Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 20 (2001), No. 4, 845–857

Existence and Regularity Results for Non-Negative Solutions of some Semilinear Elliptic Variational Inequalities via Mountain Pass Techniques

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Abstract. The main result stated in the present paper is the existence of a non-negative solution for a semilinear variational inequality through the use of some estimates for the Mountain-Pass critical points obtained for the penalized equations associated with the variational inequality. The positivity of the solution is achieved through a regularity result and the strong maximum principle.

Keywords: *Semilinear variational inequalities, penalization method, Mountain-Pass theorem*

AMS subject classification: 49J40, 35J60, 58E05

0. Introduction

Let us consider the semilinear variational inequality

$$\textbf{(VI)}\quad \begin{cases} u \in H_0^1(\Omega) : \\ \int_\Omega \nabla u(x)\nabla(v(x)-u(x)) \geq \int_\Omega p(x,u(x))(v(x)-u(x)) \;\forall\, v \in H_0^1(\Omega) \\ u(x), v(x) \leq \psi(x) \quad \text{on } \Omega \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $\psi \in H^1(\Omega)$ with $\psi|_{\partial\Omega} \geq 0$ and p satisfies some general superlinearity growth conditions at zero and at infinity. For example, p can be choosen as

$$p(x,u(x)) = p(u(x)) = |u(x)|^{\beta-2}u(x) \qquad (\beta > 2), \tag{$*$}$$

that is $p(x,t) = p(t) = P'(t)$ for any $x \in \Omega$ and $t \in \mathbb{R}$ with $P(t) = \beta^{-1}|t|^\beta$. One notes that with this choice of p the nonlinear differential operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ such that variational inequality (VI) can be equivalently written as

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Supported by MURST Project "Metodi Variazionali ed Equazioni Differenziali Non Lineari"

ISSN 0232-2064 / $ 2.50

$$\begin{cases} u \in H_0^1(\Omega) : \\ \langle A(u), v-u \rangle \geq 0 \ \forall v \in H_0^1(\Omega) \\ u(x), v(x) \leq \psi(x) \quad \text{on } \Omega \end{cases}$$

does not satisfy any coerciveness property on $H_0^1(\Omega)$. So a non-trivial solution of variational inequality (VI) (note that $u \equiv 0$ obviously solves (VI) in the case $\psi \geq 0$ on $\overline{\Omega}$) cannot be found by the simple use of the well known Hartman-Stampacchia's theorem (see [5]) related to nonlinear variational inequalities involving monotone operators.

Actually, a paper by Szulkin [16] yields some conditions on a class of functions p in order to guarantee a suitable coerciveness approach to variational inequality (VI), which enables to state various existence of (possibly positive) solutions to (VI) and some conditions characterizing the solvability of (VI). Actually, the function p given in $(*)$ does not belong to the mentioned class, which contains on the contrary the opposite function $-p$.

In the following, still Szulkin in [17] developes a general theory of minimax principles for functionals which can be written as the sum of a regular functional plus a proper convex lower semicontinuous functional that can be choosen, in particular, as the indicator function of a closed convex set K, for example, as in (VI),

$$K = \left\{ v \in H_0^1(\Omega) : \, v \leq \psi \text{ on } \Omega \right\}.$$

He exhibits, as an application of a general theorem, an existence result for a semilinear variational inequality, in the case that

$$K = \left\{ v \in H_0^1(\Omega) : \, v \geq 0 \text{ on } \Omega \right\}$$

still considering the case that p has a superlinear growth at the origin and at infinity, with a further oddness assumption, in such a way that p can be choosen as in $(*)$.

In order to apply an appropriate version of the Palais-Smale condition (adapted for the mentional class of irregular functionals) the growth of p is supposed to be subcritical, i.e. β in $(*)$ is assumed to be less than 2^*, where 2^* is the Sobolev exponent given by $2^* = \frac{2N}{N-2}$ for $N \geq 3$.

In [7] Mancini and Musina considered the critical case $\beta = 2^* - 1$ with the choice of $p(x,t) = p(t) = t|t|^{\beta-2}$ as in $(*)$ and

$$K = \left\{ v \in H_0^1(\Omega) : \, v \geq \psi \text{ on } \Omega \right\}.$$

In another paper [8] the same authors also considered the choice of K of the kind

$$K = \left\{ v \in H_0^1(\Omega) : \, v \geq 0 \text{ on } \Omega \text{ and } v \leq \psi \text{ on } C \right\}$$

where C is a closed subset of Ω with suitable properties. Actually, the main interest of the results contained in [7, 8] is connected with other important questions in the framework of semilinear elliptic problems, rather than the existence problem for variational inequalities.

In a quite different context, a paper of Passaseo [13] (see also a joint paper with Marino [9]) is devoted to a deep investigation of the structure of possible solutions of variational inequality (VI), but any existence result contained in [9, 13] requires, among other assumptions, that the function p satisfies a global Lipschitz-continuity condition with respect to the t-variable, which in case (*) is obviously satisfied only with the choice $\beta = 2$ (which corresponds to the linear case).

The aim of the present paper is to consider a completely new approach with respect to the above mentioned papers, which seems to be more natural and simpler. The assumptions on p are those mentioned at the beginning of this introduction, where the exponent growth β has to be subcritical (more precisely, it is assumed to belong to the interval $(2, \min(3, 2^*))$. Then the existence of a non-trivial non-negative solution $u = u_+$ of variational inequality (VI) is stated under some further hypotheses on ψ, that is $\psi \in H_0^1 \cap L^q$ (with $q > 2^*$ suitably connected with the superlinearity growth of p), $\psi(x) \geq 0$ and a technical assumption connecting ψ with the superquadratic growth coefficients of P in the t-variable (see conditions (5) and (7) in Section 1). Here the fact that u_+ is non-trivial means not only $u_+ \not\equiv 0$, but even that u_+ cannot be automatically got as a solution of the equation associated with (VI), i.e. with obstacle $\psi \equiv +\infty$ (see Section 1). The method is based on the consideration of a family of penalized equations associated with variational inequality (VI) in the usual way as in the linear case (see, e.g., [3]): one obtains, for any penalized problem, a Mountain Pass solution. Then some estimates from above and from below for these solutions allow, by a suitable passage to the limit as the penalization parameter ε goes to 0^+, to exhibite a non-negative solution $u = u_+ \not\equiv 0$.

The second part of the present paper is devoted to state some regularity results of any possible solution u to variational inequality (VI) as well as to guarantee the strict positivity of u in the case that u is non-negative and sufficiently regular. More precisely, by assuming some suitable further regularity conditions on ψ, we prove by a boot-strap argument and the use of so-called Lewy-Stampacchia estimates for solutions to variational inequalities (see [6, 12]) some regularity results for any solution u to (VI). The most meaning of these results yields u as a solution of a complementarity system (in a weak sense, which will be precised in Section 2). On the other hand, an appropriate use of the strong maximum principle enables to state, under further regularity assumptions on ψ, that any non-negative solution u is indeed a strictly positive solution of the complementarity system.

We point out that the regularity and the strict positivity (possible) properties of the solution u_+ given by the existence result in the first part of the present paper could be a starting point in order to interpret u_+ as the Hamilton-Jacobi function of the control problem associated with variational inequality (VI). Let us note that this is not obvious at all, since one cannot guarantee the uniqueness of a non-trivial solution u of (VI).

Finally, let us recall that some other interesting results in the framework of semilinear variational inequalities, even with some different types of constraints, were obtained in [10, 11].

1. Existence result of a non-negative non-trivial solution

Let us consider variational inequality (VI) where Ω is an open bounded subset of $\mathbb{R}^N$ with a sufficiently smooth boundary $\partial\Omega$, $H_0^1(\Omega)$ is the usual Sobolev space on Ω obtained as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|v\| = (\int_\Omega |\nabla v(x)|^2)^{1/2}$, ψ belongs to $H^1(\Omega)$ with $\psi|_{\partial\Omega} \geq 0$, and $p : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions [(1)]

$$p(x,\xi) \text{ is measurable in } x \in \overline{\Omega} \text{ and continuous in } \xi \in \mathbb{R} \tag{1}$$

$$|p(x,\xi)| \leq a_1 + a_2|\xi|^s \quad \big((x,\xi) \in \overline{\Omega} \times \mathbb{R}\big) \text{ for some } a_1, a_2 > 0$$
$$\text{with } 1 < s < \tfrac{N+2}{N-2} = 2^* - 1 \text{ if } N \geq 3 \text{ and } 1 < s \text{ if } N = 2 \tag{2}$$

$$p(x,\xi) = o(|\xi|) \text{ as } \xi \to 0. \tag{3}$$

Moreover, putting

$$P(x,\xi) = \int_0^\xi p(x,t)\,dt \qquad (x \in \mathbb{R})$$

we assume:

$$\left.\begin{array}{l} \text{There exists } r > 0 \text{ such that, for } |\xi| \geq r, \\ 0 < (s+1)\,P(x,\xi) \leq \xi\,p(x,\xi) \quad (x \in \overline{\Omega}) \end{array}\right\}. \tag{4}$$

Note that condition (4) easily yields

$$P(x,\xi) \geq a_3|\xi|^{s+1} - a_4 \quad (x \in \overline{\Omega}, \xi \in \mathbb{R}) \text{ for some } a_3, a_4 > 0. \tag{5}$$

In the case $\psi(x) \geq 0$ on $\overline{\Omega}$ it is easy to check that $u_0 \equiv 0$ is a trivial solution of variational inequality (VI). Actually, under this assumption, one can get another solution $u_- \not\equiv 0$, which is itself in some sense trivial as it can be obtained as a non-zero solution of the equation

$$\text{(E)} \quad -\Delta u_-(x) = p(x, u_-(x)) \qquad (u_- \in H_0^1(\Omega)).$$

Indeed, (1) - (5) enable to find a solution u_- of equation (E), which is negative on Ω, coinciding with a critical point of Mountain Pass type of the functional

$$I_-(v) = \tfrac{1}{2}\int_\Omega |\nabla v(x)|^2 - \int_\Omega P_-(x, v(x)) \qquad (v \in H_0^1(\Omega))$$

with

$$P_-(x,\xi) = \int_0^\xi P_-(x,t)\,dt \qquad \text{and} \qquad p_-(x,t) = \begin{cases} 0 & \text{if } t > 0 \\ p(x,t) & \text{if } t \leq 0 \end{cases}$$

(see, e.g., [14: p. 11]). Therefore, as $\psi \geq 0$ on Ω, u_- is actually a solution of variational inequality (VI).

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(1) For $N = 1$ all the statements reported here about equation (E) below still hold without assumption (2). Moreover, assumption (4) can be weakened, replacing $|\xi| \geq r > 0$ with $\xi \leq -r < 0$.

Thus the interesting problem, also in view of some possible applications to control theory, is to get a non-negative non-zero solution u_+ of variational inequality (VI). Note that the truncature argument introduced above, but replacing p_- with

$$p_+(x,t) = \begin{cases} 0 & \text{if } t < 0 \\ p(x,t) & \text{if } t \geq 0, \end{cases}$$

always yields a positive solution $\tilde{u}_+$ of equation (E), but $\tilde{u}_+$ is in general not a solution of variational inequality (VI), since one cannot state the inequality $\tilde{u}_+ \leq \psi$.

Actually, by suitably reinforcing the assumptions on ψ, one can get a non-negative non-zero solution u_+ of variational inequality (VI), even replacing condition (4) with the following weaker condition:

$$\left.\begin{array}{l} \text{There exists } r > 0 \text{ such that, for } \xi \geq r, \\ 0 < (s+1)\,P(x,\xi) \leq \xi\, p(x,\xi) \quad (x \in \overline{\Omega}) \end{array}\right\}. \tag{4'}$$

This condition implies

$$P(x,\xi) \geq a_3'(\xi)^{s+1} - a_4' \quad (\text{for a.e. } x \in \overline{\Omega}, \xi > 0) \text{ for some } a_3', a_4' > 0 \tag{5'}$$

(analogously to the case considered in order to get the negative solution u_- of equation (E), see footnote $^{(1)}$). Indeed, one can state the following

Theorem 1. *Under assumptions* $(1)-(3)$ *and* $(4')$, *let further*

$$0 \leq \psi \in H_0^1(\Omega) \cap L^{(\frac{2^*}{s})'}(\Omega) \tag{6}$$

$$\int_\Omega |\nabla \bar{v}|^2 \leq 2\big(a_3 \int_\Omega (\bar{v})^{s+1} - a_4\big) \text{ for some } 0 \leq \bar{v} \in H_0^1(\Omega), 0 \leq \bar{v}(x) \leq \psi(x) \tag{7}$$

$$s < 2 \text{ in (2)}. \tag{8}$$

Then there exists a non-negative solution $u = u_+ \not\equiv 0$ *of variational inequality* (VI).

Remark 1. Note that (7) is a really restrictive condition on ψ. For example, it implies $\int_\Omega (\psi(x))^{s+1} \geq \frac{a_4}{a_3}$ which can be seen as a condition connecting ψ with the growth coefficient p. On the other side, there are infinite many obstacles ψ satisfying (7). More precisely, if one fixes any non-negative function $0 \neq v_0 \in H_0^1(\Omega)$, one can choose $\psi = \lambda v_0$ as an obstacle with $\lambda > 0$ sufficiently large, and condition (7) is satisfied by taking $\bar{v} \equiv \psi$, due to the fact that $s+1 > 2$.

Remark 2. Note that (8) is automatically satisfied in the case $N \geq 6$, as an obvious consequence of condition (2).

We limit ourselves to deal with the case $N \geq 3$, since the cases $N = 1, 2$ can be studied in a very similar way, even by simpler arguments.

The method of finding the solution u_+ relies on the consideration of a family of penalized equations associated in a standard way with variational inequality (VI) (see, e.g., [3]). Indeed, one can prove that any penalized equation possesses a strictly positive solution of Mountain Pass type, and that a sequence choosen in this family actually converges to a non-negative solution $u_+ \not\equiv 0$ of (VI), by suitably using some estimates from below and from above for the H_0^1-norm of the penalized solutions. Neverthless, one cannot state in general the strict positivity of u_+ on the whole set Ω.

2. Proof of Theorem 1

First of all, let us introduce the penalized problem associated with variational inequality (VI), that is, for any $\varepsilon > 0$ the weak equation

$$\textbf{(E)}_\varepsilon \begin{cases} u_\varepsilon \in H^1_0(\Omega): \\ \int_\Omega \nabla u_\varepsilon \nabla v + \frac{1}{\varepsilon}\int_\Omega (u_\varepsilon - \psi)^+ v = \int_\Omega p(x, u_\varepsilon(x)) v(x) \ \forall\, v \in H^1_0(\Omega) \end{cases}$$

where g^+ denotes the positive part of the function g. Let us note that the last integral is well defined for v in $H^1_0(\Omega)$ as a consequence of condition (2) and the continuous embedding of $H^1_0(\Omega)$ into $L^{\frac{2N}{N-2}}$.

Actually, in order to look for non-negative solutions of problem $(\mathrm{E})_\varepsilon$ it is convenient to modify it with the following one:

$$(\overline{\mathrm{E}})_\varepsilon \begin{cases} u_\varepsilon \in H^1_0(\Omega): \\ \int_\Omega \nabla \bar u_\varepsilon \nabla v + \frac{1}{\varepsilon}\int_\Omega (\bar u_\varepsilon - \psi)^+ v = \int_\Omega p_+(x, \bar u_\varepsilon(x)) v(x) \ \forall\, v \in H^1_0(\Omega) \\ p_+(x,\xi) = p(x,\xi) \text{ if } \xi \ge 0 \text{ and } p_+(x,\xi) = 0 \text{ if } \xi < 0 \end{cases}$$

whose solutions coincide with the critical points of the functional

$$I_\varepsilon(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{\varepsilon}\int_\Omega \left(\int_0^{v(x)} (t - \psi(x))^+ dt \right) dx - \int_\Omega P_+(x, v(x))$$

on $H^1_0(\Omega)$ with

$$P_+(x,\xi) = \int_0^\xi p_+(x,t)\, dt \qquad (x \in \overline{\Omega}, \xi \in \mathbb{R}).$$

Indeed, one can easily check that I_ε belongs to $C^1(H^1_0(\Omega))$ and that $\langle I'_\varepsilon(\bar u'_\varepsilon), v\rangle$ as pairing between $H^1_0(\Omega)$ and its dual space coincides with the difference between the first and the second member in problem $(\overline{\mathrm{E}})_\varepsilon$.

At this point the proof of Theorem 1 starts from the check that I_ε verifies all the hypotheses of the Mountain Pass theorem by Ambrosetti and Rabinowitz [2]. Let us proceed, from now on, by steps.

Step 1. *The functional I_ε verifies for any $\varepsilon > 0$ the conditions*

$$I_\varepsilon(0) = 0 \tag{9}$$

and, for some $\alpha > 0$ and $r > 0$,

$$I_\varepsilon(v) \ge \alpha \qquad (v \in H^1_0(\Omega), \|v\| = r). \tag{10}$$

Proof. Property (9) is trivial. As for (10), let us note that the positivity of ψ on Ω yields

$$\int_\Omega \left(\int_0^{v(x)} (t - \psi(x))^+ dt \right) dx = \int_{\substack{x\in\Omega \\ v(x) \ge \psi(x)}} \left(\int_{\psi(x)}^{v(x)} (t - \psi(x)\, dt \right) dx \ge 0.$$

Thus

$$I_\varepsilon(v) \ge \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega P_+(x, v(x))\, dx. \tag{11}$$

Then (10) follows from (11), (3), (4′) and the continuous embedding of $H^1_0(\Omega)$ into $L^{s+1}(\Omega)$ (see also [14: p. 10/Proof of (I_1)]) ∎

Remark 3. Note that the numbers $\alpha > 0$ and $r > 0$ in (10) do not depend on ε, but this will not be used in the proof of Theorem 1.

Step 2. *The element $0 \neq \bar{v} \in H_0^1(\Omega)$ in (7) satisfies the property*

$$I_\varepsilon(\bar{v}) \leq 0 \tag{12}$$

for all $\varepsilon > 0$ ∎

Proof. This is an obvious consequence of the facts that, as $\bar{v} \leq \psi$, one has $\frac{1}{\varepsilon}\int_\Omega(\bar{v} - \psi)^+ = 0$ and that

$$I_\varepsilon(\bar{v}) \leq \tfrac{1}{2}\int_\Omega |\nabla \bar{v}|^2 - \left(a_3 \int_\Omega \bar{v}^{s+1} - a_4\right) \leq 0$$

by (7) and (5′) ∎

Step 3. *For any $\varepsilon > 0$, the functional I_ε satisfies the Palais-Smale condition, i.e.*

(PS) *For any sequence $\{v_n\}$ such that $\{I_\varepsilon(v_n)\}_n$ is bounded and $I'_\varepsilon(v_n) \to 0$ in the dual space of $H_0^1(\Omega)$ there exists a subsequence of $\{v_n\}$ strongly converging in $H_0^1(\Omega)$.*

Proof. The arguments are quite similar to those given in [15: Chapter II/Proof of Theorem 6.2] as the function $p_\varepsilon(x,\xi) = p_+(x,\xi) - \frac{1}{\varepsilon}(\xi - \psi(x))^+$ obviously satisfies conditions of type (1) - (2) as $p(x,\xi)$, and that $P_\varepsilon(x,\xi) = \int_0^\xi p_\varepsilon(x,t)\,dt$ satisfies the analogous relation of condition (4′) (with p_+ and P_+ replaced by p_ε and P_ε, respectively), with the same choices of s and r ∎

Step 4. *For any $\varepsilon > 0$ there exists a solution $\bar{u}_\varepsilon$ of problem $(\overline{\mathrm{E}})_\varepsilon$ such that*

$$I_\varepsilon(\bar{u}_\varepsilon) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\varepsilon(\gamma(t)) \tag{13}$$

where

$$\Gamma = \Big\{\gamma \in C^0([0,1]; H_0^1(\Omega)) : \gamma(0) = 0 \text{ and } \gamma(1) = \bar{v}\Big\}. \tag{14}$$

Moreover,

$$I_\varepsilon(\bar{u}_\varepsilon) \geq \alpha. \tag{15}$$

Proof. The assertion is a consequence of Steps 1 - 3 and the Mountain Pass theorem by Ambrosetti and Rabinowitz [2] ∎

Step 5. *One can find $\bar{u}_\varepsilon$ as a non-negative and non-trivial solution of problem* $(\mathrm{E})_\varepsilon$.

Proof. Let us choose $v(x) = (\bar{u}_\varepsilon)^-(x) = \max\{-\bar{u}_\varepsilon(x), 0\}$ in problem $(\overline{\mathrm{E}})_\varepsilon$. By the non-negativity of ψ on Ω one easily gets

$$\int_\Omega (\bar{u}_\varepsilon - \psi)^+(\bar{u}_\varepsilon)^- = 0 \tag{16}$$

as well as, by the definition of p_+,

$$\int_\Omega p_+(x, \bar{u}_\varepsilon(x))(\bar{u}_\varepsilon)^-(x) = 0. \tag{17}$$

Then $(\overline{E}_\varepsilon)$ (with $v = (\bar{u}_\varepsilon)^-$) and (16) - (17) yield $(\bar{u}_\varepsilon)^- \equiv 0$. Thus $\bar{u}_\varepsilon$ is a non-negative not identically zero solution of problem $(\mathrm{E})_\varepsilon$ ∎

Step 6. *There exists a number $c_1 > 0$ such that $\|\bar{u}_\varepsilon\| \geq c_1$ for all $\varepsilon > 0$.*

Proof. By definition of a solution of problem $(\mathrm{E})_\varepsilon$ it follows in particular

$$\int_\Omega |\nabla \bar{u}_\varepsilon|^2 + \tfrac{1}{\varepsilon}\int_\Omega (\bar{u}_\varepsilon - \psi(x))^+ \bar{u}_\varepsilon(x) = \int_\Omega p(x, \bar{u}_\varepsilon(x))\bar{u}_\varepsilon(x). \tag{18}$$

Thus, by the positivity of $\bar{u}_\varepsilon$

$$\int_\Omega |\nabla \bar{u}_\varepsilon|^2 \leq \int_\Omega p(x, \bar{u}_\varepsilon(x))\bar{u}_\varepsilon(x). \tag{19}$$

On the other hand, as a consequence of conditions (2) - (3), for any $\delta > 0$ there exists a $c(\delta) > 0$ such that

$$|\xi\, p(x,\xi)| \leq \delta|\xi|^2 + c(\delta)|\xi|^{s+1} \qquad (\text{for a.e. } x \in \Omega, \xi \in \mathbb{R})$$

which yields, using (19), the arbitrarity of δ and the continuous embedding of H_0^1 into L^2, the relation

$$\int_\Omega |\nabla \bar{u}_\varepsilon|^2 \leq \mathrm{const} \int_\Omega |\bar{u}_\varepsilon|^{s+1}.$$

Thus the assertion easily follows from the continuous embedding of H_0^1 into L^{s+1} and the assumption $s + 1 > 2$ ∎

Step 7. *There exists a number $c_2 > 0$ such that $I_\varepsilon(\bar{u}_\varepsilon) \leq c_2$ for all $\varepsilon > 0$.*

Proof. By (15) - (16) one gets

$$I_\varepsilon(\bar{u}_\varepsilon) \leq \max_{t\in[0,1]} I_\varepsilon(t\bar{v}). \tag{20}$$

Actually, as $\bar{v} \leq \psi$ one has

$$\int_\Omega \left(\int_0^{t\bar{v}(x)} (s - \psi(x)^+ ds \right) dx = 0 \qquad (t \in [0,1]). \tag{21}$$

Then (20) - (21) yield

$$I_\varepsilon(\bar{u}_\varepsilon) \leq \max_{t\in[0,1]} \left(\tfrac{t^2}{2} \int_\Omega |\nabla \bar{v}|^2 - \int_\Omega P(x, t\bar{v}(x)) \right).$$

Thus by condition $(5')$ one gets

$$I_\varepsilon(\bar{u}_\varepsilon) \leq \mathrm{const} \left(\max_{t\in[0,1]} (t^2 - t^{s+1}) \right) + \mathrm{const} \tag{22}$$

and the assertion is proved as the left member in (21) is a constant number independent on ε ∎

Step 8. *There exists a number $c_3 > 0$ such that $\|\bar u_\varepsilon\| \le c_3$ for all $\varepsilon > 0$.*

Proof. By Step 7 one gets for any $\varepsilon > 0$

$$\tfrac12 \int_\Omega |\nabla \bar u_\varepsilon|^2 + \tfrac1\varepsilon \int_\Omega \left(\int_0^{\bar u_\varepsilon(x)} (s-\psi(x))^+ ds \right) dx \le c_2 + \int_\Omega P(x, \bar u_\varepsilon(x))\, dx.$$

So, by condition (4′),

$$\tfrac12 \int_\Omega |\nabla \bar u_\varepsilon|^2 + \tfrac1\varepsilon \int_\Omega \left(\int_0^{\bar u_\varepsilon(x)} (s-\psi(x))^+ ds \right) dx \le \text{const} + \tfrac{1}{s+1} \int_\Omega p(x, \bar u_\varepsilon(x)) \bar u_\varepsilon(x)\, dx.$$

Thus, as $\bar u_\varepsilon$ solves problem $(\mathrm{E})_\varepsilon$ one gets

$$\begin{aligned} &\left(\tfrac12 - \tfrac{1}{s+1}\right) \int_\Omega |\nabla \bar u_\varepsilon|^2 \\ &\quad \le \text{const} + \tfrac{1}{(s+1)\varepsilon} \int_\Omega (\bar u_\varepsilon(x) - \psi(x))^+ \bar u_\varepsilon(x) - \tfrac1\varepsilon \int_\Omega \left(\int_0^{\bar u_\varepsilon(x)} (s-\psi(x))^+ ds \right) dx. \end{aligned} \tag{23}$$

Thus, putting $\Omega_\varepsilon = \{x \in \Omega :\ \bar u_\varepsilon(x) \ge \psi(x)\}$ one deduces from (23)

$$\left(\tfrac12 - \tfrac{1}{s+1}\right) \int_\Omega |\nabla \bar u_\varepsilon|^2 \le \text{const} + \tfrac1\varepsilon \left\{ \tfrac{1}{s+1} \int_{\Omega_\varepsilon} (\bar u_\varepsilon - \psi)\bar u_\varepsilon - \tfrac12 \int_{\Omega_\varepsilon} (\bar u_\varepsilon - \psi)^2 \right\}.$$

So, as $s+1 > 2$,

$$\int_\Omega |\nabla \bar u_\varepsilon|^2 \le \text{const} + \tfrac{1}{2\varepsilon} \int_{\Omega_\varepsilon} (\bar u_\varepsilon - \psi)\psi. \tag{24}$$

At this point, taking $v = \psi$ in problem $(\mathrm{E})_\varepsilon$ (recall that ψ belongs to $H_0^1(\Omega)$), one gets

$$\tfrac1\varepsilon \int_{\Omega_\varepsilon} (\bar u_\varepsilon - \psi)\psi = -\tfrac12 \int_\Omega \nabla \bar u_\varepsilon \nabla \psi + \int_\Omega p(x, \bar u_\varepsilon(x))\psi(x). \tag{25}$$

Using (2), one deduces from (24) - (25) and the fact that ψ belongs to $L^q(\Omega)$ with q given by (6)

$$\int_\Omega |\nabla \bar u_\varepsilon|^2 \le \text{const}\left(1 + \left(\int_\Omega |\bar u_\varepsilon|^{2^*} \right)^{\frac{s}{2^*}} \right)$$

and, by the continuous embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$, $\|\bar u_\varepsilon\|^2 \le \text{const}(1 + \|\bar u_\varepsilon\|^s)$ which gives the boundedness of $\{u_\varepsilon\}$ in $H_0^1(\Omega)$ as (8) holds ∎

Step 9. *There exists a number $c_4 > 0$ such that*

$$\|(\bar u_\varepsilon - \psi)^+\|_{L^2(\Omega)} \le c_4 \sqrt{\varepsilon}. \tag{26}$$

Proof. Since $\bar u_\varepsilon$ solves problem $(\mathrm{E})_\varepsilon$ one gets

$$\tfrac1\varepsilon \int_\Omega (\bar u_\varepsilon(x) - \psi(x))^+ \bar u_\varepsilon(x) = \int_\Omega |\nabla \bar u_\varepsilon|^2 - \int_\Omega p(x, \bar u_\varepsilon(x)) \bar u_\varepsilon(x).$$

Thus, by the positivity of ψ

$$\begin{aligned} \tfrac1\varepsilon \int_\Omega \left((\bar u_\varepsilon(x) - \psi(x))^+ \right)^2 &= \tfrac1\varepsilon \int_\Omega (\bar u_\varepsilon - \psi)^+ (\bar u_\varepsilon - \psi) \\ &\le \int_\Omega |\nabla \bar u_\varepsilon|^2 - \int_\Omega p(x, \bar u_\varepsilon(x)) \bar u_\varepsilon(x). \end{aligned} \tag{27}$$

At this point the assertion follows from condition (2) and Step 8 ∎

Step 10. *There exists a sequence* $\varepsilon_n \to 0^+$ *such that* $\{\bar{u}_{\varepsilon_n}\}_n$ *weakly converges in* $H_0^1(\Omega)$ *to some* $0 \le \bar{u} \not\equiv 0$.

Proof. First of all, by Step 8, a sequence $\{\bar{u}_{\varepsilon_n}\}_n$ with $\varepsilon_n \to 0^+$ weakly converges in $H_0^1(\Omega)$ to some $\bar{u}$, which is non-negative by the positivity of any $\bar{u}_{\varepsilon_n}$. Moreover, by the Rellich-Kondrachov theorem (ensured by the condition $s+1 < \frac{2N}{N-2}$), $\{\bar{u}_{\varepsilon_n}\}$ strongly converges to $\bar{u}$ in $L^{s+1}(\Omega)$. One claims that $\bar{u}$ is not identically zero. Indeed, $\bar{u} \equiv 0$ would imply an absurdum deduced by Step 6 and the passage to the limit as $n \to +\infty$ in relation (18) with $\varepsilon = \varepsilon_n$ ∎

Step 11. *As conclusion, the element* $u_+ = \bar{u}$ *given by Step* 10 *is a non-negative non-zero solution of variational inequality* (VI).

Proof. One has to prove only that $\bar{u}$ solves variational inequality (VI). Actually, the argument is very similar to that given in [3: p. 196/Subsection 1.4] for the linear case, but we prefer to report the details here.

First of all, at least a subsequence of $\{u_{\varepsilon_n}\}_n$ also denoted by $\{u_{\varepsilon_n}\}_n$ verifies the two convergences

$$\bar{u}_{\varepsilon_n} \to \bar{u} \text{ strongly in } L^p(\Omega) \text{ for any } p \in [2, 2^*) \tag{28}$$

$$(\bar{u}_{\varepsilon_n} - \psi)^+ \to 0 \text{ in } L^2(\Omega) \tag{29}$$

(the first was already pointed out in the proof of Step 10, the second is an obvious consequence of Step 9). Actually, both convergences yield

$$(\bar{u} - \psi)^+ = 0, \qquad \text{i.e. } \bar{u}(x) \le \psi(x) \text{ for a.e. } x \in \Omega \tag{30}$$

so that

$$\int_\Omega \nabla \bar{u}_{\varepsilon_n} \nabla v - \int_\Omega p(x, \bar{u}_{\varepsilon_n}(x))(v(x) - \bar{u}_{\varepsilon_n}(x)) \ge \int_\Omega |\nabla \bar{u}_{\varepsilon_n}|^2 \tag{31}$$

for all $v \in H_0^1(\Omega)$ with $v \le \psi$. At this point (28) implies in particular the convergence $\bar{u}_{\varepsilon_n} \to \bar{u}$ strongly in $L^{\frac{s\,2^*}{2^*-1}}$ which yields by (2) the convergence $p(\cdot, \bar{u}_{\varepsilon_n}(\cdot)) \to p(\cdot, \bar{u}(\cdot))$ strongly in $L^{\frac{2^*}{2^*-1}}$. Then, as $\bar{u}_{\varepsilon_n} \to \bar{u}$ weakly in L^{2^*}, one gets

$$\int_\Omega p(x, \bar{u}_{\varepsilon_n}(x))(v(x) - \bar{u}_{\varepsilon_n}(x)) \to \int_\Omega p(x, \bar{u}(x))(v(x) - \bar{u}(x)) \tag{32}$$

for all $v \in H_0^1(\Omega)$. On the other hand, the weak L^2-convergence $\nabla \bar{u}_{\varepsilon_n} \to \nabla \bar{u}$ implies

$$\lim_{n\to\infty} \int_\Omega |\nabla \bar{u}_{\varepsilon_n}|^2 \ge \int_\Omega |\nabla \bar{u}|^2. \tag{33}$$

Finally, (30) - (33) easily yield $\bar{u}$ as a solution of problem (VI) ∎

3. Regularity and strict positivity of solutions

As for the regularity results of any solution u of problem (VI), they can be obtained as consequences of the so-called Lewy-Stampacchia estimates for solutions of variational inequalities. At this purpose, let us recall that any solution u of a variational inequality of the kind

$$\begin{cases} u \in H_0^1(\Omega) : \\ \int_\Omega \nabla u \nabla (v-u) \geq \langle g, v-u \rangle \quad \forall\, v \in H_0^1(\Omega) \\ u, v \leq \psi \end{cases}$$

where $g \in H^{-1}(\Omega)$ (the dual space of $H_0^1(\Omega)$), $\langle \cdot, \cdot \rangle$ denotes the pairing between H_0^1 and H^{-1}, and a pair of dual order estimates of the type

(LS) $g \wedge (-\Delta\psi) \leq -\Delta u \leq g$

is assumed in the case that g and $-\Delta\psi$ are supposed to belong to the dual order of $H_0^1(\Omega)$ (that is the closed subspace of $H^{-1}(\Omega)$ given by the differences of non-negative elements in $H^{-1}(\Omega)$). [(2)] Actually, estimates of the kind of (LS), introduced in [6] for elliptic variational inequalities, were afterwards proved in a very general context for abstract variational inequalities with unilateral constraints in [12].

Indeed, some suitable further regularity assumptions on ψ enable to use appropriate (LS)-estimates in order to obtain various regularity results for the solutions of problem (VI). We have choosen to present here two main results of this kind. The first one yields the Hölder continuity and is expressed by the following

Theorem 2. *Let p be Hölder continuous on $\overline{\Omega} \times \mathbb{R}$ and let* (2) *be satisfied with $s < 1 + \frac{4}{N-2}$ if $N \geq 3$. Furthermore, let $\psi \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Then any solution u of problem* (VI) *solves the complementarity system*

$$\textbf{(C)} \begin{cases} u \in C^{0,\alpha'}(\overline{\Omega}) & \textit{for some } \alpha' \in (0,1) \\ u(x) \leq \psi(x) & \textit{for all } x \in \overline{\Omega} \\ -\Delta u(x) \leq p(x, u(x)) & \textit{for a.e. } x \in \Omega \\ -\Delta u(x) = p(x, u(x)) & \textit{for all } x \in \Omega \textit{ such that } u(x) < \psi(x) \\ u(x) = 0 & \textit{for all } x \in \partial\Omega. \end{cases}$$

Proof. The proof is obvious for $N = 1, 2$. So let us put $N \geq 3$. By the previous arguments the pair of inequalities of type (LS) yields, as $\Delta\psi$ is Hölder continuous, u is a weak solution in $H_0^1(\Omega)$ of an equation of the type

$$-\Delta u(x) = g(x) \in L^{\frac{2^*}{s}}(\Omega).$$

Actually, $-\Delta u$ belongs by (LS) to an order interval with extrema given by $p(x, u(x)) \wedge (-\Delta\psi(x))$ and $p(x, u(x))$, both belonging to $L^{\frac{2^*}{s}}(\Omega)$. Then $(-\Delta u)$ itself belongs to $L^{\frac{2^*}{s}}(\Omega)$.

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(2) Any element $v' \in H^{-1}(\Omega)$ is said to be *non-negative* if $\langle v', v \rangle \geq 0$ for all $0 \leq v \in H_0^1(\Omega)$.

Therefore, by the classical results of Agmon, Douglis and Nirenberg [1], u belongs to $H^{2,\frac{2^*}{s}}$. Then the embedding theorems by Sobolev and Morrey enable, by a bootstrap argument, to state that the α'-Hölder continuity of u on $\overline{\Omega}$ for some $\alpha' \in (0,1)$ is equivalent to the statement that for some $k_0 \in \mathbb{N}$

$$(N-2)s^{k_0} < 4 \sum_{j=0}^{k_0-1} s^j. \tag{34}$$

On the other side, this is obviously equivalent to the relation $N - 2 < \frac{4}{s-1}$, i.e. $s < 1 + \frac{4}{N-2}$. Therefore, one concludes that u itself is α'-Hölder continuous. This property, using standard arguments in the theory of variational inequalities, yields the equivalence of variational inequality (VI) with all the relations appearing in system (C) ∎

Remark 4. Let us point out that condition (8) in the existence result of u_+ (see Theorem 1) guarantees, under further regularity assumptions on p and ψ, that u_+ verifies system (C) in the case $N \le 4$.

Theorem 3. *Let all the assumptions of Theorem* 2 *be satisfied and let* $\psi \in H^{2,p}(\Omega)$ *for some* $p \ge 2$. *Then any solution* u *of* (VI) *belongs to* $H^{2,p}(\Omega)$.

Proof. The argument is similar to that given in the first part of the proof of Theorem 2. At first one finds some $k_0 \in \mathbb{N}$ such that (34) holds. At this point one takes into account that if $v_1 \in L^{q_1}$ and $v_2 \in L^{q_2}$, then $v_1 \wedge v_2 \in L^q$ with $q = \min(q_1, q_2)$. So, choosing $k \ge \max(k_0, p)$ (note that (34) holds for $k > k_0$, too) one deduces that u is the solution of an equation of the type $-\Delta u(x) = h(x) \in L^p(\Omega)$. Therefore, still by the classical results of Agmon, Douglis and Nirenberg, one gets the thesis ∎

As for the strict positivity of solutions, one can get the following result as a consequence of Theorem 2.

Theorem 4. *Let all the assumptions of Theorem* 2 *be satisfied, let* $p(x,\xi) \ge 0$ *for all* $x \in \Omega$ *and* $\xi \ge 0$, *and let* $\psi(x) > 0$ *for any* $x \in \Omega$. *Then any non-negative solution* u *of inequality* (VI) *is strictly positive on* Ω.

Proof. By Theorem 2, it follows that either one has

$$u(x) = \psi(x) \tag{35}$$

or

$$-\Delta u(x) = p(x, u(x)) \qquad \forall\, x \in \Omega_\psi = \{x \in \Omega :\, u(x) < \psi(x)\}. \tag{36}$$

For all $x \in \Omega$ such that (35) holds, u is strictly positive as $\psi(x) > 0$ for all $x \in \Omega$. For all $x \in \Omega$ such that (36) holds one can use some analogous arguments as in [4: Corollary 2.23] in order to state that the strong maximum principle applied to the problem

$$(\mathbf{P}_\psi) \begin{cases} -\Delta u(x) = p(x,u(x)) = p_+(x,u(x)) \\ u(x) = \psi(x) \quad (x \in \partial\Omega_\psi) \end{cases}$$

(recall the definition of $p_+(x,t)$ given in Section 1 and note that u is a classical solution of problem (P_ψ)) still gives the strict positivity of u on Ω_ψ. So u is strictly positive on the whole set Ω ∎

Remark 5. By Remark 4, Theorem 4 guarantees the strict positivity of the solution u_+ given by Theorem 1 under the further regularity assumptions on p and ψ given in Theorem 2 and in the case $N \leq 4$.

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Received 28.06.2000/18.02.2001; in revised form 30.08.2001