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# Wave Solutions to Reaction-Diffusion Systems in Perforated Domains

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**Abstract.** Traveling waves in periodically perforated domains are shown to exist for large classes of reaction-diffusion systems, provided the homogenized equation admits a non-degenerate traveling wave. This can be applied e.g. to a single equation with bistable non-linearity and to bistable monotone systems. The proof uses the implicit function theorem of a suitably transformed problem in the space $H^1$. Furthermore, corrector-type estimates are given for the wave profile and the wave velocity.

**Keywords:** *Reaction-Diffusion equations, front propagation, homogenization*

**AMS subject classification:** 35K57, 35B27

## 1. Introduction

Front propagation occurs in many applied problems, such as chemical kinetics, combustion, transport in porous media, and in biology. The basic phenomena can often be described by reaction-diffusion advection equations. In homogeneous media front propagation has been studied for a long time. However, the study of fronts in inhomogeneous media has begun more recently. Since heterogeneities occur in every natural environment, understanding their influence on the location of fronts, on their profile, and on their speed is of great importance. Also, for the description of moving interfaces in the large space-time scaling limit, precise information on the wave speed is needed. Since the normal velocity of the interface is equal to the speed of the wave in normal direction, even the formulation of interface motion requires a priori knowledge about the wave speed (compare [1]).

In the present paper we study a semilinear reaction-diffusion system in a periodically perforated domain. We also mention that our methods apply equally well to systems with rapidly oscillating coefficients. But since perforated domains have been much less studied and are of the same importance, we concentrate on this case.

For $\varepsilon > 0$ consider for $u(t, y) \in \mathbb{R}^m$ the system

$$\begin{aligned} \partial_t u(t,y) &= A\Delta u(t,y) + f(u(t,y)) && \text{for } t>0, y\in\Omega_\varepsilon \\ \partial_\nu u(t,y) &= 0 && \text{on } \partial\Omega_\varepsilon \end{aligned} \tag{1.1}$$

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where $A$ is a positive diagonal matrix. For the definition of $\Omega_\varepsilon$ let $\Omega$ be a connected $C^{2,\alpha}$-domain in $\mathbb{R}^n$. Let $\Omega$ be 1-periodic in each direction $e_i$, where $e_i$ denotes the standard unit vectors in $\mathbb{R}^n$. Now define the $\varepsilon$-periodic set $\Omega_\varepsilon$ as $\Omega_\varepsilon = \varepsilon\Omega$. We will denote by $\nu$ the exterior normal vector. As an application of system (1.1) one might think of a reacting medium with non-reacting inclusions. The non-linearity $f(u)$ is of class $C^2(\mathbb{R}^m)$ and has two zeros at $p \neq q$. Our results will apply to so-called bistable non-linearities. This and further assumptions on the non-linearity will be explained below.

For the definition of a traveling front connecting $p$ and $q$ let $k$ be a fixed unit vector in $\mathbb{R}^n$. The standard definition of a traveling wave has to be modified to account for the discrete invariance of the domain. A traveling wave in direction $k \in \mathbb{R}^n$ $(|k| = 1)$ with velocity $c \neq 0$ is a solution of system (1.1) satisfying for all $t, y$

$$\begin{aligned} u^\varepsilon(t - k_i \tfrac{\varepsilon}{c}, y + \varepsilon e_i) &= u^\varepsilon(t, y) \quad (1 \leq i \leq n) \\ u^\varepsilon(-\infty, y) &= p \\ u^\varepsilon(\infty, y) &= q \end{aligned} \tag{1.2}$$

where $e_i$ denotes the standard basis in $\mathbb{R}^n$.

By our method we can also handle the case of a periodically oscillating diffusion matrix and drift term, i.e.

$$\partial_t u_j = \nabla(A_j(\tfrac{y}{\varepsilon})\nabla u_j) + \tfrac{1}{\varepsilon} b_j(\tfrac{y}{\varepsilon})\nabla u_j + f_j(u) \tag{1.3}$$

in $\mathbb{R}^n$, where $b_j$ has zero divergence and zero mean value over one period cell.

There are several papers concerning the existence of traveling waves for (1.3). The known results below are all restricted to a single equation, since they rely in one or the other from on the maximum principle.

Existence results in cylindrical domains with coefficients depending only on the cross section of the cylinder can be found in [2 - 4] for different kinds of non-linearities. For the case of the whole $\mathbb{R}^n$ with oscillating diffusion coefficients and combustion-type non-linearity see [8, 9]. For a bistable non-linearity and with a diffusion matrix close to a constant existence is shown in [10]. It is known that for large deviations of the diffusion matrix from a constant matrix no traveling waves can exist, due to the existence of stationary solutions [11].

As far as we know, fronts in a perforated domain have not been studied previously. Also, the study of fronts for systems in inhomogeneous media is new.

System (1.1) - (1.2) is a non-standard boundary value problem for a parabolic equation. It can also be regarded as degenerate elliptic in $n + 1$ variables, since the boundary condition in (1.2) is nowhere characteristic. We study system (1.1) - (1.2) for small $\varepsilon$, i.e. close to the homogenization limit. Usually in homogenization problems one proves existence for $\varepsilon > 0$ first and shows convergence to the homogenized equations afterwards. This is quite difficult for the problem at hand due to the elliptic degeneracy, the unboundedness of the domain and the unknown wave velocity. Going the other way round seems easier and more natural. So, assuming the existence of a traveling wave for the homogenized problem, which is in the case studied here an ordinary differential equation, the implicit function theorem will be used to obtain in an appropriate function space a unique branch of traveling waves for small $\varepsilon$.

First we transform equation (1.1) with boundary condition (1.2) to a fixed domain. Let

$$z = \tfrac{y}{\varepsilon} \in \Omega \qquad \text{and} \qquad x = ky + ct + \varepsilon\chi(\tfrac{y}{\varepsilon}) \in \mathbb{R} \tag{1.4}$$

where $\chi = \chi(z)$ is the up to a constant unique periodic solution of the cell problem

$$\left.\begin{aligned} \Delta_z\chi = 0 &\quad \text{in } \Omega \\ \partial_\nu\chi + \nu k = 0 &\quad \text{on } \partial\Omega \end{aligned}\right\}. \tag{1.5}$$

Since $\partial\Omega$ is of class $C^{2,\alpha}$, the solution $\chi$ is bounded in $C^{2,\alpha}(\overline{\Omega})$. In the new coordinates problem (1.1) - (1.2) transform to

$$\left.\begin{aligned} A\Delta^\varepsilon u^\varepsilon - c\partial_x u^\varepsilon + f(u^\varepsilon) = 0 &\quad \text{for } x \in \mathbb{R}, z \in \Omega \\ \partial_\nu u^\varepsilon = 0 &\quad \text{for } x \in \mathbb{R}, z \in \partial\Omega \end{aligned}\right\} \tag{1.6}$$

and

$$\begin{aligned} &u^\varepsilon \ \text{ is 1-periodic in each } z_i \\ &u^\varepsilon(-\infty, z) = p, \ u^\varepsilon(\infty, z) = q \end{aligned} \tag{1.7}$$

where

$$\begin{aligned} \Delta^\varepsilon = \nabla^\varepsilon\nabla^\varepsilon &= \tfrac{1}{\varepsilon^2}\Delta_z + \tfrac{2}{\varepsilon}(k + \nabla_z\chi)\nabla_z\partial_x + |k + \nabla_z\chi|^2\partial_{xx} \\ \nabla^\varepsilon &= \tfrac{1}{\varepsilon}\nabla_z + (k + \nabla_z\chi)\partial_x. \end{aligned} \tag{1.8}$$

Observe that in (1.6) $\nu(k + \nabla_z\chi) = 0$ and in (1.8) $\Delta_z\chi = 0$ is used.

For the application of the implicit function theorem we will consider problem (1.6) - (1.7) also for negative $\varepsilon$. The term $\varepsilon\chi(z)$ in (1.4) is crucial for our approach. It accounts for the first order approximation, such that solutions of problem (1.6) - (1.7) will converge in $H^1$ strongly as $\varepsilon \to 0$. Without this term convergence takes place at best in $L^2$ only. But in this space the non-linearity is not differentiable making the use of the implicit function theorem impossible.

By periodicity it is enough to restrict (1.6) to $z \in D = \Omega \cap [0,1]^n$, considered as a subset of the flat torus.

The homogenized system for traveling waves, i.e. the limit problem of (1.6) - (1.7) for $\varepsilon \to 0$ has the form

$$\left.\begin{aligned} A^h u'' - cu' + f(u) = 0 &\ \text{ in } \mathbb{R} \\ u(-\infty) = p & \\ u(\infty) = q & \end{aligned}\right\} \tag{1.9}$$

which will be justified in Lemma 2. The homogenized diffusion matrix $A^h$ is computed as follows. With $\chi(z)$ as above we define

$$A^h = A \fint_D |k + \nabla_z\chi|^2 = A - A\fint_D |\nabla_z\chi|^2 \tag{1.10}$$

where the integral denotes the average. The last equality follows from (1.5) and partial integration. It implies $0 < A^h < A$ as matrices.

Let $H$ be the linearization of (1.9) at $(u, c) = (u^0, c^0)$, given by

$$Hv = A^h v'' - c^0 v' + D_u f(u^0) v \qquad \text{for } v \in H^2(\mathbb{R}). \tag{1.11}$$

Throughout the paper the crucial assumptions will be made that the homogenized problem (1.9) admits a solution $(u^0, c^0)$ such that:

**(A1)** $u^0$ approaches $p$ and $q$ exponentially as $x \to \pm\infty$.

**(A2)** The range of $H$ is closed in $L^2(\mathbb{R})$.

**(A3)** $H$ and and its adjoint $H^*$ have an algebraically simple eigenvalue 0.

**(A4)** $c^0 \neq 0$.

**(A5)** The nonlinearity $f$ is in $C^2(\mathbb{R}^m, \mathbb{R}^m)$ and satisfies the dissipativity condition

$$u_i f_i(u) < 0 \qquad \text{for } |u_i| > M. \tag{1.12}$$

We now comment on these assumptions. From (A1) it follows that $u^{0\prime}$ and $u^{0\prime\prime}$ decay exponentially at $\pm\infty$. This is always fulfilled if $(u, u') = (p, 0)$ and $(u, u') = (q, 0)$ are hyperbolic rest states of system (1.9), which is a generic assumption. Conditions (A2) and (A3) are difficult to verify for systems. Condition (A3) implies that the kernel of $A$ is spanned by $u_0'(x)$.

The last point allows to modify $f(u)$ such that $D_u f(u)$ and $D_{uu} f(u)$ are uniformly bounded for all $u \in \mathbb{R}^m$ preserving condition (A5). This modification will be needed for the differentiability of the mapping $u \mapsto f(u)$ in Sobolev spaces. Actually, we will construct $C^2$-solutions of system (1.6) for this modified non-linearity. The maximum principle applied to the $i$-th equation in (1.6) gives $|u_i|_\infty < M$ for the modified non-linearity. Hence we have a solution of the original problem. In the sequel this modification will always be assumed. Observe that (A1) - (A4) are just assumptions on the homogenized system (1.9), which is much easier to study than system (1.6) - (1.7).

In Chapter 4 it will be shown that all assumptions are satisfied for a single equation with cubic-like non-linearity and so-called monotone bistable systems. They are not satisfied, if there exists a continuum of wave velocities, as for Fischer-type non-linearities. In this case one might try a Lyapunov-Schmidt reduction in order to obtain a continuum of waves. The case of a standing wave, i.e. $c^0 = 0$ has to be treated differently and will not be considered here. By changing $x$ to $-x$ one can always achieve $c^0 > 0$.

Now preparations for the existence theorem will be given and its proof will be outlined. If not otherwise indicated $L^2$ means $L^2(\mathbb{R} \times D)$ with norm $|\cdot|_2$ and $H^1$ means $H^1(\mathbb{R} \times D)$ with norm $|\cdot|_{1,2}$. We decompose $u$ as

$$u(x, z) = u^0(x) + v(x, z)$$

with $v \in L^2$. Now fix $\alpha > 0$ and define for $c > 0$ and $\varepsilon \neq 0$

$$L_{c,\varepsilon} v = A \Delta^\varepsilon v - c v_x - \alpha v \tag{1.13}$$

with domain

$$D(L_{c,\varepsilon}) = \left\{ v \in H^1 \,\middle|\, \nabla_i^\varepsilon \nabla_j^\varepsilon v \in L^2 \text{ and } \partial_\nu v = 0 \text{ on } \mathbb{R} \times \partial D \right\}. \tag{1.14}$$

Similarly, define for $\varepsilon = 0$

$$L_{c,0} v = A^h v_{xx} - c v_x - \alpha v \tag{1.15}$$

with domain

$$D(L_{c,0}) = H^2(\mathbb{R}). \tag{1.16}$$

The introduction of $\alpha > 0$ avoids the zero in the spectrum of the operators above. In Lemma 2.1 it will be shown that $L_{c,\varepsilon}$ is invertible for all $\varepsilon$. In Lemma 2.2 it will be proven that for $w \in L^2$

$$\lim_{\varepsilon \to 0} L_{c,\varepsilon}^{-1} w = L_{c,0}^{-1} \fint_D w$$

holds in $H^1$. This is the convergence of the resolvents in the homogenization limit. One calculates for $\varepsilon \neq 0$

$$L_{c,\varepsilon} u^0 = |k + \nabla_z \chi|^2 A \partial_{xx} u^0 - c \partial_x u^0 - \alpha u^0.$$

For $\varepsilon \neq 0$ rewrite (1.6) - (1.7) as an equation for $v$. Using (1.9) we get

$$L_{c,\varepsilon} v = -|k + \nabla_z \chi|^2 A \partial_{xx} u^0 + c \partial_x u^0 - f(u^0 + v) - \alpha v$$

and define $K_c(v)$ as the right-hand side. Consider $K_c(v)$ as a map from $H^1$ to $L^2$. Since $f$ and $D_u f$ are bounded, $K_c$ will be of class $C^1$ (see [7]). Thus consider $G(v, c, \varepsilon) = (G_1, G_2)(v, c, \varepsilon)$ for any $c \in \mathbb{R}$ and $v \in H^1$ where

$$G_1(v, c, \varepsilon) = v - L_{c,\varepsilon}^{-1} K_c(v) \qquad (\varepsilon \neq 0) \tag{1.17}$$

$$G_1(v, c, 0) = v - L_{c,0}^{-1} \fint_D K_c(v) \tag{1.18}$$

$$G_2(v, c, \varepsilon) = \int_{\mathbb{R}^- \times D} \left( (u^0 - p + v)^2 - (u^0 - p)^2 \right) \ \forall \varepsilon. \tag{1.19}$$

The second component of $G$ will fix the shift in $x$. For fixed $\varepsilon$ the operator $G$ maps $H^1 \times \mathbb{R}$ into itself. In Lemma 2.3 it is shown that $G(v, c, \varepsilon)$ is differentiable in $(v, c)$ and $D_{(v,c)} G(v, c, \varepsilon)$ is continuous at $(v, c, \varepsilon) = (0, c^0, 0)$. The invertibility of $D_{(v,c)} G(0, c^0, 0)$ is proven in Lemma 2.4, provided the main assumptions (A1) - (A5) hold. Thus the implicit function theorem implies the existence of a unique local branch in $H^1 \times \mathbb{R}$ of zeros $(v^\varepsilon, c^\varepsilon)$ of $G(v, c, \varepsilon)$. Since $L_{c,\varepsilon}^{-1}$ actually maps into $D(L_{c,\varepsilon})$ it follows that $v^\varepsilon$ is a weak solution of problem (1.2). Regularity theory implies that $v^\varepsilon$ is a classical solution of problem (1.2). This gives the following main result:

**Theorem 1.1.** *Under assumption* (A1) - (A5), *for* $|\varepsilon| < \varepsilon_0$ *there exists a unique branch of classical solutions* $(u^\varepsilon, c^\varepsilon)$ *of problem* (1.6) – (1.7) *such that* $c^\varepsilon \to c^0$ *and* $u^\varepsilon - u^0 \to 0$ *in* $H^1(\mathbb{R} \times D)$ *as* $\varepsilon \to 0$.

In Chapter 2 the existence proof of the theorem will be given in detail. In Chapter 3 the following error estimates for the wave profile and the wave velocity will be derived.

**Theorem 1.2.** *Let $(u^\varepsilon, c^\varepsilon)$ be the unique solution branch given in Theorem* 1.1. *Assume that the non-linearity is in $C^2$. Then the estimates*

$$|u^\varepsilon - u^0|_{L^2(\mathbb{R}\times D)} \le M\varepsilon^2 \tag{1.20}$$

$$|\partial_x(u^\varepsilon - u^0)|_{L^2(\mathbb{R}\times D)} \le M\varepsilon^2 \tag{1.21}$$

$$|\nabla_z(u^\varepsilon - u^0)|_{L^2(\mathbb{R}\times D)} \le M\varepsilon^3 \tag{1.22}$$

$$|c^\varepsilon - c^0| \le M\varepsilon^2 \tag{1.23}$$

*hold for some constant $M$ independent of $\varepsilon < \varepsilon_0$.*

The estimates are of second order by our choice of the coordinate transformation (1.4). For the scalar case and monotone systems all the assumptions (A1) - (A5) will be verified in Chapter 4.

## 2. Proof of existence

In a series of lemmas all the requirements for the application of the implicit function theorem will be verified. At first, in order to set up an equation in $H^1$, the existence of $L_{c,\varepsilon}^{-1}$ is needed. In order to simplify the notation in this chapter the diffusion matrix $A$ is assumed to be the identity.

**Lemma 2.1.** *The operators*

$$\begin{aligned} &L_{c,0} : D(L_{c,0}) \to L^2(\mathbb{R}) \\ &L_{c,\varepsilon} : D(L_{c,\varepsilon}) \to L^2(\mathbb{R}\times D) \quad (\varepsilon \neq 0) \end{aligned}$$

*are invertible for all $c > 0$ and $\alpha > 0$. There is a constant $M$ independent of $\varepsilon$ such that*

$$|L_{c,\varepsilon}^{-1} g|_{1,2} \le M|g|_2 \tag{2.1}$$

*holds.*

**Proof.** Consider only the case $\varepsilon \neq 0$, since the case $\varepsilon = 0$ is simpler. Let $v$ be in the kernel of $L_{c,\varepsilon}$. Taking $v$ as a test function in $L_{c,\varepsilon} v = 0$ we get

$$\int_{\mathbb{R}\times D} (|\nabla^\varepsilon v|^2 + \alpha v^2) = 0.$$

Hence $v = 0$. It is easy to see that the adjoint $L_{c,\varepsilon}^*$ of $L_{c,\varepsilon}$ is the same as $L_{c,\varepsilon}$, except $c$ has to be replaced by $-c$. The same calculation as above shows that the kernel of $L_{c,\varepsilon}^*$ is zero. It remains to show that the graph of $L_{c,\varepsilon}$ and the range $R(L_{c,\varepsilon})$ are closed. The first property will be obtained as a byproduct of the proof of the second. Let $L_{c,\varepsilon} v_n = g_n \in R(L_{c,\varepsilon})$ such that the right-hand side has a limit $g \in L^2$. Testing with $v_n$ gives

$$\int_{\mathbb{R}\times D} (|\nabla^\varepsilon v_n|^2 + \alpha v_n^2) = \int_{\mathbb{R}\times D} g_n v_n$$

and hence

$$\int_{\mathbb{R}\times D}(|\nabla^\varepsilon v_n|^2 + \tfrac{\alpha}{2}v_n^2) \leq \tfrac{1}{2\alpha}\int_{\mathbb{R}\times D} g_n^2. \tag{2.2}$$

Now let

$$D_h v_n = \frac{1}{2h}\big(v_n(x+h,z) - v_n(x-h,z)\big) \in H^1$$

denote the symmetric difference quotient in $x$-direction and use it as a test function in (1.6) to obtain

$$\int_{\mathbb{R}\times D}\big(\nabla^\varepsilon D_h v_n \nabla^\varepsilon v_n + cD_h v_n \partial_x v_n + \alpha D_h v_n v_n\big) = -\int_{\mathbb{R}\times D} g_n D_h v_n.$$

The first and third term on the left vanish after a shift in $x$. Difference quotients of $H^1$-functions converge weakly in $L^2$ to the derivative as $h \to 0$. Hence one concludes

$$c\int_{\mathbb{R}\times D}|\partial_x v_n|^2 = -\int_{\mathbb{R}\times D} g_n \partial_x v_n \tag{2.3}$$

$$|\partial_x v_n|_2^2 \leq \tfrac{1}{c^2}|g_n|_2^2. \tag{2.4}$$

Adding (2.2) and (2.4) implies the uniform $L^2$-estimates for $\nabla_z v_n$

$$|\nabla_z v_n|_2 \leq \varepsilon\left(\frac{1}{\sqrt{2\alpha}} + \frac{|k+\nabla_z\chi|_\infty}{c}\right)|g_n|_2 \tag{2.5}$$

for $\nabla_z v_n$. Hence we may extract a subsequence of $v_n$ such that $v_n \rightharpoonup v$ weakly in $H^1$. Thus we can pass to the limit in the weak formulation of $L_{c,\varepsilon}v_n = g_n$ and obtain $L_{c,\varepsilon}v = g$ weakly in $H^1$. Since $N(L_{c,\varepsilon})$ is trivial, the limit $v$ is unique and the whole sequence converges.

It remains to show that $v \in D(L_{c,\varepsilon})$. For obtaining $L^2$-estimates of $\nabla_i^\varepsilon\nabla_j^\varepsilon v$ one proceeds as usual in regularity theory (see [6: Theorems 8.8 and 8.12]). Omitting details, only modifications are indicated. For $w \in H^1(\mathbb{R}\times D)$ such that the support of $w$ has a positive distance to $\mathbb{R}\times\partial D$, let

$$D_i^h w(x,z) = \frac{1}{h\varepsilon}\Big(w\big(x+\varepsilon h k_i + \varepsilon\chi(z+he_i) - \varepsilon\chi(z), z+he_i\big) - w(x,z)\Big)$$

denote the difference quotient corresponding to $\nabla_i^\varepsilon w$. Now use $D_j^{-h}w$ for $w = \eta^2 D_i^{+h}v$, with $\eta \in C_0^1(D)$, as a test function in (1.6). In this way $L^2(\mathbb{R}\times D')$-bounds for $\nabla_i^\varepsilon\nabla_j^\varepsilon v$ are obtained for any $D' \subset D$ which has a positive distance to $\partial D$. Near the boundary $\partial D$ the estimates are more conveniently derived in the original coordinates $(t,y)$, using the standard procedure of straightening the boundary. Therefore, $\nabla_i^\varepsilon\nabla_j^\varepsilon v \in L^2$ holds. Furthermore, this implies that the boundary condition $\nu\nabla_z v = \nu\nabla^\varepsilon v = 0$ on $\mathbb{R}\times\partial D$ is preserved for the limit $v$. Thus $v \in D(L_{c,\varepsilon})$ and $R(L_{c,\varepsilon})$ is closed. The above calculations also prove (2.1) ∎

The next lemma essentially shows the validity of the homogenization limit.

**Lemma 2.2.** *Let $c > 0$ and $\alpha > 0$. Then:*

**(i)** *In $H^1$, $L_{c,\varepsilon}^{-1} g \to L_{c,0}^{-1} ⨍_D g$ holds for all $g \in L^2$ as $\varepsilon \to 0$.*

**(ii)** *$G : H^1 \times \mathbb{R}^+ \times \mathbb{R} \to H^1 \times \mathbb{R}$ is continuous.*

**Proof.** Estimates (2.2), (2.4) and (2.5) in Lemma 2.1 show that, as $\varepsilon \to 0$, $v^\varepsilon = L_{c,\varepsilon}^{-1} g$ converges for a subsequence strongly in $L^2_{loc}$ and weakly in $H^1$ to some $v^0$ and $\nabla_z v^0 = 0$ holds. Now take $\phi \in H^1(\mathbb{R})$ as a test function in (1.6). Then

$$\int_{\mathbb{R}\times D} \Big(\partial_x \phi (\nabla_z \chi + k)\big(\tfrac{1}{\varepsilon}\nabla_z v^\varepsilon + (\nabla_z \chi + k)\partial_x v^\varepsilon\big) + c\phi\partial_x v^\varepsilon + \alpha\phi v^\varepsilon\Big) = -\int_{\mathbb{R}\times D} \phi g.$$

The term with $\frac{1}{\varepsilon}$ vanishes after partial integration, by the definition (1.5) of $\chi$. Hence the limit $v^0$ satisfies

$$\int_{\mathbb{R}\times D} \big(\partial_x \phi |\nabla_z \chi + k|^2 \partial_x v^0 + c\phi\partial_x v^0 + \alpha\phi v^0\big) = -\int_{\mathbb{R}\times D} \phi g.$$

Thus, by the definition of $A^h$ in (1.10), $L_{c,0} v^0 = ⨍_D g$ holds weakly and $v^0 \in H^2$ follows.

In order to prove strong convergence in $H^1$ let $\widetilde{v}^\varepsilon = v^\varepsilon - v^0$ and calculate

$$L_{c,\varepsilon}\widetilde{v}^\varepsilon = g - ⨍_D g + (A^h - |\nabla_z \chi + k|^2) v^0_{xx} =: a.$$

Observe that $a$ has zero mean value with respect to the $z$-variable. Now define

$$\overline{\widetilde{v}^\varepsilon}(x) = ⨍_D \widetilde{v}^\varepsilon(x,z)dz.$$

Testing (1.6) with $\widetilde{v}^\varepsilon$ and using the Poincare inequality and estimate (2.5) gives

$$\int_{\mathbb{R}\times D} (|\nabla^\varepsilon \widetilde{v}^\varepsilon|^2 + \alpha|\widetilde{v}^\varepsilon|^2) = -\int_{\mathbb{R}\times D} a(\widetilde{v}^\varepsilon - \overline{\widetilde{v}^\varepsilon}) \leq M_1 |a|_{L^2} |\nabla_z \widetilde{v}^\varepsilon|_{L^2} \leq M_2 \varepsilon.$$

Hence $v^\varepsilon \to v^0$ and $\nabla_z v^\varepsilon \to 0$ strongly in $L^2$. Now use (2.3) for $v^\varepsilon$ and $v^0$ and the weak convergence in $H^1$ to conclude

$$c\int_{\mathbb{R}\times D} |\partial_x v^\varepsilon|^2 = \int_{\mathbb{R}\times D} g\partial_x v^\varepsilon \to \int_{\mathbb{R}\times D} g v^0_x = c\int_{\mathbb{R}\times D} |v^0_x|^2,$$

i.e. the $H^1$-norm converges. Together with the weak convergence this implies the strong convergence in $H^1$, proving assertion (i).

The continuity of $G_1$ at $\varepsilon = 0$ follows easily from assertion (i), since the nonlinearity $f$ maps $H^1$ continuously into $L^2$. The continuity of $G_1(v,c,\varepsilon)$ for $\varepsilon \neq 0$ is proved similarly, but simpler, as above. At last, the continuity of $G_2$ is obvious ∎

Next the Fréchet differentiability of $G(v,c,\varepsilon)$ with respect to $(v,c)$ will be proven. Remark that Lemma 2.2 only implies the continuity of the Gâteaux derivatives as $\varepsilon \to 0$. At $(v,c,\varepsilon) = (0,c^0,0)$ higher regularity properties of $u^0$ will be used to prove the continuity of $D_{(v,c)}G(v,c,\varepsilon)$ at $(0,c^0,\varepsilon)$. This will suffice for the application of the implicit function theorem.

**Lemma 2.3.** *If the solution $u^0$ of system* (1.6) – (1.7) *and the non-linearity $f$ are of $C^2(\mathbb{R}^m, \mathbb{R}^m)$-type, then for fixed $\varepsilon \in \mathbb{R}$ the operator $G(v, c, \varepsilon)$ is Fréchet differentiable with respect to $(v, c) \in H^1(\mathbb{R} \times D) \times \mathbb{R}^+$ and $D_{(v,c)}G(v, c, \varepsilon)$ is continuous at $(0, c^0, 0)$.*

**Proof.** Since $f$ and $D_u f$ are bounded, the mapping $f$ is continuously Fréchet differentiable from $H^1$ to $L^2$ (see [7]). The operators $L_{c,\varepsilon}^{-1}$ and $L_{c,0}^{-1} \fint_D$ map $L^2$ into $H^1$. Hence for fixed $\varepsilon \in \mathbb{R}$ the operator $G(v, c, \varepsilon)$ is continuously Fréchet differentiable from $H^1 \times \mathbb{R}^+$ to $H^1 \times \mathbb{R}$ with derivative

$$D_{(v,c)}G(v, c, \varepsilon)\binom{v}{\widetilde{c}} = \begin{pmatrix} \widetilde{v} - L_{c,\varepsilon}^{-1}\big(D_v K_c(v)\widetilde{v} + \widetilde{c}\partial_x(L_{c,\varepsilon}^{-1} K_c(v) + u^0)\big) \\ 2\int_{\mathbb{R}^- \times D}(u^0 + v - p)\widetilde{v} \end{pmatrix} \tag{2.6}$$

for $\varepsilon \neq 0$ and

$$D_{(v,c)}G(v, c, 0)\binom{\widetilde{v}}{\widetilde{c}} = \begin{pmatrix} \widetilde{v} - L_{c,0}^{-1}\big(\fint_D D_v K_c(v)\widetilde{v} + \widetilde{c}\partial_x\big(L_{c,0}^{-1} \fint_D K_c(v) + u^0\big)\big) \\ 2\int_{\mathbb{R}^- \times D}(u^0 + v - p)\widetilde{v} \end{pmatrix} \tag{2.7}$$

with $v, \widetilde{v} \in H^1$ and $D_v K_c(v)\widetilde{v} = -(D_u f(u^0 + v) + \alpha)\widetilde{v}$. Observe that the first components in (2.6) - (2.7) make sense since $\partial_x L_{c,\varepsilon}^{-1} K_c(v) \in L^2$ and $\partial_x L_{c,0}^{-1} \fint_D K_c(v) \in L^2(\mathbb{R})$. We have used the identities $D_c L_{c,\varepsilon}^{-1} = L_{c,\varepsilon}^{-1}\partial_x L_{c,\varepsilon}^{-1}$ and $D_c K_c = \partial_x u_0$. In particular, at $(v, c, \varepsilon) = (0, c^0, 0)$ we obtain, using $G(0, c^0, 0) = 0$, that

$$M\binom{\widetilde{v}}{\widetilde{c}} := D_{(v,c)}G(0, c^0, 0)\binom{\widetilde{v}}{\widetilde{c}} = \begin{pmatrix} \widetilde{v} - L_{c_0,0}^{-1}\big(\fint_D D_v K_{c_0}(0)\widetilde{v} + \widetilde{c}\partial_x u^0\big) \\ 2\int_{\mathbb{R}^- \times D}(u^0 - p)\widetilde{v} \end{pmatrix} \tag{2.8}$$

holds.

In order to prove continuity at $(0, c^0, 0)$ consider first the case $\widetilde{c} = 0$ in (2.6) - (2.7). Lemma 2.2 implies that $D_v G_1(v, c, \varepsilon)\widetilde{v} \to D_v G_1(v, c, 0)\widetilde{v}$ in $H^1$, but gives no uniform convergence in $\widetilde{v}$ as $\varepsilon \to 0$. Uniform convergence will now be proven as $(v, c, \varepsilon) \to (0, c^0, 0)$. For this define

$$w^\varepsilon = L_{c,\varepsilon}^{-1} D_v K_c(v)\widetilde{v} \qquad \text{and} \qquad w^0 = L_{c_0,0}^{-1} \fint_D D_v K_{c_0}(0)\widetilde{v}$$

for $\varepsilon \neq 0$. Let $w = w^\varepsilon - w^0$ and calculate

$$L_{c,\varepsilon} w = a + b \tag{2.9}$$

with

$$a = \big(A^h - |\nabla_z \chi + k|^2\big)w_{xx}^0 - \big(D_u f(u^0) + \alpha\big)\widetilde{v} + \fint_D \big(D_u f(u^0) + \alpha\big)\widetilde{v}$$
$$b = -\big(D_u f(u^0 + v) - D_u f(u^0)\big)\widetilde{v} + (c^0 - c)\partial_x w^0.$$

The definition of $w^0$ is equivalent to

$$A^h w_{xx}^0 - c^0 w_x^0 - \alpha w^0 = -\fint_D (D_u f(u^0) + \alpha)\widetilde{v}.$$

Since $f \in C^2(\mathbb{R})$, $u^0 \in H^3(\mathbb{R}) \cap C^3(\mathbb{R})$ and $\widetilde{v} \in H^1$ one concludes that $\partial_x^{(3)} w^0 \in L^2$ and hence $a, \partial_x a \in L^2$. This fails, if one tries to prove continuity of $D_v G_1$ in a full neighborhood of $(v, c, \varepsilon) = (0, c^0, 0)$. Furthermore, $a$ and $\partial_x a$ have zero mean with respect to $D$. Since $f$ is continuously differentiable from $H^1$ to $L^2$ and $L_{c,0}^{-1}$ is bounded from $L^2$ to $H^1$, it follows $|w_x^0|_2 \leq M|\widetilde{v}|_{1,2}$ and $|b|_2 \leq \eta|\widetilde{v}|_{1,2}$ where $\eta \to 0$ as $|v|_{1,2} \to 0$ and $c \to c^0$. Testing (2.9) with $w$ and using the Poincare inequality with constant $\gamma$ yields

$$|\nabla^\varepsilon w|_2^2 + \tfrac{\alpha}{2}|w|_2^2 \leq \gamma|a|_2|\nabla_z w|_2 + \tfrac{1}{2\alpha}|b|_2^2. \tag{2.10}$$

Estimate (2.3) gives, using again the Poincare inequality,

$$\tfrac{c}{2}|\partial_x w|_2^2 \leq \gamma|\partial_x a|_2|\nabla_z w|_2 + \tfrac{1}{2c}|b|_2^2. \tag{2.11}$$

Adding suitable multiples of (2.10) and (2.11) implies

$$\tfrac{1}{2\varepsilon^2}|\nabla_z w|_2^2 \leq \gamma\big(|a|_2 + \tfrac{2}{c}|\partial_x a|_2\big)|\nabla_z w|_2 + \big(\tfrac{1}{c^2} + \tfrac{1}{2\alpha}\big)|b|_2^2. \tag{2.12}$$

Hence $|\nabla_z w|_2$ is small, if $|v|_{1,2}$, $c - c^0$ and $\varepsilon$ are small. Now (2.11) and (2.12) imply the same for $|w|_2$ and $|\partial_x w|_2$. This holds uniformly for $|\widetilde{v}|_{1,2} \leq 1$, proving the continuity of $D_v G_1(v, c, \varepsilon)$ at $(0, c^0, 0)$.

Now let $\widetilde{v} = 0$ in (2.6) - (2.7). For all $\varepsilon \neq 0$,

$$\begin{aligned} D_c G_1(v, c, \varepsilon) &= L_{c,\varepsilon}^{-1} \circ \partial_x \circ L_{c,\varepsilon}^{-1} K_c(v) - L_{c,\varepsilon}^{-1} u^{0\prime} \\ D_c G_1(v, c, 0) &= L_{c,0}^{-1} \circ \partial_x \circ L_{c,0}^{-1} \fint_D K_c(v) - L_{c,0}^{-1} u^{0\prime} \end{aligned}$$

holds. Lemma 2.2 implies that $L_{c,\varepsilon}^{-1} K_c(v)$ converges in $H^1$ to $L_{c,0}^{-1} \fint_D K_c(v)$. Therefore, $\partial_x L_{c,\varepsilon}^{-1} K_c(v)$ converges in $L^2$ to $\partial_x L_{c,0}^{-1} \fint_D K_c(v)$. Lemma 2.2 implies then the continuity of $D_c G_1(v, c, \varepsilon)$ as $\varepsilon \to 0$. The second components of $D_{(v,c)} G$ in (2.6) - (2.7) are obviously continuous. This completes the proof of the lemma ∎

It remains to show that $M = D_{(v,c)} G(0, c^0, 0)$ defined in (2.8) is invertible.

**Lemma 2.4.** *If assumptions* (A1) - (A5) *are satisfied, then the operator $M$ is invertible as a map from $H^1 \times \mathbb{R}$ into itself.*

**Proof.** At first let us prov that $M$ has a closed range $R(M)$. For this let

$$M(\widetilde{v}_n, \widetilde{c}_n) = (\widetilde{g}_n, \widetilde{d}_n) \to (\widetilde{g}, \widetilde{d}) \qquad \text{in } H^{1,2} \times \mathbb{R}$$

and let $v_n = \widetilde{v}_n - \widetilde{g}_n$. Then (2.8) implies $v_n \in D(L_{c_0,0})$. Furthermore, with the definition of $H$ in (1.11),

$$Hv_n = L_{c_0,0} v_n + (D_u f(u^0) + \alpha) v_n = -\fint_D (D_u f(u^0) + \alpha)\widetilde{g}_n + \widetilde{c}_n u^{0\prime} =: g_n$$

holds. Testing with $0 \neq w \in N(H^*)$ implies

$$\widetilde{c}_n \int_{\mathbb{R}\times D} w u^{0\prime} = \int_{\mathbb{R}\times D} w(D_u f(u^0) + \alpha)\widetilde{g}_n.$$

The left-hand side is non-zero, since zero is an algebraically simple eigenvalue of $H$, by condition (A3). Therefore, $\widetilde{c}_n$ converges to some $\widetilde{c}$. Furthermore, $g_n$ converges in $L^2(\mathbb{R})$ to some $g$. Using the closedness of $R(H)$, from condition (A2) it follows that $Hv = g$ for some $v \in H^2(\mathbb{R})$, which is unique up to adding a multiple of $u^{0\prime}$. Let $\widetilde{v} = v + \gamma u^{0\prime} + g$, where $\gamma$ is chosen such that

$$2 \int_{\mathbb{R}^- \times D} (u^0 - p)v = \widetilde{d}.$$

It is easy to check that $M(\widetilde{v}, \widetilde{c}) = (\widetilde{g}, \widetilde{d})$ and that $R(M)$ is closed.

Now suppose that $M(\widetilde{v}, \widetilde{c}) = 0$. From (2.8) $\nabla_z \widetilde{v} = 0$ follows since $L^{-1}_{c_0,0}$ maps into $D(L_{c_0,0}) = H^2(\mathbb{R})$. Furthermore,

$$L_{c_0,0}\widetilde{v} + \alpha\widetilde{v} + D_u f(u^0)\widetilde{v} = \widetilde{c}u^{0\prime}.$$

Observe that the left-hand side is just the linearization $H$ at the homogenized wave solution. The eigenvalue zero is algebraically simple by condition (A3). This implies $\widetilde{c} = 0$ and $\widetilde{v} = \gamma u^{0\prime}$. Now (2.8) implies

$$0 = 2\gamma \int_{\mathbb{R}^- \times D} (u^0 - p)u^{0\prime} = \gamma(u^0(0) - p)^2.$$

Using the assumption $u^0(0) \neq p$ we get $\gamma = 0$. Hence M has a trivial kernel. The adjoint of the operator $M$ is given by

$$M^* \binom{\widetilde{v}}{\widetilde{c}} = \begin{pmatrix} \widetilde{v} + (D_u f^T(u^0) + \alpha) L^{*-1}_{c_0,0} \fint_D \widetilde{v} + 2\widetilde{c}(u^0 - p)\chi_{\mathbb{R}^-} \\ - \int_{\mathbb{R}} u^{0\prime} L^{*-1}_{c_0,0} \fint_D \widetilde{v} \end{pmatrix} \tag{2.13}$$

where $\chi_{\mathbb{R}^-}$ is the characteristic function of $\mathbb{R}^-$. If $(\widetilde{v}, \widetilde{c})$ is in the kernel of $M^*$, then the first equation in (2.13) implies $\nabla_z \widetilde{v} = 0$ since $L^{*-1}_{c_0,0}$ maps into $D(L^*_{c_0,0}) = H^2(\mathbb{R})$. Using $v = L^{*-1}_{c_0,0}\widetilde{v}$ gives

$$L^*_{c_0,0}v + (D_u f(u^0)^T + \alpha)v = -2\widetilde{c}(u^0 - p)\chi_{\mathbb{R}^-}.$$

The left-hand side is the adjoint $H^*$ of the linearization of (1.9) at $(u^0, c^0)$. Testing with $u^{0\prime}$ which is in the kernel $N(H)$ of $H$ gives $0 = \widetilde{c}(u^0(0) - p)^2$ and therefor $\widetilde{c} = 0$. Thus $v$ is in $N(H^*)$. The second equation in (2.13) implies that v is orthogonal to $N(H)$. Hence $v$ is in $R(H^*)$ by the closed range theorem. But 0 is an algebraically simple eigenvalue of the adjoint, implying $v = \widetilde{v} = 0$ and $M^*$ has a trivial kernel. Thus $M$ is invertible, proving the lemma ∎

Now all assumptions needed for the application of the implicit function theorem are fulfilled, completing the proof of the theorem.

## 3. Error estimates

In this chapter corrector-type estimates are derived. They will be of second order, since the first order is incorporated in the coordinate transformation (1.4.)

**Proof of Theorem 1.2.** Again suppose for simplicity that the diffusion matrix is the identity. The formal second order expansion of $v^\varepsilon = u^\varepsilon - u^0$ shows that the first order term vanishes and the second order term is given by $v_2(x,z) = \varepsilon^2 u^0_{xx}(x)\psi(z)$ where $\psi \in C^{2,\alpha}$, up to a constant unique, is the periodic solution of the problem

$$\left.\begin{array}{ll} \Delta_z \psi = \fint_D |k+\nabla_z\chi|^2 - |k+\nabla_z\chi|^2 & \text{in } D \\ \partial_N \psi = 0 & \text{on } \partial D \end{array}\right\}. \tag{3.1}$$

In order to obtain an estimate for $v^\varepsilon - v_2$ let $w^\varepsilon = G_1(v_2, c^0, \varepsilon)$, or equivalently, using equation (1.9) for $u^0$,

$$\begin{aligned} L_{c,\varepsilon} w^\varepsilon = &\big(\Delta_z\psi - A^h + |k+\nabla_z\chi|^2\big)u^0_{xx} \\ &+ 2\varepsilon(k+\nabla_z\chi)\nabla_z\psi\partial_x^{(3)}u^0 \\ &+ \varepsilon^2\psi\partial_x^{(4)}u^0 - f(u^0) + f(u^0+\varepsilon^2 u^0_{xx}\psi). \end{aligned} \tag{3.2}$$

Observe that $u_0 \in C^4(\mathbb{R})$ since $f \in C^2(\mathbb{R})$ and the derivatives decay exponentially at infinity. The definition of $\psi$ in (3.1) and that of $A^h$ in (1.10) implies that the term in front of $u^0_{xx}$ vanishes. Thus (3.2) gives

$$L_{c,\varepsilon} w^\varepsilon = \varepsilon r + \varepsilon^2 s^\varepsilon \tag{3.3}$$

with

$$\begin{aligned} r &= 2\partial_x^{(3)}u^0(k+\nabla_z\chi)\nabla_z\psi \\ s^\varepsilon &= \partial_x^{(4)}u^0\psi + \tfrac{1}{\varepsilon^2}\big(f(u^0+\varepsilon^2 u^0_{xx}\psi) - f(u^0)\big). \end{aligned}$$

Since $f$ is uniformly bounded in $C^2$, then $s^\varepsilon, r, r_x$ are uniformly bounded in $L^2$. Definition (1.5) of $\chi$ implies $\int_D r = \int_D r_x = 0$. Let $\overline{w^\varepsilon}(x) = \fint_D w^\varepsilon(x,z)dz$ and let $M_i$ denote appropriate positive constants independent of $\varepsilon$. Testing (3.3) with $w^\varepsilon$ yields

$$\begin{aligned} \int_{\mathbb{R}\times D} \big(|\nabla^\varepsilon w^\varepsilon|^2 + \alpha|w^\varepsilon|^2\big) &= -\varepsilon\int_{\mathbb{R}\times D} r(w^\varepsilon - \overline{w^\varepsilon}) - \varepsilon^2\int_{\mathbb{R}\times D} s^\varepsilon w^\varepsilon \\ &\le M_1\varepsilon|\nabla_z w^\varepsilon|_2|r|_2 + \varepsilon^2|s^\varepsilon|_2|w^\varepsilon|_2. \end{aligned} \tag{3.4}$$

Equation (2.3) in the proof of Lemma 2.2 gives

$$\begin{aligned} c^\varepsilon\int_{\mathbb{R}\times D} |w^\varepsilon_x|^2 &= \varepsilon\int_{\mathbb{R}\times D} r_x(w^\varepsilon - \overline{w^\varepsilon}) - \varepsilon^2\int_{\mathbb{R}\times D} s^\varepsilon w^\varepsilon_x \\ &\le \varepsilon M_1|\nabla_z w^\varepsilon|_2|\partial_x r|_2 + \varepsilon^2|s^\varepsilon_x|_2|w^\varepsilon|_2. \end{aligned} \tag{3.5}$$

In (3.4) - (3.5) the Poincare inequality has been used. Adding a suitable multiple of (3.4) - (3.5) yields

$$\int_{\mathbb{R}\times D} \big(\tfrac{M_2}{\varepsilon^2}|\nabla_z w^\varepsilon|^2 + \alpha|w^\varepsilon|^2\big) \le M_3\varepsilon^4.$$

This and (3.5) imply

$$|w^\varepsilon|_{L^2(\mathbb{R}\times D)} \le M_4\varepsilon^2$$
$$|\nabla_z w^\varepsilon|_{L^2(\mathbb{R}\times D)} \le M_5\varepsilon^3$$
$$|\partial_x w^\varepsilon|_{L^2(\mathbb{R}\times D)} \le M_6\varepsilon^2.$$

Using $G_1(v^\varepsilon, c^\varepsilon, \varepsilon) = 0$ implies now

$$\left|G_1(v^\varepsilon, c^\varepsilon, \varepsilon) - G_1(\varepsilon^2 u^0_{xx}\psi, c^0, \varepsilon)\right|_{1,2} \le M_7\varepsilon^2.$$

It is easy to see that

$$\left|G_2(v^\varepsilon, c^\varepsilon, \varepsilon) - G_2(\varepsilon^2 u^0_{xx}\psi, c^0, \varepsilon)\right| \le M_8\varepsilon^2.$$

Since $D_{(v,c)}G(0, c^0, 0)$ is invertible and $D_{(v,c)}G$ is continuous at $(0, c^0, 0)$, $D_{(v,c)}G$ has an inverse for small $\varepsilon$. This implies $|v^\varepsilon - v_2|_{1,2} \le M_9\varepsilon^2$ and $|c^\varepsilon - c^0| \le M_{10}\varepsilon^2$. The first inequality gives easily $|v^\varepsilon|_{1,2} \le M_{11}\varepsilon^2$ and (1.20), (1.21) and (1.23) are proved. Now applying the same procedure as above to the equation $L_{c,\varepsilon}v^\varepsilon = K_c(v^\varepsilon)$ and using the estimates for $v^\varepsilon$ and $c^\varepsilon$ gives the improved estimate (1.22) for $\nabla_z v^\varepsilon$, completing the proof ∎

We remark that an expansion can be proved for $(u^\varepsilon, c^\varepsilon)$ up to order $\varepsilon^j$, if $f$ is of class $C^j$ and if assumptions (A1) - (A5) hold. The second order coefficient in the expansion of $c^\varepsilon = c^0 + \varepsilon^2 c_2 + O(\varepsilon^3)$ turns out to be given by

$$c_2 \int_{\mathbb{R}} u^0_x v = \fint_D |\nabla\psi|^2 \int_{\mathbb{R}} \partial_x^{(4)} u^0 v \tag{3.6}$$

where $v \ne 0$ is in the kernel of $H^*$ and $\psi$ is defined in (3.5). Observe that the integral on the left-hand side does not vanish since zero is an algebraically simple eigenvalue of $H$.

## 4. The scalar case and monotone systems

It will be shown that in the case of a scalar equation all assumptions (A1) - (A5) are satisfied for a cubic-like non-linearity:

$$f \in C^2([0,1], \mathbb{R})$$
$$f'(0) < 0 \;\text{ and }\; f'(1) < 0$$
$$\textstyle\int_0^1 f(u)du > 0$$
$$f(u) \;\text{ has exactly three zeros at } 0, 1 \text{ and at some } a \in (0,1).$$

A typical example is $f(u) = u(u-a)(1-u)$ with $0 < a < \frac{1}{2}$. It is well known [5] that for this kind of non-linearity (1.9) admits a unique traveling wave $(u^0, c^0)$ with $u^0(-\infty) = 0, u^0(+\infty) = 1, c^0 > 0$ and $u^{0\prime} > 0$. The linearization $H$ is given by

$$Hv = A^h v'' - c^0 v' + f'(u^0)v.$$

The linearization of (1.9) at $\pm\infty$ shows that $u^{0\prime}$ decays exponentially. Thus condition (A1) is fulfilled. Hence $u^{0\prime}$ is in the null space $N(H)$ of $H$ and zero is a geometrically simple eigenvalue of $H$. Now suppose that

$$Hv = u^{0\prime}. \tag{4.1}$$

From the linearization at $-\infty$ it follows that $u^0(x)$ and $u^{0\prime}(x)$ decay like $e^{\lambda x}$ with

$$\lambda = \frac{c}{2A^h} + \Big(\frac{c^2}{4A^{h2}} - f'(0)\Big)^{\frac{1}{2}} > \frac{c}{A^h}.$$

Hence $w^0 = e^{-xc/A^h} u^{0\prime} \in L^2(\mathbb{R})$. One checks that $w^0 \in N(H^*)$ where

$$H^* w = A^h w'' + c^0 w' + f'(u^0) w.$$

Testing (4.1) with $w^0$ yields

$$0 = \int_{\mathbb{R}} v H^* w^0 = \int_{\mathbb{R}} w^0 H v = \int_{\mathbb{R}} e^{-xc/A^h} u^{0\prime 2}$$

which is impossible. Hence 0 is an algebraically simple eigenvalue of $H$. If $H^* w = w^0$, then testing with $u^{0\prime}$ implies that 0 is also for $H^*$ an algebraically simple eigenvalue. This implies condition (A3).

In order to show the closedness of $R(H)$ in condition (A2) let $Hv_n = g_n \to g$ in $L^2(\mathbb{R})$. Suppose that $v_n$ is orthogonal to $u^{0\prime}$ and that $|v_n|_{L^2} \to \infty$ in $L^2(\mathbb{R})$. If $w_n = \frac{v_n}{|v_n|_{L^2}}$, then

$$L_{c_0,0} w_n = -(f'(u^0) + \alpha) w_n + \frac{g_n}{|v_n|_{L^2}}$$

is bounded in $L^2(\mathbb{R})$. Hence $w_n$ is bounded in $H^2(\mathbb{R})$ and a subsequence converges in $L^2_{loc}(\mathbb{R})$ to $w$. Furthermore, $Hw = 0$ and hence $w = \gamma u^{0\prime}$. But $\gamma = 0$ since $w$ is orthogonal to $u^{0\prime}$. In particular, $w_n$ tends weakly to zero in $L^2(\mathbb{R})$. Choose $N$ so large that $f'(u^0(x)) < \frac{f'(1)}{2} < 0$ for $x \in (N, \infty)$. Multiplying $Hw_n = \frac{g_n}{|v_n|_{L^2}} \to 0$ by $w_n$ and integrating over $(N, \infty)$ implies

$$\int_N^\infty \big( - A^h (w_n')^2 + f'(u^0) w_n^2 \big) \to 0.$$

since by regularity theory boundary terms tend to zero as $n \to \infty$. By the choice of $N$ and the local convergence this implies convergence to zero in $L^2(0, \infty)$. Using the same reasoning at $-\infty$ implies that $w_n$ tends to zero strongly in $L^2(\mathbb{R})$, which contradicts $|w_n|_{L^2(\mathbb{R})} = 1$. Hence $v_n$ and $L_{c_0,0} v_n = -(f'(u^0) + \alpha) v_n + g_n$ are bounded in $L^2(\mathbb{R})$. Using Lemma 2.1 this implies that $v_n$ is bounded in $H^1$ and a subsequence converges weakly in $H^1(\mathbb{R})$ to $v$ and $Hv = g$ weakly. This implies that $v \in H^2(\mathbb{R})$ and hence $R(H)$ is closed.

At last, condition (A5) is satisfied for a cubic-like non-linearity. Thus all the assumptions are verified and Theorem 1.1 applies.

Now in the expansion of $c^\varepsilon$ the second order term (3.6) $c_2$, in particular its sign, is computed in the scalar case. The kernel of $H^*$ is spanned by $v = u_x^0 e^{-xc^0/A^h}$. Hence (3.6) implies

$$\operatorname{sign} c_2 = \operatorname{sign} \int_{\mathbb{R}} \partial_x^{(4)} u^0 \partial_x u^0 e^{-xc^0/A^h}.$$

Only in some cases, e.g. if $f(u)$ is a cubic function, $u(x)$ and $c$ are explicitly known and the integral can be evaluated explicitly. It is found in this case that $\operatorname{sign} c_2 = -\operatorname{sign} c_0$. Thus the speed is slowed down by the inhomogeneities compared to the homogenized medium. We conjecture this to hold for general bistable non-linearities.

The method in this chapter can be extended to so-called monotone systems with bistable non-linearity. These are systems admitting a maximum principle. Existence of monotone traveling waves for the homogenized system and their spectral properties have been studied in [12]. For convenience we state the assumptions such that conditions (A1) - (A5) hold. Let $p < q$ where inequalities between vectors are understood to hold for each component.

**1.** (*Monotonicity*) For all $u \in \mathbb{R}^n$ with $p \leq u \leq q$ and all $k \neq j$, $\frac{\partial f_k}{\partial u_j}(u) \geq 0$.

**2.** (*Bistability*) The eigenvalues of $D_u f(p)$ and of $D_u f(q)$ lie in the left half-plane. Furthermore, $f(u)$ vanishes at a finite number of points $r_i$ $(p < r_i < q)$ only and there exist vectors $v_i \geq 0$ such that $D_u f(r_i)$ has at least one eigenvalue in the right half plane.

**3.** $D_u f(u)$ is irreducible for all $u$ between $p$ and $q$. This means that $D_u f(u)$ leaves none of the hyperplanes invariant spanned by some of the standard basis vectors.

By [12: Chapter 3/Theorem 3.2] Properties 1 and 2 guarantee the existence of a monotone wave. The spectral properties of the linearization follow from [12: Chapter 3/Theorem 5.1].

## References


[1] Barles, G. and P. E. Souganidis: *A new approach to front propagation problems: theory and applications.* Arch. Rat. Mech. Anal. 141 (1998), 237 – 296.

[2] Berestycki, H. and B. Larrouturou: *A semilinear elliptic equation in a strip arising in a two-dimensional flame propagation model.* J. reine ang. Math. 396 (1989), 14 – 40.

[3] Berestycki, H., Larrouturou, B. and P. L. Lions: *Multi-dimensional traveling wave solutions of a flame propagation model.* Arch. Rat. Mech. Anal. 111 (1990), 33 – 49.

[4] Berestycki, H. and L. Nirenberg: *Traveling fronts in cylinders.* Ann. Inst. Henri Poincare, Anal. Non Lineaire 9 (1992), 497 – 572.

[5] Fife, P. and J. B. McLeod: *The approach of solutions of nonlinear diffusion equations to traveling front solutions.* Arch. Rat. Mech. Anal. 65 (1977), 335 – 361.

[6] Gilbarg, D. and N. S. Trudinger: *Elliptic Partial Differential Equations of Second Order*, 2nd ed. (Grundl. der math. Wiss.: Bd. 224). Berlin - Heidelberg: Springer-Verlag 1983.

[7] Krasnoselskii, M. A., Zabreiko, P. P., Pustylnik, E. I. and P. E. Sbolevskii: *Integral Operators in Spaces of Summable Functions.* Groningen: Noordhoff Int. Publ. 1976.

[8] Vol'pert, A. I., Vol'pert, Vit. and Vl. A. Vol'pert: *Traveling Wave Solutions of Parabolic Systems* (Transl. Math. Mon.: Vol. 140). Providence (RI, USA): Amer. Math. Soc. 1994.

[9] Xin, J.: *Existence of planar flame fronts in convective periodic media.* Arch. Rat. Mech. Anal. 121 (1992), 205 – 233.

[10] Xin, J.: *Existence and uniqueness of traveling waves in a reaction-diffusion equation with combustion nonlinearity.* Indiana Univ. Math. J. 40 (1991), 985 – 1008.

[11] Xin, J.: *Multidimensional stability of traveling waves in a bistable reaction- diffusion equation*, Part I. Com. Part. Diff. Eq. 17 (1992), 1889 – 1899.

[12] Xin, J. and J. Zhu: *Quenching and propagation of bistable reaction-diffusion front in multidimensional periodic media.* Physica D 81 (1995), 94 – 110.

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