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On the Cauchy Problem for a Degenerate Parabolic Equation

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Abstract. Existence and uniqueness of global positive solutions to the degenerate parabolic problem

$$\left.\begin{aligned} u_t &= f(u)\Delta u \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} &= u_0 \end{aligned}\right\}$$

with $f \in C^0([0,\infty)) \cap C^1((0,\infty))$ satisfying $f(0) = 0$ and $f(s) > 0$ for $s > 0$ are investigated. It is proved that, without any further conditions on f, decay of u_0 in space implies uniform zero convergence of $u(t)$ as $t \to \infty$. Furthermore, for a certain class of functions f explicit decay rates are established.

Keywords: *Degenerate diffusion, large-time behaviour*

AMS subject classification: 35K55, 35K65, 35B40

0. Introduction

We are concerned with positive solutions to the Cauchy problem for a class of degenerate parabolic equations

$$\left.\begin{aligned} u_t &= f(u)\Delta u \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} &= u_0 \end{aligned}\right\} \tag{0.1}$$

where $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is positive in $\mathbb{R}^n$, while the given function f is required to be in $C^0([0,\infty)) \cap C^1((0,\infty))$ with $f(s) > 0$ for $s > 0$ and, which makes the equation *degenerate* parabolic, $f(0) = 0$.

So far, to the best of our knowledge, a detailed study on problem (0.1) has been done only for the special case $f(s) = s^p$ $(0 < p < 1)$ or – more or less – slight perturbations thereof. In this case, namely, the substitution $U(x,t) = (1-p)^{\frac{1-p}{p}} u^{1-p}(x,t)$ transforms (0.1) into the Cauchy problem for the porous medium equation $U_t = \Delta U^m$ with $m = \frac{1}{1-p} > 1$ which has been studied by several authors (see [1, 2, 7], for example).

In order to motivate the question of qualitative behavior of solutions of problem (0.1), let us assume for a moment that f increases to ∞ as $s \nearrow \infty$. If then we investigate instead of (0.1) the corresponding initial boundary value problem with zero Dirichlet

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data in some smooth bounded domain, it is easy to see by a comparison argument that all solutions tend to zero uniformly in Ω as $t \to \infty$. More generally, replacing in the latter problem $-\Delta$ by any second-order linear elliptic operator A (with sufficiently smooth coefficients) having first eigenvalue λ_1, we achieve global existence and large time decay as before whenever $\lambda_1 > 0$. On the other hand, $\lambda_1 < 0$ implies finite time blow-up for any positive solution (see, e.g., [5] or [12]). The borderline case $\lambda_1 = 0$ has been investigated only under special circumstances so far; in [9] there is proved for $f(s) = s^p$ $(p > 0)$ that if u_0 decays fast enough near $\partial\Omega$, then u exists globally, while if u_0 decreases sufficiently slowly, then u is bounded away from zero uniformly on compact subsets of Ω for all times. It is not known, however, whether there are initial data which cause unboundedness or zero decay of solutions.

In the situation of $A = -\Delta$, increasing Ω to $\mathbb{R}^n$ means taking $\lambda_1 \searrow 0$, thus in problem (0.1) we formally have exactly the borderline case, so the question is whether one of the tendencies towards blow-up on the one hand or stabilization to zero on the other hand will win, or if intermediate effects occur. The main results of the present note are that u exists globally and tends to zero, provided merely that u_0 vanishes at infinity (cf. Section 3), and if u_0 and f enjoy further properties, then upper bounds for the decay rate can be given (cf. Section 2).

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1. Existence and uniqueness

Assuming throughout that

(H1) $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is positive

we are first of all interested in whether problem (0.1) has a solution at all. As we are familiar with the Dirichlet problem for $u_t = f(u)\Delta u$ on bounded domains, we are led to the idea to construct a solution on $\mathbb{R}^n$ via approximation by a sequence of solutions on, say, $B_R = B_R(0)$. More precisely, let, for $k \in \mathbb{N}$, $u_{0,k} \in C^1(\bar{B}_k)$ be such that $0 < u_{0,k} < u_{0,k+1}$ in B_k, $u_{0,k}|_{\partial B_k} = 0$ and $u_{0,k} \nearrow u_0$ in $\mathbb{R}^n$. Concerning solvability of the corresponding initial-boundary value problem in B_k with zero boundary values and initial data $u_{0,k}$, we have

Lemma 1.1. *The problem*

$$\left.\begin{aligned} \partial_t u_k &= f(u_k)\Delta u_k \quad \text{in } B_k \times (0,\infty) \\ u_k|_{\partial B_k} &= 0 \\ u_k|_{t=0} &= u_{0,k} \end{aligned}\right\} \tag{1.1}$$

is uniquely solvable in $C^0(\bar{B}_k \times [0,\infty)) \cap C^{2,1}(B_k \times (0,\infty))$. *The solution can be obtained as the* $C^0_{loc}(\bar{B}_k \times [0,\infty)) \cap C^{2,1}_{loc}(B_k \times (0,\infty))$*-limit of a decreasing sequence of solutions* $u_{k,\varepsilon}$ *of problem* (1.1) *with* $u_{k,\varepsilon}|_{\partial B_k} = \varepsilon$ *and* $u_{k,\varepsilon}|_{t=0} = u_{0,k} + \varepsilon$ *for* $\varepsilon \searrow 0$.

Proof. Local existence of $u_{\varepsilon,k}$ and monotonic convergence to a limit function u_k is proved in a standard way using arguments pointed out in detail in [11: Theorem 1.2.2] (cf. also [10: Theorem 3.2]). To see that the solution actually exists for $t \in (0,\infty)$ we only have to note that by comparison $\varepsilon \le u_{k,\varepsilon} \le \|u_{0,k}\|_{L^\infty(B_k)} + \varepsilon$ as long as $u_{k,\varepsilon}$ exists, so that $u_{k,\varepsilon}$ and hence u_k can be extended for all times ∎

Taking $k \to \infty$, we in fact obtain a solution to the original problem.

Lemma 1.2. *Problem* (0.1) *admits a positive classical solution* $u \in C^0(\mathbb{R}^n \times [0,\infty)) \cap C^{2,1}(\mathbb{R}^n \times (0,\infty)) \cap L^\infty(\mathbb{R}^n \times (0,\infty))$. *If* u_k *denotes the solution of problem* (1.1)*, we have* $u_k \to u$ *in* $C^0_{loc}(\mathbb{R}^n \times [0,\infty)) \cap C^{2,1}_{loc}(\mathbb{R}^n \times (0,\infty))$.

Proof. As $u_{0,k+1} \ge u_{0,k}$ in B_k and $u_{0,k+1}|_{\partial B_k} \ge 0$, we have $u_{k+1,\varepsilon} \ge u_k$ for all ε and thus $u_{k+1} \ge u_k$ in $B_k \times (0,\infty)$. Consequently, as $k \to \infty$, the u_k monotonically increase to some limit u which is easily seen to fulfil $0 < u \le \|u_0\|_{L^\infty(\mathbb{R}^n)}$. To find a uniform local bound from below, let $k_0 \in \mathbb{N}$ be given. Then there exists a constant $c_{k_0} > 0$ depending on k_0 only such that $u_{0,k} \ge u_{0,k_0+1} \ge c_{k_0}\Theta_{k_0}$ in B_{k_0} for all $k > k_0$, where Θ_{k_0} denotes the Dirichlet eigenfunction of $-\Delta$ in B_{k_0} with $\max\Theta_{k_0} = 1$, corresponding to the first eigenvalue $\lambda_{1,k_0} > 0$. Setting

$$y(t) = c_{k_0} e^{-\alpha t} \qquad \text{with } \alpha = \lambda_{1,k_0}\|f\|_{L^\infty((0,c_{k_0}))}$$

we find that

$$\begin{aligned} \partial_t(y\Theta_{k_0}) - f(y\Theta_{k_0})\Delta(y\Theta_{k_0}) &= y'\Theta_{k_0} + \lambda_{1,k_0} f(y\Theta_{k_0}) y\Theta_{k_0} \\ &\le (y' + \alpha y)\Theta_{k_0} \\ &\le 0 \quad \text{in } B_{k_0} \times (0,\infty) \end{aligned}$$

which yields by comparison

$$u_k \ge y(t)\Theta_{k_0}(x) \qquad \text{in } B_{k_0} \times (0,\infty) \text{ for all } k > k_0.$$

Thus for all $K \times [0,T] \subset\subset \mathbb{R}^n \times [0,\infty)$ there is a constant $c_{K,T} > 0$ such that for k large (depending on K)

$$u_k \ge c_{K,T} \qquad \text{in } K \times [0,T].$$

Together with $u_k \le \|u_0\|_{L^\infty(\mathbb{R}^n)}$, this provides uniform local two-sided bounds on the coefficients $f(u_k)$ in (1.1). Hence, parabolic Hölder and Schauder estimates (see [6: Theorems V.1.1 and IV.10.1]) together with the Arzelà-Ascoli theorem show that all derivatives of u_k up to order two converge uniformly to those of u in any compact subset of $\mathbb{R}^n \times (0,\infty)$ and u solves $u_t = u^p\Delta u$. Moreover, if $u_0 \in C^1(\mathbb{R}^n)$, the same estimates show that $u_k \to u$ even in $C^0_{loc}(\mathbb{R}^n \times [0,\infty))$ and $u|_{t=0} = u_0$.

If u_0 is merely continuous, we use the result just obtained in the following way: Let us fix $k_0 \in \mathbb{N}$ and $\varepsilon > 0$. We take $\tilde u_0 \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $u_0 \le \tilde u_0$ in $\mathbb{R}^n$ and $\tilde u_0 \le u_0 + \varepsilon$ in B_{k_0}. By what we have just shown, there is a solution $\tilde u$ of problem (0.1) with $\tilde u|_{t=0} = \tilde u_0$ which is continuous down to $t = 0$. In particular, $\tilde u \le u_0 + 2\varepsilon$ in $B_{k_0} \times (0,\tau)$ for some sufficiently small $\tau > 0$. By comparison, $u_k \le \tilde u$ for all k and hence

$$u \le u_0 + 2\varepsilon \qquad \text{in } B_{k_0} \times (0,\tau). \tag{1.2}$$

On the other hand, by Dini's theorem, $u_{0,k} \to u_0$ holds uniformly in B_{k_0}, hence there is $k_1 \in \mathbb{N}$ such that $u_{0,k_1} \geq u_0 - \varepsilon$ in B_{k_0}. Continuity of u_{k_1} now gives $u_{k_1} \geq u_0 - 2\varepsilon$ in $B_{k_0} \times (0,\tau)$ after diminishing τ if necessary. Thus by monotonicity,

$$u_k \geq u_0 - 2\varepsilon \qquad \text{in } B_{k_0} \times (0,\tau) \ \text{ for all } k \geq k_1. \tag{1.3}$$

Combining (1.2) and (1.3), we end up with $0 \leq u - u_k \leq 4\varepsilon$ in $B_{k_0} \times (0,\tau)$ for all $k \geq k_1$ which implies $u \in C^0(\mathbb{R}^n \times [0,\infty))$ and $u|_{t=0} = u_0$ ∎

In the sequel, most of the assertions on u essentially rely on the fact that $u = \lim u_k$; thus, the question of uniqueness is of great importance. Before answering it we assert that if u_0 vanishes at $|x| = \infty$, then so does $u(t)$.

Lemma 1.3. *Suppose that, in addition to condition* (H1),

$$\|u_0\|_{L^\infty(\partial B_R)} \to 0 \qquad \text{as } R \to \infty. \tag{1.4}$$

Then

$$\|u(t)\|_{L^\infty(\partial B_R)} \to 0 \qquad \text{as } R \to \infty \ \forall\, t > 0. \tag{1.5}$$

Proof. i) Starting with the radially symmetric case, we first suppose $u_0(x) = U_0(|x|)$ in $\mathbb{R}^n$ with some non-increasing $U_0 \in C^2([0,\infty))$. Then the $u_{0,k}$ clearly can be chosen radially symmetric, that is $u_{0,k}(x) = U_{0,k}(|x|)$ where we may assume $U_{0,k} \in C^2([0,R])$ to be non-increasing. Then the $u_{k,\varepsilon}$ from Lemma 1.1 and thus u are also radially symmetric, i.e. $u_{k,\varepsilon}(x,t) = U_{k,\varepsilon}(|x|,t)$ and $u(x,t) = U(|x|,t)$.

I) We assert that $r \mapsto U_{k,\varepsilon}(r,t)$ is non-increasing on $(0,R)$ for all $t > 0$ which will imply that $r \mapsto U(r,t)$ does not increase on $(0,\infty)$ for $t > 0$. Indeed, by parabolic regularity theory, $z(r,t) = \partial_r U_{k,\varepsilon}(r,t)$ is in $C^0(\bar{Q}) \cap C^{2,1}(Q)$ with $Q = (0,R) \times (0,\infty)$ and satisfies the linear parabolic equation

$$z_t = f(U_{k,\varepsilon}) z_{rr} + \Big[\tfrac{n-1}{r} f(U_{k,\varepsilon}) + f'(U_{k,\varepsilon})\big((U_{k,\varepsilon})_{rr} + \tfrac{n-1}{r}(U_{k,\varepsilon})_r\big)\Big] z_r - \tfrac{n-1}{r^2} f(U_{k,\varepsilon}) z$$

in Q with coefficients in $C^0(Q)$, and as $u_{k,\varepsilon} \geq \varepsilon$, we have $z(R,t) \leq 0$ as well as $z(0,t) = 0$ for all t. Since also $z(r,0) \leq 0$ by assumtion on $U_{0,k}$, we have $z \leq 0$ in Q by comparison.

II) If (1.5) were false, there would be $t > 0$ and $\varepsilon_0 > 0$ such that

$$u(t) > \varepsilon_0 \qquad \text{in } \mathbb{R}^n \tag{1.6}$$

due to the monotonicity property of $u(t)$. We choose a non-decreasing $f_0 \in C^\infty([0,M])$ with $M = \|u_0\|_{L^\infty(\mathbb{R}^n)}$ such that $f_0 \leq f$, $f_0(s) = 0$ for $s < \frac{\varepsilon_0}{4}$ and $f_0(s) > 0$ for $s > \frac{\varepsilon_0}{2}$. Finally, we set

$$\Phi(s) = \int_0^s \frac{f_0(\sigma)}{f(\sigma)}\, d\sigma \qquad (s > 0)$$

and test (1.1) with the smooth function $\frac{f_0}{f}(u_k)$ having compact support in $B_k \times [\tau, t]$ $(0 < \tau < t)$, to obtain

$$-\int_\tau^t\!\!\int_{B_k} f_0'(u_k)|\nabla u_k|^2 = \int_\tau^t\!\!\int_{B_k} \partial_t \Phi(u_k) = \int_{B_k} \Phi(u_k(t)) - \int_{B_k} \Phi(u_k(\tau)),$$

and hence upon letting $\tau \searrow 0$,

$$\int_{B_k} \Phi(u_k(t)) + \int_0^t\int_{B_k} f_0'(u_k)|\nabla u_k|^2 \le \int_{B_k} \Phi(u_{0,k})$$

by Fatou's lemma so that

$$\int_{\mathbb{R}^n} \Phi(u(t)) + \int_0^t\int_{\mathbb{R}^n} f_0'(u)|\nabla u|^2 \le \int_{\mathbb{R}^n} \Phi(u_0). \tag{1.7}$$

But by (1.6), $\Phi(u(t)) \ge \Phi(\varepsilon_0) > 0$ in $\mathbb{R}^n$, hence the left-hand side equals $+\infty$, while $\int_{\mathbb{R}^n} \Phi(u_0)$ is finite due to (1.4), which is a contradiction.

ii) For general u_0, we define a continuous and non-increasing function φ on $[0,\infty)$ by $\varphi(R) = \|u_0\|_{L^\infty(\mathbb{R}^n\setminus B_R)}$ and fix any non-increasing C^2-function $\widetilde{U}_0$ with $\varphi(R) < \widetilde{U}_0(R) < \varphi(R) + \frac{1}{R}$ in $[0,\infty)$. Then comparison shows that each u_k and hence u is majorized by the corresponding solution $\tilde{u}$ evolving from $\tilde{u}_0(x) = \widetilde{U}_0(|x|)$, whence (1.5) follows from part i) ∎

We are now ready to show uniqueness.

Lemma 1.4.

i) *If $n \le 2$, then the solution $u = \lim_{k\to\infty} u_k$ constructed above is unique within the class $\mathcal{C}$ of non-negative classical solutions of problem* (0.1) *from $C^0(\mathbb{R}^n \times [0,\infty)) \cap C^{2,1}(\mathbb{R}^n \times (0,\infty)) \cap L^\infty(\mathbb{R}^n \times (0,\infty))$.*

ii) *If $n \ge 3$ and, in addition to condition* (H1), (1.4) *holds, then u is unique among all solutions from $\mathcal{C}$ sharing the spatial decay property*

$$\|u(t)\|_{L^\infty(\partial B_R)} \to 0 \qquad \text{as } R \to \infty\ \forall\, t > 0. \tag{1.8}$$

Proof. We begin by constructing suitable functions to test (0.1) with. For $R > 1$, let $\varphi_R(x) = f_R(|x|)$ be the solution of the problem

$$\left.\begin{aligned} -\Delta\varphi_R(x) &= \chi(|x|) \ \text{ in } B_R \\ \varphi_R|_{\partial B_R} &= 0 \end{aligned}\right\} \tag{1.9}$$

where $\chi \in C_0^\infty([0,1))$ with $\chi_{[0,\frac{1}{2}]} \le \chi \le \chi_{[0,1]}$. Expressed in terms of f_R, problem (1.9) transforms into

$$\left.\begin{aligned} -f_R''(r) - \tfrac{n-1}{r} f_R'(r) &= \chi(r) \ \text{ in } (0,R) \\ f_R(R) &= 0 \end{aligned}\right\}$$

which is explicitly solved by

$$f_R(r) = \int_r^R \int_0^\varrho \left(\frac{\xi}{\varrho}\right)^{n-1} \chi(\xi)\, d\xi d\varrho.$$

Observe that f_R is non-decreasing in R and

$$f_R'(R) = -\int_0^R \left(\frac{\xi}{R}\right)^{n-1} \chi(\xi)\, d\xi \ge -\frac{1}{R^{n-1}} \int_0^1 \xi^{n-1} d\xi = -\frac{1}{nR^{n-1}}$$

which implies

$$\partial_N \varphi_R|_{\partial B_R} \geq -\frac{1}{nR^{n-1}}. \tag{1.10}$$

Abbreviating $Hs = \int_1^s \frac{d\sigma}{f(\sigma)}$ for $s > 0$, we rewrite problem (0.1) in the form

$$\left.\begin{aligned} \partial_t Hu - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} = u_0 \end{aligned}\right\}$$

and assume v is another solution of (0.1) from the indicated class. By comparison, $v \geq u_k$ in $B_k \times (0,\infty)$ for all k, hence $v \geq u$. Multiplying $\partial_t(Hv - Hu) - \Delta(v-u) = 0$ by φ_R and integrating over $B_R \times [\tau, t]$ $(0 < \tau < t)$ we get

$$\begin{aligned} I_1 + I_2 + I_3 & \\ := \int_{B_R} (Hv - Hu)(t)\cdot\varphi_R + \int_\tau^t \int_{B_R} (v-u)\cdot\chi(|x|) + \int_\tau^t \int_{\partial B_R} (v-u)\cdot\partial_N\varphi_R & \\ = \int_{B_R} (Hv - Hu)(\tau)\cdot\varphi_R =: I_4. & \end{aligned} \tag{1.11}$$

Both Hu and Hv are continuous in $\bar{B}_R \times [0,\infty)$ and equal Hu_0 for $t = 0$, thus

$$I_4 \to 0 \qquad \text{as } \tau \to 0. \tag{1.12}$$

As $v \geq u$,

$$I_2 \geq 0 \tag{1.13}$$

while by (1.10)

$$I_3 \geq -\frac{1}{nR^{n-1}} \int_\tau^t \int_{\partial B_R} (v-u) \geq -c\int_0^t \|v(s)\|_{L^\infty(\partial B_R)} ds. \tag{1.14}$$

Now in the case $n \geq 3$, (1.8) and Lebesgue's dominated convergence theorem show that

$$\int_0^t \|v(s)\|_{L^\infty(\partial B_R)} ds \to 0 \qquad \text{as } R \to \infty \tag{1.15}$$

which in view of (1.12) - (1.14) immediately gives $Hv(t) \leq Hu(t)$ or $v(t) \leq u(t)$ on $\mathbb{R}^n$ since φ_R is non-decreasing in R and positive in B_R. If $n \leq 2$, however, it follows from

$$f_R(r) \geq \tfrac{1}{n}\left(\tfrac{1}{2}\right)^n \int_{\min\{1,r\}}^R \varrho^{1-n} d\varrho$$

that $\lim_{R\to\infty} \varphi_R = \infty$ uniformly on compact subsets of $\mathbb{R}^n$, so that using the trivial estimate $I_3 \geq -c\|v\|_{L^\infty(\mathbb{R}^n \times (0,\infty))}$ (instead of (1.15)) and taking $R \to \infty$ in (1.11) we infer that $|\{Hv(t) > Hu(t)\}| = 0$ and thus u is unique within the set of classical solutions without satisfying any further decay condition ∎

Remark. By a slight modification in the proof (using Hölder's inequality to guarantee that $\liminf_{R\to\infty} I_3 = 0$) it is possible to replace (1.8) by a 'decay in mean' requirement $u \in L^\infty_{loc}\big([0,\infty); L^q(\mathbb{R}^n)\big)$ for any $q \geq 1$, provided of course that $u = \lim u_k$ enjoys this property at all (cf. Lemma 2.1).

2. Large time decay: the case of nice f

In this section we asssume that f, apart from being merely positive for positive arguments, satisfies in addition

(H2) For all $M > 0$ there exists $\beta = \beta(M) > 0$ such that $\frac{sf'(s)}{f(s)} \geq \beta$ on $(0, M)$.

Note that condition (H2) is fulfilled, e.g., by $f(s) = s^p \ \ (p > 0)$ (with $\beta = p$), as well as by $f(s) = e^{-\frac{1}{s^p}} \ \ (p > 0)$ (with $\beta(M) = \frac{p}{M^p}$), but neither by non-monotonic functions nor by those approaching zero very slowly as $s \searrow 0$, such as $f(s) = \frac{1}{1+|\ln s|^p} \ \ (p > 0)$.

In particular, we shall see in a minute that condition (H2) endows the solution $u = \lim u_k$ from Lemma 1.2 with one first important feature, namely the one of non-increasing distance to zero in any of the spaces $L^q(\mathbb{R}^n) \ \ (0 < q < \infty)$. Consequently, due to the remark following Lemma 1.4, it is unique in the class of non-negative bounded classical solutions from $L^\infty_{loc}([0,\infty); L^q(\mathbb{R}^n))$ provided that, besides condition (H1), u_0 fulfils

(H3) $u_0 \in L^q(\mathbb{R}^n)$ for some $q > 0$ with $q \geq 1 - \beta, \beta = \beta(\|u_0\|_{L^\infty(\mathbb{R}^n)})$.

Throughout this section, whenever the parameter q arises it will be assumed implicitly that condition (H3) holds for this q.

Lemma 2.1. *For all $q > 0$ with $q \geq 1 - \beta$,*

$$\int_{\mathbb{R}^n} u^q(t) \leq \int_{\mathbb{R}^n} u_0^q \qquad \forall\, t > 0. \tag{2.1}$$

Proof. The procedure is similar to the one used in part i)/II) of the proof of Lemma 1.3, but we have to be a bit more careful here since our integrals cover regions where u is small. In virtue of Fatou's lemma, it suffices to prove that for all k and all $t > 0$

$$\int_{B_k} u_k^q(t) \leq \int_{B_k} u_{0,k}^q. \tag{2.2}$$

To this end let, for some sequence $\delta = (\delta_j)$ with $\delta_j \searrow 0$, $\varphi_\delta \in C^\infty([0,\infty))$ be such that $\chi_{[\delta,\infty)} \leq \varphi_\delta \leq \chi_{[\frac{\delta}{2},\infty)}$, $\varphi_\delta' \geq 0$ and $\varphi_\delta \nearrow 1$ on $(0,\infty)$ as $\delta \searrow 0$. Then for all $0 < \tau < t < \infty$ the function $\psi = \varphi_\delta(u_k) \cdot u_k^{q-1}$ is smooth and has compact support in $B_k \times [\tau, t]$, hence testing (1.1) with ψ gives

$$0 = \int_\tau^t\!\!\int_{B_k} \varphi_\delta(u_k) u_k^{q-1} \partial_t u_k + \int_\tau^t\!\!\int_{B_k} \nabla u_k \cdot \nabla\big(f(u_k) u_k^{q-1} \varphi_\delta(u_k)\big) =: I_1 + I_2.$$

Here I_2 is non-negative since $F(s) = s^{q-1} f(s) \varphi_\delta(s)$ has its derivative

$$F'(s) \geq s^{q-2} f(s) \big((q - 1 + \beta)\varphi(s) + s\varphi'(s)\big)$$

non-negative due to the choice of q. Setting

$$\Phi_\delta(s) = \int_0^s \varphi_\delta(\sigma) \sigma^{q-1} d\sigma$$

we have $\Phi_\delta(s) \nearrow \frac{1}{q}s^q$ as $\delta \searrow 0$ and thus

$$I_1 = \int_{B_k} \Phi_\delta(u_k(t)) - \int_{B_k} \Phi_\delta(u_k(\tau)) \to \tfrac{1}{q}\int_{B_k} u_k^q(t) - \tfrac{1}{q}\int_{B_k} u_k^q(\tau)$$

as $\delta \to 0$ by Beppo Levi's theorem, where we notice that both terms on the right are finite for fixed k. Thus,

$$\int_{B_k} u_k^q(t) \le \int_{B_k} u_k^q(\tau) \qquad \forall\, 0 < \tau < t < \infty$$

which implies (2.2) as $\tau \to 0$ since u_k is continuous on $\bar{B}_k \times [0,\infty)$ ∎

The monotonicity hypothesis (H2) translates into a monotonicity property of our solution.

Lemma 2.2. *For all $k \in \mathbb{N}$ we have*

$$\frac{\partial_t u_k}{u_k} \ge -\frac{1}{\beta t} \qquad \text{in } \mathbb{R}^n \times (0,\infty). \tag{2.3}$$

Consequently,

$$\frac{u_t}{u} \ge -\frac{1}{\beta t} \qquad \text{in } \mathbb{R}^n \times (0,\infty). \tag{2.4}$$

Proof. For fixed $\tau > 0$, classical regulartiy theory tells us that the approximate solutions $u_{k,\varepsilon}$ from Lemma 1.1 are in $C^{2,1}(\bar{B}_k \times [\tau,\infty))$, hence the function

$$z_{k,\varepsilon} = \frac{\partial_t u_{k,\varepsilon}}{u_{k,\varepsilon}} = f(u_{k,\varepsilon})u_{k,\varepsilon}\Delta u_{k,\varepsilon} =: F(u_{k,\varepsilon})\Delta u_{k,\varepsilon}$$

is in $C^0(\bar{B}_k \times [\tau,\infty))$ and fulfils

$$\begin{aligned}\partial_t z_{k,\varepsilon} &= \Big(\frac{f'(u_{k,\varepsilon})u_{k,\varepsilon} - f(u_{k,\varepsilon})}{u_{k,\varepsilon}^2}\Big)u_{k,\varepsilon}z_{k,\varepsilon}\Delta u_{k,\varepsilon} + \frac{f(u_{k,\varepsilon})}{u_{k,\varepsilon}}\Delta(u_{k,\varepsilon}z_{k,\varepsilon})\\ &= \frac{f'(u_{k,\varepsilon})u_{k,\varepsilon}}{f(u_{k,\varepsilon})}z_{k,\varepsilon}^2 + \frac{f(u_{k,\varepsilon})}{u_{k,\varepsilon}}\big(u_{k,\varepsilon}\Delta z_{k,\varepsilon} + 2\nabla u_{k,\varepsilon}\cdot\nabla z_{k,\varepsilon}\big).\end{aligned}$$

$z_{k,\varepsilon}$ vanishes at $\partial B_k \times [\tau,\infty)$, while at $t = \tau$, $z_{k,\varepsilon} \ge -M$ for all $M \ge M_\varepsilon$ and some sufficiently large $M_\varepsilon > 0$. Hence, by comparison, $z_{k,\varepsilon} \ge \varphi_M$ on $B_k \times (\tau,\infty)$ for all $M \ge M_\varepsilon$, where $\varphi_M(t)$ is the solution of $\varphi_M' = \beta\varphi_M^2$ on (τ,∞), $\varphi_M(\tau) = -M$, i.e. $\varphi_M(t) = -\frac{1}{\beta(t-\tau)+M^{-1}}$. Consequently, $z_{k,\varepsilon} \ge -\frac{1}{\beta(t-\tau)}$ on $B_k \times (\tau,\infty)$ for all $\tau > 0$, hence also $z_{k,\varepsilon} \ge -\frac{1}{\beta t}$ on $B_k \times (0,\infty)$. Taking successively $\varepsilon \to 0$ and then $k \to \infty$, we arrive at (2.3) and (2.4), respectively ∎

Via Lemma 2.2, condition (H2) (together with condition (H3)) will imply additional regularity properties of the solution which are not a priori obvious in the context of degenerate parabolic equations. At the same time, it provides a quantitative homogenization rate.

Lemma 2.3. *For all $q > 0$ with $q > 1-\beta$, the estimate*

$$\int_{\mathbb{R}^n} |\nabla h_q(u(t))|^2 \le \tfrac{1}{t}\, \tfrac{1}{\beta(q-1+\beta)} \int_{\mathbb{R}^n} u_0^q \tag{2.5}$$

holds for $t \in (0,\infty)$ where $h_q(s) = \int_0^s \sigma^{\frac{q}{2}-1}\sqrt{f(\sigma)}\,d\sigma \quad (s \ge 0)$.

Proof. For fixed t we gain from Lemma 2.2 the inequality $-\Delta u_k \le \frac{1}{\beta t}\frac{u_k}{f(u_k)}$ which we test with the compactly supported function $u_k^{q-1} f(u_k)\varphi_\delta(u_k) \in C^2(\bar{B}_k)$, φ_δ as in the proof of Lemma 2.1, to obtain

$$I := \int_{B_k} \nabla u_k \cdot \nabla\big(u_k^{q-1} f(u_k)\varphi_\delta(u_k)\big) \le \tfrac{1}{\beta t}\int_{B_k} u_k^q \varphi_\delta(u_k) \le \tfrac{1}{\beta t}\int_{\mathbb{R}^n} u_0^q$$

where we have made use of Lemma 2.1. Using again

$$F(s) = s^{q-1} f(s)\varphi_\delta(s) \qquad \text{with } F'(s) \ge (q-1+\beta)s^{q-2} f(s)\varphi(s)$$

we observe

$$\begin{aligned} I &= \int_{B_k} F'(u_k)|\nabla u_k|^2 \\ &\ge (q-1+\beta)\int_{B_k} \varphi_\delta(u_k) u_k^{q-2} f(u_k)|\nabla u_k|^2 \\ &= (q-1+\beta)\int_{B_k} \varphi_\delta(u_k)|\nabla h_q(u_k)|^2 \end{aligned}$$

and complete the proof upon letting $\delta \searrow 0$ and then $k \to \infty$, each time employing Fatou's lemma ∎

In order to derive decay estimates for u itself (rather than its gradient), we employ the Gagliardo-Nirenberg inequality, a suitable formulation of which is given for convenience in the following lemma. Note that integrability powers $\mu < 1$ are involved.

Lemma 2.4. *Suppose $s \in (1, n^\star)$ where $n^\star = \frac{2n}{n-2}$ for $n \ge 3$ and $n^\star = \infty$ for $n \le 2$. Then for all $\mu \in (0,s)$, there is a constant $c_0 = c_0(s,\mu)$ such that the estimate*

$$\|\varphi\|_{L^s(\mathbb{R}^n)} \le c_0 \|\nabla\varphi\|^a_{L^2(\mathbb{R}^n)} \|\varphi\|^{1-a}_{L^\mu(\mathbb{R}^n)} \tag{2.6}$$

holds for all $\varphi \in L^\mu(\mathbb{R}^n)$ with $\nabla\varphi \in L^2(\mathbb{R}^n)$, the number $a \in (0,1)$ being defined by

$$-\tfrac{n}{s} = \big(1 - \tfrac{n}{2}\big)a - \tfrac{n}{\mu}(1-a). \tag{2.7}$$

Proof. For $\mu \ge 1$, (2.6) is the standard Gagliardo-Nirenberg inequality proved, e.g., in [8: Chapter 3.4]. For $\mu \in (0,1)$, we first apply this – with μ replaced by 1 – to obtain

$$\|\varphi\|_{L^s(\mathbb{R}^n)} \le c_1 \|\nabla\varphi\|^b_{L^2(\mathbb{R}^n)} \|\varphi\|^{1-b}_{L^1(\mathbb{R}^n)} \qquad \text{where } -\tfrac{n}{s} = \big(1-\tfrac{n}{2}\big)b - n(1-b).$$

By standard interpolation, using Hölder's inequality,

$$\|\varphi\|_{L^1(\mathbb{R}^n)} \le \|\varphi\|^c_{L^s(\mathbb{R}^n)} \|\varphi\|^{1-c}_{L^\mu(\mathbb{R}^n)} \qquad \text{with } c = s\tfrac{1-\mu}{s-\mu}.$$

Now (2.6) follows upon combining these inequalities and using that $\frac{b}{1-(1-b)c}$ coincides with a which follows from an elementary calculation ∎

The main result of the present section is

Theorem 2.5. *Let $q > 0$ be such that $q > 1 - \beta$.*

i) *For all $r \in [\max\{q, \frac{2q}{n^\star} - \beta\}, \infty)$ and all $s \in [1, \infty)$ with $s > \frac{2q}{r+\beta}$, there are constants $\alpha > 0$ and $c > 0$ such that*

$$\|h_r(u(t))\|_{L^s(\mathbb{R}^n)} \le ct^{-\alpha} \qquad \text{for all } t > 0. \tag{2.8}$$

ii) *If $n = 1$, then in addition*

$$\|h_r(u(t))\|_{L^\infty(\mathbb{R})} \le ct^{-\alpha} \qquad \text{for all } t > 0 \tag{2.9}$$

with $\alpha = \frac{1}{2 + \frac{2q}{r+\beta}}$.

Proof.

i) As $r \ge q$, we have by interpolation

$$\|u_0\|_{L^r(\mathbb{R}^n)} \le \|u_0\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{q}{r}} \|u_0\|_{L^q(\mathbb{R}^n)}^{\frac{q}{r}} < \infty.$$

Hence Lemma 2.3 applies to give

$$\|\nabla h_r(u(t))\|_{L^2(\mathbb{R}^n)} \le ct^{-\frac{1}{2}} \|u_0\|_{L^r(\mathbb{R}^n)}^{\frac{r}{2}}.$$

On the other hand, we obtain from an integration of (H2) that $f(\sigma) \le c\sigma^\beta$, so that $h_r(\sigma) \le c\sigma^{\frac{r+\beta}{2}}$. Thus, setting $\mu = \frac{2q}{r+\beta}$, we employ Lemma 2.1 to see that

$$\|h_r(u(t))\|_{L^\mu(\mathbb{R}^n)} \le C\big(\|u_0\|_{L^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)}\big).$$

Now the Gagliardo-Nirenberg inequality yields

$$\|h_r(u(t))\|_{L^s(\mathbb{R}^n)} \le c\|\nabla h_r(u(t))\|_{L^2(\mathbb{R}^n)}^a \|h_r(u(t))\|_{L^\mu(\mathbb{R}^n)}^{1-a} \tag{2.10}$$

for all $s \in (\max\{\mu, 1\}, n^\star)$, with

$$a = \frac{\frac{1}{\mu} - \frac{1}{s}}{\frac{1}{\mu} + \frac{1}{n} - \frac{1}{2}} \tag{2.11}$$

which is in $(0, 1)$ since $\mu < s < n^\star$.

If $n \ge 3$, (2.10) continues to hold for $a = 1$ and $s = n^\star < \infty$, and for $s > n^\star$, interpolation between $n^\star$ and ∞ gives

$$\|h_r(u(t))\|_{L^s(\mathbb{R}^n)} \le c\|h_r(u(t))\|_{L^{n^\star}(\mathbb{R}^n)}^{\frac{n^\star}{s}} \le ct^{-\frac{n^\star}{2s}}$$

and thus (2.8) follows.

ii) In one space dimension, $s = \infty$ is allowed in (2.10), where now $a = \frac{2}{2+\mu}$. The proof is complete ∎

Corollary 2.6. *Suppose $f(s) = s^p$ $(p \geq 1)$ and $\vartheta \in (\frac{p}{2}, \frac{p}{2}n^\star]$ with the exception $\vartheta < \infty$ for $n = 2$. Then for all $u_0 \in \bigcap_{q>0} L^q(\mathbb{R}^n)$ and all $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that the corresponding solution u of problem (0.1) satisfies*

$$\|u(t)\|_{L^\vartheta(\mathbb{R}^n)} \leq c_\varepsilon t^{-\frac{1}{p}+\varepsilon}. \tag{2.12}$$

Proof. Noting that $\beta = p$ and $h_r(\sigma) = \frac{2}{r+p}\sigma^{\frac{r+p}{2}}$ in this case, we choose $r = q$ small such that

$$r < 2\vartheta - p, \quad \mu = \frac{2q}{r+p} < 1, \quad \text{and} \quad \frac{a}{r+p} \geq \frac{1}{p} - \varepsilon, \quad \text{where } a = \frac{1 - \frac{\mu}{s}}{1 + \frac{2-n}{n}\mu}.$$

We set $s = \frac{2\vartheta}{r+p}$ to obtain $1 < s \leq n^\star$ and $s < n^\star$ if $n = 2$. Going back to the proof of Theorem 2.5, (2.11) now reads

$$\|u(t)\|^{\frac{r+p}{2}}_{L^{\frac{r+p}{2}s}(\mathbb{R}^n)} \leq ct^{-\frac{a}{2}} \qquad \text{or} \qquad \|u(t)\|_{L^\vartheta(\mathbb{R}^n)} \leq ct^{-\frac{a}{r+p}} \leq ct^{-\frac{1}{p}+\varepsilon}$$

which is exactly the claim ∎

Remark. It is easy to see that if $f(s) = s^p$ $(p \geq 1)$ and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, then a family of solutions of problem (0.1) is given by $u_\gamma(x,t) = (\gamma + pt)^{-\frac{1}{p}}W(x)$, where $\gamma > 0$ and W is the positive solution of $\Delta W + W^{1-p} = 0$ in Ω, $W|_{\partial\Omega} = 0$ (cf. [10]). Accordingly, estimate (2.12) is not far away from being sharp.

One might ask whether *uniform* decay can be achieved also for space dimensions higher than one, possibly not at a fixed rate. A positive answer to this question will be the subject of the following section.

3. Large time decay: the case of general *f*

Although we have seen that condition (H2) covers a not too tiny class of diffusion coefficient functions f (including positive powers as the most frequently mentioned representants), we do so far have no guaranty that solutions might behave completely different if we perturb such an f so as to violate (H2). One particular question is whether or not we may admit f' to change sign (or touch zero) at least for s bounded away from zero. Keeping in mind that large time decay surely takes place in case of the familiar heat equation (where $f \equiv 1$ and hence condition (H2) is hurt), one might conjecture that, if existing at all, something like a 'no decay phenomenon' should be caused by a bad behavior of f near zero rather than for larger values. The problems even seem to increase if we admit that for general f with $f(0) = 0$ we do not know how to control the derivatives of u near points where u is small (and these will be quite a lot if u is to vanish asymptotically), so that the decay arguments from Section 2, basing upon homogenization in space, seem to be little adequate in the present situation. Searching for an alternative approach, we note that due to the comparison principle, once we have shown decay of one special solution u, we at the same time have proved zero convergence of any other solution with initial value less than u_0. Therefore the key to the main result of this section will be to find out under which assumptions on u_0 a *radially symmetric* solution decays. Fortunately, the weakest possible spatial decay hypothesis $\lim_{R\to\infty}\|u_0\|_{L^\infty(\partial B_R)}=0$ (that is sharp in the sense that admitting $\liminf_{R\to\infty}\|u_0\|_{L^\infty(\partial B_R)}>0$ would allow constant initial data which, however, trivially solve problem (0.1)) turns out to be sufficient for uniform decay in the case of arbitrary function f.

Theorem 3.1. *Suppose that, in addition to condition* (H1), $\|u_0\|_{L^\infty(\partial B_R)}\to 0$ *as* $R\to\infty$. *Then the solution* u *of problem* (0.1) *satisfies*

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)}\to 0 \qquad \textit{as } t\to\infty. \tag{3.1}$$

Proof. i) We first note that by a reduction argument similar to the one used in the proof of Lemma 1.3, we may assume without loss of generality that $u(x,t)=U(|x|,t)$ is radially symmetric and $r\mapsto U(r,t)$ non-increasing on $(0,\infty)$ for all $t\ge 0$.

ii) We claim that for all $\varepsilon>0$ there is a constant $T_0>0$ such that

$$u(t)<\varepsilon \qquad \text{on } \partial B_1 \ \text{ for all } t\ge T_0. \tag{3.2}$$

Suppose on the contrary that for some $\varepsilon_0>0$ and a sequence of times $t_k\nearrow\infty$ we had $u(t_k)\ge\varepsilon_0$ on ∂B_1. Let Θ denote the first Dirichlet eigenfunction of $-\Delta$ in B_1 corresponding to the first eigenvalue $\lambda_1>0$ with $\max\Theta=1$, and set

$$z(x,t)=y(t)\Theta(x) \qquad \text{with } y(t)=\varepsilon_0 e^{-\gamma(t-t_k)} \text{ and } \gamma=\lambda_1\|f(u)\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}$$

in $B_1\times[t_k,\infty)$. Then $z\le u$ on ∂B_1 and, as $u(t)|_{\bar B_1}$ takes its minimum on ∂B_1 by step i), also at $t=t_k$. Moreover, $z_t-f(u)\Delta z=y'\Theta+\lambda_1 f(u)y\Theta\le 0$, so that $u\ge z$ in $B_1\times[t_k,\infty)$ by comparison, which implies the existence of numbers $\delta>0$ and $\rho>0$ such that

$$u\ge\frac{\varepsilon_0}{2} \qquad \text{in } B_\rho\times[t_k,t_k+\delta]\ \forall\, k\in\mathbb{N}. \tag{3.3}$$

Next, we choose a non-decreasing $f_0\in C^\infty([0,M])$, $M=\|u_0\|_{L^\infty(\mathbb{R}^n)}$, such that $f_0(s)=0$ for $s<\frac{\varepsilon_0}{8}$ and $f_0(s)=\alpha(s-\frac{\varepsilon_0}{8})$ for $s>\frac{\varepsilon_0}{4}$, where $\alpha=\min_{s\in[\frac{\varepsilon_0}{8},M]}f(s)>0$, so that $f_0<f$ and $f_0'=\alpha>0$ on $[\frac{\varepsilon_0}{4},M]$. As in the proof of Lemma 1.3, we set

$$\Phi(s)=\int_0^s\frac{f_0(\sigma)}{f(\sigma)}\,d\sigma$$

and test (1.1) with $\frac{f_0}{f}(u_k)$ to obtain after taking $k\to\infty$

$$\int_{\mathbb{R}^n}\Phi(u(t))+\int_0^t\!\!\int_{\mathbb{R}^n}|\nabla h(u)|^2\le\int_{\mathbb{R}^n}\Phi(u_0) \qquad \forall\, t>0 \tag{3.4}$$

where $h(s)=\int_0^s\sqrt{f_0'(\sigma)}\,d\sigma$. Note that the right-hand side in (3.4) is finite since by assumption the set $\{u_0\ge\frac{\varepsilon_0}{8}\}$ is compact.

Let us now fix R large such that $|B_R|\Phi(\frac{\varepsilon_0}{3}) \geq 2\int_{\mathbb{R}^n}\Phi(u_0)$ and define a function $v(r,t)$ on $[0,\infty)^2$ by $v(|x|,t) = h(u(x,t))$. By (3.4), $\int_0^\infty \int_0^\infty r^{n-1}|v_r|^2 drdt < \infty$, hence for each k there is $\tilde{t}_k \in [t_k, t_k+\delta]$ such that

$$\int_{\frac{\varrho}{2}}^{R} |v_r(r,\tilde{t}_k)|^2 dr \to 0, \qquad \text{that is } \|v_r(\cdot,\tilde{t}_k)\|_{L^2((\frac{\varrho}{2},R))} \to 0 \ \text{ as } k \to \infty.$$

We thus find a subsequence such that $v(\cdot,\tilde{t}_k) \to w$ in $C^0([\frac{\varrho}{2},R])$, where w must be a constant. By (3.3), $w \geq h(\frac{\varepsilon_0}{2})$, whence by uniform convergence we have $u(\tilde{t}_k) \geq \frac{\varepsilon_0}{3}$ in B_R for some large $\tilde{t}_k$. But as $\Phi' \geq 0$, this implies

$$\int_{\mathbb{R}^n} \Phi(u(\tilde{t}_k)) \geq \int_{B_R} \Phi(u(\tilde{t}_k)) \geq |B_R|\Phi\big(\tfrac{\varepsilon_0}{3}\big) > \int_{\mathbb{R}^n} \Phi(u_0)$$

which is absurd in view of (3.4).

iii) Now let $\varepsilon > 0$ be given and fix T_0 such that (3.2) holds. If $e \in C^2(\bar{B}_1)$ denotes the solution of $-\Delta e = 1$ in B_1, $e|_{\partial B_1} = 1$, we have $e \geq 1$ in B_1 and therefore the function

$$z(x,t) = \varepsilon + y(t)e(x) \qquad \text{with } y(t) = \|u(T_0)\|_{L^\infty(\mathbb{R}^n)} e^{-\mu(t-T_0)},$$

$\mu > 0$ small to be fixed soon, majorizes u at $t = T_0$ and, by (3.2), also on $\partial B_1 \times [T_0,\infty)$. As

$$z_t - f(z)\Delta z = [-\mu e + f(\varepsilon + ye)]y$$

is non-negative in $B_1 \times (T_0,\infty)$ if we choose

$$\mu = \frac{1}{\|e\|_{L^\infty(B_1)}} \min_{s\in[\varepsilon,M]} f(s), \qquad \text{where } M = \varepsilon + \|u(T_0)\|_{L^\infty(\mathbb{R}^n)}\|e\|_{L^\infty(B_1)}$$

we infer from the comparison principle that $u \leq z$ in $B_1 \times (T_1,\infty)$ and therefore $u(t) < 2\varepsilon$ in B_1 (and thus in all of $\mathbb{R}^n$ by monotonicity) for all sufficiently large t ∎

Remark. Except for the uniqueness proof in Lemma 1.4, none of our arguments actually required that the domain under consideration has no boundary. In fact, all of the existence and decay assertions remain valid if $\mathbb{R}^n$ is replaced by any (bounded or unbounded) domain $\Omega \subset \mathbb{R}^n$ with, e.g., Lipschitz boundary.

Finally, we mention that all the results from Sections 1 and 3 remain valid without any change if we drop the degeneracy condition $f(0) = 0$ – note that then Section 2 becomes obsolete since condition (H2) implies $f(s) \leq f(1)s^\beta$ for $s \in (0,1)$, hence $f(0) = 0$. Indeed, reviewing the proofs shows that the degenerate case (to which we have restricted ourselves) seems to be the most critical among all cases im which $f(s) > 0$ for $s > 0$ is required.

Accordingly, we obtain as a corollary to Theorem 3.1 that for *any quasilinear equation* $u_t = f(u)\Delta u$ with $f \in C^0([0,\infty)) \cap C^1((0,\infty))$ positive on $(0,\infty)$, every solution evolving from positive initial data decaying arbitrarily slowly in space decays uniformly as time tends to infinity.

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