Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 20 (2001), No. 3, 755–760

Crack Detection in Plane Semilinear Elasticity

D. D. Trong

Abstract. Let Ω be a two-dimensional semilinear elastic body limited by a known outer boundary Γ represented by a Jordan curve and an unknown inner boundary γ represented by a finite disjoint union of piecewise C^1 Jordan curves. Plane stress is considered. We assume that the Lamé coefficient λ depends on the spacial variables x, y and the displacements u, v. Our main result asserts that γ is uniquely determined by the displacements and stresses prescribed on an open portion Γ_0 of Γ.

Keywords: *Crack detection, Lamé coefficient, plane stress, semilinear elastic body*

AMS subject classification: 35R30, 73C02, 73C15

Let Ω be a plane solid body bounded by a known outer boundary Γ and an unknown inner boundary γ, represented by a disjoint union of Jordan curves. The domain bounded by γ can be seen as cracks. If the solid body is electrical conducting, then it has been shown in [5, 8] (the linear case) and in [20, 21] (the semilinear case) that the cracks are uniquely determined by values of the electrical potential and flux described on an open portion Γ_0 of Γ. If the domain Ω described above is a linear elastic body (i.e., the Lamé coefficient λ, μ depends only on the spacial variables x, y) and if the inner boundary γ is a C^1 Jordan curve stress free, then it is shown in [7] that the location and the shape of a crack are uniquely determined by the values of the displacements and stresses specified on an open portion Γ_0 of Γ.

In the present paper, we consider the problem of identifiability of the unknown cracks (assumed to be finite in number) in a semilinear elastic body having the boundary data mentioned at the end of the preceding paragraph. In fact, letting u and v be the displacements and stresses in the $x-$ and $y-$directions, respectively, we shall assume that the Lamé coefficient λ depends on x, y and u, v, i.e.,

$$\lambda = \lambda(x, y, u, v). \tag{1}$$

Letting σ_1, σ_2, τ be the stresses (see [19]), we have the system

$$\left.\begin{aligned} \sigma_{1x} + \tau_y + X = 0 \\ \sigma_{2y} + \tau_x + Y = 0 \end{aligned}\right\} \tag{2}$$

Dang Duc Trong: Hochiminh City Nat. Univ., Dept. Math. & Comp. Sci., 227 Nguyen Van Cu, Q5, Hochiminh City, Vietnam; `ddtrong@mathdep.hcmuns.edu.vn`

ISSN 0232-2064 / $ 2.50

where $\phi_x = \frac{\partial \phi}{\partial x}$ and $\phi_y = \frac{\partial \phi}{\partial y}$. Assuming plane stresses, we have the relations

$$\left.\begin{aligned} \tau &= \tfrac{\mu}{2}(u_y + v_x) \\ \sigma_1 &= \lambda e + \mu u_x \\ \sigma_2 &= \lambda e + \mu v_y \\ e &= u_x + v_y \end{aligned}\right\} \tag{3}$$

where λ, μ are the positive Lamé coefficients. We assume that the displacements and the surface stresses are given on a portion Γ_0 of Γ, i.e.

$$(u, v)|_{\Gamma_0} = (f_1, f_2) \tag{4}$$

and

$$\left.\begin{aligned} \ell\sigma_1 + m\tau &= \overline{X} \\ m\sigma_2 + \ell\tau &= \overline{Y} \end{aligned}\right\} \qquad \text{on } \Gamma_0 \tag{5}$$

where (ℓ, m) is the exterior unit normal to $\partial\Omega$.

Let $\omega_1, ..., \omega_n$ be the unknown internal cracks in Ω. We shall assume that $\partial\omega_i$ $(i = 1, 2, ..., m)$ are Jordan curves piecewise of C^1-type and that

$$\overline{\omega}_i \cap \overline{\omega}_j = \emptyset \qquad \text{for } i \neq j. \tag{6}$$

The set $\gamma = \bigcup_{i=1}^{n} \partial\omega_i$ is the inner boundary of Ω. On γ we assume that the surface stresses vanish except at a finite set of points $\{y_1, ..., y_k\}$ in γ, i.e. on $\gamma_* = \gamma\backslash\{y_1, ..., y_k\}$,

$$\left.\begin{aligned} \ell\sigma_1 + m\tau &= 0 \\ m\sigma_2 + \ell\tau &= 0 \end{aligned}\right\}. \tag{7}$$

One has

Theorem. *Let* (3) – (6) *hold. If* $X = Y = 0$ *and* $(\overline{X}, \overline{Y}) \not\equiv (0, 0)$, *then system* (2) *subject to conditions* (4) – (7) *has at most one solution* $(\Omega, (u, v))$ *with* u, v *in* $C^3(\Omega \cup \Gamma_0) \cap H^1(\Omega)$ *and* $\Gamma, \partial\omega_i$ $(i = 1, ..., n)$ *are piecewise of* C^1*-type.*

This result is, to our knowledge, new. The key of the proof is the unique continuation for the Lamé system (see [9, 11, 22]). We also refer to the book [10] and the paper of Andrieux, Abda and Bui [4] dealing with the problem of rectilinear or planars crack in elastic bodies from boundary measurements in terms of a functional introduced by the authors. In the present paper, we only consider open simply connected cracks. The case of infinitely thin cracks will be the object of a future study. We refer to the papers [1 - 3, 12, 14 - 16] studying the problem of detection of infinitely thin cracks for elliptic equations.

We now turn to the

Proof of Theorem. Let $(\Omega^1, (u^1, v^1))$ and $(\Omega^2, (u^2, v^2))$ satisfy (2), (4) - (7). Let γ^1 and γ^2 be the inner boundaries of Ω^1 and Ω^2, respectively. By assumptions, γ^i $(i = 1, 2)$ is C^1-smooth except at a finite set of points $\{y_1^i, ..., y_{k_i}^i\}$ in γ^i. Suppose by contradiction that $\Omega^1 \neq \Omega^2$. Without loss of generality, we assume that $\Omega^1 \setminus \overline{\Omega}^2 \neq \emptyset$. Denote by W the connected component of $\Omega^1 \cap \Omega^2$ such that $\Gamma \subset \partial W$. One has the following lemma (which will be proved later) related to the uniqueness of solutions of system (2) satisfying conditions (4) - (7).

Lemma 1. *Let Γ_0 be C^1-smooth, let λ and μ be in $C^2(\mathbb{R}^2 \times \mathbb{R}^2)$ and $C^2(\mathbb{R}^2)$, respectively. Then*

$$(u^1, v^1) = (u^2, v^2) \qquad \text{on } W. \tag{8}$$

Using the results of [21], we can find an open subset $U_0 \subset \Omega^1 \setminus \overline{\Omega}^2$ such that $U_0 \neq \emptyset$, ∂U_0 is piecewise of C^1-type and

$$\partial U_0 \subset (\partial W \setminus \Gamma) \cup \gamma^1. \tag{9}$$

Let B_1 be a finite set of points such that $\left\{y_1^1, ..., y_{k_1}^1, y_1^2, ..., y_{k_2}^2\right\} \subset B_1$ and $\partial U_0 \setminus B_1$ is a finite union of open C^1-curves. From (9), for $z \in \partial U_0 \setminus B_1$, one has to consider two cases

(i) $z \in \gamma^1 \setminus B_1$

(ii) $z \in \partial W \cap \gamma^2 \setminus B_1$

(note that $\partial W \subset \partial\Omega^1 \cup \partial\Omega^2 = \Gamma \cup \gamma^1 \cup \gamma^2$). If (i) holds, then (7) holds for σ_1, τ replaced by σ^1, τ^1 where $\sigma_1^i, \sigma_2^i, \tau^i \ \ (i = 1, 2)$ can be calculated from u^i, v^i by (3). In the case (ii), (8) gives in view of (3)

$$\left.\begin{aligned} &\sigma_j^1(z) = \sigma_j^2(z) \quad (j = 1, 2) \\ &\tau^1(z) = \tau^2(z). \end{aligned}\right\} \tag{10}$$

Since $z \in \partial W \cap \gamma^2 \setminus B_1 \subset \gamma^2 \setminus \{y_1^2, ..., y_{k_2}^2\}$, relations (7) imply

$$\left.\begin{aligned} \ell(z)\sigma_1^2(z) + m(z)\tau^2(z) = 0 \\ m(z)\sigma_2^2(z) + \ell(z)\tau^2(z) = 0. \end{aligned}\right\}$$

From (10), the latter equalities implies that (7) holds for σ_1, σ_2 and τ replaced by σ_1^1, σ_2^1 and τ^1, respectively. This gives, for $z \in \partial U_0 \setminus B_1$,

$$\left.\begin{aligned} \ell(z)\sigma_1^1(z) + m(z)\tau^1(z) = 0 \\ m(z)\sigma_2^1(z) + \ell(z)\tau^1(z) = 0. \end{aligned}\right\} \tag{11}$$

Multiplying (1) (corresponding to (u^1, v^1)) by u^1, intergrating over U_0 and applying $(11)_1$ we get

$$\int_{U_0} (\sigma_1^1 u_x^1 + \tau^1 u_y^1)\, dxdy = 0. \tag{12}$$

Similarly, multiplying $(2)_1$ (in (u^1, v^1)) by v^1, integrating over U_0 and applying $(17)_1$ give

$$\int_{U_0} (\sigma_2^1 v_y^1 + \tau^1 v_x^1)\, dxdy = 0. \tag{13}$$

Adding together (2) and (3) and using $(3)_{2-3}$, we get after some rearrangements

$$\int_{U_0} \Big(\lambda(e^1)^2 + \mu((u_x^1)^2 + (v_y^1)^2) + \tfrac{\mu}{2}(u_y^1 + v_x^1)^2\Big) dxdy = 0$$

where $e^1 = u^1_x + v^1_y$. This gives $u^1_x = v^1_y = u^1_y + v^1_x = 0$ in U_0. Letting $\tilde{U}$ be an open ball in U_0, we can show by elementary techniques that, for $(x,y) \in \tilde{U}$,

$$\left.\begin{aligned} u^1(x,y) &= cy + d \\ v^1(x,y) &= -cx + d' \end{aligned}\right\}$$

where c, d, d' are constants. Put

$$\tilde{u} = u^1 - cy - d \qquad \tilde{v} = v^1 + cx - d' \qquad \tilde{e} = u^1_x + v^1_y$$
$$\tilde{\sigma}_1 = \lambda\tilde{e} + \mu\tilde{u}_x \qquad \tilde{\sigma}_2 = \lambda\tilde{e} + \mu\tilde{v}_y \qquad \tilde{\tau} = \tfrac{\mu}{2}(\tilde{u}_y + \tilde{v}_x).$$

Then $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\tau}$ satisfy system (1) with $X = Y = 0$. Moreover, $\tilde{u} = \tilde{v} = \tilde{\sigma}_1 = \tilde{\sigma}_2 = 0$ in $\tilde{U}$. Hence, using Lemma, one gets $\tilde{\sigma}_1 = \tilde{\sigma}_2 = \tilde{\tau} = 0$ in Ω^1. It follows that $(\overline{X}, \overline{Y}) = (0,0)$ on Γ_0, which is a contradiction. The proof of the theorem completes once Lemma is proved.

In [22], the 3-dimensional case of Lemma was proved. The proof given in there is carried almost verbatim to the one of the 2-dimensional case. Hence, we only give here an

Outline of the proof of Lemma. By direct computation, one has for $i = 1, 2$

$$\left.\begin{aligned} \mu\Delta u^i + F^i &= 0 \\ \mu\Delta v^i + G^i &= 0 \end{aligned}\right\} \tag{14}$$

where

$$\begin{aligned} F^i &= (2\lambda^i e^i)_x + \mu_x e^i + \mu e^i_x + \mu_x u^i_x + \mu_y u^i_y + \mu_y v^i_x - \mu_x u^i_x + X \\ G^i &= (2\lambda^i e^i)_y + \mu_y e^i + \mu e^i_y + \mu_y v^i_y + \mu_x v^i_x + \mu_x u^i_y - \mu_y v^i_x + Y \\ \lambda^i(x,y) &= \lambda(x, y, u^i(x,y), v^i(x,y)). \end{aligned}$$

Differentiating (14) with respect to x and y, respectively, and adding the results thus obtained, we get

$$2\Delta((\lambda + \mu)e) + H^i = 0 \tag{15}$$

where

$$\begin{aligned} H^i = &-e^i\Delta\mu - 2\mu_x e^i_x - 2\mu_y e^i_y + 2\mu_x \Delta u^i + 2\mu_y \Delta v^i \\ &+ 2(\mu_x e^i_x + \mu_y e^i_y) + 2\mu_{xx} v^i_x + 2\mu_{yy} u^i_y + X_x + Y_y. \end{aligned}$$

Put

$$\begin{aligned} \varphi_1 &= u^1 - u^2 \\ \varphi_2 &= v^1 - v^2 \\ \varphi_3 &= (\lambda^1 + \mu)e^1 - (\lambda^1 + \mu)e^2. \end{aligned}$$

By (14), (15) and the mean value theorem of Lagrange, we can find continuous functions a_{ijk}, b_{ip} in $C(\Omega \cup \Gamma_0)$ $(j,k = 1,2;\ i,p = 1,2,3)$ such that on W

$$\Delta\varphi_i + \sum_{i=1}^{2}(a_{ij1}\varphi_{jx} + a_{ij2}\varphi_{jy}) + \sum_{p=1}^{3} b_{ip}\varphi_p = 0. \tag{16}$$

On the other hand, by direct computation on Γ_0, one has in view of $(5)_1$ and (6) that

$$\varphi_i = \frac{\partial \varphi_i}{\varphi n} = 0 \quad \text{on } \Gamma_0 \qquad (i = 1, 2, 3). \tag{17}$$

Since the principal part of system (16) is the Laplacian, we can use Carleman's estimate (see, e.g., [13, 17]) to prove that the functions φ_i $(i = 1, 2, 3)$ satisfying (16) and (17) vanish on W. This completes the proof of Lemma and the proof of Theorem ∎

Acknowledgment. This work was supported by the Natural Sciences Council of Vietnam.

References

[1] Alessandrini, G.: *Stable determination of a crack from boundary measurements.* Proc. Royal Soc. Edinburgh 123A (1993), 497 – 516.

[2] Alessandrini, G.: *Examples of instability in inverse boundary-value problems.* Inverse Problem 13 (1997), 887 – 897.

[3] Alessandrini, G. and E. DiBenedetto: *Determining 2-dimensional cracks in 3-dimensional body: uniqueness & stability.* Indiana Univ. Math. J. 46 (1987), 1 – 83.

[4] Andrieux, S., BenAbda, A. and H. D. Bui: *Sur l'Identification de fissures planes via le concept d'Écart à la réciprocité en elasticité.* C.R. Acad. Sci. Paris, Série I 324 (1997), 1431 – 1438.

[5] Ang, D. D. and D. D. Trong: *Crack detection by the electric method: uniqueness and approximation.* Int. J. Fracture 93 (1998), 63 – 86.

[6] Ang, D. D., Trong, D. D. and M. Yamamoto: *Unique continuation and identification of boundary of an elastic body.* J. Inv. & Ill-Pos. Probl. 3 (1996), 417 – 428.

[7] Ang, D. D., Trong, D. D. and M. Yamamoto: *Identification of cavities inside two-dimensional heterogeneous isotropic elastic bodies.* J. elasticity 56 (1999), 199 – 212.

[8] Ang, D. D. and L. K. Vy: *Domain identification for harmonic functions.* Acta Appl. Math. 38 (1995), 217 – 238.

[9] Ang, D. D., Ikehata, M., Trong, D. D. and M. Yamamoto: *Unique continuation for a stationary isotropic Lamé system with variable coefficients.* Comm. Part. Diff. Equ. 23 (1997), 371 – 385.

[10] Bui, H. D.: *Inverse Problem in the Mechanic of Materials: an Introduction.* Boca Raton et al.: CRC Press Inc. 1994.

[11] Dehman, B. and L. Robbiano: *La propiété du prolongement unique pour un système elliptique: le système de Lamé.* J. Math. Pures Appl. 72 (1993), 475 – 492.

[12] Friedman, A. and M. Vogelius: *Determining cracks by boundary measurements.* Indiana Univ. Math. J. 38 (1989), 527 – 556.

[13] Hörmander, L.: *Linear Partial Differential Operators.* Berlin: Springer-Verlag 1984.

[14] McIver, M.: *An inverse problem in electro-magnetic crack detection.* IMA J. Appl. Math. 47 (1991), 127 – 141.

[15] Kim, H. and J. K. Seo: *Uniqueness of determination of a collate of finite cracks from two boundary measurements.* SIAM J. Math. Analysis 27 (1996), 1336 – 1340.

[16] Kubo, S.: *Requirements for uniqueness crack identification from electric potential distribution.* In: Inverse Problems in Engineering Sciences (eds.: M. Yamaguchi et al.). Tokyo et al.: Springer-Verlag 1990, pp. 52 – 58.

[17] Nirenberg, L.: *Uniqueness in Cauchy problems for differential equations with constant leading coefficients.* Comm. Pure Appl. Math. 10 (1957), 89 – 105.

[18] Santosa, F. and M. Vogelius: *A complete algorithm to determine cracks from electrical body measurements.* J. Eng. Sci. 29 (1991), 913 – 937.

[19] Timosenko, S. P. and J. N. Goodier: *Theory of Elasticity.* New York: Mc. Graw-Hill 1970.

[20] Trong, D. D.: *Domain identification for a nonlinear elliptic equation.* Z. Anal. Anw. 17 (1998), 1021 – 1024.

[21] Trong, D. D. and D. D. Ang: *Domain identification for semilinear elliptic equations in the plane: the zero flux case.* Z. Anal. Anw. 19 (2000), 109 – 120.

[22] Trong, D. D. and N. V. Huy: *Uniqueness of elastic continuation in a semilinear elastic body.* ZAMM (submitted).

Received 14.08.2000; in revised form 19.03.2001