to be *associated* with v. We will state a positive result provided some additional assumptions on the data are valid but also show an instance of non-existence of delta waves for a particular choice of data.

Existence, uniqueness and regularity of generalized solutions to the Cauchy problem for hyperbolic $(n \times n)$-systems in two variables have been investigated in [4, 10, 13]. The existence of delta waves for such problems has been studied in [4 - 6, 11 - 16]. In particular, the sources [14, 15] are devoted to propagation of strong singularities in Cauchy problems for multidimensional constant coefficient semilinear hyperbolic systems and in a nonlinear boundary value problem (with a smooth function of (x, t, U) defining the boundary condition) for a hyperbolic equation, respectively. A main result of these papers is that, when nonlinear parts of the problem are bounded functions, a delta-wave exists and splits up into the sum of a linear singular part and a nonlinear regular part.

The contribution of our paper is that it treats non-local (non-separable and integral) boundary conditions with singular coefficients. Similarly to the aforementioned results, we show that a delta wave, if it exists, splits up into a singular and a regular part.

2. Existence and uniqueness of generalized solutions in the sense of Colombeau

In this section we focus on the questions of existence and uniqueness of a generalized solution to problem (1) - (3). To formulate the main result of this section we need two definitions concerning growth properties of Colombeau functions.

Definition 1 (see [11: Definition 17.4(a)]). Let $\Omega \subset \mathbb{R}^n$. An element $V \in \mathcal{G}(\overline{\Omega})$ is called *globally bounded*, if it has a representative $v \in \mathcal{E}_M(\overline{\Omega})$ with the following property: There is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$ there exist $C > 0$ and $\eta > 0$ with $\sup_{y \in \overline{\Omega}} |v(\varphi_\varepsilon, y)| \le C$ for $0 < \varepsilon < \eta$.

The following definition generalizes [11: Definition 17.4(b)].

Definition 2. Let $\Omega \subset \mathbb{R}^n$. Suppose we have a function $\gamma : (0,1) \mapsto (0, \infty)$. An element $V \in \mathcal{G}(\overline{\Omega})$ is called *locally of γ-growth*, if it has a representative $v \in \mathcal{E}_M(\overline{\Omega})$ with the following property: For every compact subset $K \subset \Omega$ there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$ there exist $C > 0$ and $\eta > 0$ with $\sup_{y \in K} |v(\varphi_\varepsilon, y)| \le C\gamma(\varepsilon)$ for $0 < \varepsilon < \eta$.

In addition to all the assumptions on λ, f, A, B, C, D and H made in Section 1 we impose the following conditions:

1. The mapping $U \mapsto f(x, t, U)$ and all its derivatives are polynomially bounded, uniformly for (x, t) varying in compact subsets of $\overline{\Pi}$.

2. The mapping $U \mapsto \nabla_U f(x, t, U)$ is globally bounded, uniformly for (x, t) varying in compact subsets of $\overline{\Pi}$.

3. The determinant of the matrix

$$R(t) = \begin{pmatrix} B_{1,k+1} & \dots & B_{1n} & C_{11} & \dots & C_{1k} \\ B_{2,k+1} & \dots & B_{2n} & C_{21} & \dots & C_{2k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{n,k+1} & \dots & B_{nn} & C_{n1} & \dots & C_{nk} \end{pmatrix}$$

has a representative $r \in \mathcal{E}_M(\overline{\Pi})$ with the property: There is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N(\mathbb{R})$ there exist $C > 0$ and $\eta > 0$ with $\inf_{t \in \overline{\Pi}} |r(\varphi_\varepsilon, t)| \geq C$ for $0 < \varepsilon < \eta$.

4. All elements $R_{ij}(t)$ of the matrix $R(t)$ are globally bounded.

5. $\operatorname{supp} A(x) \subset (-L, L)$, $\operatorname{supp} B_{ij}(t)$, $\operatorname{supp} C_{is}(t) \subset (0, \infty)$ for $1 \leq i \leq n$, $1 \leq j \leq k$ and $k+1 \leq s \leq n$; $\operatorname{supp} D_{im}(x,t) \subset (0,\infty) \times [-L, L]$ for $1 \leq i, m \leq n$.

6. $B_{ij}(t), C_{is}(t)$ and $D_{im}(x,t)$ for $1 \leq i, m \leq n, 1 \leq j \leq k, k+1 \leq s \leq n$ are locally of γ-growth with respect to a function $\gamma(\varepsilon)$ satisfying the condition $\gamma(\varepsilon)^{\gamma(\varepsilon)} = O(\varepsilon^{-N})$ for some $N \in \mathbb{N}$.

Assumption 1 ensures that the composition $f(x,t,U)$ is a well defined within the Colombeau algebra $\mathcal{G}(\overline{\Pi})$ for arbitrary $U \in \mathcal{G}(\overline{\Pi})$. Assumption 2 guarantees that problem (1) - (3) with smooth initial and boundary conditions has a global classical solution. Next, assumption 3 allows us to transform problem (1) - (3) into an equivalent system of integral equations. By assumption 5 we have compatibility conditions of any desired order between (2) and (3). Note that the conditions 1 - 6 are not particularly restrictive from the viewpoint of applications. In particular, condition 5 is in correspondence with mathematical models of the aforementioned applications in mathematical biology concerning continuous models of discrete structures.

The x-component of the characteristic flow according to the i-th equation in system (1) satisfies an ordinary differential equation of the form

$$\frac{dx}{dt} = \lambda_i(x(t), t).$$

Let $\omega_i(t; x_0, t_0)$ denote the (x-component of the) i-th characteristic curve of (1) that passes through a point $(x_0, t_0) \in \overline{\Pi}$ at the parameter value $t = t_0$. The smallest value of $t \geq 0$ at which the characteristic $t \mapsto (\omega_i(t; x_0, t_0), t)$ intersects the boundary of Π will be denoted by $t_i(x_0, t_0)$. We illustrate typical cases in the following figure ($j \leq k$ and $l > k$).

Let us transform problem (1) - (3) for a function $U \in \mathcal{G}(\overline{\Pi})$ into an equivalent integral operator form (recall that all integrations over finite intervals are carried out on the level of representatives). We have

$$\begin{aligned} U_i(x,t) = (R_i U)(x,t) + \int_{t_i(x,t)}^{t} \Big[& U(\omega_i(\tau; x, t), \tau) \\ & \times \int_0^1 \nabla_U f_i(\omega_i(\tau; x,t), \tau, \sigma U)\, d\sigma + f_i(\omega_i(\tau; x, t), \tau, 0) \Big] d\tau \\ & (1 \leq i \leq n) \end{aligned} \tag{7}$$

where

$$(R_i U)(x,t) = \begin{cases} M_i(t_i(x,t)) & \text{if } t_i(x,t) > 0 \\ A_i(\omega_i(0; x,t)) & \text{if } t_i(x,t) = 0 \end{cases}$$

and

$$M_i(t) = U_i|_{x=-L} \quad (k+1 \leq i \leq n)$$
$$M_i(t) = U_i|_{x=L} \quad (1 \leq i \leq k).$$

It follows from assumption 4 and [10] that the matrix $R(t)$ has an inverse with entries in $\mathcal{G}(\overline{\Pi})$, in particular there is a unique element $\det R^{-1} \in \mathcal{G}(\overline{\Pi})$ such that $\det R \det R^{-1} = 1$. Therefore

$$M_i(t) = \frac{1}{\det R(t)} \sum_{j=1}^{n} R_{ji}^{ad}(t) \Big[H_j(t) - \sum_{s=1}^{k} B_{js}(t) U_s(-L,t) - \sum_{s=k+1}^{n} C_{js}(t) U_s(L,t) - \sum_{s=1}^{n} \int_{-L}^{L} D_{js}(x,t) U_s(x,t)\, dx \Big].$$

Theorem 3. *Suppose that assumptions 1 - 6 are met for $A \in \mathcal{G}[-L,L]$, $D \in \mathcal{G}(\overline{\Pi})$ and $B, C, H \in \mathcal{G}(\mathbb{R}_+)$. Then problem $(1)-(3)$ has a unique solution $U \in \mathcal{G}(\overline{\Pi})$.*

Proof. We will make use of a priori estimates for global smooth solutions U to problem (1) - (3) with smooth initial and boundary data. We obtain these global estimates by iterating the a priori estimates on local smooth solutions in a finite number of steps. In order to prove existence of a generalized solution we need to take care of the norms of the coefficients $B_{ij}(t)$ and $C_{is}(t)$ $(1 \le i \le n,\ 1 \le j \le k,\ k+1 \le s \le n)$ in each step of the iteration process.

Given $T > 0$, denote

$$\Pi^T = \Big\{ (x,t) \Big| -L < x < L \text{ and } 0 < t < T \Big\}.$$

The existence will follow from three intermediate claims.

Claim 1: *Given $m \in \mathbb{N} \cup \{0\}$, there exists a unique solution $U \in \mathcal{C}^m(\overline{\Pi}^{t(m)})$ to problem $(1)-(3)$ with smooth initial and boundary data for some $t(m) > 0$.* By condition

5 the compatibility conditions of any order between (2) and (3) are fulfilled. Since λ is globally bounded, there exists $t_0 > 0$ such that

$$\omega_n(t; -L, \tau) < \omega_1(t; L, \tau) \qquad \forall\, \tau \geq 0, t \in [\tau, \tau + t_0]. \tag{8}$$

Let us fix t_0 with property (8). We will choose $t(m)$ so that

$$t(m) \leq t_0 \qquad \forall\, m \in \mathbb{N}_0. \tag{9}$$

For $t \in [0, t_0]$ we can convert the above expression for $M_i(t)$ into the form

$$\begin{aligned}
M_i(t) = \frac{1}{\det R(t)} \sum_{j=1}^{n} R_{ji}^{ad}(t) \Bigg\{ & H_j(t) - \sum_{s=1}^{k} B_{js}(t) \Bigg[A_s(\omega_s(0; -L, t)) \\
& + \int_0^t \Bigg(U(\omega_s(\tau; -L, t), \tau) \int_0^1 \nabla_U f_s(\omega_s(\tau; -L, t), \tau, \sigma U)\, d\sigma \\
& + f_s(\omega_s(\tau; -L, t), \tau, 0) \Bigg) d\tau \Bigg] - \sum_{s=k+1}^{n} C_{js}(t) \Bigg[A_s(\omega_s(0; L, t)) \\
& + \int_0^t \Bigg(U(\omega_s(\tau; L, t), \tau) \int_0^1 \nabla_U f_s(\omega_s(\tau; L, t), \tau, \sigma U)\, d\sigma \\
& + f_s(\omega_s(\tau; L, t), \tau, 0) \Bigg) d\tau \Bigg] - \sum_{s=k+1}^{n} \int_{-L}^{L_s(t)} D_{js}(x,t) U_s(x,t)\, dx \\
& - \sum_{s=k+1}^{n} \int_{L_s(t)}^{L} D_{js}(x,t) \Bigg[A_s(\omega_s(0; x, t)) \\
& + \int_0^t \Bigg(U(\omega_s(\tau; x, t), \tau) \int_0^1 \nabla_U f_s(\omega_s(\tau; x, t), \tau, \sigma U)\, d\sigma \\
& + f_s(\omega_s(\tau; x, t), \tau, 0) \Bigg) d\tau \Bigg] dx - \sum_{s=1}^{k} \int_{L_s(t)}^{L} D_{js}(x,t) U_s(x,t)\, dx \\
& - \sum_{s=1}^{k} \int_{-L}^{L_s(t)} D_{js}(x,t) \Bigg[A_s(\omega_s(0; x, t)) + \int_0^t \Bigg(U(\omega_s(\tau; x, t), \tau) \\
& \times \int_0^1 \nabla_U f_s(\omega_s(\tau; x, t), \tau, \sigma U)\, d\sigma + f_s(\omega_s(\tau; x, t), \tau, 0) \Bigg) d\tau \Bigg] dx \Bigg\}
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
L_s(t) &= \omega_s(t; L, 0) \qquad && (1 \leq s \leq k) \\
L_s(t) &= \omega_s(t; -L, 0) \qquad && (k+1 \leq s \leq n)
\end{aligned}$$

and, therefore,

$$\begin{aligned}
|L - L_s(t)| &\leq t \max_{\overline{\Pi}} |\lambda_s| \qquad && (1 \leq s \leq k) \\
|L + L_s(t)| &\leq t \max_{\overline{\Pi}} |\lambda_s| \qquad && (k+1 \leq s \leq n).
\end{aligned}$$

We see that (7) is a system of Volterra integral equations of the second kind to which we can apply the contraction principle. We apply the operator defined by the right-hand side of (7) to two continuous functions U^1 and U^2 having the same boundary and initial values and compare their difference

$$\begin{aligned}
&\big|U_i^1(x,t) - U_i^2(x,t)\big| \\
&\quad \le \big|(R_iU^1)(x,t) - (R_iU^2)(x,t)\big| + n\int_{t_i(x,t)}^{t} \max_{1\le i\le n} |U_i^1 - U_i^2| \\
&\qquad \times \int_0^1 \big|\nabla_U f_i(\omega_i(\tau;x,t),\tau,\sigma U^1 + (1-\sigma)U^2)\big| d\sigma d\tau \\
&\quad \le t(0)q_0 \max_{(x,t)\in\overline{\Pi}^T, 1\le i\le n} \big|U_i^1(x,t) - U_i^2(x,t)\big|
\end{aligned}$$

where

$$\begin{aligned}
q_0 = n^2 &\max_{\substack{t\in[0,T]\\ 1\le i,j\le n}} \Big|\frac{R_{ji}^{ad}(t)}{R(t)}\Big| \\
&\times \Bigg[\Bigg(\max_{\substack{t\in[0,T], 1\le j\le n\\ 1\le s\le k, k+1\le r\le n}} \big\{|B_{js}(t)|, |C_{jr}(t)|\big\} + 2L \max_{\substack{(x,t)\in\overline{\Pi}^T\\ 1\le i,j\le n}} |D_{ij}(x,t)|\Bigg) \\
&\times n \max_{\substack{(x,t,y)\in\overline{\Pi}^T\times\mathbb{R}\\ 1\le i\le n}} \big|\nabla_U f_i(x,t,y)\big| \\
&+ \max_{\substack{(x,t)\in\overline{\Pi}^T\\ 1\le i,j\le n}} |D_{ij}(x,t)| \max_{\substack{(x,t)\in\overline{\Pi}^T\\ 1\le i\le n}} |\lambda_i(x,t)|\Bigg] \\
&+ n \max_{\substack{(x,t,y)\in\overline{\Pi}^T\times\mathbb{R}\\ 1\le i\le n}} \big|\nabla_U f_i(x,t,y)\big|.
\end{aligned}$$

Assuming in addition to condition (9) that $t_0 < \frac{1}{q}$ we have thus proved the contraction property and hence existence and uniqueness of a continuous solution U. We have the following estimate of U in $\Pi^{t(0)}$:

$$\begin{aligned}
|U_i(x,t)| \le \frac{1}{1-q_0t(0)}\Bigg[&\max_{\substack{x\in[-L,L]\\ 1\le i\le n}} |A_i(x)| \Bigg(1 + n^2E(1+2L) \max_{\substack{t\in[0,T]\\ 1\le i,j\le n}} \Big|\frac{R_{ji}^{ad}(t)}{R(t)}\Big|\Bigg) \\
&+ n \max_{\substack{t\in[0,T]\\ 1\le i\le n}} |H_i(t)| \max_{\substack{t\in[0,T]\\ 1\le i,j\le n}} \Big|\frac{R_{ji}^{ad}(t)}{R(t)}\Big| \\
&+ n^2t(0)E(1+2L) \max_{\substack{t\in[0,T]\\ 1\le i,j\le n}} \Big|\frac{R_{ji}^{ad}(t)}{R(t)}\Big| \max_{\substack{(x,t)\in\overline{\Pi}^T\\ 1\le i\le n}} |f_i(x,t,0)|\Bigg]
\end{aligned}$$

where

$$E = \max_{\substack{(x,t)\in\overline{\Pi}^T, 1\le i,j\le n\\ 1\le s\le k, k+1\le r\le n}} \big\{|B_{js}(t)|, |C_{jr}(t)|, |D_{ij}(x,t)|\big\}.$$

To proceed with derivatives of higher order let us consider the following initial boundary problem for $\partial_x U$:

$$\begin{aligned}\partial_x U_i(x,t) = (R'_{ix}U)(x,t) + \int_{t_i(x,t)}^{t} \Big[\nabla_U f_i(\xi,\tau,U)\cdot\partial_x U(\xi,\tau) \\ - (\partial_x\lambda_i)(\xi,\tau)\partial_x U_i(\xi,\tau) + (\partial_x f_i)(\xi,\tau,U)\Big]\Big|_{\xi=\omega_i(\tau;x,t)} d\tau \\ (1\le i\le n)\end{aligned} \tag{12}$$

where

$$(R'_{ix}U)(x,t) = \begin{cases} \frac{1}{\lambda_i(x,\tau)}\Big(f_i(x,\tau,U) - M'_i(\tau)\Big)\Big|_{\tau=t_i(x,t)} & \text{if } t_i(x,t)>0 \\ A'_i(\omega_i(0;x,t)) & \text{if } t_i(x,t)=0 \end{cases}$$

$$\begin{aligned}M'_i(t) = \widetilde{H}_i(t) - \frac{1}{\det R(t)}\sum_{j=1}^{n} R^{ad}_{ji}(t)\Big[\sum_{s=1}^{k} B_{js}(t)(\partial_t U_s)(-L,t) \\ - \sum_{s=k+1}^{n} C_{js}(t)(\partial_t U_s)(L,t) - \sum_{s=1}^{n}\int_{-L}^{L} D_{js}(x,t)(\partial_t U_s)(x,t)\,dx\Big]\end{aligned}$$

$$\begin{aligned}\widetilde{H}_i(t) = \frac{1}{\det R(t)}\sum_{j=1}^{n} R^{ad}_{ji}(t) \\ \times\Big[H'_j(t) - \sum_{s=1}^{k} B'_{js}(t)U_s(-L,t) - \sum_{s=k+1}^{n} C'_{js}(t)U_s(L,t) \\ - \sum_{s=1}^{n}\int_{-L}^{L}(\partial_t D_{js})(x,t)U_s(x,t)\,dx\Big] \\ + \sum_{j=1}^{n}\Big(\frac{R^{ad}_{ji}(t)}{\det R(t)}\Big)'\Big[H_j(t) - \sum_{s=1}^{k} B_{js}(t)U_s(-L,t) \\ - \sum_{s=k+1}^{n} C_{js}(t)U_s(L,t) - \sum_{s=1}^{n}\int_{-L}^{L} D_{js}(x,t)U_s(x,t)\,dx\Big].\end{aligned}$$

Since U has been already estimated, $\widetilde{H}_i(t)$ is known. Recalling that $\partial_t U = f(x,t,U) - \lambda(x,t)\partial_x U$ (system (1)), we see that problem (12) with respect to $\partial_x U$ has the same structure as problem (7) has with respect to U.

Estimating the function $\partial_x U$ in the same manner as U, we obtain continuity and an estimate of type (11) for $\partial_x U$ with the value of

$$\begin{aligned}q_1 = \Big(q_0 - n\max_{\substack{(x,t,y)\in\overline{\Pi}^T\times\mathbb{R}\\ 1\le i\le n}}\big|\nabla_U f_i(x,t,y)\big|\Big)\max_{\substack{(x,t)\in\overline{\Pi}^T\\ 1\le i\le n}}|\lambda_i(x,t)|\Big(\min_{\substack{(x,t)\in\overline{\Pi}^T\\ 1\le i\le n}}|\lambda_i(x,t)|\Big)^{-1} \\ + n\max_{\substack{(x,t,y)\in\overline{\Pi}^T\times\mathbb{R}\\ 1\le i\le n}}\big|\nabla_U f_i(x,t,y)\big| + \max_{\substack{(x,t)\in\overline{\Pi}^T\\ 1\le i\le n}}\big|(\partial_x\lambda_i)(x,t)\big|\end{aligned}$$

in place of q_0 and $t(1) < \frac{1}{q_1}$ in place of $t(0)$.

Proceeding further by induction, we estimate all higher derivatives of U with respect to x. So, for $\partial_x^m U$ we obtain an estimate of type (11) with the value of

$$q_m = \Big(q_0 - n \max_{\substack{(x,t,y)\in\overline{\Pi}^T\times\mathbb{R}\\ 1\le i\le n}} \big|\nabla_U f_i(x,t,y)\big|\Big)\Big(\max_{\substack{(x,t)\in\overline{\Pi}^T\\ 1\le i\le n}} |\lambda_i(x,t)|\Big)^m \Big(\min_{\substack{(x,t)\in\overline{\Pi}^T\\ 1\le i\le n}} |\lambda_i(x,t)|\Big)^{-m}$$
$$+ n \max_{\substack{(x,t,y)\in\overline{\Pi}^T\times\mathbb{R}\\ 1\le i\le n}} \big|\nabla_U f_i(x,t,y)\big| + m \max_{\substack{(x,t)\in\overline{\Pi}^T\\ 1\le i\le n}} |(\partial_x\lambda_i)(x,t)|$$

in place of q_0 and $t(m) < \frac{1}{q_m}$ in place of $t(0)$ (in addition to condition (9)). Using system (1) and its suitable differentiations, we estimate all derivatives with respect to t and all mixed derivatives.

Claim 2: *For an arbitrarily fixed $T > 0$, in the domain Π^T there exists a unique smooth solution U to problem* (1)−(3) *with smooth initial and boundary data.* We prove this claim for $U \in \mathcal{C}(\overline{\Pi}^T)$ in $\lceil T/t(0)\rceil$ steps ($\lceil x\rceil$ denoting the smallest integer $n \ge x$) by iterating the local existence and uniqueness result in domains

$$(\Pi^{kt(0)} \cap \Pi^T) \setminus \overline{\Pi}^{(k-1)t(0)} \qquad (1 \le k \le \lceil T/t(0)\rceil).$$

We can do so since q_0 depends on T and does not depend on $t(0)$. Using estimate (11) $\lceil T/t(0)\rceil$ times and each time starting with the final value U from the previous step, we obtain the a priori estimate

$$\max_{\substack{(x,t)\in\overline{\Pi}^T\\ 1\le i\le n}} |U_i(x,t)| \le \Big(\frac{1}{1-q_0t(0)}\Big)^{\lceil T/t(0)\rceil} P(E)\Big(1 + \max_{\substack{x\in[-L,L]\\ 1\le i\le n}} |A_i(x)| + \max_{\substack{t\in[0,T]\\ 1\le i\le n}} |H_i(t)|\Big)$$

where $P(E)$ is a polynomial of degree $\lceil T/t(0)\rceil$ with positive coefficients depending on $f(x,t,0), R(t), n, L$ and T. By Claim 1, a similar global estimate is true for all derivatives of U.

Claim 3: *For an arbitrarily fixed $T > 0$, a prospective representative $u \in \mathcal{E}(\overline{\Pi}^T)$ of the solution U to problem* (1) − (3) *is moderate.* We consider all initial and boundary data as elements of the corresponding Colombeau algebras (according to the assumptions of the theorem). We choose representatives a, b, c, d and h of A, B, C, D and H, respectively, with the properties required in the theorem. Let $\phi = \varphi \otimes \varphi \in \mathcal{A}_0(\mathbb{R}^2)$. Consider a prospective representative $u = u(\phi, x, t)$ of U which is the classical smooth solution to problem (1) - (3) with initial data $a(\varphi, x)$ and boundary data $b(\varphi, t), c(\varphi, t), d(\phi, x, t), h(\varphi, t)$. Our goal is to show that u is moderate, i.e. that $u \in \mathcal{E}$. Let ε be small enough and $\phi \in \mathcal{A}_N$ with N chosen so large that the following conditions are true:

a) The moderation property holds for $a(\varphi_\varepsilon, x)$ and $h(\varphi_\varepsilon, t)$.

b) The global-boundedness estimate (see Definition 1) holds for $b_{ij}(\varphi_\varepsilon, t)$ and $c_{is}(\varphi_\varepsilon, t)$, where $1 \le i \le n, k+1 \le j \le n, 1 \le s \le k$.

c) The local-γ-growth estimate (see Definition 2) holds for $b_{ij}(\varphi_\varepsilon, t), c_{is}(\varphi_\varepsilon, t)$ and $d_{im}(\phi, x, t)$, where $1 \le i, m \le n, 1 \le j \le k, k+1 \le s \le n$, with function γ specified in assumption 7.

Since $q_0 \le C_1\gamma(\varepsilon)$ for sufficiently small ε, we can choose $t(0) = \frac{1}{2C_1\gamma(\varepsilon)} < \frac{1}{2q_0}$. Taking into account the inequality

$$2^{\lceil 2TC_1\gamma(\varepsilon)\rceil} \le \gamma(\varepsilon)^{\lceil 2TC_1\gamma(\varepsilon)\rceil} \le C_0\varepsilon^{-N_0}$$

for small enough ε, we now rewrite estimate (13) for the function $u^\varepsilon = u(\phi_\varepsilon, x, t)$ as

$$\max_{\substack{(x,t)\in\overline{\Pi}^T\\ 1\le i\le n}} |u_i^\varepsilon(x,t)| \le 2^{\lceil 2TC_1\gamma(\varepsilon)\rceil} C_2\varepsilon^{-N}\big(1 + C_3\varepsilon^{-N} + C_4\varepsilon^{-N}\big) \le C\varepsilon^{-N_1}.$$

By Claim 2, a similar estimate is true for all derivatives of $u_i^\varepsilon(x,t)$. As T is an arbitrary fixed positive real number, the existence part of the theorem follows.

The proof of the uniqueness part follows the same scheme. The only difference is that now we consider problem (1) - (3) with respect to a solution with right hand sides of (2) and (3) in $\mathcal{N}$. Note that it is sufficient then to check negligibility at order zero [7].

3. Case studies: existence and non-existence of delta waves

3.1 Existence of delta wave solutions. Under appropriate assumptions we wish to prove that a generalized solution to the problem (1) - (3) in $\mathcal{G}(\Pi)$ admits an associated distribution. To simplify notation we consider the case of a (2×2)-system (the same results hold for $(n\times n)$-systems of this kind). Namely,

$$\begin{aligned}(\partial_t + \lambda_-(x,t)\partial_x)u_1 &= f_1(x,t,u) = p_1(x,t)g_1(u)\\ (\partial_t + \lambda_+(x,t)\partial_x)u_2 &= f_2(x,t,u) = p_2(x,t)g_2(u)\end{aligned} \tag{14}$$

$$u|_{t=0} = a_s(x) + a_r(x) \tag{15}$$

$$\begin{aligned}u_1|_{x=L} &= \big(b_{1s}(t) + b_{1r}(t)\big)u_1|_{x=-L} + \big(c_{1s}(t) + c_{1r}(t)\big)u_2|_{x=L} + h_{1s}(t) + h_{1r}(t)\\ u_2|_{x=-L} &= \big(b_{2s}(t) + b_{2r}(t)\big)u_1|_{x=-L} + \big(c_{2s}(t) + c_{2r}(t)\big)u_2|_{x=L} + h_{2s}(t) + h_{2r}(t)\end{aligned} \tag{16}$$

where $a_s(x) \in \mathcal{E}'(-L,L)$ has point support at finitely many points $-L < x_1^* < x_2^* < \ldots < x_m^* < L$, $b_s, c_s \in \mathcal{E}'(0,\infty)$ have point support at points $0 < t_1^* < t_2^* < \ldots < t_l^*$ and $h_s \in \mathcal{E}'(0,\infty)$ have point support at points $0 < t_1^{**} < t_2^{**} < \ldots < t_q^{**}$. This assumption means that for every x_i^* $(1 \le i \le m)$ at least one of the functions a_{1s} or a_{2s} has its singular support at $x = x_i^*$; for every t_j^* $(1 \le j \le l)$ at least one of the functions b_{1s}, b_{2s}, c_{1s} or c_{2s} has the singular support at $t = t_j^*$; and for every t_k^{**} $(1 \le k \le q)$ at least one of the functions h_{1s} or h_{2s} has its singular support at $t = t_k^{**}$. Furthermore, a_s, b_s, c_s, h_s are sums of Dirac measures concentrated at various points. Moreover, we suppose that $\lambda_- < 0 < \lambda_+$, the function $a_r \in C^1[-L,L]$ is the regular part of the initial conditions (15) and the functions $b_r, c_r, h_r \in C^1[0,\infty]$ are the regular parts of the boundary conditions (16). Besides, the functions λ and f are continuous with respect to all their arguments, are C^1 with respect to x, and λ is globally bounded in $\overline{\Pi}$. We assume also that zero-order and first-order compatibility conditions between (15) and (16) are satisfied:

$$
\begin{aligned}
a_{1r}(L) &= b_{1r}(0)a_{1r}(-L) + c_{1r}(0)a_{2r}(L) + h_{1r}(0)\\
a_{2r}(-L) &= b_{2r}(0)a_{1r}(-L) + c_{2r}(0)a_{2r}(L) + h_{2r}(0)
\end{aligned}
$$

and

$$
\begin{aligned}
&p_1(L,0)g_1(a_r(L)) - \lambda_-(L,0)a'_{1r}(L)\\
&\quad = b'_{1r}(0)a_{1r}(-L) + c'_{1r}(0)a_{2r}(L) + h'_{1r}(0)\\
&\qquad + b_{1r}(0)\big[p_1(-L,0)g_1(a_r(-L)) - \lambda_-(-L,0)a'_{1r}(-L)\big]\\
&\qquad + c_{1r}(0)\big[p_2(L,0)g_2(a_r(L)) - \lambda_+(L,0)a'_{2r}(L)\big]\\
&p_2(-L,0)g_2(a_r(-L)) - \lambda_+(-L,0)a'_{2r}(-L)\\
&\quad = b'_{2r}(0)a_{1r}(-L) + c'_{2r}(0)a_{2r}(L) + h'_{2r}(0)\\
&\qquad + b_{2r}(0)\big[p_1(-L,0)g_1(a_r(-L)) - \lambda_-(-L,0)a'_{1r}(-L)\big]\\
&\qquad + c_{2r}(0)\big[p_2(L,0)g_2(a_r(L)) - \lambda_+(L,0)a'_{2r}(L)\big].
\end{aligned} \tag{17}
$$

Denote by J^* the set of all points $(x_i^*,0),(-L,t_j^*),(L,t_p^*)$ such that $t=t_j^*$ is a point of support for $b_s(t)$ and $t=t_p^*$ is a point of support for $c_s(t)$; by J^{**} the set of all points $(-L,t_k^{**}),(L,t_n^{**})$ such that $t=t_k^{**}$ is a point of support for $h_{2s}(t)$ and $t=t_l^{**}$ is a point of support for $h_{1s}(t)$. Let $J=J^*\cup J^{**}$. The union of the characteristic curves issuing from all points of J^* (respectively, J) and their "reflections" at the boundary $\partial\Pi$ in the direction of decreasing time up to $t=0$ (respectively, in the direction of increasing time) in accordance with (14) and (16) is denoted by I_- (respectively, I_+). The set I_- will keep track of the appearance of singularities issuing from points in $J^*\setminus([-L,L]\times\{0\})$ and the set I_+ describes the propagation of all singularities.

We denote by I_-^ε (respectively, I_+^ε) the union of all characteristic curves issuing from the ε-neighborhoods of all points included in J^* (respectively, J) and their "reflections" at the boundary $\partial\Pi$ accordingly with (14) and (16). In a similar way we introduce the sets $I_+^{(x_i^*,0)}, I_\pm^{(-L,t_j^*)}, I_\pm^{(L,t_p^*)}, I_+^{(-L,t_n^{**})}, I_+^{(L,t_l^{**})}$ the unions of all characteristic curves issuing from the corresponding points of J, with positive resp. negative time orientation, and their "reflections" at the boundary $\partial\Pi$. We denote by $I_+^{(x_i^*,0),\varepsilon}$, and similarly for the other sets, the neighborhoods of $I_+^{(x_i^*,0)}$ of "thickness" ε. Let $I = I_- \cup I_+$. Observe that $I = \cap_{\varepsilon>0} I^\varepsilon$. So, for any $T > 0$ the domain Π^T is divided by I into finitely many disjoint subdomains Π_i, i.e. $\Pi \setminus I = \cup_i \Pi_i$.

Our aim is to show that a delta wave, if it exists, splits up into the sum $v + w$ of a regular part w and a singular part v. The function w is a classical solution to the nonlinear problem

$$(\partial_t + \lambda(x,t)\partial_x)w = f(x,t,w) \tag{18}$$

$$w|_{t=0} = a_r(x) \tag{19}$$

$$\begin{aligned} w_1|_{x=L} &= b_{1r}(t)w_1|_{x=-L} + c_{1r}(t)w_2|_{x=L} + h_{1r}(t) \\ w_2|_{x=-L} &= b_{2r}(t)w_1|_{x=-L} + c_{2r}(t)w_2|_{x=L} + h_{2r}(t). \end{aligned} \tag{20}$$

The function v is equal to $\lim_{\varepsilon\to 0} v^\varepsilon$ in $\mathcal{D}'(\Pi)$, where v^ε for every $\varepsilon > 0$ is the classical solution to the linear problem

$$(\partial_t + \lambda(x,t)\partial_x)v^\varepsilon = f^\varepsilon(x,t,w) \tag{21}$$

$$v^\varepsilon|_{t=0} = a_s^\varepsilon(x) + a_r^\varepsilon(x) \tag{22}$$

$$\begin{aligned} v_1^\varepsilon|_{x=L} &= \big(b_{1s}^\varepsilon(t) + b_{1r}^\varepsilon(t)\big)v_1^\varepsilon|_{x=-L} + \big(c_{1s}^\varepsilon(t) + c_{1r}^\varepsilon(t)\big)v_2^\varepsilon|_{x=L} + h_{1s}^\varepsilon(t) + h_{1r}^\varepsilon(t) \\ v_2^\varepsilon|_{x=-L} &= \big(b_{2s}^\varepsilon(t) + b_{2r}^\varepsilon(t)\big)v_1^\varepsilon|_{x=-L} + \big(c_{2s}^\varepsilon(t) + c_{2r}^\varepsilon(t)\big)v_2^\varepsilon|_{x=L} + h_{2s}^\varepsilon(t) + h_{2r}^\varepsilon(t) \end{aligned} \tag{23}$$

where

$$f_i^\varepsilon(x,t,w) = p_i^\varepsilon(x,t)g_i(w)$$

and

$$a_s^\varepsilon = a_s * \varphi_\varepsilon, \quad b_s^\varepsilon = b_s * \varphi_\varepsilon, \quad c_s^\varepsilon = c_s * \varphi_\varepsilon, \quad h_s^\varepsilon = h_s * \varphi_\varepsilon.$$

Here $\{\varphi_\varepsilon\}_{\varepsilon>0}$ is a model delta net, i.e.

$$\varphi_\varepsilon(x) = \tfrac{1}{\varepsilon}\varphi\big(\tfrac{x}{\varepsilon}\big)$$

where φ is an arbitrarily fixed function of $\mathcal{D}(\mathbb{R})$ with $\int \varphi(x) = 1$. The functions $a_r^\varepsilon, b_r^\varepsilon, c_r^\varepsilon, h_r^\varepsilon$ and p^ε are defined as follows. Suppose that

(ι) When restricted to $\partial\Pi$, each pair of the sets $I_+^{(x_i^*,0)}$ and $I_-^{(-L,t_j^*)}$, $I_+^{(x_i^*,0)}$ and $I_-^{(L,t_p^*)}$, $I_+^{(-L,t_j^*)}$ and $I_-^{(L,t_p^*)}$ (if $t_j^* < t_p^*$), $I_-^{(-L,t_j^*)}$ and $I_+^{(L,t_p^*)}$ (if $t_j^* > t_p^*$) do not intersect.

In particular, this means that propagating singularities do not hit the singularity points at the boundary (to avoid multiplication of delta distributions there). As a consequence, there exists ε_0 such that condition (ι) holds true for the sets $I_+^{(x_i^*,0),\varepsilon}, I_\pm^{(-L,t_j^*),\varepsilon}, I_\pm^{(L,t_p^*),\varepsilon}$ for all $\varepsilon < \varepsilon_0$. For the rest of this section we assume $\varepsilon < \frac{1}{2}\varepsilon_0$. We consider the set

$I^\varepsilon_- \cap ([-L,L]\times\{0\})$ consisting of a finite number of intervals with centers $(\bar{x}_i, 0)$ $(\bar{x}_1 < \bar{x}_2 < \ldots < \bar{x}_{N_1})$ and of lengths $2\tilde{\varepsilon}_i(\varepsilon,\lambda)$ $(1 \le i \le N_1)$, respectively; similarly $I^\varepsilon \cap (\partial\Pi^T \setminus ([-L,L]\times\{0\}))$ consists of a finite number of intervals with centers $(-L,\bar{t}_j)$ and/or $(L,\bar{t}_j)$ $(\bar{t}_1 < \bar{t}_2 < \ldots < \bar{t}_{N_2})$ and of lengths $2\varepsilon_j(\varepsilon,\lambda)$ $(1 \le j \le N_2)$, respectively. It is clear that $\tilde{\varepsilon}_i(\varepsilon,\lambda) = \varepsilon$ if $\bar{x}_i = x_i^*$, that $\varepsilon_j(\varepsilon,\lambda) = \varepsilon$ if $\bar{t}_j = t_j^*$ or $\bar{t}_j = t_j^{**}$, and that $\tilde{\varepsilon}_i$ and ε_j depends on ε and λ. Furthermore, $\tilde{\varepsilon}_i \to 0$ and $\varepsilon_j \to 0$ as $\varepsilon \to 0$. Suppose that $\bar{t}_1 > 0, \bar{x}_1 > -L, \bar{x}_{N_1} < L$ and choose ε so small that $\bar{t}_1 - \varepsilon_1 > 0, \bar{x}_1 - \tilde{\varepsilon}_1 > -L, \bar{x}_{N_1} + \tilde{\varepsilon}_{N_1} < L$.

We choose a_r^ε and h_r^ε with the properties

$$\begin{aligned}
a_r^\varepsilon &\in C^1[-L,L] \\
a_r^\varepsilon(x) &= a_r(x) && \big(x \in [\bar{x}_i - \tilde{\varepsilon}_i, \bar{x}_i + \tilde{\varepsilon}_i]\big) \\
0 \le a_r^\varepsilon(x) &\le \max_{x_1 \in [\bar{x}_i - 2\tilde{\varepsilon}_i, \bar{x}_i + 2\tilde{\varepsilon}_i]} a_r(x_1) && \big(x \in (\bar{x}_i - 2\tilde{\varepsilon}_i, \bar{x}_i - \tilde{\varepsilon}_i] \cup [\bar{x}_i + \tilde{\varepsilon}_i, \bar{x}_i + 2\tilde{\varepsilon}_i)\big) \\
a_r^\varepsilon(x) &= 0 && \big(x \notin [\bar{x}_i - 2\tilde{\varepsilon}_i, \bar{x}_i + 2\tilde{\varepsilon}_i]\big)
\end{aligned}$$

and

$$\begin{aligned}
h_r^\varepsilon &\in C^1[0,T] \\
h_r^\varepsilon(t) &= h_r(t) && \big(t \in [\bar{t}_j - \varepsilon_j, \bar{t}_j + \varepsilon_j]\big) \\
0 \le h_r^\varepsilon(t) &\le \max_{t_1 \in [\bar{t}_j - 2\varepsilon_j, \bar{t}_j + 2\varepsilon_j]} h_r(t_1) && \big(t \in (\bar{t}_j - 2\varepsilon_j, \bar{t}_j - \varepsilon_j] \cup [\bar{t}_j + \varepsilon_j, \bar{t}_j + 2\varepsilon_j)\big) \\
h_r^\varepsilon(t) &= 0 && \big(t \notin [\bar{t}_j - 2\varepsilon_j, \bar{t}_j + 2\varepsilon_j]\big).
\end{aligned}$$

In a similar way we define b_r^ε and c_r^ε. Then we choose p^ε with the properties

$$\begin{aligned}
p^\varepsilon &\in C^{1,0}_{x,t}(\overline{\Pi}^T) \\
p^\varepsilon(x,t) &= p(x,t) && \big((x,t) \in \bar{I}^\varepsilon_-\big) \\
0 \le p^\varepsilon(x,t) &\le \max_{(x_1,t_1) \in \bar{I}^{2\varepsilon}_-} p(x_1,t_1) && \big((x,t) \in I^{2\varepsilon}_- \setminus I^\varepsilon_-\big) \\
p^\varepsilon(x,t) &= 0 && \big((x,t) \notin I^{2\varepsilon}_-\big).
\end{aligned}$$

Problem (21) - (23) can be expressed in an equivalent integral form. Namely, in the domain

$$\Pi_0 = \Big\{(x,t)\Big|\ \omega_2(t;-L,0) < x < \omega_1(t;L,0) \text{ and } 0 \le t < T_0\Big\}$$

system (21) - (23) is equivalent to

$$v^\varepsilon(x,t) = a_s^\varepsilon(\omega(0;x,t)) + a_r^\varepsilon(\omega(0;x,t)) + \int_0^t f^\varepsilon(\omega(\tau;x,t),\tau,w)\,d\tau, \tag{24}$$

in the domain

$$\Pi_1 = \Big\{(x,t)\Big|\omega_2(T_0;-L,0) < x < L \text{ and } \tilde{\omega}_1(x;L,0) \le t < \tilde{\omega}_2(x;-L,0)\Big\}$$

provided $\tilde{\omega}_2(L;-L,0) \le \tilde{\omega}_1(-L;L,0)$ it is equivalent to

$$\begin{aligned} v_1^\varepsilon(x,t) = \Big[&\big(b_{1s}^\varepsilon(t) + b_{1r}^\varepsilon(t)\big)\Big(a_{1s}^\varepsilon(\omega_1(0;-L,t)) \\ &+ a_{1r}^\varepsilon\big(\omega_1(0;-L,t)\big) + \int_0^t f_1^\varepsilon\big(\omega_1(\tau;-L,t),\tau,w\big)d\tau\Big) \\ &+ \big(c_{1s}^\varepsilon(t) + c_{1r}^\varepsilon(t)\big)\Big(a_{2s}^\varepsilon(\omega_2(0;L,t)) + a_{2r}^\varepsilon(\omega_2(0;L,t)) \\ &+ \int_0^t f_2^\varepsilon\big(\omega_1(\tau;L,t),\tau,w\big)d\tau\Big) + h_{1r}^\varepsilon(t) + h_{1s}^\varepsilon(t)\Big]\Big|_{t=t_1(x,t)} \\ &+ \int_{t_1(x,t)}^t f_1^\varepsilon\big(\omega_1(\tau;x,t),\tau,w\big)d\tau \\ v_2^\varepsilon(x,t) = a_{2s}^\varepsilon&(\omega_2(0;x,t)) + a_{2r}^\varepsilon(\omega_2(0;x,t)) + \int_0^t f_2^\varepsilon(\omega_2(\tau;x,t),\tau,w)\,d\tau \end{aligned} \tag{25}$$

where T_0 is the unique solution to the functional equation $\omega_2(T_0;-L,0) = \omega_1(T_0;L,0)$ and $\tilde{\omega}_i(x;x_0,t_0)$ is the unique solution to the Cauchy problem

$$\left.\begin{aligned} \tfrac{dt}{dx} &= \tfrac{1}{\lambda_i(x,t(x))} \\ t(x_0) &= t_0 \end{aligned}\right\}.$$

In general, in Π^T we have

$$v_i^\varepsilon(x,t) = (R_i v^\varepsilon)(x,t) + \int_{t_i(x,t)}^t f_i^\varepsilon(\omega_i(\tau;x,t),\tau,w)\,d\tau \tag{26}$$

where

$$\begin{aligned} &(R_i v^\varepsilon)(x,t) = \\ &\begin{cases} a_{is}^\varepsilon(\omega_i(0;x,t)) + a_{ir}^\varepsilon(\omega_i(0;x,t)) & \text{if } t_i(x,t) = 0 \\ \big[(b_{is}^\varepsilon + b_{ir}^\varepsilon)v_1^\varepsilon(-L,t) + (c_{is}^\varepsilon + c_{ir}^\varepsilon)v_2^\varepsilon(L,t) + h_{ir}^\varepsilon + h_{is}^\varepsilon\big]\big|_{t=t_i(x,t)} & \text{if } t_i(x,t) > 0. \end{cases} \end{aligned} \tag{27}$$

The initial and boundary conditions of problems (18) - (20) and (21) - (23) are compatible by condition (17) and by our assumptions on ε, respectively. ¿From the proof of Theorem 3 (Section 2) we immediately obtain existence and uniqueness of solutions $w \in C^1(\overline{\Pi}^T)$ and $v^\varepsilon \in C^1(\overline{\Pi}^T)$ for any fixed $\varepsilon > 0$ and $T > 0$.

Let u^ε be the classical solution to the problem

$$(\partial_t + \lambda(x,t)\partial_x)u^\varepsilon = f(x,t,u^\varepsilon) \tag{28}$$

$$u^\varepsilon|_{t=0} = a_s^\varepsilon(x) + a_r(x) \tag{29}$$

$$\begin{aligned} u_1^\varepsilon|_{x=L} &= \big(b_{1s}^\varepsilon(t) + b_{1r}(t)\big)u_1^\varepsilon|_{x=-L} + \big(c_{1s}^\varepsilon(t) + c_{1r}(t)\big)u_2^\varepsilon|_{x=L} + h_{1s}^\varepsilon(t) + h_{1r}(t) \\ u_2^\varepsilon|_{x=-L} &= \big(b_{2s}^\varepsilon(t) + b_{2r}(t)\big)u_1^\varepsilon|_{x=-L} + \big(c_{2s}^\varepsilon(t) + c_{2r}(t)\big)u_2^\varepsilon|_{x=L} + h_{2s}^\varepsilon(t) + h_{2r}(t). \end{aligned} \tag{30}$$

By (17) we have zero-order and first-order compatibility of (29) and (39).

To prove the main result of this section we need the following lemma.

Lemma 4. *Assume that condition* (ι) *is satisfied, and the functions* g *and* $\operatorname{grad} g$ *are globally bounded. Then* $u^\varepsilon - v^\varepsilon - w$ *when restricted to* $\partial\Pi$ *tends to 0 in* $L^1_{loc}(\partial\Pi)$ *as* $\varepsilon \to 0$.

Proof. The proof of the lemma consists of 5 steps. In what follows T is arbitrary positive real. Let $\partial\Pi^T = \partial\Pi \cap \overline{\Pi}^T$.

Claim 1: *The functions* v^ε *are bounded on* $\overline{\Pi^T \setminus I^\varepsilon}$, *uniformly in* ε. Since $\Pi^T \setminus \bar{I}^\varepsilon = \cup_j \Pi_j^\varepsilon$, where $\Pi_j^\varepsilon = (\Pi_j \setminus \bar{I}^\varepsilon) \cap \Pi^T$, it will be sufficient to show that v^ε is bounded on every $\overline{\Pi}_j^\varepsilon$ uniformly in ε. This follows from (24) - (27) restricted to $\overline{\Pi}_j^\varepsilon$, which causes $a_s^\varepsilon = b_s^\varepsilon = c_s^\varepsilon = 0$ there. Formulas (24) - (27) include a finite number of integral terms, each of which is bounded by $(1+\varepsilon)C_0(\lambda)\max_{x,t,u}|f|$ with a constant $C_0(\lambda)$ depending only on λ. Indeed, for $(x,t) \in \Pi_j^\varepsilon$

$$v_i^\varepsilon = (R_i v^\varepsilon)(x,t) + \int_{\Delta_i^\varepsilon(x,t)} f_i^\varepsilon(\omega_i(\tau;x,t),\tau,w)\,d\tau \tag{31}$$

where

$$(R_i v^\varepsilon)(x,t) = \begin{cases} \left[b_{ir}^\varepsilon v_1^\varepsilon(-L,t) + c_{ir}^\varepsilon(v_2^\varepsilon(L,t) + h_{ir}^\varepsilon\right]\big|_{t=t_i(x,t)} & \text{if } t_i(x,t) > 0 \\ a_{ir}^\varepsilon(\omega_i(0;x,t)) & \text{if } t_i(x,t) = 0 \end{cases} \tag{32}$$

and

$$\Delta_i^\varepsilon(x,t) = \left\{\tau \in [0,t] \,\middle|\, (\omega_i(\tau;x,t),\tau) \in I^{2\varepsilon}\right\}. \tag{33}$$

The integral summand in (31) is bounded by $C_0(\lambda)\max_{x,t,u}|f|$ if $\Delta_i^\varepsilon(x,t) = [t_i(x,t),t]$, and by $\varepsilon C_0(\lambda)\max_{x,t,u}|f|$ otherwise. Gronwall's argument and boundedness of g give the desired ε-uniform a priori estimate.

Claim 2: *The functions* $u^\varepsilon - v^\varepsilon$ *are bounded on* $\overline{\Pi^T \setminus I^\varepsilon}$, *uniformly in* ε. The initial-boundary value problem with respect to $u^\varepsilon - v^\varepsilon$ can be obtained by using systems (21) - (23) and (28) - (30) and expressed in the form

$$\begin{aligned} u_i^\varepsilon - v_i^\varepsilon &= (R_i(u^\varepsilon - v^\varepsilon))(x,t) \\ &\quad + \int_{t_i(x,t)}^t \Big(f_i(\omega_i(\tau;x,t),\tau,u^\varepsilon) - f_i^\varepsilon(\omega_i(\tau;x,t),\tau,w)\Big)d\tau \end{aligned} \tag{34}$$

where

$$(R_i(u^\varepsilon - v^\varepsilon))(x,t) =$$
$$\begin{cases} a_{ir}^{-\varepsilon}(\omega_i(0;x,t)) & \text{if } t_i(x,t) = 0 \\ (b_{is}^\varepsilon + b_{ir}^\varepsilon)(u_1^\varepsilon - v_1^\varepsilon)|_{x=-L} + (c_{is}^\varepsilon + c_{ir}^\varepsilon)(u_2^\varepsilon - v_2^\varepsilon)|_{x=L} & \\ +b_{ir}^{-\varepsilon}(u_1^\varepsilon - v_1^\varepsilon)|_{x=-L} + c_{ir}^{-\varepsilon}(u_2^\varepsilon - v_2^\varepsilon)|_{x=L} & \\ +b_{ir}^{-\varepsilon}v_1^\varepsilon|_{x=-L} + c_{ir}^{-\varepsilon}v_2^\varepsilon|_{x=L} + h_{ir}^{-\varepsilon} & \text{if } t_i(x,t) > 0 \end{cases}$$

and

$$a_r^{-\varepsilon} = a_r - a_r^\varepsilon, \quad b_r^{-\varepsilon} = b_r - b_r^\varepsilon, \quad c_r^{-\varepsilon} = c_r - c_r^\varepsilon, \quad h_r^{-\varepsilon} = h_r - h_r^\varepsilon. \tag{35}$$

Following the idea of the 1-st step, we consider this problem on every $\overline{\Pi}_j^\varepsilon$. Restricting system (34) - (35) to $\overline{\Pi}_j^\varepsilon$, we conclude that all summands with $a_s^\varepsilon, b_s^\varepsilon, c_s^\varepsilon$ vanish. From Claim 1 it follows that $b_{ir}^{-\varepsilon} v_1^\varepsilon|_{x=-L}$ and $c_{ir}^{-\varepsilon} v_2^\varepsilon|_{x=L}$ are bounded uniformly in ε. As in Claim 1, the uniform a priori estimate follows.

Claim 3: *For the functions $u^\varepsilon - w$ on $\overline{\Pi^T \setminus I_+^\varepsilon}$ the estimate*

$$|u^\varepsilon - w| \le \varepsilon C(\lambda) \tag{36}$$

with a constant $C(\lambda)$ depending on λ holds. We consider a similar initial-boundary value problem for $u^\varepsilon - w$ as we did in Claim 2 for $u^\varepsilon - v^\varepsilon$. Restricting to $\Pi^T \setminus I_+^\varepsilon$, we obtain

$$\begin{aligned} u_i^\varepsilon(x,t) - w_i(x,t) = {} & (R_i(u^\varepsilon - w))(x,t) \\ & + \int_{[t_i(x,t),t]\setminus\Delta_i^\varepsilon(x,t)} p_i(x,\tau) G_i(x,\tau)\cdot(u^\varepsilon - w)\big|_{x=\omega_i(\tau;x,t)} d\tau \\ & + \int_{\Delta_i^\varepsilon(x,t)} p_i(x,\tau)\big(g_i(u^\varepsilon) - g_i(w)\big)\big|_{x=\omega_i(\tau;x,t)} d\tau \end{aligned} \tag{37}$$

where

$$\begin{aligned} & (R_i(u^\varepsilon - w))(x,t) \\ & \qquad = \begin{cases} \big[b_{ir}(u_1^\varepsilon - w_1)|_{x=-L} + c_{ir}(u_2^\varepsilon - w_2)|_{x=L}\big]\big|_{t=t_i(x,t)} & \text{if } t_i(x,t) > 0 \\ 0 & \text{if } t_i(x,t) = 0 \end{cases} \\ & G_i(x,\tau) = \int_0^1 \nabla g_i\big(\sigma u^\varepsilon(x,\tau) + (1-\sigma) w(x,\tau)\big) d\sigma \\ & \Delta_i^\varepsilon(x,t) = \big\{\tau \in [0,t] \big| \, (\omega_i(\tau;x,t),\tau) \in I_+^\varepsilon\big\}. \end{aligned} \tag{38}$$

In the right-hand side of (37) we used the mean value theorem for g. Observe that the second integral in (37) is bounded by $\varepsilon C_1(\lambda)\max_{x,t,u}|f|$. By Gronwall's inequality we easily obtain estimate (36).

Claim 4: *The functions $u^\varepsilon - v^\varepsilon$ are bounded on $I^\varepsilon \cap (\partial\Pi^T)$, uniformly in ε.* According to condition (ι) we have $b_s^\varepsilon = c_s^\varepsilon = 0$ on $I^{(x_i^*,0),\varepsilon}$. From this and the boundedness of g we conclude that the claim is true for the set $\cup_i (I^{(x_i^*,0),\varepsilon} \cap \partial\Pi^T)$.

Let us consider $u_1^\varepsilon - v_1^\varepsilon$ on $I_-^\varepsilon \cap (\{-L\}\times[0,T])$ and $u_2^\varepsilon - v_2^\varepsilon$ on $I_-^\varepsilon \cap (\{L\}\times[0,T])$. Restricting (34) - (35) to $I_-^\varepsilon \cap \partial\Pi^T$, we obtain the system

$$\begin{aligned} u_i^\varepsilon(x,t) - v_i^\varepsilon(x,t) = {} & (R_i(u^\varepsilon - v^\varepsilon))(x,t) \\ & + \int_{[t_i(x,t),t]\setminus\Delta_i^\varepsilon(x,t)} p_i(x,\tau) G_i(x,\tau)\cdot(u^\varepsilon - w)\big|_{x=\omega_i(\tau;x,t)} d\tau \\ & + \int_{\Delta_i^\varepsilon(x,t)} p_i(x,\tau)\big(g_i(u^\varepsilon) - g_i(w)\big)\big|_{x=\omega_i(\tau;x,t)} d\tau \end{aligned} \tag{39}$$

where

$$\begin{aligned} & (R_i(u^\varepsilon - v^\varepsilon))(x,t) \\ & \qquad = \begin{cases} \big[b_{ir}(u_1^\varepsilon - v_1^\varepsilon)|_{x=-L} + c_{ir}(u_2^\varepsilon - v_2^\varepsilon)|_{x=L}\big]\big|_{t=t_i(x,t)} & \text{if } t_i(x,t) > 0 \\ 0 & \text{if } t_i(x,t) = 0 \end{cases} \end{aligned} \tag{40}$$

and $\Delta_i^\varepsilon(x,t)$ is given by formula (38). To get the desired estimate

$$\left|u^\varepsilon(x,t) - v^\varepsilon(x,t)\right| \le \varepsilon \widetilde{C}(\lambda) \tag{41}$$

for all $(x,t) \in I_-^\varepsilon \cap \partial\Pi^T$, we use the upper bound $\varepsilon C_1(\lambda)$ for the integrals on $\Delta_i^\varepsilon(x,t)$ and employ Claim 3 for the integrals on $[t_i(x,t), t] \setminus \Delta_i^\varepsilon(x,t)$.

It is obvious that for any fixed $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\int \varphi(x) = 1$ we have

$$b_s^\varepsilon = O\left(\tfrac{1}{\varepsilon}\right) \qquad \text{and} \qquad c_s^\varepsilon = O\left(\tfrac{1}{\varepsilon}\right). \tag{42}$$

Estimating the right-hand side of the boundary conditions in (35) by (41) and (42) we obtain Claim 4.

Claim 5: *The function $u^\varepsilon - v^\varepsilon - w \to 0$ when restricted to $\partial\Pi$ tends to 0 in $L^1_{loc}(\partial\Pi)$ as $\varepsilon \to 0$.* We consider the function $u^\varepsilon - v^\varepsilon - w$ on $\partial\Pi^T$ for some $T > 0$. From systems (19) - (20), (22) - (23) and (29) - (30) we obtain

$$(u^\varepsilon - v^\varepsilon - w)|_{t=0} = -a_r^\varepsilon \tag{43}$$

$$\begin{aligned}
(u_1^\varepsilon - v_1^\varepsilon - w_1)|_{x=L} &= (b_{1s}^\varepsilon + b_{1r}^\varepsilon)(u_1^\varepsilon - v_1^\varepsilon)|_{x=-L} + (c_{1s}^\varepsilon + c_{1r}^\varepsilon)(u_2^\varepsilon - v_2^\varepsilon)|_{x=L} \\
&\quad + b_{1r}^{-\varepsilon} u_1^\varepsilon|_{x=-L} + c_{1r}^{-\varepsilon} u_2^\varepsilon|_{x=L} + h_{1r}^{-\varepsilon} \\
&\quad - \left(b_{1r} w_1|_{x=-L} + c_{1r} w_2|_{x=L} + h_{1r}\right) \\
(u_2^\varepsilon - v_2^\varepsilon - w_2)|_{x=-L} &= (b_{2s}^\varepsilon + b_{2r}^\varepsilon)(u_1^\varepsilon - v_1^\varepsilon)|_{x=-L} + (c_{2s}^\varepsilon + c_{2r}^\varepsilon)(u_2^\varepsilon - v_2^\varepsilon)|_{x=L} \\
&\quad + b_{2r}^{-\varepsilon} u_1^\varepsilon|_{x=-L} + c_{2r}^{-\varepsilon} u_2^\varepsilon|_{x=L} + h_{2r}^{-\varepsilon} \\
&\quad - \left(b_{2r} w_1|_{x=-L} + c_{2r} w_2|_{x=L} + h_{2r}\right).
\end{aligned} \tag{44}$$

It is not difficult to show that the function $u^\varepsilon(x,t) - v^\varepsilon(x,t) - w(x,t)$ converges to zero pointwise on $\partial\Pi^T \setminus I$. Indeed, if $(x,t) \in \overline{\Pi}^T \setminus I$ and ε is sufficiently small, then

$$\begin{gathered}
a_s^\varepsilon = b_s^\varepsilon = c_s^\varepsilon = h_s^\varepsilon = 0 \\
a_r^\varepsilon = b_r^\varepsilon = c_r^\varepsilon = h_r^\varepsilon = 0 \\
b_r^{-\varepsilon} = b_r, \quad c_r^{-\varepsilon} = c_r, \quad h_r^{-\varepsilon} = h_r.
\end{gathered}$$

¿From Claim 3 it follows that $u^\varepsilon(x,t)$ converges to the solution of problem (18) - (20) pointwise in $\Pi^T \setminus I$.

Moreover, the function $u^\varepsilon(x,t) - v^\varepsilon(x,t) - w(x,t)$ is bounded uniformly in ε in $\partial\Pi^T$. This fact follows from Claims 1, 2, 4 and boundedness of $w(x,t)$. More precisely, applying Claims 1, 2, 4 and the equalities

$$\begin{aligned}
(u_1^\varepsilon - v_1^\varepsilon)|_{x=L} &= (b_{1s}^\varepsilon + b_{1r}^\varepsilon)(u_1^\varepsilon - v_1^\varepsilon)|_{x=-L} + (c_{1s}^\varepsilon + c_{1r}^\varepsilon)(u_2^\varepsilon - v_2^\varepsilon)|_{x=L} \\
&\quad + b_{1r}^{-\varepsilon}(u_1^\varepsilon - v_1^\varepsilon)|_{x=-L} + c_{1r}^{-\varepsilon}(u_2^\varepsilon - v_2^\varepsilon)|_{x=L} \\
&\quad + b_{1r}^{-\varepsilon} v_1^\varepsilon|_{x=-L} + c_{1r}^{-\varepsilon} v_2^\varepsilon|_{x=L} + h_{1r}^{-\varepsilon} \\
(u_2^\varepsilon - v_2^\varepsilon)|_{x=-L} &= (b_{2s}^\varepsilon + b_{2r}^\varepsilon)(u_1^\varepsilon - v_1^\varepsilon)|_{x=-L} + (c_{2s}^\varepsilon + c_{2r}^\varepsilon)(u_2^\varepsilon - v_2^\varepsilon)|_{x=L} \\
&\quad + b_{2r}^{-\varepsilon}(u_1^\varepsilon - v_1^\varepsilon)|_{x=-L} + c_{2r}^{-\varepsilon}(u_2^\varepsilon - v_2^\varepsilon)|_{x=L} \\
&\quad + b_{2r}^{-\varepsilon} v_1^\varepsilon|_{x=-L} + c_{2r}^{-\varepsilon} v_2^\varepsilon|_{x=L} + h_{2r}^{-\varepsilon}
\end{aligned}$$

we conclude that the functions

$$(b_{is}^\varepsilon + b_{ir}^\varepsilon)(u_1^\varepsilon - v_1^\varepsilon)|_{x=-L} + (c_{is}^\varepsilon + c_{ir}^\varepsilon)(u_2^\varepsilon - v_2^\varepsilon)|_{x=L} \qquad (i = 1, 2)$$

are uniformly bounded in ε. That $b_{ir}^{-\varepsilon} u_1^\varepsilon|_{x=-L}$ and $c_{ir}^{-\varepsilon} u_2^\varepsilon|_{x=L}$ are uniformly bounded follows from Claims 1 and 2. Claim 5 follows ∎

Theorem 5. *Assume that all conditions of Lemma* 4 *are satisfied. Then:*

1. $v^\varepsilon \to v$ *in* $\mathcal{D}'(\Pi)$ *as* $\varepsilon \to 0$.

2. $v^\varepsilon(x,t) \to 0$ *pointwise for* $(x,t) \in \Pi \setminus I$ *as* $\varepsilon \to 0$.

3. $u^\varepsilon - v^\varepsilon - w \to 0$ *in* $L^1_{loc}(\Pi)$ *as* $\varepsilon \to 0$.

4. $u^\varepsilon \to v + w$ *in* $\mathcal{D}'(\Pi)$ *as* $\varepsilon \to 0$.

Proof. To prove Claim 1 we use equations (23) - (25). Note that in these formulas the compositions of $a_r^\varepsilon, b_r^\varepsilon, c_r^\varepsilon$ and f^ε with ω are in fact independent of ε near the points where the boundary deltas sit and constitute continuous functions with respect to t. The simplest terms appearing in the expressions for v^ε are continuous functions (independent of ε) and nets converging to measures concentrated on I_+ (corresponding to the regularized deltas in the data). The boundary operator in equation (25) involves also pull-backs by the C^1 map $(x,t) \mapsto t_1(x,t)$ and products thereof. The same is true for the boundary operator terms in equation (26) in general. We remark only on the convergence of the terms of the kind pull-back by t_1 of $\alpha_1 \cdot \alpha_2^\varepsilon$, where α_1 is continuous and $\alpha_2^\varepsilon \to \delta_{t_0}$. Explicitly writing the action $\langle \alpha_1 \cdot \alpha_2^\varepsilon(t_1(x,t)), \psi(x,t)\rangle$ on a test function ψ as an integral and transforming variables $(y,s) = (x, t_1(x,t))$, we easily obtain the limit $\alpha_1(t_0) \int \psi(y, t_2(y,t_0))\, dy$, where $t_2(x, t_1(x,t)) = t$.

We observe that for any fixed $(x_0,t_0) \in \Pi^T \setminus I$ and sufficiently small ε

$$(R_i v^\varepsilon)(x_0,t_0) = 0.$$

Therefore

$$v_i^\varepsilon(x_0,t_0) = \int_{\Delta_i^\varepsilon(x_0,t_0)} f_i^\varepsilon(\omega_i(\tau; x_0,t_0), \tau, w)\, d\tau$$

where $\Delta_i^\varepsilon(x,t)$ is defined by formula (33) and $\Delta_i(x_0,t_0) \neq [t_i(x_0,t_0), t_0]$. Thus

$$|v^\varepsilon(x,t)| \le \varepsilon C_0(\lambda) \max_{x,t,u} |f|.$$

This proves Claim 2.

To prove Claim 3, observe that the function $u^\varepsilon(x,t) - v^\varepsilon(x,t) - w(x,t)$ is a solution to the system of differential equations

$$\begin{aligned}
&(\partial_t + \lambda(x,t)\partial_x)(u^\varepsilon - v^\varepsilon - w) \\
&\quad = f(x,t,u^\varepsilon) - f^\varepsilon(x,t,w) - f(x,t,w) \\
&\quad = F(x,t) \cdot (u^\varepsilon - v^\varepsilon - w) + f(x,t,u^\varepsilon) - f(x,t,u^\varepsilon - v^\varepsilon) - f^\varepsilon(x,t,w)
\end{aligned} \tag{45}$$

with initial conditions (43) and boundary conditions (44), where $F(x,t)$ is the gradient of f evaluated at an intermediate point. It is clear that the functions $f(x,t,u^\varepsilon) - f(x,t,u^\varepsilon - v^\varepsilon)$ and $f^\varepsilon(x,t,w)$ are bounded uniformly in ε and converge to 0 pointwise off I. Therefore these functions converge to 0 in L^1_{loc}-norm. By Lemma 4 the same conclusion holds true for the function $u^\varepsilon(x,t) - v^\varepsilon(x,t) - w(x,t)$ on $\partial\Pi^T$. By Lebesgue's dominated theorem we conclude that $u^\varepsilon - v^\varepsilon - w \to 0$ in $L^1(\Pi^T)$ as $\varepsilon \to 0$. This proves Claim 3. Claim 4 follows from Claims 1 and 3 due to the embedding $L^1_{loc}(\Pi) \hookrightarrow \mathcal{D}'(\Pi)$ ∎

We conclude that in situation (14) - (16) a delta wave exists and splits up into a sum of a regular part w and a singular part v as described in Theorem 5.

3.2 Non-existence of delta waves. We consider simple situations where the characteristic flow is given by straight lines and all distributional data are Dirac measures at single points.

Example 1 We consider a situation where a propagating delta singularity exactly hits a Dirac measure at the boundary. This leads to a divergent distributional interaction of the type δ^2. It is described by the equation

$$(\partial_t - \partial_x)u = 0$$

with initial value

$$u|_{t=0} = \delta_{x_1^*}$$

and boundary condition

$$u|_{x=L} = \delta_{t_1^*} u|_{x=-L} + h(t).$$

The characteristics are $(\omega(\tau; x, t), \tau) = (x + t - \tau, \tau)$. The corresponding function of "departure time" at the boundary for the characteristic to reach the point (x, t) is given by

$$t_1(x, t) = \max(x + t - L, 0).$$

We regularize the appearing delta distributions by convolution with the mollifiers $\varphi_\varepsilon(x)$ for the initial data and $\psi_\varepsilon(x)$ for the boundary data. The corresponding smooth solution of the regularized initial boundary value problem is denoted by u^ε and is given by

$$u^\varepsilon(x,t) = \begin{cases} \varphi_\varepsilon(x + t - x_1^*) & \text{if } t_1(x,t) = 0 \\ \psi_\varepsilon(x + t - L - t_1^*)u^\varepsilon(-L, x + t - L) + h(x + t - L) & \text{if } t_1(x,t) > 0 \end{cases} \tag{46}$$

We can compute u^ε explicitly in the domain

$$\Pi_1 = \Big\{(x,t) \in \Pi \Big| -L < x < L \text{ and } L - x < t < 2L - x\Big\},$$

i.e. where $t_1 > 0$, in two steps. First,

$$u^\varepsilon(x,t) = \psi_\varepsilon(x + t - L - t_1^*)u^\varepsilon(-L, x + t - L) + h(x + t - L)$$

by the second case in equation (46). Since $(-L, x + t - L)$ then lies on the boundary of the domain

$$\Pi_0 = \Big\{(x,t) \in \Pi\Big|; -L < x < L \text{ and } 0 < t < L - x\},$$

where $t_1 = 0$, we have

$$u^\varepsilon(-L, x + t - L) = \varphi_\varepsilon((-L) + (x + t - L) - x_1^*) = \varphi_\varepsilon(x + t - 2L - x_1^*).$$

Therefore u^ε is the sum of the ε-independent term $h(x + t - L)$ and the term $\psi_\varepsilon(x + t - L - t_1^*)\varphi_\varepsilon(x + t - 2L - x_1^*)$ which represents the product of measures concentrated on lines.

If $L + t_1^* \neq 2L + x_1^*$, then on a fixed compact subset of Π_1 the supports of the factors will be separated for ε small enough and the product vanishes. If $L + t_1^* = 2L + x_1^*$, we show that no distributional limit exists.

Let χ be a test function on $\mathbb{R}^2$ and set $a = L + t_1^* = 2L + x_1^*$. We have to investigate the convergence properties of

$$\big\langle \psi_\varepsilon(x+t-a)\varphi_\varepsilon(x+t-a), \chi(x,t)\big\rangle = \int \psi_\varepsilon(x+t-a)\varphi_\varepsilon(x+t-a)\chi(x,t)\, d(x,t)$$

as $\varepsilon \to 0$. Upon the change of variables $(z,s) = (\frac{x+t-a}{\varepsilon}, s)$ and using the definitions of φ_ε and ψ_ε, we can rewrite this in the form

$$\tfrac{1}{\varepsilon} \int \psi(z)\varphi(z)\chi(\varepsilon z + a - s, s)\, d(z,s).$$

The absolute value of the integrand is dominated by the function

$$\|\psi\|_{L^\infty}\Big(|\varphi(z)|\,|\chi(\varepsilon z + a - s, s) - \chi(a-s,s)| + |\varphi(z)|\,|\chi(a-s,s)|\Big)$$

which has compact support in a fixed compact set K independently of ε if ε is small. Since $|\chi(\varepsilon z + a - s, s) - \chi(a-s,s)| \to 0$ uniformly on K if $\varepsilon \to 0$, we can choose $\varepsilon_0 > 0$ and estimate the integrand by the integrable function

$$\|\psi\|_{L^\infty}\Big(|\varphi(z)|\,|\chi(\varepsilon_0 z + a - s, s) - \chi(a-s,s)| + |\varphi(z)|\,|\chi(a-s,s)|\Big).$$

The pointwise limit of the integrand as $\varepsilon \to 0$ is $\psi(z)\varphi(z)\chi(a-s,s)$, therefore by the dominated convergence the above integral (without the factor $\frac{1}{\varepsilon}$) has the limit

$$\int \psi(z)\varphi(z)\, dz \int \chi(a-s,s)\, ds$$

as $\varepsilon \to 0$. If ψ and ϕ are chosen so that the first integral is non-zero, then by the unboundedness of $\frac{1}{\varepsilon}$ we conclude that $\big\langle \psi_\varepsilon(x+t-a)\varphi_\varepsilon(x+t-a), \chi(x,t)\big\rangle$ does not converge.

Example 2 A similar distributional divergence can be produced by a quadratic (hence unbounded) right-hand side that eventually picks up a singularity from the boundary conditions. We consider the equation

$$\left.\begin{aligned} (\partial_t + \partial_x)u_1 &= u_2^2 \\ (\partial_t - \partial_x)u_2 &= 0 \end{aligned}\right\}$$

with initial values

$$\begin{aligned} u_1|_{t=0} &= 0 \\ u_2|_{t=0} &= 1 \end{aligned}$$

and boundary conditions

$$u_2|_{x=L} = (\delta_{t_1^*} + 1)\, u_2|_{x=-L}$$
$$u_1|_{x=-L} = u_2|_{x=L}$$

where $0 < t_1^* < 2L$. We regularize the delta distribution for the boundary data by convolution with the mollifiers $\varphi_\varepsilon(x)$. It is easy to compute explicitly

$$u_2^\varepsilon(x,t) = \varphi_\varepsilon(\max\{-t_1^*, x+t-L-t_1^*\}) + 1$$

in the domain

$$\Big\{(x,t) \in \Pi \Big| -L < x < L \text{ and } 0 < t < 2L - x\Big\}$$

and

$$u_1^\varepsilon(x,t) = \int_0^t \left(\varphi_\varepsilon^2(\max\{-t_1^*, \xi(\tau)+t-L-t_1^*\}) + 1\right)\Big|_{\xi(\tau)=\tau+x-t} d\tau$$

in the domain

$$\Big\{(x,t) \in \Pi \Big| -L < x < L \text{ and } 0 < t < x + L\Big\}.$$

Obviously, if $t > L + t_1^* - x$, no distributional limit of $u_1^\varepsilon(x,t)$ as $\varepsilon \to 0$ exists.

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