Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 20 (2001), No. 3, 691–708

Parametric Weighted Integral Inequalities for A-Harmonic Tensors

S. Ding

Abstract. We prove the $A_r(\Omega)$-weighted Hardy-Littlewood inequality, the $A_r(\Omega)$-weighted weak reverse Hölder inequality and the $A_r(\Omega)$-weighted Caccioppoli-type estimate for A-harmonic tensors all being generalizations of classical results.

Keywords: *A_r-weights, inequalities, A-harmonic equation, differential forms*

AMS subject classification: Primary 31C45, secondary 58A10, 35B45, 26D10

1. Introduction

The purpose of this paper is to develop parametric versions of the $\mathrm{A}_r(\Omega)$-weighted integral inequalities for A-harmonic tensors. These results are of interest in nonlinear potential theory, degenerate elliptic equations, continuum mechanics, and the L^p theory. They can be used to study the integrability of A-harmonic tensors and to estimate the integrals for A-harmonic tensors. A-harmonic tensors are differential forms which satisfy the A-harmonic equation. They are interesting and important extensions of p-harmonic tensors. In the meantime, p-harmonic tensors are extensions of harmonic functions and p-harmonic functions, $p > 1$. Many interesting results of A-harmonic tensors and their applications in different fields, such as quasiregular mappings and the theory of elasticity, have been found recently (see [1 - 4, 8 - 12, 14]).

We always assume that Ω is a connected open subset of $\mathbb{R}^n$. We write $\mathbb{R} = \mathbb{R}^1$. Balls are denoted by B, and σB is the ball with the same center as B and with $\mathrm{diam}(\sigma B) = \sigma \mathrm{diam}(B)$. We do not distinguish the balls from cubes throughout this paper. The n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is denoted by $|E|$. We call w a weight if $w \in L^1_{loc}(\mathbb{R}^n)$ and $w > 0$ a.e. Also, in general $d\mu = w\,dx$ where w is a weight. For $0 < p < \infty$ we denote the weighted L^p-norm of a measurable function f over E by

$$\|f\|_{p,E,w} = \left(\int_E |f(x)|^p w(x)\,dx \right)^{\frac{1}{p}}.$$

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard unit basis of $\mathbb{R}^n$. Assume that $\wedge^l = \wedge^l(\mathbb{R}^n)$ is the linear space of l-vectors spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$,

Shusen Ding: Seattle Univ., Dept. Math., Seattle, WA 98122, USA; sding@seattleu.edu
Research supported by Seattle University Faculty Fellowships for Summer of 2000.

ISSN 0232-2064 / $ 2.50

corresponding to all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$ $(1 \le i_1 < i_2 < \ldots < i_l \le n;\ l = 0, 1, \ldots, n)$. The Grassman algebra $\wedge = \oplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$ the inner product in $\wedge$ is given by

$$\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$$

with summation over all l-tuples $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$. We define the Hodge star operator

$$\star : \wedge \to \wedge$$

by the rule

$$\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n \qquad \text{and} \qquad \alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle (\star 1)$$

for all $\alpha, \beta \in \wedge$. The norm of $\alpha \in \wedge$ is given by the formula

$$|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \wedge^0 = \mathbb{R}.$$

The Hodge star is an isometric isomorphism on $\wedge$ with $\star : \wedge^l \to \wedge^{n-l}$ and $\star\star(-1)^{l(n-l)} : \wedge^l \to \wedge^l$.

A differential l-form ω on Ω is a de Rham current (see [13: Chapter III]) on Ω with values in $\wedge^l(\mathbb{R}^n)$. We use $D'(\Omega, \wedge^l)$ to denote the space of all differential l-forms and $L^p(\Omega, \wedge^l)$ to denote the l-forms

$$\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$$

with $\omega_I \in L^p(\Omega, \mathbb{R})$ for all ordered l-tuples I. Thus $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx \right)^{\frac{1}{p}} = \left(\int_\Omega \left(\sum_I |\omega_I(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}.$$

Similarly, $W_p^1(\Omega, \wedge^l)$ are the differential l-forms on Ω whose coefficients are in $W_p^1(\Omega, \mathbb{R})$. The notations $W_{p,loc}^1(\Omega, \mathbb{R})$ and $W_{p,loc}^1(\Omega, \wedge^l)$ are self-explanatory. We denote the exterior derivative by

$$d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$$

for $l = 0, 1, \ldots, n$. Its formal adjoint operator

$$d^\star : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$$

is given by

$$d^\star = (-1)^{nl+1} \star d \star \quad \text{on } D'(\Omega, \wedge^{l+1}) \qquad (l = 0, 1, \ldots, n).$$

Many interesting results have been established in the study of the A-harmonic equation

$$d^\star A(x, d\omega) = 0 \tag{1.1}$$

for differential forms, where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ satisfies the conditions

$$|A(x,\xi)| \le a|\xi|^{p-1} \qquad \text{and} \qquad \langle A(x,\xi),\xi\rangle \ge |\xi|^p \tag{1.2}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with equation (1.1). A solution to equation (1.1) is an element of the Sobolev space $W^1_{p,loc}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi\rangle = 0$$

for all $\varphi \in W^1_p(\Omega, \wedge^{l-1})$ with compact support.

Definition 1.1. We call u an *A-harmonic tensor* in Ω if u satisfies the A-harmonic equation (1.1) in Ω.

A differential l-form $u \in D'(\Omega, \wedge^l)$ is called a *closed form* if $du = 0$ in Ω. Similarly, a differential $(l+1)$-form $v \in D'(\Omega, \wedge^{l+1})$ is called a *co-closed form* if $d^\star v = 0$. The equation

$$A(x, du) = d^\star v \tag{1.3}$$

is called the *conjugate* A-harmonic equation. For example, $du = d^* v$ is an analogue of a Cauchy-Riemann system in $\mathbb{R}^n$. Clearly, the A-harmonic equation is not affected by adding a closed form to u and co-closed form to v. Therefore, any type of estimates between u and v must be modulo such forms. Suppose that u is a solution to equation (1.1) in Ω. Then, at least locally in a ball B, there exists a form $v \in W^1_q(B, \wedge^{l+1})$ $(\frac{1}{p} + \frac{1}{q} = 1)$ such that (1.3) holds. Throughout this paper, we always assume that $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 1.2. When u and v satisfy (1.3) in Ω and A^{-1} exists in Ω, we call u and v *conjugate A-harmonic tensors* in Ω.

Iwaniec and Lutoborski prove the following result in [9]:

Let $Q \subset \mathbb{R}^n$ be a cube or a ball. To each $y \in Q$ there corresponds a linear operator

$$K_y : C^\infty(Q, \wedge^l) \to C^\infty(Q, \wedge^{l-1})$$

defined by

$$(K_y\omega)(x;\xi_1,\ldots,\xi_l) = \int_0^1 t^{l-1}\omega\big(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1}\big)\,dt$$

and the decomposition $\omega = d(K_y\omega) + K_y(d\omega)$.

We define another linear operator

$$T_Q : C^\infty(Q, \wedge^l) \to C^\infty(Q, \wedge^{l-1})$$

by averaging K_y over all points y in Q:

$$T_Q\omega = \int_Q \varphi(y) K_y\omega\, dy$$

where $\varphi \in C_0^\infty(Q)$ is normalized by $\int_Q \varphi(y)\,dy = 1$. We define the l-form $\omega_Q \in D'(Q, \wedge^l)$ by

$$\omega_Q = \begin{cases} |Q|^{-1}\int_Q \omega(y)\,dy & \text{if } l = 0 \\ d(T_Q\omega) & \text{if } l = 1, 2, \ldots, n \end{cases}$$

for all $\omega \in L^p(Q, \wedge^l)$ $(1 \le p < \infty)$.

2. The $A_r(\Omega)$-weighted Hardy-Littlewood inequality

In this section, we prove different versions of the $A_r(\Omega)$-weighted Hardy-Littlewood inequality.

Definition 2.1. A weight $w = w(x)$ is called an A_r*-weight* for some $r > 1$ in a domain Ω, write $w \in A_r(\Omega)$, if $w > 0$ a.e. and

$$\sup_B \left(\frac{1}{|B|} \int_B w\, dx \right) \left(\frac{1}{|B|} \int_B \left(\frac{1}{w} \right)^{\frac{1}{r-1}} dx \right)^{r-1} < \infty \tag{2.1}$$

for any ball $B \subset \Omega$.

See [5, 7] for properties of $A_r(\Omega)$-weights. We will need the following generalized Hölder inequality.

Lemma 2.2. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $\frac{1}{s} = \frac{1}{\alpha} + \frac{1}{\beta}$. If f and g are measurable functions on $\mathbb{R}^n$, then*

$$\|fg\|_{s,\Omega} \le \|f\|_{\alpha,\Omega} \|g\|_{\beta,\Omega}$$

for any $\Omega \subset \mathbb{R}^n$.

We also need the following lemma [5].

Lemma 2.3. *If $w \in A_r(\Omega)$, then there exist constants $\beta > 1$ and $C > 0$, independent of w, such that*

$$\|w\|_{\beta,B} \le C|B|^{\frac{1-\beta}{\beta}} \|w\|_{1,B}$$

for all balls $B \subset \mathbb{R}^n$.

Hardy and Littlewood prove the following inequality for conjugate harmonic functions in the unit disk D in [6]:

Theorem A. *For each $p > 0$, there is a constant $C > 0$ such that*

$$\int_D |u - u(0)|^p dxdy \le C \int_D |v - v(0)|^p dxdy$$

for all analytic functions $f = u + iv$ in the unit disk D.

The above Hardy-Littlewood inequality has been generalized into different versions. In [12] Nolder proves the following version of it.

Theorem B. *Let u and v be conjugate A-harmonic tensors in $\Omega \subset \mathbb{R}^n$, $\sigma > 1$, and $0 < s, t < \infty$. Then there exists a constant $C > 0$, independent of u and v, such that*

$$\|u - u_B\|_{s,B} \le C|B|^\beta \|v - c\|_{t,\sigma B}^{\frac{q}{p}}$$

for all balls B with $\sigma B \subset \Omega$. Here c is any form in $W^1_{p,loc}(\Omega, \Lambda)$ with $d^\star c = 0$ and $\beta = \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{t} + \frac{1}{n})q}{p}$.

Now we prove the following parametric weighted Hardy-Littlewood inequality.

Theorem 2.4. *Let u and v be conjugate A-harmonic tensors in a domain $\Omega \subset \mathbb{R}^n$ and $w \in A_r(\Omega)$ for some $r > 1$. Let $0 < s, t < \infty$. Then there exists a constant $C > 0$, independent of u and v, such that*

$$\left(\int_B |u - u_B|^s w^\alpha dx\right)^{\frac{1}{s}} \le C|B|^\gamma \left(\int_{\sigma B} |v - c|^t w^{\frac{pt\alpha}{qs}} dx\right)^{\frac{q}{pt}} \tag{2.2}$$

for all balls B with $\sigma B \subset \Omega \subset \mathbb{R}^n$, $\sigma > 1$ and $0 < \alpha \le 1$. Here c is any form in $W^1_{q,loc}(\Omega, \Lambda)$ with $d^* c = 0$ and $\gamma = \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{t} + \frac{1}{n})q}{p}$.

As mentioned in Section 1, the A-harmonic equation is not affected by adding a closed form to u and co-closed form to v. Therefore, any type of estimates between u and v must be modulo such forms. Thus, (2.2) is equivalent to

$$\left(\int_B |u|^s w^\alpha dx\right)^{\frac{1}{s}} \le C|B|^\gamma \left(\int_{\sigma B} |v - c|^t w^{\frac{pt\alpha}{qs}} dx\right)^{\frac{q}{pt}} \tag{2.2$'$}$$

Note that (2.2) can also be written as the symmetric form

$$\left(\frac{1}{|B|}\int_B |u - u_B|^s w^\alpha dx\right)^{\frac{1}{qs}} \le C|B|^{\frac{\frac{1}{q} - \frac{1}{p}}{n}} \left(\frac{1}{|B|}\int_{\sigma B} |v - c|^t w^{\frac{pt\alpha}{qs}} dx\right)^{\frac{1}{pt}}. \tag{2.2$''$}$$

Proof of Theorem 2.4. We first show that (2.2) holds for $0 < \alpha < 1$. Let $k = \frac{s}{1-\alpha}$. Using Lemma 2.2 we have

$$\begin{aligned}\left(\int_B |u - u_B|^s w^\alpha dx\right)^{\frac{1}{s}} &= \left(\int_B \left(|u - u_B| w^{\frac{\alpha}{s}}\right)^s dx\right)^{\frac{1}{s}} \\ &\le \|u - u_B\|_{k,B} \left(\int_B w^{\frac{k\alpha}{k-s}} dx\right)^{\frac{k-s}{ks}} \\ &= \|u - u_B\|_{k,B} \left(\int_B w dx\right)^{\frac{\alpha}{s}}.\end{aligned} \tag{2.3}$$

Choose $m = \frac{qst}{qs + \alpha pt(r-1)}$. Then $m < t$. By Theorem B we have

$$\|u - u_B\|_{k,B} \le C_1 |B|^\beta \|v - c\|^{\frac{q}{p}}_{m,\sigma B} \tag{2.4}$$

where $\beta = \frac{1}{k} + \frac{1}{n} - \frac{(\frac{1}{m} + \frac{1}{n})q}{p}$. Substituting (2.4) into (2.3) yields

$$\left(\int_B |u - u_B|^s w^\alpha dx\right)^{\frac{1}{s}} \le C_1 |B|^\beta \|v - c\|^{\frac{q}{p}}_{m,\sigma B} \left(\int_B w dx\right)^{\frac{\alpha}{s}}. \tag{2.5}$$

Since $\frac{1}{m} = \frac{1}{t} + \frac{t-m}{mt}$, by Lemma 2.2 again we find that

$$\begin{aligned}\|v - c\|_{m,\sigma B} &= \left(\int_{\sigma B} \left(|v - c| w^{\frac{p\alpha}{qs}} w^{-\frac{p\alpha}{qs}}\right)^m dx\right)^{\frac{1}{m}} \\ &\le \left(\int_{\sigma B} |v - c|^t w^{\frac{pt\alpha}{qs}} dx\right)^{\frac{1}{t}} \left(\int_{\sigma B} \left(\frac{1}{w}\right)^{\frac{pmt\alpha}{qs(t-m)}} dx\right)^{\frac{t-m}{mt}} \\ &= \left(\int_{\sigma B} |v - c|^t w^{\frac{pt\alpha}{qs}} dx\right)^{\frac{1}{t}} \left(\int_{\sigma B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{\frac{p\alpha(r-1)}{qs}}.\end{aligned}$$

Hence

$$\|v-c\|_{m,\sigma B}^{\frac{q}{p}} \le \left(\int_{\sigma B}\left(\frac{1}{w}\right)^{\frac{1}{r-1}}dx\right)^{\frac{\alpha(r-1)}{s}}\left(\int_{\sigma B}|v-c|^t w^{\frac{pt\alpha}{qs}}dx\right)^{\frac{q}{pt}}. \tag{2.6}$$

Combining (2.5) and (2.6) we obtain

$$\begin{aligned}&\left(\int_B |u-u_B|^s w^\alpha dx\right)^{\frac{1}{s}}\\ &\le C_1|B|^\beta\left(\int_B wdx\right)^{\frac{\alpha}{s}}\left(\int_{\sigma B}\left(\frac{1}{w}\right)^{\frac{1}{r-1}}dx\right)^{\frac{\alpha(r-1)}{s}}\left(\int_{\sigma B}|v-c|^t w^{\frac{pt\alpha}{qs}}dx\right)^{\frac{q}{pt}}.\end{aligned} \tag{2.7}$$

Using the condition that $w\in A_r(\Omega)$ yields

$$\begin{aligned}&\left(\int_B wdx\right)^{\frac{\alpha}{s}}\left(\int_{\sigma B}\left(\frac{1}{w}\right)^{\frac{1}{r-1}}dx\right)^{\frac{\alpha(r-1)}{s}}\\ &\le|\sigma B|^{\frac{\alpha r}{s}}\left(\left(\frac{1}{|\sigma B|}\int_B wdx\right)\left(\frac{1}{|\sigma B|}\int_{\sigma B}\left(\frac{1}{w}\right)^{\frac{1}{r-1}}dx\right)^{r-1}\right)^{\frac{\alpha}{s}}\\ &\le C_2|\sigma B|^{\frac{\alpha r}{s}}\\ &=C_3|B|^{\frac{\alpha r}{s}}.\end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.7) and noting that $\beta+\frac{\alpha r}{s}=\frac{1}{k}+\frac{1}{n}-\frac{(\frac{1}{m}+\frac{1}{n})q}{p}+\frac{r}{\alpha s}=\frac{1}{s}+\frac{1}{n}-\frac{(\frac{1}{t}+\frac{1}{n})q}{p}$, we have

$$\left(\int_B |u-u_B|^s w^\alpha dx\right)^{\frac{1}{s}}\le C_4|B|^\gamma\left(\int_{\sigma B}|v-c|^t w^{\frac{pt\alpha}{qs}}dx\right)^{\frac{q}{pt}}$$

where $\gamma=\frac{1}{s}+\frac{1}{n}-\frac{(\frac{1}{t}+\frac{1}{n})q}{p}$.

Next, we prove that Theorem 2.4 holds if $\alpha=1$. By Lemma 2.3 there exist constants $\beta_1>1$ and $C_5>0$, independent of w, such that

$$\|w\|_{\beta_1,\sigma B}\le C_5|B|^{\frac{1-\beta_1}{\beta_1}}\|w\|_{1,\sigma B}. \tag{2.9}$$

Since $\frac{1}{\beta_1 s}+\frac{\beta_1-1}{\beta_1 s}=\frac{1}{s}$, then by Lemma 2.2 we have

$$\|u-u_B\|_{s,B,w}\le\|w\|_{\beta_1,B}^{\frac{1}{s}}\|u-u_B\|_{\frac{\beta_1 s}{\beta_1-1},B}. \tag{2.10}$$

By Theorem B, there is a constant $C_6>0$, independent of u and v, such that for any $t'>0$ we have

$$\|u-u_B\|_{\frac{\beta_1 s}{\beta_1-1},B}\le C_6|B|^{\beta'}\|v-c\|_{t',\sigma B}^{\frac{q}{p}} \tag{2.11}$$

where $\beta' = \frac{\beta_1 - 1}{\beta_1 s} + \frac{1}{n} - \frac{(\frac{1}{t'} + \frac{1}{n})q}{p}$. Combining (2.10) and (2.11) we obtain

$$\|u - u_B\|_{s,B,w} \le C_6 |B|^{\beta'} \|w\|^{\frac{1}{s}}_{\beta_1, B} \|v - c\|^{\frac{q}{p}}_{t', \sigma B}. \tag{2.12}$$

Now, choose $t' = \frac{t}{k_1}$ where k_1 is to be determined later. Since $|v - c| = w^{-\frac{p}{qs}} |v - c| w^{\frac{p}{qs}}$, by Lemma 2.2 we obtain

$$\|v - c\|_{t', \sigma B} \le \left\| \left(\frac{1}{w}\right)^{\frac{pt}{qs}} \right\|^{\frac{1}{t}}_{\frac{1}{k_1 - 1}, \sigma B} \left(\int_{\sigma B} |v - c|^t w^{\frac{pt}{qs}} dx \right)^{\frac{1}{t}}. \tag{2.13}$$

From (2.9), (2.12) and (2.13) we have

$$\begin{aligned} &\|u - u_B\|_{s,B,w} \\ &\le C_7 |B|^{\beta' + \frac{1 - \beta_1}{\beta_1 s}} \|w\|^{\frac{1}{s}}_{1, \sigma B} \left\| \left(\frac{1}{w}\right)^{\frac{pt}{qs}} \right\|^{\frac{q}{pt}}_{\frac{1}{k_1 - 1}, \sigma B} \left(\int_{\sigma B} |v - c|^t w^{\frac{pt}{qs}} dx \right)^{\frac{q}{pt}}. \end{aligned} \tag{2.14}$$

Set $k_1 = 1 + \frac{pt(r-1)}{qs}$, then $\frac{(k_1 - 1)qs}{pt} = r - 1$. By $w \in A_r(\Omega)$ we know that

$$\begin{aligned} &\|w\|^{\frac{1}{s}}_{1, \sigma B} \left\| \left(\frac{1}{w}\right)^{\frac{pt}{qs}} \right\|^{\frac{q}{pt}}_{\frac{1}{k_1 - 1}, \sigma B} \\ &\quad = |\sigma B|^{\frac{1}{s} + \frac{(k_1 - 1)q}{pt}} \left(\frac{1}{|\sigma B|} \int_{\sigma B} w dx \left(\frac{1}{|\sigma B|} \int_{\sigma B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx \right)^{r-1} \right)^{\frac{1}{s}} \\ &\quad \le C_8 |B|^{\frac{1}{s} + \frac{(k_1 - 1)q}{pt}}. \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15) we have

$$\|u - u_B\|_{s,B,w} \le C_9 |B|^{\gamma} \left(\int_{\sigma B} |v - c|^t w^{\frac{pt}{qs}} dx \right)^{\frac{q}{pt}}$$

where

$$\gamma = \beta' + \frac{1 - \alpha}{\alpha s} + \frac{1}{s} + \frac{q(k-1)}{pt} = -\frac{nq + t(q-p)}{npt} + \frac{1}{s} = \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{t} + \frac{1}{n})q}{p}.$$

Therefore, (2.2) holds if $\alpha = 1$. We have completed the proof of Theorem 2.4 ∎

We need the following properties of the Whitney covers appearing [12].

Lemma 2.5. *Each Ω has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that*

$$\cup_i Q_i = \Omega, \qquad \sum_{Q \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}} Q} \le N \chi_\Omega$$

for all $x \in \mathbb{R}^n$ and some $N > 1$, and if $Q_i \cap Q_j \ne \phi$, then there exists a cube R (this cube does not need be a member of $\mathcal{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if Ω is δ-John, then there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \ldots, Q_k = Q$ from $\mathcal{V}$ and such that $Q \subset \rho Q_i$ $(i = 0, 1, 2, \ldots, k)$ for some $\rho = \rho(n, \delta)$.

As applications of Theorem 2.4 we prove the following global $A_r(\Omega)$-weighted Hardy-Littlewood inequality.

Theorem 2.6. *Let $u \in D'(\Omega, \Lambda^{l-1})$ and $v \in D'(\Omega, \Lambda^{l+1})$ be conjugate A-harmonic tensors. Let $q \le p$, $v - c \in L^t(\Omega, \Lambda^{l+1})$ $(l = 1, 2, \dots, n-1)$ and $w \in A_r(\Omega)$. If s is defined by*

$$s = \frac{npt}{nq + t(q-p)} \qquad (0 < t < \infty), \tag{2.16}$$

then there exists a constant $C > 0$, independent of u and v, such that

$$\left(\int_\Omega |u|^s w^\alpha dx\right)^{\frac{1}{s}} \le C \left(\int_\Omega |v - c|^t w^{\frac{pt\alpha}{qs}} dx\right)^{\frac{q}{pt}}$$

for any domain $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$. Here c is any form in $W^1_{q,loc}(\Omega, \Lambda)$ with $d^*c = 0$.

Proof. From (2.2)$'$ we have

$$\left(\int_Q |u|^s w^\alpha dx\right)^{\frac{1}{s}} \le C|Q|^\gamma \left(\int_{\sigma Q} |v - c|^t w^{\frac{pt\alpha}{qs}} dx\right)^{\frac{q}{pt}} \tag{2.17}$$

where $\gamma = \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{t} + \frac{1}{n})q}{p}$. Substituting (2.16) into the expression of γ we get

$$\gamma = \frac{1}{s} + \frac{1}{n} - \left(\frac{q}{pt} + \frac{q}{np}\right) = \frac{nq + t(q-p)}{npt} + \frac{1}{n} - \left(\frac{q}{pt} + \frac{q}{np}\right) = 0. \tag{2.18}$$

Thus we find that (2.17) reduces to

$$\left(\int_Q |u|^s w^\alpha dx\right)^{\frac{1}{s}} \le C \left(\int_{\sigma Q} |v - c|^t w^{\frac{pt\alpha}{qs}} dx\right)^{\frac{q}{pt}}. \tag{2.19}$$

Combining (2.19) and Lemma 2.5, we get

$$\begin{aligned}
\left(\int_\Omega |u|^s w^\alpha dx\right)^{\frac{1}{s}} &\le \sum_{Q \in \mathcal{V}} \left(\int_Q |u|^s w^\alpha dx\right)^{\frac{1}{s}} \\
&\le \sum_{Q \in \mathcal{V}} \left(\int_Q |u|^s w^\alpha \chi_{\sqrt{\frac{5}{4}}Q} dx\right)^{\frac{1}{s}} \\
&\le \sum_{Q \in \mathcal{V}} \left(\int_Q |u|^s w^\alpha dx\right)^{\frac{1}{s}} \chi_{\sqrt{\frac{5}{4}}Q} \\
&\le \sum_{Q \in \mathcal{V}} C_1 \left(\int_{\sigma Q} |v - c|^t w^{\frac{pt\alpha}{qs}} dx\right)^{\frac{q}{pt}} \chi_{\sqrt{\frac{5}{4}}Q} \\
&\le \sum_{Q \in \mathcal{V}} C_1 \left(\int_\Omega |v - c|^t w^{\frac{pt\alpha}{qs}} dx\right)^{\frac{q}{pt}} \chi_{\sqrt{\frac{5}{4}}Q} \\
&\le C_1 \left(\int_\Omega |v - c|^t w^{\frac{pt\alpha}{qs}} dx\right)^{\frac{q}{pt}} \sum_{Q \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q} \\
&\le C_2 \left(\int_\Omega |v - c|^t w^{\frac{pt\alpha}{qs}} dx\right)^{\frac{q}{pt}}.
\end{aligned}$$

The proof of Theorem 2.6 has been completed ∎

Note that $\alpha \in (0,1]$ is arbitrary in Theorem 2.4. Hence, if we choose α to be some special values, we will have some different versions of the Hardy-Littlewood inequality. For example, if we let $\alpha = qs$, $qs \le 1$. By Theorem 2.4, we have the following symmetric version of the Hardy-Littlewood inequality.

Corollary 2.7. *Let u and v be conjugate A-harmonic tensors in a domain $\Omega \subset \mathbb{R}^n$ and $w \in A_r(\Omega)$ for some $r > 1$. Let $0 < t < \infty$ and $qs \le 1$. Then there exists a constant $C > 0$, independent of u and v, such that*

$$\left(\int_B |u - u_B|^s w^{qs} dx\right)^{\frac{1}{qs}} \le C|B|^\gamma \left(\int_{\sigma B} |v - c|^t w^{pt} dx\right)^{\frac{1}{pt}}$$

for all balls B with $\sigma B \subset \Omega \subset \mathbb{R}^n$ and $\sigma > 1$. Here c is any form in $W^1_{q,loc}(\Omega, \Lambda)$ with $d^*c = 0$ and $\gamma = \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{t}+\frac{1}{n})q}{p}$.

If we choose $\alpha = \frac{1}{pt}$ and $pt \ge 1$ in Theorem 2.4, we obtain the following symmetric version.

Corollary 2.8. *Let u and v be conjugate A-harmonic tensors in a domain $\Omega \subset \mathbb{R}^n$ and $w \in A_r(\Omega)$ for some $r > 1$. Let $0 < t < \infty$ and $pt \ge 1$. Then there exists a constant $C > 0$, independent of u and v, such that*

$$\left(\int_B |u - u_B|^s w^{\frac{1}{pt}} dx\right)^{\frac{1}{qs}} \le C|B|^\gamma \left(\int_{\sigma B} |v - c|^t w^{\frac{1}{qs}} dx\right)^{\frac{1}{pt}}$$

for all balls B with $\sigma B \subset \Omega \subset \mathbb{R}^n$ and $\sigma > 1$. Here c is any form in $W^1_{q,loc}(\Omega, \Lambda)$ with $d^*c = 0$ and $\gamma = \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{t}+\frac{1}{n})q}{p}$.

If we choose $\alpha = \frac{1}{p}$ in Theorem 2.4, we obtain the following result.

Corollary 2.9. *Let u and v be conjugate A-harmonic tensors in a domain $\Omega \subset \mathbb{R}^n$ and $w \in A_r(\Omega)$ for some $r > 1$. Let $0 < s, t < \infty$. Then there exists a constant $C > 0$, independent of u and v, such that*

$$\left(\int_B |u - u_B|^s w^{\frac{1}{p}} dx\right)^{\frac{1}{qs}} \le C|B|^\gamma \left(\int_{\sigma B} |v - c|^t w^{\frac{t}{qs}} dx\right)^{\frac{1}{pt}}$$

for all balls B with $\sigma B \subset \Omega \subset \mathbb{R}^n$ and $\sigma > 1$. Here c is any form in $W^1_{q,loc}(\Omega, \Lambda)$ with $d^*c = 0$ and $\gamma = \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{t}+\frac{1}{n})q}{p}$.

If we choose $\alpha = 1$ in Theorem 2.4, we have the following corollary.

Corollary 2.10. *Let u and v be conjugate A-harmonic tensors in a domain $\Omega \subset \mathbb{R}^n$ and $w \in A_r(\Omega)$ for some $r > 1$. Let $0 < s, t < \infty$. Then there exists a constant $C > 0$, independent of u and v, such that*

$$\left(\int_B |u - u_B|^s w dx\right)^{\frac{1}{qs}} \le C|B|^\gamma \left(\int_{\sigma B} |v - c|^t w^{\frac{pt}{qs}} dx\right)^{\frac{1}{pt}}$$

for all balls B with $\sigma B \subset \Omega \subset \mathbb{R}^n$ and $\sigma > 1$. Here c is any form in $W^1_{q,loc}(\Omega, \Lambda)$ with $d^*c = 0$ and $\gamma = \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{t}+\frac{1}{n})q}{p}$.

Remark. By making different choices for α in Theorem 2.6, we shall have different versions of the global Hardy-Littlewood inequality. Considering the length of the paper, we do not list them here.

3. The $A_r(\Omega)$-weighted weak reverse Hölder inequality

In [12], Nolder obtains the following Caccioppoli-type inequality.

Theorem C. *Let u be an A-harmonic tensor in Ω and let $\sigma > 1$. Then there exists a constant $C > 0$, independent of u, such that*

$$\|du\|_{s,B} \le C \mathrm{diam(B)}^{-1} \|u - c\|_{s,\sigma B}$$

for all balls or cubes B with $\sigma B \subset \Omega$ and all closed forms c. Here $1 < s < \infty$.

The following weak reverse Hölder inequality appears in [12].

Theorem D. *Let u be an A-harmonic tensor in Ω, $\sigma > 1$ and $0 < s, t < \infty$. Then there exists a constant $C > 0$, independent of u, such that*

$$\|u\|_{s,B} \le C|B|^{\frac{t-s}{st}} \|u\|_{t,\sigma B}$$

for all balls or cubes B with $\sigma B \subset \Omega$.

Using the same method as those used in Section 2, we prove the following $A_r(\Omega)$-weighted weak reverse Hölder inequality with parameter α for A-harmonic tensors.

Theorem 3.1. *Let $u \in D'(\Omega, \wedge^l)$ $(l = 0, 1, \dots, n)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$, $\sigma > 1$. Assume that $0 < s, t < \infty$ and $w \in A_r(\Omega)$ for some $r > 1$. Then there exists a constant $C > 0$, independent of u, such that*

$$\left(\frac{1}{|B|} \int_B |u|^s w^\alpha dx \right)^{\frac{1}{s}} \le C \left(\frac{1}{|B|} \int_{\sigma B} |u|^t w^{\frac{\alpha t}{s}} dx \right)^{\frac{1}{t}} \tag{3.1}$$

for all balls B with $\sigma B \subset \Omega$ and any real number α with $0 < \alpha \le 1$.

Proof. First, we suppose that $0 < \alpha < 1$. Let $k = \frac{s}{1-\alpha}$. From Lemma 2.2 we find that

$$\begin{aligned} \left(\int_B |u|^s w^\alpha dx \right)^{\frac{1}{s}} &= \left(\int_B (|u| w^{\frac{\alpha}{s}})^s dx \right)^{\frac{1}{s}} \\ &\le \left(\int_B |u|^k dx \right)^{\frac{1}{k}} \left(\int_B (w^{\frac{\alpha}{s}})^{\frac{ks}{k-s}} dx \right)^{\frac{k-s}{ks}} \qquad (3.2) \\ &= \|u\|_{k,B} \left(\int_B w \, dx \right)^{\frac{\alpha}{s}} \end{aligned}$$

for all balls B with $\sigma B \subset \Omega$. Let $m = \frac{st}{s+\alpha t(r-1)}$. By Theorem D we obtain

$$\|u\|_{k,B} \le C_1 |B|^{\frac{m-k}{km}} \|u\|_{m,\sigma B}. \tag{3.3}$$

Using the Hölder inequality with $\frac{1}{m} = \frac{1}{t} + \frac{t-m}{mt}$ yields

$$\begin{aligned} \|u\|_{m,\sigma B} &= \left(\int_{\sigma B} (|u| w^{\frac{\alpha}{s}} w^{-\frac{\alpha}{s}})^m dx\right)^{\frac{1}{m}} \\ &\le \left(\int_{\sigma B} |u|^t w^{\frac{\alpha t}{s}} dx\right)^{\frac{1}{t}} \left(\int_{\sigma B} \left(\frac{1}{w}\right)^{\frac{\alpha m t}{s(t-m)}} dx\right)^{\frac{t-m}{mt}} \\ &= \left(\int_{\sigma B} |u|^t w^{\frac{\alpha t}{s}} dx\right)^{\frac{1}{t}} \left(\int_{\sigma B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{\frac{\alpha(r-1)}{s}}. \end{aligned} \tag{3.4}$$

Combining (3.2) - (3.4) we find that

$$\begin{aligned} &\left(\int_B |u|^s w^\alpha dx\right)^{\frac{1}{s}} \\ &\le C_1 |B|^{\frac{m-k}{km}} \left(\int_B w\, dx\right)^{\frac{\alpha}{s}} \left(\int_{\sigma B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{\frac{\alpha(r-1)}{s}} \left(\int_{\sigma B} |u|^t w^{\frac{\alpha t}{s}} dx\right)^{\frac{1}{t}}. \end{aligned} \tag{3.5}$$

Since $w \in A_r(\Omega)$, then we have

$$\begin{aligned} &\left(\int_B w\, dx\right)^{\frac{\alpha}{s}} \left(\int_{\sigma B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{\frac{\alpha(r-1)}{s}} \\ &= \left(\left(\int_B w\, dx\right) \left(\int_{\sigma B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{r-1}\right)^{\frac{\alpha}{s}} \\ &\le |\sigma B|^{\frac{\alpha r}{s}} \left(\left(\frac{1}{|\sigma B|}\int_B w\, dx\right) \left(\frac{1}{|\sigma B|}\int_{\sigma B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{r-1}\right)^{\frac{\alpha}{s}} \\ &\le C_2 |\sigma B|^{\frac{\alpha r}{s}} \\ &= C_3 |B|^{\frac{\alpha r}{s}}. \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5) we obtain

$$\left(\int_B |u|^s w^\alpha dx\right)^{\frac{1}{s}} \le C_4 |B|^{\frac{t-s}{st}} \left(\int_{\sigma B} |u|^t w^{\frac{\alpha t}{s}} dx\right)^{\frac{1}{t}}.$$

Then (3.1) holds if $0 < \alpha < 1$.

For the case $\alpha = 1$, by Lemma 2.3, there exist constants $\beta > 1$ and $C_5 > 0$ such that

$$\|w\|_{\beta,B} \le C_5 |B|^{\frac{1-\beta}{\beta}} \|w\|_{1,B} \tag{3.7}$$

for any cube or any ball $B \subset \mathbb{R}^n$. Choose $k = \frac{s\beta}{\beta-1}$. Then $s < k$ and $\beta = \frac{k}{k-s}$. By (3.7) and Lemma 2.2 we have

$$\begin{aligned}\left(\int_B |u|^s w\,dx\right)^{\frac{1}{s}} &\le \left(\int_B |u|^k dx\right)^{\frac{1}{k}} \left(\int_B (w^{\frac{1}{s}})^{\frac{sk}{k-s}} dx\right)^{\frac{k-s}{sk}} \\ &= \|u\|_{k,B} \|w\|_{\beta,B}^{\frac{1}{s}} \\ &\le C_6 |B|^{\frac{1-\beta}{\beta s}} \|w\|_{1,B}^{\frac{1}{s}} \|u\|_{k,B} \\ &= C_6 |B|^{-\frac{1}{k}} \|w\|_{1,B}^{\frac{1}{s}} \|u\|_{k,B}.\end{aligned} \tag{3.8}$$

Selecting $m = \frac{st}{s+t(r-1)}$ and repeating the same procedure as the case $0 < \alpha < 1$, we see that (3.1) is also true for $\alpha = 1$. This ends the proof of Theorem 3.1 ∎

As application of Theorem 3.1, we choose the parameter $\alpha = 1$ in Theorem 3.1. Then, we have the following version of the reverse Hölder inequality.

Corollary 3.2. *Let $u \in D'(\Omega, \wedge^l)$ $(l = 0, 1, \ldots, n)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$, $\sigma > 1$. Assume that $0 < s, t < \infty$ and $w \in A_r(\Omega)$ for some $r > 1$. Then there exists a constant $C > 0$, independent of u, such that*

$$\left(\frac{1}{|B|}\int_B |u|^s w\,dx\right)^{\frac{1}{s}} \le C \left(\frac{1}{|B|}\int_{\sigma B} |u|^t w^{\frac{t}{s}} dx\right)^{\frac{1}{t}}$$

for all balls B with $\sigma B \subset \Omega$.

Let $\alpha = s$ with $0 < s \le 1$ in Theorem 3.1. We obtain the following symmetric version.

Corollary 3.3. *Let $u \in D'(\Omega, \wedge^l)$ $(l = 0, 1, \ldots, n)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$, $\sigma > 1$. Assume that $0 < t < \infty$, $0 < s \le 1$ and $w \in A_r(\Omega)$ for some $r > 1$. Then there exists a constant $C > 0$, independent of u, such that*

$$\left(\frac{1}{|B|}\int_B |u|^s w^s dx\right)^{\frac{1}{s}} \le C \left(\frac{1}{|B|}\int_{\sigma B} |u|^t w^t dx\right)^{\frac{1}{t}}$$

for all balls B with $\sigma B \subset \Omega$.

Let $\alpha = \frac{1}{t}$ with $t \ge 1$ in Theorem 3.1. Then we have the following

Corollary 3.4. *Let $u \in D'(\Omega, \wedge^l)$ $(l = 0, 1, \ldots, n)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$, $\sigma > 1$. Assume that $t \ge 1$, $0 < s < \infty$ and $w \in A_r(\Omega)$ for some $r > 1$. Then there exists a constant $C > 0$, independent of u, such that*

$$\left(\frac{1}{|B|}\int_B |u|^s w^{\frac{1}{t}} dx\right)^{\frac{1}{s}} \le C \left(\frac{1}{|B|}\int_{\sigma B} |u|^t w^{\frac{1}{s}} dx\right)^{\frac{1}{t}}$$

for all balls B with $\sigma B \subset \Omega$.

We prove the following global result.

Theorem 3.5. *Let $u \in D'(\Omega, \wedge^l)$ $(l = 0, 1, \ldots, n)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$. Assume that $0 < s \le t < \infty$ and $w \in A_r(\Omega)$ for some $r > 1$. Then*

$$\left(\frac{1}{|\Omega|}\int_\Omega |u|^s w^\alpha dx\right)^{\frac{1}{s}} \le \left(\frac{1}{|\Omega|}\int_\Omega |u|^t w^{\frac{\alpha t}{s}} dx\right)^{\frac{1}{t}} \tag{3.9}$$

for any real number α with $0 < \alpha \le 1$.

Proof. It is clear that (3.9) is true if $s = t$. Now we assume that $s < t$. Using Lemma 2.2 with $\frac{1}{s} = \frac{1}{t} + \frac{t-s}{st}$, we have

$$\begin{aligned}\left(\int_\Omega |u|^s w^\alpha dx\right)^{\frac{1}{s}} &= \left(\int_\Omega \left(|u| w^{\alpha/s}\right)^s dx\right)^{\frac{1}{s}} \\ &\le \left(\int_\Omega 1\, dx\right)^{\frac{t-s}{st}} \left(\int_\Omega (|u| w^{\frac{\alpha}{s}})^t dx\right)^{\frac{1}{t}} \\ &= |\Omega|^{\frac{t-s}{st}} \left(\int_\Omega |u|^t w^{\frac{\alpha t}{s}} dx\right)^{\frac{1}{t}}\end{aligned}$$

which is equivalent to (3.9). The proof of Theorem 3.5 is completed ∎

Remark. Theorem 3.5 can be proved by using Theorem 3.1 directly (see [11: Proof of Theorem 2.3]). Here we have the stronger condition $0 < s \le t < \infty$. But the result is also stronger: the constant C in Theorem 3.1 now reduces to $C = 1$. By choosing α to be some special values in (3.9), we have some global results as we did for the local case.

4. The $A_r(\Omega)$-weighted Caccioppoli-type estimate

We prove the following $A_r(\Omega)$-weighted Caccioppoli-type estimate with parameter α for A-harmonic tensors.

Theorem 4.1. *Let $u \in D'(\Omega, \wedge^l)$ $(l = 0, 1, \ldots, n)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation and $w \in A_r(\Omega)$ for some $r > 1$. Then there exists a constant $C > 0$, independent of u, such that*

$$\left(\int_B |du|^s w^\alpha dx\right)^{\frac{1}{s}} \le \frac{C}{\operatorname{diam}(B)} \left(\int_{\rho B} |u - c|^s w^\alpha dx\right)^{\frac{1}{s}} \tag{4.1}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c. Here α is any constant with $0 < \alpha \le 1$.

Proof. First, we assume that $0 < \alpha < 1$. Choose $t = \frac{s}{1-\alpha}$. Since $\frac{1}{s} = \frac{1}{t} + \frac{t-s}{st}$, using Lemma 2.2 and Theorem C, we obtain

$$\begin{aligned}\left(\int_B |du|^s w^\alpha dx\right)^{\frac{1}{s}} &= \left(\int_B (|du| w^{\frac{\alpha}{s}})^s dx\right)^{\frac{1}{s}} \\ &\le \left(\int_B |du|^t dx\right)^{\frac{1}{t}} \left(\int_B (w^{\frac{\alpha}{s}})^{\frac{st}{t-s}} dx\right)^{\frac{t-s}{st}} \\ &\le \|du\|_{t,B} \left(\int_B w\, dx\right)^{\frac{\alpha}{s}} \\ &= C_1 \mathrm{diam}(B)^{-1} \|u-c\|_{t,\sigma B} \left(\int_B w\, dx\right)^{\frac{\alpha}{s}}\end{aligned} \tag{4.2}$$

for all balls B with $\sigma B \subset \Omega$ and all closed forms c. Since c is a closed form and u is an A-harmonic tensor, then $u - c$ is still an A-harmonic tensor. Taking $m = \frac{s}{1+\alpha(r-1)}$, then $m < s < t$. By Theorem D we have

$$\|u-c\|_{t,\sigma B} \le C_2 |B|^{\frac{m-t}{mt}} \|u-c\|_{m,\sigma^2 B} = C_2 |B|^{\frac{m-t}{mt}} \|u-c\|_{m,\rho B} \tag{4.3}$$

where $\rho = \sigma^2$. Substituting (4.3) into (4.2) we get

$$\left(\int_B |du|^s w^\alpha dx\right)^{\frac{1}{s}} \le C_3 \mathrm{diam}(B)^{-1} |B|^{\frac{m-t}{mt}} \|u-c\|_{m,\rho B} \left(\int_B w\, dx\right)^{\frac{\alpha}{s}}. \tag{4.4}$$

Using Lemma 2.2 with $\frac{1}{m} = \frac{1}{s} + \frac{s-m}{sm}$ we obtain

$$\begin{aligned}\|u-c\|_{m,\rho B} &= \left(\int_{\rho B} |u-c|^m dx\right)^{\frac{1}{m}} \\ &= \left(\int_{\rho B} (|u-c| w^{\frac{\alpha}{s}} w^{-\frac{\alpha}{s}})^m dx\right)^{\frac{1}{m}} \\ &\le \left(\int_{\rho B} |u-c|^s w^\alpha dx\right)^{\frac{1}{s}} \left(\int_{\rho B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{\frac{\alpha(r-1)}{s}}\end{aligned} \tag{4.5}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c. Substituting (4.5) into (4.4) we obtain

$$\begin{aligned}&\left(\int_B |du|^s w^\alpha dx\right)^{\frac{1}{s}} \\ &\quad\le C_3 \mathrm{diam}(B)^{-1} |B|^{\frac{m-t}{mt}} \|w\|_{1,B}^{\frac{\alpha}{s}} \left\|\frac{1}{w}\right\|_{\frac{1}{r-1},\rho B}^{\frac{\alpha}{s}} \left(\int_{\rho B} |u-c|^s w^\alpha dx\right)^{\frac{1}{s}}.\end{aligned} \tag{4.6}$$

Now $w \in A_r(\Omega)$ yields

$$\begin{aligned}\|w\|_{1,B}^{\frac{\alpha}{s}} \left\|\frac{1}{w}\right\|_{\frac{1}{r-1},\rho B}^{\frac{\alpha}{s}} &\le \left(\left(\int_{\rho B} w\, dx\right) \left(\int_{\rho B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{r-1}\right)^{\frac{\alpha}{s}} \\ &= \left(|\rho B|^r \left(\frac{1}{|\rho B|}\int_{\rho B} w\, dx\right) \left(\frac{1}{|\rho B|}\int_{\rho B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{r-1}\right)^{\frac{\alpha}{s}} \\ &\le C_4 |B|^{\frac{\alpha r}{s}}.\end{aligned} \tag{4.7}$$

Combining (4.7) and (4.6) we find that

$$\left(\int_B |du|^s w^\alpha dx\right)^{\frac{1}{s}} \le \frac{C_5}{\operatorname{diam}(B)} \left(\int_{\rho B} |u-c|^s w^\alpha dx\right)^{\frac{1}{s}} \tag{4.8}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c. We have proved that (4.1) is true if $0 < \alpha < 1$.

For the case $\alpha = 1$, by Lemma 2.3 there exist constants $\beta > 1$ and $C_6 > 0$ such that

$$\|w\|_{\beta,B} \le C_6 |B|^{\frac{1-\beta}{\beta}} \|w\|_{1,B} \tag{4.9}$$

for any cube or any ball $B \subset \mathbb{R}^n$. Choose $t = \frac{s\beta}{\beta-1}$. Then $1 < s < t$ and $\beta = \frac{t}{t-s}$. Since $\frac{1}{s} = \frac{1}{t} + \frac{t-s}{st}$, by Lemma 2.2, Theorem C and (4.9) we have

$$\begin{aligned}
\left(\int_B |du|^s w\, dx\right)^{\frac{1}{s}} &= \left(\int_B (|du| w^{\frac{1}{s}})^s dx\right)^{\frac{1}{s}} \\
&\le \left(\int_B |du|^t dx\right)^{\frac{1}{t}} \left(\int_B (w^{\frac{1}{s}})^{\frac{st}{t-s}} dx\right)^{\frac{t-s}{st}} \\
&\le C_7 \|du\|_{t,B} \|w\|_{\beta,B}^{\frac{1}{s}} \\
&\le C_8 \operatorname{diam}(B)^{-1} \|u-c\|_{t,\sigma B} \|w\|_{\beta,B}^{\frac{1}{s}} \\
&\le C_9 \operatorname{diam}(B)^{-1} |B|^{\frac{1-\beta}{\beta s}} \|w\|_{1,B}^{\frac{1}{s}} \|u-c\|_{t,\sigma B} \\
&= C_9 \operatorname{diam}(B)^{-1} |B|^{-\frac{1}{t}} \|w\|_{1,B}^{\frac{1}{s}} \|u-c\|_{t,\sigma B}
\end{aligned}$$

which is similar to (4.2). Now, choosing $m = \frac{s}{r}$ and repeating the same procedure as the case $0 < \alpha < 1$, we can also obtain (4.1) if $\alpha = 1$. This ends the proof of Theorem 4.1 ∎

Note that the parameter α in Theorem 4.1 is any real number with $0 < \alpha \le 1$. Therefore, we can have different versions of the Caccioppoli-type inequality by choosing α to be different values. For example, setting $t = 1 - \alpha$ in Theorem 4.1 we obtain the following result.

Corollary 4.2. *Let $u \in D'(\Omega, \wedge^l)$ $(l = 0, 1, \ldots, n)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation and $w \in A_r(\Omega)$ for some $r > 1$. Then there exists a constant $C > 0$, independent of u, such that*

$$\left(\int_B |du|^s w^{-t} d\mu\right)^{\frac{1}{s}} \le \frac{C}{\operatorname{diam}(B)} \left(\int_{\rho B} |u-c|^s w^{-t} d\mu\right)^{\frac{1}{s}} \tag{4.10}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c. Here t is any real number with $0 \le t < 1$ and $d\mu = w(x)\, dx$.

Choosing $\alpha = \frac{1}{r}$ in Theorem 4.1 we have the following result.

Corollary 4.3. *Let $u \in D'(\Omega, \wedge^l)$ $(l = 0, 1, \ldots, n)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation and $w \in A_r(\Omega)$ for some $r > 1$. Then there exists a constant $C > 0$, independent of u, such that*

$$\left(\int_B |du|^s w^{\frac{1}{r}} dx\right)^{\frac{1}{s}} \le \frac{C}{\mathrm{diam}(B)} \left(\int_{\rho B} |u - c|^s w^{\frac{1}{r}} dx\right)^{\frac{1}{s}} \tag{4.11}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c.

If we choose $\alpha = \frac{1}{s}$ in Theorem 4.1, then $0 < \alpha < 1$ since $1 < s < \infty$. Thus, Theorem 4.1 reduces to the following symmetric version.

Corollary 4.4. *Let $u \in D'(\Omega, \wedge^l)$ $(l = 0, 1, \ldots, n)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation and $w \in A_r(\Omega)$ for some $r > 1$. Then there exists a constant $C > 0$, independent of u, such that*

$$\left(\int_B |du|^s w^{\frac{1}{s}} dx\right)^{\frac{1}{s}} \le \frac{C}{\mathrm{diam}(B)} \left(\int_{\rho B} |u - c|^s w^{\frac{1}{s}} dx\right)^{\frac{1}{s}} \tag{4.12}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c.

If we choose $\alpha = 1$ in Theorem 4.1, we have the following result.

Corollary 4.5. *Let $u \in D'(\Omega, \wedge^l)$ $(l = 0, 1, \ldots, n)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation and $w \in A_r(\Omega)$ for some $r > 1$. Then there exists a constant $C > 0$, independent of u, such that*

$$\|du\|_{s,B,w} \le C\mathrm{diam}(B)^{-1} \|u - c\|_{s,\rho B,w} \tag{4.13}$$

or

$$\left(\int_B |du|^s d\mu\right)^{\frac{1}{s}} \le \frac{C}{\mathrm{diam}(B)} \left(\int_{\rho B} |u - c|^s d\mu\right)^{\frac{1}{s}} \tag{4.14}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c.

Finally, we prove the following global $A_r(\Omega)$-weighted Caccioppoli-type estimate for A-harmonic tensors.

Theorem 4.6. *Let $u \in D'(\Omega, \wedge^l)$ $(l = 0, 1, \ldots, n)$ be an A-harmonic tensor in a bounded domain $\Omega \subset \mathbb{R}^n$ which has a finite open cover $\mathcal{V} = \{B_1, B_2, \ldots, B_m\}$ consistimg of open balls. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation and $w \in A_r(\cup_i^m B_i)$ for some $r > 1$. Then there exists a constant $C > 0$, independent of u, such that*

$$\left(\int_\Omega |du|^s w^\alpha dx\right)^{\frac{1}{s}} \le \frac{C}{\mathrm{diam}(\Omega)} \left(\int_\Omega |u - c|^s w^\alpha dx\right)^{\frac{1}{s}} \tag{4.15}$$

for all closed forms c *and any constant* α *with* $0 < \alpha \le 1$.

Proof. Let $\mathcal{V} = \{B_1, B_2, \dots, B_m\}$ be an open cover of the bounded domain $\Omega \subset \mathbb{R}^n$ and $d_i = \operatorname{diam}(B_i) > 0 \ \ (i = 1, 2, \dots, m)$. Assume that $d = \min\{d_1, d_2, \dots, d_m\}$. Since Ω is bounded, then there exists a constant $C_1 > 0$ such that

$$\frac{1}{d} \le \frac{C_1}{\operatorname{diam}(\Omega)}. \tag{4.16}$$

Using (4.16) and Theorem 4.1, we have

$$\begin{aligned}\left(\int_\Omega |du|^s w^\alpha dx\right)^{\frac{1}{s}} &\le \sum_{B\in\mathcal{V}} \left(\int_B |du|^s w^\alpha dx\right)^{\frac{1}{s}} \\ &\le \sum_{B\in\mathcal{V}} \frac{C_2}{\operatorname{diam}(B)} \left(\int_{\rho B} |u-c|^s w^\alpha dx\right)^{\frac{1}{s}} \\ &\le \sum_{B\in\mathcal{V}} \frac{C_2}{d} \left(\int_{\rho B} |u-c|^s w^\alpha dx\right)^{\frac{1}{s}} \\ &\le \sum_{B\in\mathcal{V}} \frac{C_3}{\operatorname{diam}(\Omega)} \left(\int_{\Omega} |u-c|^s w^\alpha dx\right)^{\frac{1}{s}} \\ &\le \frac{C_4}{\operatorname{diam}(\Omega)} \left(\int_{\Omega} |u-c|^s w^\alpha dx\right)^{\frac{1}{s}}.\end{aligned}$$

Hence (4.15) follows. The proof of Theorem 4.6 has been completed ∎

Remark. Choosing α to be some special values in (4.15), we shall have some corresponding global results.

Acknowledgements. I would like to thank the referee and Prof. R. Finn for their precious and thoughtful suggestions on this paper.

References

[1] Ball, J. M.: *Convexity conditions and existence theorems in nonlinear elasticity]* Arch. Rat. Mech. Anal. 63 (1977), 337 – 403.

[2] Ball, J. M. and F. Murat: *$W^{1,p}$-quasi-convexity and variational problems for multiple integrals.* J. Funct. Anal. 58 (1984), 225 – 253.

[3] Ding, S.: *Some examples of conjugate p-harmonic differential forms.* J. Math. Anal. Appl. 227 (1998), 251 – 270.

[4] Ding, S. and C. A. Nolder: *$L^s(\mu)$-averaging domains.* J. Math. Anal. Appl. (to appear).

[5] Garnett, J. B.: *Bounded Analytic Functions.* New York: Acad. Press 1970.

[6] Hardy, G. H. and J. E. Littlewood: *Some properties of conjugate functions.* J. Reine Angew. Math. 167 (1932), 405 – 423.

[7] Heinonen, J., Kilpelainen, K. and O. Martio: *Nonlinear Potential Theory of Degenerate Elliptic Equations.* Oxford: Clarendon Press 1993.

[8] Iwaniec, T.: *p-harmonic tensors and quasiregular mappings.* Ann. Math. 136 (1992), 589 – 624.

[9] Iwaniec, T. and A. Lutoborski: *Integral estimates for null Lagrangians.* Arch. Rat. Mech. Anal. 125 (1993), 25 – 79.

[10] Iwaniec, T. and G. Martin: *Quasiregular mappings in even dimensions.* Acta Math. 170 (1993), 29 – 81.

[11] Liu, B. and S. Ding: *The monotonic property of the $L^s(\mu)$-averaging domains.* J. Math. Anal. Appl. 237 (1999), 730 – 739.

[12] Nolder, C. A.: *Hardy-Littlewood theorems for A-harmonic tensors.* Illinois J. Math. 43 (1999), 613 – 631.

[13] De Rham, G.: *Differential Manifolds.* Berlin: Springer-Verlag 1980.

[14] Stroffolini, B.: *On weakly A-harmonic tensors.* Studia Math. 114 (1995), 289 – 301.

Received 05.10.2000