Orthogonality and Completeness
of q-Fourier Type Systems

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Abstract. We establish orthogonality and completeness of the system of q-exponential functions $\{\mathcal{E}_q(\,\cdot\,; i\omega_n)\}$ using orthogonality and dual orthogonality of a q-analogue of Lommel polynomials. We also set up a very general procedure by which one can produce similar orthogonal systems using bilinear generating functions formed by products of two complete orthogonal function systems.

Keywords: *Continuous q-ultraspherical polynomials, q-exponential functions, orthogonality, completeness, discrete orthogonal polynomials, dual orthogonality*

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1. Introduction

This paper deals with q-Fourier series and their generalizations. One reason for writing this paper is to make the material accessible to some one with little knowledge about q-series. In many instances the technical use of transformations of basic hypergeometric series exemplified in [6] is replaced by simple analytic arguments or more elementary arguments making this paper easy to follow. The only identity we need is Euler's theorem (1.1) below and it will be used only to identify the constants in a specific example.

Throughout this paper we will always assume $|q| < 1$. We follow the notation in Gasper and Rahman [9] or in Andrews, Askey and Roy [2] for q-shifted factorials. Euler's theorem is

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} z^n}{(q;q)_n} = (-z;q)_\infty. \tag{1.1}$$

The q-exponential function alluded to in the abstract is

$$\mathcal{E}_q(\cos\theta, \cos\phi; \alpha)$$
$$= \frac{(\alpha^2; q^2)_\infty}{(q\alpha^2; q^2)_\infty} \sum_{n=0}^{\infty} \big(-e^{i(\phi+\theta)} q^{\frac{1-n}{2}}, -e^{i(\phi-\theta)} q^{\frac{1-n}{2}}; q \big)_n \frac{(\alpha e^{-i\phi})^n}{(q;q)_n} q^{\frac{n^2}{4}}.$$

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Note that θ herein is not necessarily real. In fact, by $x = \cos\theta$ we mean $e^{\pm i\theta} = x \pm \sqrt{x^2 - 1}$ where the $\pm$ sign is chosen to make $\sqrt{x^2 - 1} \approx x$ as $x \to \infty$. If we let $\mathcal{E}_q(x; \alpha) = \mathcal{E}_q(x, 0; \alpha)$, then $\mathcal{E}_q(0; \alpha) = 1$ and $\lim_{q \to 1} \mathcal{E}_q(x; (1-q)\frac{\alpha}{2}) = \exp(\alpha x)$. The notation for $\mathcal{E}_q$ adopted here is different from the original notation in [14].

In their original paper [14] where Ismail and Zhang studied the $\mathcal{E}_q$ function they also introduced q-analogues of the sine and cosine functions and used transformation formulas to analytically continue them to entire functions. Bustoz and Suslov [6] derived several interesting properties of $\mathcal{E}_q$ and the q-sine and q-cosine functions including the important orthogonality property of the $\mathcal{E}_q$ functions. This paved the way towards a comprehensive study of q-Fourier series, where q-analogues of some results in classical Fourier series have been proved but many more are under investigation.

One purpose of this paper is to give a new proof of the orthogonality property of the q-exponential functions. We also give a new and very short proof of their completeness in a certain weighted L^2 space. Furthermore, we show that the orthogonality and completeness of the q-exponentials is just one instance of a very general structure which remains to be explored.

Ismail and Zhang [14] established the q-plane wave expansion

$$
\begin{aligned}
&\mathcal{E}_q\left(x; i\tfrac{\alpha}{2}\right) \\
&= \frac{\left(\tfrac{2}{\alpha}\right)^\nu (q;q)_\infty}{\left(-q\tfrac{\alpha^2}{4}; q^2\right)_\infty (q^{\nu+1}; q)_\infty} \sum_{n=0}^{\infty} \frac{(1 - q^{n+\nu})}{(1 - q^\nu)} q^{\frac{n^2}{4}} i^n J_{\nu+n}^{(2)}(\alpha; q)\, C_n(x; q^\nu|q)
\end{aligned}
\tag{1.2}
$$

where $C_n(\cdot; \beta|q)$ are the continuous q-ultraspherical polynomials [3, 9] and

$$
J_\nu^{(2)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (\tfrac{z}{2})^{\nu+2n}}{(q, q^{\nu+1}; q)_n} q^{n(\nu+n)}
\tag{1.3}
$$

is one of Jackson's three q-Bessel functions. We note that the series here converges for all z and that the poles in q^ν of the series are removed by the zeros of the infinite product $(q^{\nu+1}; q)_\infty$. This notation for q-Bessel functions is from our work [11] and was adopted by Gasper and Rahman [9] but is different from Jackson's original notation. Formula (1.2) turned out to be very useful in deriving addition theorems for q-exponential functions [13] and in mathematical physics [8].

In Section 2 we state few facts about the q-analogue of Lommel polynomials we introduced in [11]. In Section 3 we use a special case of (1.2) and recognize orthogonality of q-exponentials as nothing but the dual orthogonality for the q-analogue of Lommel polynomials. In this section we also give a proof of the completeness of the q-exponential functions $\mathcal{E}_q(\cdot; \alpha_n)$ where $\{\alpha_n\}$ is a certain sequence related to the zeros of $J_{\frac{1}{2}}^{(2)}$. In Section 4 we show how the system of q-exponential functions used in the so-called q-Fourier series is just one example of a general class of functions which are orthogonal and complete in certain weighted L^2 spaces. The idea is to consider functions of the form

$$
\mathcal{F}(x; x_k) = \sum_{n=0}^{\infty} \zeta_n\, r_n(x_k)\, p_n(x)
\tag{1.4}
$$

where $\{p_n\}$ is an complete orthonormal system and $\{r_n\}$ is a complete discrete orthonormal system. Let the orthogonality relations of $\{p_n\}$ and $\{r_n\}$ be

$$\left.\begin{array}{c} \displaystyle\int_a^b p_m(x)p_n(x)w(x)\,dx = \delta_{m,n} \\[2mm] \displaystyle\sum_{k=1}^{\infty} \rho(x_k)\,r_m(x_k)r_n(x_k) = \delta_{m,n} \end{array}\right\} \tag{1.5}$$

respectively. In (1.4) it is assumed that $\{\zeta_n\}$ is a sequence of points lying on the unit circle. We also assume that the Hamburger moment problems associated with both the p_n's and the r_n's are determinate, that is $\{p_n\}$ and $\{r_n\}$ are orthogonal with respect to unique positive measures. In this generality we prove in Section 4 that $\{\mathcal{F}(\cdot; x_k)\}$ is a complete orthogonal system in $L^2[a, b; w]$. The interval $[a, b]$ may or may not be bounded. In the case of the functions $\mathcal{E}(\cdot; \alpha)$ the special values of α used make the q-Bessel functions in (1.2) equal to certain multiples of q-Lommel polynomials evaluated at the points supporting ρ masses. Note that this more general class of functions (1.4) is reminiscent of the kernels in the theory of reproducing kernel spaces.

The continuous q-ultraspherical polynomials are generated (see [3]) by

$$\left.\begin{array}{c} C_0(x; \beta|q) = 1 \\[2mm] C_1(x; \beta|q) = 2x\frac{1-\beta}{1-q} \\[2mm] 2x(1 - \beta q^n)C_n(x; \beta|q) = (1 - q^{n+1})\,C_{n+1}(x; \beta|q) + (1 - \beta^2 q^{n-1})\,C_{n-1}(x; \beta|q) \end{array}\right\}.$$

Their orthogonality relation is (see [3] and [9: Formula (7.4.15)])

$$\int_{-1}^{1} C_m(x; \beta|q)\,C_n(x; \beta|q)\,w(x; \beta|q)\,dx = \frac{2\pi(\beta, q\beta; q)_\infty}{(q, \beta^2; q)_\infty}\,\frac{(1-\beta)(\beta^2; q)_n}{(1 - \beta q^n)(q; q)_n}\,\delta_{m,n} \tag{1.6}$$

where the weight function is defined by

$$w(\cos\theta; \beta|q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\sin\theta\,(\beta e^{2i\theta}, \beta e^{-2i\theta}; q)_\infty} \qquad (0 < \theta < \pi).$$

2. Dual orthogonality and *q*-Lommel polynomials

Before introducing the q-Lommel polynomials we remind the reader of the concept of dual orthogonality. Let $\{p_n\}$ be a system of discrete orthogonal polynomials and let their orthogonality relation be

$$\sum_{k=0}^{\infty} w(x_k)p_m(x_k)p_n(x_k) = \tfrac{1}{\pi_n}\,\delta_{m,n}.$$

It is known that such orthogonal polynomials may also satisfy the dual orthogonality relation

$$\sum_{n=0}^{\infty} \pi_n p_n(x_j)p_n(x_k) = \tfrac{1}{w(x_k)}\,\delta_{k,j}.$$

We now give a precise statement and proof of this dual orthogonality relation.

Theorem 2.1. *Let $\{p_n\}$ be a sequence of orthonormal polynomials, with measure of orthogonality μ, which satisfy a three term recurrence relation*

$$x p_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x). \tag{2.1}$$

Assume that the sequence $\{a_n\}$ is bounded and that the corresponding moment problem is determinate [1]. *If the orthogonality measure μ has point masses at α and β (β may be equal α), then*

$$\sum_{n=0}^{\infty} p_n(\alpha)\, p_n(\beta) = \tfrac{1}{\mu(\{\alpha\})}\, \delta_{\alpha,\beta}.$$

Proof. For a determinate moment problem the series $\sum_{n=0}^{\infty} p_n^2(x)$ converges if and only if $\mu(\{x\}) > 0$, in which case the series converges to $\frac{1}{\mu(\{x\})}$ (see [1, 15]). This settles the case $\alpha = \beta$. If $\alpha \neq \beta$, the Christoffel-Darboux formula (see [17: Subsection 3.2]) implies

$$\sum_{n=0}^{\infty} p_n(\alpha)\, p_n(\beta) = \lim_{m \to \infty} a_m \frac{p_{m+1}(\alpha)\, p_m(\beta) - p_{m+1}(\beta)\, p_m(\alpha)}{\alpha - \beta}$$

which implies the result since $p_n(x) \to 0$ as $n \to \infty$ and $\{a_n\}$ is bounded ∎

Theorem 2.1 is very likely to be known, but we have been unable to find it stated explicitly in the standard sources. Essentially, the proof uses the orthogonality of eigenvectors of a symmetric operator corresponding to different eigenvalues und uses the moment problem to find the normalization. Observe that if $\{p_n\}$ is a finite family of orthogonal polynomials, then the dual orthogonality is the linear algebra fact that for square matrices A, $AA^T = I$ implies $A^T A = I$. The connection between orthogonality and dual orthogonality for systems of finitely many polynomials is well-explained in Atkinson [5] and is exploited in the construction of the orthogonality measure from a three term recurrence relations of polynomials.

In [11] we proved that the q-Bessel functions satisfy the recurrence relation

$$q^\nu J_{\nu+1}^{(2)}(z;q) = \frac{2(1 - q^\nu)}{z} J_\nu^{(2)}(z;q) - J_{\nu-1}^{(2)}(z;q) \tag{2.2}$$

and observed that this relation implies $J_{\nu+m}^{(2)}$ is expressed in terms of $J_\nu^{(2)}$ and $J_{\nu-1}^{(2)}$ as

$$q^{m\nu + \frac{m(m-1)}{2}} J_{\nu+m}^{(2)}(z;q)$$
$$= h_{m,\nu}(\tfrac{1}{z};q) J_\nu^{(2)}(z;q) - h_{m-1,\nu+1}(\tfrac{1}{z};q) J_{\nu-1}^{(2)}(z;q) \qquad (m \in \mathbb{N}). \tag{2.3}$$

The polynomials $h_{n,\nu}(\cdot\,;q)$ are q-analogues of the Lommel polynomials [18]. They can be generated by

$$2(1 - q^{n+\nu}) x\, h_{n,\nu}(x;q) = h_{n+1,\nu}(x;q) + q^{n+\nu-1} h_{n-1,\nu}(x;q) \tag{2.4}$$

for $n > 0$ and the initial conditions $h_{0,\nu}(x;q) = 1$ and $h_{1,\nu}(x;q) = 2x(1 - q^\nu)$.

In [11] we have proved that the zeros of $z^{-\nu} J_\nu^{(2)}(z; q)$, which are symmetric about $z = 0$, are real and simple for $\nu > -1$. Denote by $\{j_{\nu,k}(q)\}$ the sequence of positive zeros of $J_\nu^{(2)}(\cdot; q)$. In [11] we have established the Mittag-Leffler expansion

$$\sum_{k=1}^{\infty} A_k(\nu+1) \left[\frac{1}{z - j_{\nu,k}(q)} + \frac{1}{z + j_{\nu,k}(q)} \right] = -2 \frac{J_{\nu+1}^{(2)}(z; q)}{J_\nu^{(2)}(z; q)}$$

with coefficients A_n as in

$$\frac{d}{dz} J_\nu^{(2)}(z; q) \Big|_{z=j_{\nu,n}(q)} = -2 \frac{J_{\nu+1}^{(2)}(j_{\nu,n}(q); q)}{A_n(\nu+1)} = 2q^{-\nu} \frac{J_{\nu-1}^{(2)}(j_{\nu,n}(q); q)}{A_n(\nu+1)}. \tag{2.5}$$

The second equality follows from (2.2). With these notations we have the orthogonality relation (see [11])

$$\sum_{k=1}^{\infty} \frac{A_k(\nu+1)}{j_{\nu,k}^2(q)} h_{n,\nu+1}\left(\frac{1}{j_{\nu,k}(q)}; q\right) h_{m,\nu+1}\left(\frac{1}{j_{\nu,k}(q)}; q\right)$$

$$+ \sum_{k=1}^{\infty} \frac{A_k(\nu+1)}{j_{\nu,k}^2(q)} h_{n,\nu+1}\left(-\frac{1}{j_{\nu,k}(q)}; q\right) h_{m,\nu+1}\left(-\frac{1}{j_{\nu,k}(q)}; q\right) = \frac{q^{\nu n + \frac{n(n+1)}{2}}}{1 - q^{n+\nu+1}} \delta_{m,n} \tag{2.6}$$

for $\nu > -1$. This indicates that the polynomials $h_{n,\nu}(\cdot; q)$ are orthogonal with respect to a purely discrete measure with compact support when $\nu > 0$.

For future reference we record the dual orthogonality relation of the *q*-Lommel polynomials. Let for $a = \pm \frac{1}{j_{\nu,k}}$

$$\sum_{n=0}^{\infty} (1 - q^{n+\nu+1}) q^{-n\nu - \frac{n(n+1)}{2}} h_{n,\nu+1}(a; q) h_{n,\nu+1}(b; q) = \frac{j_{\nu,k}^2}{A_k(\nu+1)} \delta_{a,b} \tag{2.7}$$

where $\frac{1}{b}$ is any zero of $z^{-\nu} J_\nu^{(2)}(z; q)$. Formula (2.7) follows from Theorem 2.1 because the three term recurrence relation (2.4) identifies the coefficients in (2.1) as given by

$$\left. \begin{array}{l} b_n = 0 \\ 2a_n = q^{\frac{n+\nu}{2}} \left[(1 - q^{n+\nu})(1 - q^{n+\nu+1}) \right]^{-\frac{1}{2}} \end{array} \right\}.$$

In the next part of this section we reconcile the standard notation used so far with the notation used by Bustoz and Suslov [6] in the case $\nu = \frac{1}{2}$ in order to compare the results. Note that (2.5) can be written in the form

$$\frac{d}{dz} \frac{J_\nu^{(2)}(z; q)}{J_{\nu-1}^{(2)}(z; q)} \Big|_{z=j_{\nu,n}(q)} = \frac{2q^{-\nu}}{A_n(\nu+1)}. \tag{2.8}$$

When $\nu = \pm\frac{1}{2}$, things simplify dramatically due to Euler's theorem (1.1). From (1.3) it follows that $J^{(2)}_{\frac{1}{2}}(z;q)$ is

$$\frac{(q^{\frac{3}{2}};q)_\infty}{(q;q)_\infty} \left(\frac{z}{2}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+\frac{1}{2})}}{(q,q^{\frac{3}{2}};q)_n} \left(\frac{z}{2}\right)^{2n}$$

$$= \frac{(q^{\frac{1}{2}};q)_\infty}{i(q;q)_\infty} \left(\frac{2}{z}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{q^{\frac{2n(2n+1)}{4}}}{(q^{\frac{1}{2}};q^{\frac{1}{2}})_{2n+1}} \left(i\frac{z}{2}\right)^{2n+1}$$

$$= \frac{(q^{\frac{1}{2}};q)_\infty}{2i(q;q)_\infty} \left(\frac{2}{z}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left[\frac{q^{\frac{n(n-1)}{4}}}{(q^{\frac{1}{2}};q^{\frac{1}{2}})_n} \left(i\frac{z}{2}\right)^n - \frac{q^{\frac{n(n-1)}{4}}}{(q^{\frac{1}{2}};q^{\frac{1}{2}})_n} \left(-i\frac{z}{2}\right)^n \right].$$

By Euler's theorem (1.1) we get

$$J^{(2)}_{\frac{1}{2}}(z;q) = \frac{(q^{\frac{1}{2}};q)_\infty}{2i(q;q)_\infty} \left(\frac{2}{z}\right)^{\frac{1}{2}} \left[\left(-i\frac{z}{2};q^{\frac{1}{2}}\right)_\infty - \left(i\frac{z}{2};q^{\frac{1}{2}}\right)_\infty \right]. \tag{2.9}$$

Similarly,

$$J^{(2)}_{-\frac{1}{2}}(z;q) = \frac{(q^{\frac{1}{2}};q)_\infty}{2(q;q)_\infty} \left(\frac{2}{z}\right)^{\frac{1}{2}} \left[\left(-i\frac{z}{2};q^{\frac{1}{2}}\right)_\infty + \left(i\frac{z}{2};q^{\frac{1}{2}}\right)_\infty \right]. \tag{2.10}$$

The observations (2.9) - (2.10) are due to Ismail and Stanton as acknowledged by Bustoz and Suslov just before formulas (5.37) in [6]. Bustoz and Suslov [6] used the notation

$$\left.\begin{array}{l} \omega_0 = 0 \\ \omega_n = \frac{1}{2}j_{\frac{1}{2},n}(q) \quad (n \in \mathbb{N}). \end{array}\right\}$$

Thus

$$\frac{J^{(2)}_{\frac{1}{2}}(z;q)}{J^{(2)}_{-\frac{1}{2}}(z;q)} = i\,\frac{1-g(z)}{1+g(z)} \qquad \text{where } g(z) = \frac{\left(-i\frac{z}{2};q^{\frac{1}{2}}\right)_\infty}{\left(i\frac{z}{2};q^{\frac{1}{2}}\right)_\infty}. \tag{2.11}$$

Hence $J^{(2)}_{\frac{1}{2}}(z;q) = 0$ if and only if $g(z) = 1$. Now (2.8) implies

$$\frac{q^{-\frac{1}{2}}}{A_n(\frac{3}{2})} = \frac{g'(2\omega_n)}{4i} = -i\frac{1}{4}\frac{d}{dz}\ln g(z)\Big|_{z=2\omega_n}. \tag{2.12}$$

3. Orthogonality and completeness

We first establish orthogonality of the *q*-exponentials. Using (1.2), (1.6) and the completeness of the *q*-ultraspherical polynomials in $L^2[-1, 1; w(\,\cdot\,; \beta|q)]$ we find

$$\int_{-1}^{1} \mathcal{E}(x; i\tfrac{\alpha}{2}) \, \overline{\mathcal{E}_q(x; i\tfrac{\beta}{2})} \, w(x; q^\nu|q) \, dx$$

$$= \frac{2\pi(\frac{\alpha\beta}{4})^{-\nu}(q; q)_\infty}{(-q\frac{\alpha^2}{4}, -q\frac{\beta^2}{4}; q^2)_\infty (q^{2\nu}; q)_\infty} \tag{3.1}$$

$$\times \sum_{n=0}^{\infty} (1 - q^{n+\nu}) \, q^{\frac{n^2}{2}} \frac{(q^{2\nu}; q)_n}{(q; q)_n} \, J_{\nu+n}^{(2)}(\alpha; q) \, J_{\nu+n}^{(2)}(\beta; q).$$

If α and β are zeros of $J_\nu^{(2)}(\,\cdot\,; q)$, then the sum in (3.1) starts from $n = 1$. If $\alpha = 0$ or $\beta = 0$ but $\alpha \neq \beta$, then the sum in (3.1) is obviously zero. If $\alpha\beta \neq 0$, we apply (2.3) to reduce the sum in (3.1) to

$$J_{\nu-1}^{(2)}(\alpha; q) \, J_{\nu-1}^{(2)}(\beta; q) q^{-2\nu+\frac{1}{2}}$$

$$\times \sum_{n=0}^{\infty} (1 - q^{n+\nu+1}) \, q^{-2n\nu-\frac{n^2}{2}} \frac{(q^{2\nu}; q)_{n+1}}{(q; q)_{n+1}} \, h_{n,\nu+1}(\tfrac{1}{\alpha}; q) \, h_{n,\nu+1}(\tfrac{1}{\beta}; q).$$

We recognize the above sum as the sum in the dual orthogonality relation (2.7) if we can get rid of the term $\frac{(q^{2\nu}; q)_{n+1}}{(q; q)_{n+1}}$. This forces the choice $\nu = \frac{1}{2}$ and now even the powers of *q* match. In the notation of (2.11) we set $\alpha = 2\omega_m$ and $\beta = 2\omega_n$ and obtain

$$\int_{-1}^{1} \mathcal{E}_q(x; i\omega_m) \, \overline{\mathcal{E}_q(x; i\omega_n)} \, w(x; q^{\frac{1}{2}}|q) \, dx = \frac{8\pi|\omega_n|q^{-\frac{1}{2}}}{A_n(\frac{3}{2})} \left[\frac{J_{-\frac{1}{2}}^{(2)}(2|\omega_n|; q)}{(-q\omega_n^2; q^2)_\infty} \right]^2 \delta_{m,n} \tag{3.2}$$

for all $m, n \in \mathbb{N}_0$ with $m^2 + n^2 \neq 0$. If $m = n = 0$, then $\mathcal{E}_q(x; i\omega_m) = \mathcal{E}_q(x; i\omega_n) = 1$ and (1.6) shows that the right-hand side of (3.2) becomes $\frac{2\pi(q^{\frac{1}{2}}, q^{\frac{3}{2}}; q)_\infty}{(q, q; q)_\infty}$. Observe that, as $q \to 1^-$, $\frac{j_{\nu,n}(q)}{1-q} \to j_{\nu,n}$ and $\frac{A_n(\nu)}{1-q} \to 2$. Moreover, $w(x; q^{\frac{1}{2}}) \to 2$ as $q \to 1^-$.

We now reduce the right-hand side of (3.2) to the form used in Bustoz and Suslov [6]. Formulas (2.11) and (2.12) imply that the right-hand side of (3.2) is

$$\frac{2\pi(q^{\frac{1}{2}}; q)_\infty^2 (i\omega_n, -i\omega_n; q^{\frac{1}{2}})_\infty}{i \, (q; q)_\infty^2 (-q^2\omega_n^2; q^2)_\infty^2} \frac{d}{dz} \ln g(z) \Big|_{z=2\omega_n}.$$

The above expression is the same as (6.20) of [6] since

$$(i\omega_n, -i\omega_n; q^{\frac{1}{2}})_\infty = (-\omega_n^2; q)_\infty = (-\omega_n^2, -q\omega_n^2; q^2)_\infty.$$

The outcome is that (3.2) becomes

$$\int_{-1}^{1} \mathcal{E}_q(x; i\omega_m) \, \overline{\mathcal{E}_q(x; i\omega_n)} \, w(x; q^{\frac{1}{2}}|q) \, dx$$

$$= \frac{2\pi(q^{\frac{1}{2}}; q)_\infty^2 (-\omega_n^2; q^2)_\infty}{(q, q; q)_\infty (-q\omega_n^2; q^2)_\infty} \left(\sum_{k=0}^{\infty} \frac{q^{\frac{k}{2}}}{1 + \omega_n^2 q^k} \right) \delta_{m,n} \tag{3.3}$$

for all *m* and *n*. The case $m = n = 0$ where $\omega_0 = 0$ is also covered by (3.3) since the right-hand sides of (3.3) and (1.6) are equal when $m = n = 0$.

Remark. It is clear from the argument presented here that the family of functions

$$\left(\frac{2}{\alpha}\right)^\nu \sum_{n=0}^\infty (1 - q^{\frac{n+1}{2}}) \zeta_n q^{\frac{n^2}{4}} J^{(2)}_{\nu+n}(\alpha; q) \, C_n(x; q^{\frac{1}{2}}|q)$$

will also satisfy (3.3) when $|\zeta_n| = 1$ for all n. Of course, the $\mathcal{E}_q$ correspond to the case $\zeta_n = i^n$. This makes it important to write the orthogonality relations as in (3.3) and not as in [6] where $\overline{\mathcal{E}_q(x; i\omega_n)}$ is written as $\mathcal{E}_q(x; -i\omega_n)$, which is certainly true for the case $\zeta_n = i^n$ considered in [6]. This situation will occur again in Section 4.

We now come to the completeness of the q-exponentials.

Theorem 3.1. *The system $\{\mathcal{E}_q(x; i\omega_n)\}_{n\in\mathbb{N}_0}$ is complete in $L^2[-1, 1; w(\cdot; q^{\frac{1}{2}}|q)]$.*

Proof. For $f \in L^2[-1, 1; w(\cdot; q^{\frac{1}{2}}|q)]$ define f_k and ϕ_k by

$$\left.\begin{aligned}
f_k &= \int_{-1}^1 f(x) \overline{\mathcal{E}_q(x; iw_k)} \, w(x; q^{\frac{1}{2}}|q) \, dx \\
\phi_k &= \int_{-1}^1 f(x) \, C_k(x; q^{\frac{1}{2}}|q) \, w(x; q^{\frac{1}{2}}|q) \, dx.
\end{aligned}\right\} \tag{3.4}$$

Now assume that $f_k = 0$ $(k \in \mathbb{N}_0)$. Since the weight function $w(\cdot; q^{\frac{1}{2}}|q)$ is continuous and bounded in $[-1, 1]$ and positive on $(-1, 1)$, then f is also in $L[-1, 1; w(\cdot; q^{\frac{1}{2}}|q)]$. The fact $\omega_0 = 0$ makes $\mathcal{E}_q(x; \omega_0) = C_0(x; q^{\frac{1}{2}}|q) = 1$ and (3.4) gives $f_0 = \phi_0$ and we get $\phi_0 = 0$. It is known that the set of polynomials $\{C_n(\cdot; q^{\frac{1}{2}}|q)\}$ is complete in $L^2[-1, 1; w(\cdot; q^{\frac{1}{2}}|q)]$. For $k > 0$, Parseval's theorem and (1.2) give

$$0 = \sum_{n=0}^\infty (1 - q^{n+\frac{1}{2}}) q^{\frac{n^2}{4}} (-i)^n J^{(2)}_{n+\frac{1}{2}}(2\omega_k; q) \, \phi_n$$

$$= -J^{(2)}_{-\frac{1}{2}}(2\omega_k; q) \sum_{n=1}^\infty (1 - q^{n+\frac{1}{2}}) q^{\frac{n^2}{4}} (-i)^n q^{-\frac{n^2}{2}} h_{n-1,\frac{3}{2}}\left(\tfrac{1}{2\omega_k}; q\right) \phi_n$$

where we used (2.3). Taking into account $h_{n,\nu}(-x; q) = (-1)^n h_{n,\nu}(x; q)$ we establish

$$0 = \sum_{n=0}^\infty \phi_{n+1}(1 - q^{n+\frac{3}{2}})(\mp i)^n q^{\frac{-n(n+2)}{4}} h_{n,\frac{3}{2}}\left(\pm\tfrac{1}{2\omega_k}; q\right).$$

As we saw in (2.7) the q-Lommel polynomials are orthogonal on a compact set. Hence the corresponding Hamburger moment problem is determinate [1, 15] and the set of polynomials is complete in $l^2[\mu]$, where μ is the measure with respect to which they are orthogonal. Here μ is purely discrete and has masses at $\pm\frac{1}{2\omega_m}$ for $m > 0$. Moreover, the orthonormal polynomials at a point x belong to l^2 if and only if $\mu(\{x\}) > 0$. From (2.6) we see that $q^{-\frac{n(n+2)}{4}} h_{n,\frac{3}{2}}(x; q) \in l^2$ for all $x = \pm\frac{1}{2\omega_n}$ $(n > 0)$. Furthermore, $f \in L^2[-1, 1; w(\cdot; q^{\frac{1}{2}}|q)]$ and (1.6) imply $\{\phi_k\} \in l^2$. Therefore the partial sums of the series

$$\sum_{n=0}^\infty \phi_{n+1}(1 - q^{n+\frac{3}{2}})(-i)^n q^{-\frac{n(n+2)}{4}} h_{n,\frac{3}{2}}(x; q)$$

converge in $l^2[\mu]$ to a function $u \in l^2[\mu]$. But this function u vanishes at all points supporting positive masses, so it must be zero μ-a.e. which implies $\phi_n = 0$ for $n > 0$. Thus we have already established that $\phi_n = 0$ for all $n \geq 0$. The completeness of the continuous q-ultraspherical polynomials in $L^2[-1, 1; w(\cdot\, ; q^{\frac{1}{2}}|q)]$ guarantees that $f = 0$ almost every where on $[-1, 1]$ and the proof is complete ∎

4. A generalization

At a first glance it seems that one can generalize the results in Section 3 in a straight forward way by introducing functions of type (1.4). As noted in the introduction, $\{p_n\}$ is a complete orthonormal system and $\{r_n\}$ is a complete discrete orthonormal system, and their orthogonality relations are given in (1.5). In (1.4) it was assumed that $\{\zeta_n\}$ is a sequence of points on the unit circle. The definition (1.4) will work but it is not clear how to define $\mathcal{F}(\cdot\, ; \alpha)$ when α does not support a discrete mass for the r_n's. In fact, the series $\sum_{n=0}^{\infty} \zeta_n \, r_n(\alpha) \, p_n(x)$ will diverge if α is not a ρ-mass point. The reason is that $\{r_n(\alpha)\} \in l^2$ if and only if α is a mass point for the measure of orthogonality of the r_n's, hence $r_n(\alpha)$ is small where the masses lie but is large at the rest of the complex plane. By reexamining (1.2) we now realize that the presence of the q-Bessel functions achieved this goal of interpolating through the mass points and the key is the recurrence relations (2.2) and (2.3). Luckily for us, analogues of (2.2) and (2.3) have been developed in some generality in our paper with Rahman and Zhang [12]. Before mentioning details of this interpolating property we show how to expand polynomials in the system $\{\mathcal{F}(\cdot\, ; \alpha)\}$.

Let f be a polynomial. We now expand $f(x)$ as $\sum_{k=0}^{m} f_k p_k(x)$. From the definition of the Fourier coefficients write

$$p_k(x) = \sum_{j=0}^{\infty} F(x, x_j) \, r_k(x_j)\rho(x_j), \qquad r_n(x_j) = \int_a^b F(x, x_j)p_n(x)w(x)\,dx.$$

Thus we find

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{m} f_k p_k(x) \\
&= \sum_{k=0}^{m} f_k \sum_{j=0}^{\infty} F(x, x_j)r_k(x_j)\rho(x_j) \\
&= \sum_{j=0}^{\infty} F(x, x_j)\rho(x_j) \sum_{k=0}^{m} f_k r_k(x_j) \\
&= \sum_{j=0}^{\infty} F(x, x_j)\rho(x_j) \sum_{k=0}^{m} f_k \int_a^b F(x, x_j)p_k(x)w(x)\,dx.
\end{aligned}
$$

Thus we have proved

$$f(x) = \sum_{j=0}^{\infty} F(x, x_j)\rho(x_j) \int_a^b F(x, x_j)f(x)w(x)\,dx.$$

The above analysis was done for the Fourier-ultraspherical case in an early version of [13] dated and circulated in July/August 1998. It was also pointed out that a q-analogue is similar.

We now proceed with the details of this interpolating property.

Theorem 4.1 (see [12]). *Let $\mathcal{J}(\cdot\,;\nu)$ satisfy*

$$C_\nu \mathcal{J}(x;\nu+1) = \frac{B_\nu}{x}\mathcal{J}(x;\nu) - \mathcal{J}(x;\nu-1) \tag{4.1}$$

and let $\{f_{n,\nu}\}$ be a sequence of polynomials recursively defined by

$$\left.\begin{aligned} f_{0,\nu}(x) &= 1 \\ f_{1,\nu}(x) &= xB_\nu \\ f_{n+1,\nu}(x) &= [xB_{n+\nu}]f_{n,\nu}(x) - C_{n+\nu-1}f_{n-1,\nu}(x) \end{aligned}\right\}. \tag{4.2}$$

Then

$$C_\nu C_{\nu+1}\cdots C_{\nu+n-1}\mathcal{J}(x;\nu+n) = \mathcal{J}(x;\nu)f_{n,\nu}\!\left(\tfrac{1}{x}\right) - \mathcal{J}(x;\nu-1)f_{n-1,\nu+1}\!\left(\tfrac{1}{x}\right). \tag{4.3}$$

The reader will immediately realize that (4.3) is (2.3) in this general setting. The version in [12] contains an arbitrary number of parameters and allows for another term independent of x in the coefficient of $\mathcal{J}(x;\nu)$ in (4.1) but Theorem 4.1 is sufficient for our needs here.

We need one more result due to H. M. Schwartz [16].

Theorem 4.2 (see [16]). *Let $\{s_{n,\nu}\}$ be a family of monic polynomials generated by*

$$\left.\begin{aligned} s_{0,\nu}(x) &= 1 \\ s_{1,\nu}(x) &= x + \beta_\nu \\ s_{n+1,\nu}(x) &= [x - \beta_{n+\nu}]s_{n,\nu}(x) + C_{n+\nu}s_{n-1,\nu}(x) \quad (n>0) \end{aligned}\right\}. \tag{4.4}$$

If both series $\sum_{n=0}^\infty |\beta_{n+\nu} - a|$ and $\sum_{n=1}^\infty |C_{n+\nu}|$ converge, then $x^n s_{n,\nu}(a + \tfrac{1}{x})$ converge uniformly on compact subsets of the complex plane to an entire function.

From the general theory of orthogonal polynomials it is known that when the positivity condition $B_{n+\nu}B_{n+\nu+1}C_{n+\nu} > 0$ $(n \geq 0)$ holds, then the polynomials $f_{n,\nu}$ are orthogonal with respect to a positive measure. Moreover, the monic polynomials will be $\left\{\frac{f_{n,\nu}}{B_\nu\cdots B_{\nu+n-1}}\right\}$.

Theorem 4.3. *Assuming the positivity condition $B_{n+\nu}B_{n+\nu+1}C_{n+\nu} > 0$ $(n \geq 0)$ and that the series $\sum_{n=0}^\infty \frac{C_{n+\nu}}{B_{n+\nu}B_{n+\nu+1}}$ converges, then*

$$\lim_{n\to\infty} \frac{z^{\nu+n}f_{n,\nu+1}\!\left(\frac{1}{z}\right)}{B_{\nu+1}\cdots B_{\nu+n}} = H(z;\nu) \tag{4.5}$$

exists and is uniform on compact subsets of the complex z-plane and $z^{-\nu}H(z;\nu)$ is an entire function. Moreover, the polynomials $f_{n,\nu}$ are orthogonal with respect to a compact supported discrete measure ρ_ν whose Stieltjes transform is given by

$$\int_a^b \frac{d\rho_\nu(t)}{z-t} = \frac{H(\frac{1}{z};\nu)}{H(\frac{1}{z};\nu-1)} \qquad \left(z \notin \mathrm{supp}\{\rho_\nu\}\right). \tag{4.6}$$

Furthermore, ρ_ν is normalized by $\int_a^b d\rho_\nu(t) = 1$ and $f_{n,\nu}$ satisfy the orthogonality relation

$$\int_a^b f_{m,\nu}(x) f_{n,\nu}(x)\, d\rho_\nu(x) = \lambda_n(\nu)\, \delta_{m,n}, \quad \lambda_n(\nu) = \tfrac{B_\nu}{B_{n+\nu}}\, C_\nu \cdots C_{\nu+n-1}. \tag{4.7}$$

Proof. The first part follows from Theorem 4.2. To prove the second part first recall that the numerator polynomials associated with (4.2), $f_{n,\nu}^*$ satisfy (4.2) with the initial conditions $f_{0,\nu}^* = 0$ and $f_{1,\nu}^* = B_\nu$ (see [4]). Therefore $f_{n,\nu}^* = B_\nu f_{n-1,\nu+1}$. From Markov's theorem (see [4, 17]), $\frac{f_{n,\nu}^*(x)}{f_{n,\nu}(x)}$ tends to the left-hand side of (4.6) uniformly on compact subsets of the complex x-plane not intersecting the support of ρ_ν. This and (4.5) establish (4.6). To see that the measure ρ_ν is purely discrete just note that the right-hand side of (4.6) is meromorphic. Then invoke the Perron-Stieltjes inversion formula for the Stieltjes transform. Orthogonality relations (4.7) follow from standard difference equation identities (see [4]) ∎

It must be noted that Schwartz knew that the continued J-fraction whose denominators are $\{f_{n,\nu}\}$ converges to the right-hand side of (4.6) under conditions weaker than the positivity condition. We required the positivity condition since we are interested in the case when the polynomials $f_{n,\nu}$ are orthogonal. Dickinson, Pollack and Wannier [7] proved Theorem 4.3 when $C_\nu = 1$.

What is left is to relate $H(\cdot,\nu)$ to a solution of (4.1). To do so first note that $f_{n,\nu}$ and $f_{n\pm 1,\nu\mp 1}$ are solutions to the second order difference equation (4.2), hence they must be linear dependent, that is $f_{n,\nu}(x)$ is $A(x)f_{n+1,\nu-1}(x) + B(x)f_{n-1,\nu+1}(x)$. By matching the initial conditions we find

$$x\, B_{\nu-1} f_{n,\nu}(x) = f_{n+1,\nu-1}(x) + C_{\nu-1} f_{n-1,\nu+1}(x). \tag{4.8}$$

Replace x by $\frac{1}{x}$, multiply by $\frac{x^{n+\nu-1}}{B_\nu \cdots B_{n+\nu-1}}$ and then let $n \to \infty$. After applying (4.5), the result after $\nu \to \nu+1$ is

$$\frac{1}{x} B_\nu B_{\nu+1} H(x;\nu) = B_\nu B_{\nu+1} H(x;\nu-1) + C_\nu H(x;\nu+1). \tag{4.9}$$

Let g be a function such that

$$g(\nu+1) = B_\nu g(\nu). \tag{4.10}$$

From (4.9) and (4.10) it follows that $\frac{H(x;\nu)}{g(\nu+1)}$ satisfies (4.1).

Definition. Define $\mathcal{J}(\cdot\,;\nu)$ to be the function $\frac{H(\cdot\,;\nu)}{g(\nu+1)}$ constructed above.

The functional relations (4.8) - (4.10) are in [7, 10] when $C_\nu = 1$. The sentence before Theorem 4.3 (4.2) and Theorem 4.3 show that

$$z^{-\nu}H(z;\nu) = \sum_{n=0}^{\infty}(-1)^n a_n(\nu)z^{2n}$$

where

$$\left.\begin{array}{l} a_0(\nu) = 1 \\[2mm] a_n(\nu) = a_n(\nu-1) + \dfrac{a_{n-1}(\nu+1)}{B_\nu B_{\nu+1}} \quad (n \geq 1) \end{array}\right\}.$$

Furthermore, $a_n \geq 0$. At this stage we make the additional assumption

$$\lim_{\nu\to\infty} z^{-\nu}H(z;\nu) = 1. \tag{4.11}$$

This assumption is satisfied in the cases of Bessel and q-Bessel functions. We do not have precise necessary and sufficient conditions on the coefficients C_ν and B_ν which imply (4.11).

Theorem 4.4. *Under the assumptions of Theorem 4.3 and (4.11), $z^{-\nu}\mathcal{J}(z;\nu)$ is an even function with only real and simple zeros and does not vanish at $z = 0$. The zeros of $z^{1-\nu}\mathcal{J}(z;\nu-1)$ and $z^{-\nu}\mathcal{J}(z;\nu)$ interlace. Let*

$$0 < x_{\nu,1} < \ldots < x_{\nu,n} < \ldots \tag{4.12}$$

be the positive zeros of $z^{-\nu}\mathcal{J}(z;\nu)$. Then the measure ρ_ν of (4.6) satisfies $\rho_\nu(\{x\}) > 0$ for $x \neq 0$ if and only if $x = \pm\frac{1}{x_{\nu,n}}$ $(n \in \mathbb{N})$. Moreover, $\rho_\nu(\{0\})$ is $\mathcal{M}_0(\nu)$, where

$$\frac{1}{\mathcal{M}_0(\nu)} = \sum_{n=1}^{\infty} \frac{B_{n+\nu}}{B_\nu} \prod_{k=0}^{n-1} \frac{C_{\nu+2k}}{C_{\nu+2k+1}}.$$

In Theorem 4.4, $\frac{1}{\mathcal{M}_0(\nu)}$ is a sum of positive terms, and if the series diverges, then ρ_ν does not have a mass at $x = 0$, but $x = 0$ always belongs to the support of ρ_ν. Theorem 4.4 in the case $C_\nu = 1$ was proved in [7] where the case of mass at zero was incorrect. This was corrected later by Goldberg in [10]. Both [7, 10] further assume that ν in (4.1) - (4.3) is a non-negative integer.

Proof of Theorem 4.4. From (4.6) we see that ρ_ν has a mass at x if and only if x is a pole or an essential singularity of the right-hand side. Because $H(\cdot\,;\nu)$ satisfies a three-term recurrence relation, if two consecutive H's have a common zero, then all the function will have a common zero, which in view of (4.11) is a contradiction. Because the support of ρ_ν is bounded, it is compact and the Hamburger moment problem has a unique solution. In this case the theory of the Hamburger moment problem [1, 15] asserts that $x = 0$ supports a positive ρ_ν-mass if and only if the series $\sum_{n=0}^{\infty} \frac{f_{n,\nu}^2(0)}{\lambda_n(\nu)}$ converges in which case the mass will be the reciprocal of the sum of the series. Now (4.2) shows that $f_{n,\nu}(0) = 0$ for n odd and $|f_{2n,\nu}(0)|$ is $\prod_{k=1}^{n} |C_{\nu+2k-2}|$ ∎

Definition. Assume that $\{p_n\}$ is a family of orthonormal polynomials satisfying (1.5) and that the corresponding moment problem is determined. A *generalized exponential function* $\mathcal{F}(\,\cdot\,; \alpha)$ is

$$\mathcal{F}(x; \alpha) = \alpha^{-\nu} \sum_{n=0}^{\infty} \zeta_n \sqrt{\lambda_n(\nu)} \, \tfrac{B_{n+\nu}}{B_\nu} \, \mathcal{J}(\alpha; n + \nu) \, p_n(x) \tag{4.13}$$

where $|\zeta_n| = 1$ for all n, provided the series converges.

We have not been able to derive a large ν asymptotics to prove convergence of the series in (4.13) under general assumptions on the sequences β_ν and C_ν. This is not needed for orthogonality or completeness.

Before establishing the orthogonality and completeness of the $\mathcal{F}$'s we need to identify the measure ρ_ν. As before we let $x_{\nu,n}$ be the positive zeros of $\mathcal{J}(\,\cdot\,; \nu)$. Let $\frac{B_\nu \mathcal{A}_n(\nu)}{2x_{\nu-1,n}^2}$ be the mass of ρ_ν at $\pm \frac{1}{x_{\nu-1,n}}$. Then

$$2\frac{\mathcal{M}_0(\nu+1)}{B_{\nu+1}} \, z - 2\frac{\mathcal{J}(z; \nu+1)}{\mathcal{J}(z; \nu)} = \sum_{k=1}^{\infty} \mathcal{A}_k(\nu+1) \left[\frac{1}{z + x_{\nu,k}} + \frac{1}{z - x_{\nu,k}} \right]. \tag{4.14}$$

Note that this is the equation between (2.4) and (2.5) extended to a general setting. To prove (4.14) use (4.10) to rewrite (4.6), with (z, ν) replaced by $(\frac{1}{z}, \nu + 1)$, in the form

$$\mathcal{M}_0(\nu+1) \, z + \sum_{k=1}^{\infty} \frac{\mathcal{A}_k(\nu+1) B_{\nu+1}}{2x_{\nu,k}} \left[\frac{z}{x_{\nu,k} - z} + \frac{z}{x_{\nu,k} + z} \right] = \frac{B_{\nu+1} \mathcal{J}(z; \nu+1)}{\mathcal{J}(z; \nu)}$$

which implies (4.14).

A consequence of (4.1) and (4.14) is

$$\frac{d}{dz} \mathcal{J}(z; \nu) \Big|_{z = x_{\nu,n}} = -2\frac{\mathcal{J}(x_{\nu,n}; \nu+1)}{\mathcal{A}_n(\nu+1)} = 2\frac{\mathcal{J}(x_{\nu,n}; \nu-1)}{C_\nu \mathcal{A}_n(\nu+1)}$$

which corresponds to (2.5) in the *q*-Lommel case. With this notation the dual orthogonality relation of (4.7) is

$$\sum_{n=0}^{\infty} \frac{B_{\nu+1}}{2\lambda_n(\nu+1)} \, f_{n,\nu+1}\!\left(\frac{1}{x_{\nu,k}}\right) f_{n,\nu+1}\!\left(\frac{1}{x_{\nu,j}}\right) = \frac{x_{\nu,j}^2}{\mathcal{A}_j(\nu+1)} \, \delta_{j,k}. \tag{4.15}$$

Theorem 4.5. *Let $x_{\nu,0} = 0$ and $x_{\nu,n}$ be as in (4.12) and assume that $\mathcal{M}_0(\nu+1) = 0$. In addition to the assumptions in Theorem 4.3 let the system $\{p_n\}$ be orthonormal on $[a, b]$ with respect to a weight function w and assume that the system $\{p_n\}$ is complete in $L^2[a, b; w]$. Then $\{\mathcal{F}(\,\cdot\,; x_{\nu,n})\}$ is a complete orthogonal system in $L^2[a, b; w]$ and its orthogonality relation is given by*

$$\int_a^b \mathcal{F}(x; x_{\nu,m}) \, \overline{\mathcal{F}(x; x_{\nu,n})} \, w(x) \, dx = \frac{2\mathcal{J}^2(x_{\nu,n}; \nu-1) \, \delta_{m,n}}{(x_{\nu,n})^{2\nu-2} \, B_\nu C_\nu \mathcal{A}_n(\nu+1)}. \tag{4.16}$$

Proof. Use Parseval's theorem to see that the left-hand side of (4.16) is a single sum, then apply (4.3) and (4.9) to recognize the sum as the sum in the dual orthogonality relation (4.15). The dual orthogonality holds since $a_n = \frac{C_{\nu+n-1}}{B_{\nu+n} B_{\nu+n+1}}$, hence $a_n \to 0$ as $n \to \infty$. Simple calculations establish (4.16), hence the orthogonality follows. The proof of the completeness is exactly the same as our proof of Theorem 3.1 and will be omitted ∎

Let us consider the example of q-Lommel polynomials. In this case $B_\nu = 2(1 - q^\nu)$ and $C_\nu = q^\nu$. Also,

$$\left. \begin{aligned} p_n(x) &= \sqrt{\frac{(1 - q^{n+\nu})(q; q)_n}{(1 - q^\nu)(q^{2\nu}; q)_n}} \, C_n(x; q^\nu | q) \\ \lambda_n(\nu) &= \frac{(1 - q^\nu)}{(1 - q^{n+\nu})} \, q^{\nu n + \frac{n(n-1)}{2}} \end{aligned} \right\}.$$

Thus

$$\mathcal{F}(x; \alpha) = \alpha^{-\nu} \sum_{n=0}^\infty \zeta_n q^{n(2\nu+n-1)/4} \frac{(1 - q^{n+\nu})}{(1 - q^\nu)} \sqrt{\frac{(q; q)_n}{(q^{2\nu}; q)_n}} \, J_{\nu+n}^{(2)}(\alpha; q) \, C_n(x; q^\nu | q). \quad (4.17)$$

This formula makes it evident that if we want a complete system of orthogonal functions with respect to $w(\cdot; q^\nu | q)$, we must use the $\mathcal{F}$ in it. Unlike expansion (1.2) where $\mathcal{E}_q$ does not depend on ν, the $\mathcal{F}$ in (4.17) depends on ν.

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