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Some Distributional Products of Mikusiński Type in the Colombeau Algebra $\mathcal{G}(R^m)$

B. Damyanov

Abstract. Particular products of Schwartz distributions on the Euclidean space $\mathbb{R}^m$ are derived when the latter have coinciding point singularities and the products are 'balanced' so that their sum to give an ordinary distribution. These products follow the pattern of a known distributional product published by Jan Mikusiński in 1966. The results are obtained in the Colombeau algebra $\mathcal{G}(\mathbb{R}^m)$ of generalized functions. $\mathcal{G}(\mathbb{R}^m)$ is a relevant algebraic construction, with the distribution space linearly embedded, which by the notion of 'association' allows the results to be evaluated on the level of distributions.

Keywords: *Schwartz distributions, Colombeau generalized functions, multiplication of distributions*

AMS subject classification: 46F10, 46F30

0. Introduction

In 1966, Jan Mikusiński published his famous result in [7] that

$$x^{-1} \cdot x^{-1} - \pi^2 \delta(x) \cdot \delta(x) = x^{-2} \qquad (x \in \mathbb{R}). \tag{1}$$

Although neither of the products on the left-hand side here exists, their difference still has a correct meaning in the distribution space $\mathcal{D}'(\mathbb{R})$. Formulas of this type can be found in mathematical and physical literature. We think it relevant to name such equations *products of Mikusiński type* or *M-type products*, for short. In a previous paper [5] we derived a generalization of the basic Mikusiński product (1) in the Colombeau algebra of tempered generalized functions (see equation (2) below).

Recall that the Colombeau algebra $\mathcal{G}$ introduced in [1] has followed several constructions of differential algebras that include distributions, proposed by König, Berg, Antonevich and Radyno, Egorov and other authors. Lately, the algebra $\mathcal{G}$ became very popular since it has almost optimal properties, as long as Schwartz distributions and the problem of their multiplication are concerned. $\mathcal{G}$ is an associative differential algebra with the distributions linearly embedded into it, and the multiplication is compatible with differentiation and multiplication by C^∞-differentiable functions. Moreover, the

B. Damyanov: Bulgarian Acad. Sci., INRNE - Theory Group, 72 Tzarigradsko Shosse, 1784 Sofia, Bulgaria; damyanov@netel.bg

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so-called *association* in $\mathcal{G}$, being a faithful generalization of the equality of distributions, leads to results in terms of distributions (and numerical factors).

In this paper, we continue the study in [2]. Following the above approach, we obtain results on singular M-type products of distributions of several variables, as embedded into the Colombeau algebra, when the products admit associated distributions.

1. Definitions and preliminary results

We introduce first some basic notions from Colombeau theory.

Notation 1.1.

(a) If $\mathbb{N}_0$ stands for the set of all non-negative integers and $p = (p_1, ..., p_m)$ is a multi-index in $\mathbb{N}_0^m$, we let $|p| = \sum_{i=1}^m p_i$ and $p! = p_1! \cdots p_m!$. Then, if $x = x_1 \cdots x_m \in \mathbb{R}^m$, we denote

$$x^p = x_1^{p_1} \cdots x_m^{p_m} \qquad \text{and} \qquad \partial^p = \frac{\partial^{|p|}}{\partial x_1^{p_1} \cdots \partial x_m^{p_m}}.$$

Also, $x < 0$ means $x_1 \leq 0, ..., x_m \leq 0$ and $x \neq 0$.

(b) If $q \in \mathbb{N}_0$, we put

$$A_q(\mathbb{R}) = \left\{ \varphi \in \mathcal{D}(\mathbb{R}) \;\middle|\; \begin{array}{l} \int_{\mathbb{R}} x^j \varphi(x)\, dx = \delta_{0j} \text{ for } 0 \leq j \leq q \\ \text{where } \delta_{00} = 1 \text{ and } \delta_{0j} = 0 \text{ for } j > 0 \end{array} \right\}.$$

This definition extends to $\mathbb{R}^m$ as an m-fold product:

$$A_q(R^m) = \left\{ \varphi \in \mathcal{D}(\mathbb{R}^m) : \varphi(x_1, ..., x_m) = \prod_{i=1}^m \chi(x_i) \text{ for some } \chi \in A_q(\mathbb{R}) \right\}.$$

Finally, we denote $\varphi_\varepsilon(x) = \varepsilon^{-m} \varphi(\varepsilon^{-1} x)$ $(x \in \mathbb{R}^m)$ for $\varphi \in A_q(\mathbb{R}^m)$ and $\varepsilon > 0$.

Definition 1.2. Let $\mathcal{E}[\mathbb{R}^m]$ be the algebra of functions $f(\varphi, x) : A_0(\mathbb{R}^m) \times \mathbb{R}^m \to \mathbb{C}$ that are infinitely differentiable, by a fixed 'parameter' φ. The generalized functions of Colombeau are elements of the quotient algebra $\mathcal{G} = \mathcal{G}(\mathbb{R}^m) = \mathcal{E}_M[\mathbb{R}^m]/\mathcal{I}[\mathbb{R}^m]$. Here $\mathcal{E}_M[\mathbb{R}^m]$ is the subalgebra of 'moderate' functions such that for each compact subset $K \subset \mathbb{R}^m$ and $p \in \mathbb{N}_0^m$ there is a $q \in \mathbb{N}$ such that, for each $\varphi \in A_q(\mathbb{R}^m$, $\sup_{x \in K} |\partial^p f(\varphi_e, x)| = O(\varepsilon^{-q})$ as $\varepsilon \to 0_+$. In turn, the ideal $\mathcal{I}[\mathbb{R}^m]$ of $\mathcal{E}_M[\mathbb{R}^m]$ is the set of functions such that for each compact subset $K \subset \mathbb{R}^m$ and any $p \in \mathbb{N}_0^m$ there is a $q \in \mathbb{N}$ such that, for every $r \geq q$ and $\varphi \in A_r(\mathbb{R}^m)$, $\sup_{x \in K} |\partial^p f(\varphi_\varepsilon, x)| = O(\varepsilon^{r-q})$ as $\varepsilon \to 0_+$.

The algebra $\mathcal{G}$ contains the distributions on $\mathbb{R}^m$, canonically embedded as a $\mathbb{C}$-vector subspace by the map $i : \mathcal{D}'(\mathbb{R}^m) \to \mathcal{G}$, $u \mapsto \widetilde{u} = \{\widetilde{u}(\varphi, x) = (u * \check{\varphi})(x)\}$ where $\check{\varphi}(x) = \varphi(-x)$ and φ is running the set $A_q(\mathbb{R}^m)$.

Definition 1.3.

(a) Generalized functions $f, g \in \mathcal{G}$ are said to be *associated*, denoted $f \approx g$, if for some representatives $f(\varphi_\varepsilon, x), g(\varphi_\varepsilon, x)$ and any $\psi \in \mathcal{D}(\mathbb{R}^m)$ there is a $q \in \mathbb{N}_0$ such that, for any $\varphi \in A_q(\mathbb{R})$, $\lim_{\varepsilon \to 0_+} \int_{\mathbb{R}} [f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x)] \psi(x)\, dx = 0$.

(b) An $f \in \mathcal{G}$ is said to admit some $u \in \mathcal{D}'(\mathbb{R})^m$ as *associated distribution*, denoted $f \approx u$, if for some representative $f(\varphi_\varepsilon, x)$ of f and any $\psi \in \mathcal{D}(\mathbb{R}^m)$ there is a $q \in \mathbb{N}_0$ such that, for any $\varphi \in A_q(\mathbb{R})$, $\lim_{\varepsilon\to 0_+} \int_{\mathbb{R}} f(\varphi_\varepsilon, x)\psi(x)\,dx = \langle u, \psi\rangle$.

Definitions 1.3 are independent of the representative chosen. The distribution associated, if it exists, is unique. The image in $\mathcal{G}$ of every distribution is associated with the latter, the association thus being a faithful generalization of the equality of distributions.

Then, by the product of some distributions in the algebra $\mathcal{G}$ (called 'Colombeau product' sometimes), it is meant the product of their embeddings into $\mathcal{G}$, whenever the result admits an associated distribution. The following interpretation is in order: If we reduce the information contained in the product down to the level of distribution theory, i.e. consider its limiting action on test functions ψ, then the product behaves like the distribution associated. A characterization of Colombeau product and comparison with other distributional products can be found in [4].

Now, let $\widetilde{x^{-p}}$ and $\widetilde{\delta^{(p)}}(x)$ be the embeddings into $\mathcal{G}(\mathbb{R})$ of the distributions x^{-p} and $\delta^{(p)}(x)$ $(p \in \mathbb{N})$. The following M-type product was proved in [3], which generalizes the basic Mikusiński formula (1) for arbitrary $p, q \in \mathbb{N}$:

$$\widetilde{x^{-p}} \cdot \widetilde{x^{-q}} - \pi^2 \frac{(-1)^{p+q}}{(p-1)!\,(q-1)!} \widetilde{\delta^{(p-1)}}(x) \cdot \widetilde{\delta^{(q-1)}}(x) \approx x^{-p-q} \qquad (x \in \mathbb{R}). \qquad (2)$$

We recall next some necessary results, starting with an extension of the multi-index notation.

Notation 1.4.

(a) If $a = (a_1, \ldots, a_m)$ are ordered m-tuples in $\mathbb{R}^m$ with the vector operations, and $k \in \mathbb{Z}$ (the set of all integers), we specify that $a + k$ stands for $(a_1 + k, \ldots, a_m + k)$, $a > k$ amounts to $a_i > k$ $(i = 1, ..., m)$ and 0 is the zero vector in $\mathbb{R}^m$. We will also use the short-hand notations $x^a = x_1^{a_1} \cdots x_m^{a_m}$ and $\Gamma(a) = \prod_{i=1}^m \Gamma(a_i)$ $(= p!$ whenever $a = p + 1 \in \mathbb{N}_o^m)$. Finally, we let $\Omega = \{a \in \mathbb{R} : a \neq -1, -2, \ldots\}$ and $\Omega^m = \Omega \times \cdots \times \Omega$ to be the m-fold tensor product.

(b) Denote further the *normed* powers of the variable $x \in \mathbb{R}^m$ for arbitrary $a \in \Omega$ that are supported only in one quadrant of the Euclidean space $\mathbb{R}^m$ as follows:

$$\nu_+^a \equiv \nu_+^a(x) = \begin{cases} \frac{x^a}{\Gamma(a+1)} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\nu_-^a \equiv \nu_-^a(x) = \begin{cases} \frac{(-x)^a}{\Gamma(a+1)} & \text{for } x < 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Here, in accordance with Notation 1.1/(a), $\frac{x^a}{\Gamma(a+1)} = \prod_{i=1}^m \frac{x_i^{a_i}}{\Gamma(a_i)}$. Since $x \to \nu_\pm^a(x)$ is a locally integrable function for $a > -1$, we can define the distributions $\nu_\pm^a$, for arbitrary $a \in \Omega^m$, if we choose some $k \in \mathbb{N}_0$ subject to the condition $a + k + 1 > 0$ and then put

$$\nu_+^a = \partial_x^k \nu_+^{a+k}(x)$$

$$\nu_-^a = (-1)^k \partial_x^k \nu_-^{a+k}(x)$$

where the derivatives are in distributional sense. Note that, due to the equations $\partial_x \nu_\pm^a = \pm\nu_\pm^{a-1}$ (with no number coefficients) the calculations with these distributions are facilitated.

With the above notation, the following Colombeau products were proved in [2].

Theorem 1.

(i) *For an arbitrary* $p \in \mathbb{N}_0^m$, *the embeddings* $\widetilde{\nu_\pm^p}$ *and* $\widetilde{\delta^{(p)}}(x)$ *into* $\mathcal{G}(\mathbb{R}^m)$ *of the distributions* $\nu_\pm^p$ *and* $\delta^{(p)}(x)$ *satisfy*

$$\left.\begin{aligned} \widetilde{\nu_+^p} \cdot \widetilde{\delta^{(p)}}(x) \approx (-1)^{|p|} 2^{-m} \delta(x) \\ \widetilde{\nu_-^p} \cdot \widetilde{\delta^{(p)}}(x) \approx 2^{-m} \delta(x) \end{aligned}\right\} \qquad (x \in \mathbb{R}^m). \tag{3}$$

(ii) *For any* $a, b \in \Omega^m$ *such that* $a + b + 1 = 0$ *the embeddings* $\widetilde{\nu_\pm^a}$ *into* $\mathcal{G}(\mathbb{R}^m)$ *of the distributions* $\nu_\pm^a$ *satisfy*

$$\widetilde{\nu_+^a} \cdot \widetilde{\nu_-^b} = \widetilde{\nu_-^a} \cdot \widetilde{\nu_+^b} \approx 2^{-m} \delta(x) \qquad (x \in \mathbb{R}^m). \tag{4}$$

Let us remark that equations (3) and (4) are known in distribution theory, but they have been only derived in dimension one as regularized products, using symmetric mollifiers.

2. Results on Mikusiński type distributional products in $\mathcal{G}(\mathbb{R}^m)$

We now proceed to singular M-type products of the distributions $\nu_\pm^a$ and $\delta^{(p)}$ in the Colombeau algebra $\mathcal{G}(\mathbb{R}^m)$ that extend further the results obtained in [2]. The following general property of certain M-type products of distributions on $\mathbb{R}^m$ having tensor-product structure will be employed.

Theorem 2. *Let* $u_k, v_k \ \ (k = 1, 2)$ *be distributions in* $\mathcal{D}'(\mathbb{R}^m)$ *such that* $u_k(x) = \prod_{i=1}^m u_k^i(x_i)$ *and* $v_k(x) = \prod_{i=1}^m v_k^i(x_i)$. *If all* u_k^i, v_k^i *are distributions in* $\mathcal{D}'(\mathbb{R}^m)$ *and their embeddings into* $\mathcal{G}(\mathbb{R}^m)$ *satisfy* $\widetilde{u_1^i} \cdot \widetilde{v_1^i} - \widetilde{u_2^i} \cdot \widetilde{v_2^i} \approx 0 \ \ (i = 1, ..., m)$, *then the embeddings into* $\mathcal{G}(\mathbb{R}^m)$ *of the tensor-product distributions* u_k, v_k *satisfy*

$$\widetilde{u_1} \cdot \widetilde{v_1} - \widetilde{u_2} \cdot \widetilde{v_2} \approx 0. \tag{5}$$

Proof. By the linearity of Definition 1.3 we have $\widetilde{u_1^i} \cdot \widetilde{v_1^i} \approx \widetilde{u_2^i} \cdot \widetilde{v_2^i}$ which holds in $\mathcal{G}(\mathbb{R}^m)$ for $i = 1, ...m$. Suppose that we have restricted ourselves to the subspace of test functions $\psi(x) = \prod_{i=1}^m \psi_i(x_i)$ with each $\psi_i \in \mathcal{D}(\mathbb{R})$. Then, in view of the tensor-product structure both of the distributions $u_k, v_k \in \mathcal{D}'(\mathbb{R}^m)$ and the parameter functions $\varphi \in A_0(\mathbb{R}^m)$, on applying a Fubini-type theorem for tensor-product distributions [5: Section 4.3], we get for the functional value

$$\begin{aligned} \left\langle \widetilde{u_1}(\varphi_\varepsilon, x)\, \widetilde{v_1}(\varphi_\varepsilon, x), \psi(x) \right\rangle &= \left\langle \prod_{i=1}^m \widetilde{u_1}^i(\chi_\varepsilon, x_i) \prod_{i=1}^m \widetilde{v_1}^i(\chi_\varepsilon, x_i), \prod_{i=1}^m \psi_i(x_i) \right\rangle \\ &= \prod_{i=1}^m \left\langle \widetilde{u_1}^i(\chi_\varepsilon, x_i)\, \widetilde{v_1}^i(\chi_\varepsilon, x_i), \psi_i(x_i) \right\rangle \\ &= \prod_{i=1}^m \left[\left\langle \widetilde{u_2}^i(\chi_\varepsilon, x_i)\, \widetilde{v_2}^i(\chi_\varepsilon, x_i), \psi_i(x_i) \right\rangle + o^i(1) \right] \\ &= \left\langle \widetilde{u_2}(\varphi_\varepsilon, x)\, \widetilde{v_2}(\varphi_\varepsilon, x), \psi(x) \right\rangle + o(1). \end{aligned}$$

Here each Landau symbol $o(1)$ stands for an arbitrary function of asymptotic order less than any constant, or equivalently, that tends to 0 as $\varepsilon \to 0$. Therefore, we have

$$\lim_{\varepsilon\to 0}\left\langle \left[\widetilde{u_1}(\varphi_\varepsilon,x)\,\widetilde{v_1}(\varphi_\varepsilon,x) - \widetilde{u_2}(\varphi_\varepsilon,x)\,\widetilde{v_2}(\varphi_\varepsilon,x)\right],\ \psi(x)\right\rangle = 0.$$

Now, since the set of test functions $\psi(x) = \prod_{i=1}^m \psi_i(x_i)$ is a dense subset of $\mathcal{D}(\mathbb{R}^m)$ [5: Section 4.3], it follows by Definition 1.3 that the M-type product (5) holds for the distributions u_k and v_k ∎

We are now in position to prove the following

Theorem 3. *For any $a, b \in \Omega^m$ such that $a + b > -2$ the M-type product*

$$\widetilde{\nu_+^a}\cdot\widetilde{\nu_-^b} - \widetilde{\nu_-^{a+b+1}}\cdot\widetilde{\delta}(x) \approx 0 \qquad (x\in\mathbb{R}^m) \tag{6}$$

holds for the embeddings into $\mathcal{G}(\mathbb{R}^m)$ of the distributions $\nu_\pm^a$ and $\delta(x)$.

Proof. Consider first the one-variable case, i.e. $x \in \mathbb{R}$ and $a \in \Omega$. Suppose that $k \in \mathbb{N}_0$ is such that $k > \max\{-a-1, -b-1\}$. Then, to get the embedding into $\mathcal{G}(\mathbb{R})$ of the distribution $\widetilde{\nu_+^a}$ we use the derivative in the Colombeau algebra, which gives

$$\begin{aligned}
\widetilde{\nu_+^a}(\varphi_\varepsilon,x) &= \frac{\varepsilon^{-1}}{\Gamma(a+k+1)}\partial_x^k\left(\int_0^\infty (y^{a+k})\,\varphi\left(\frac{y-x}{\varepsilon}\right)dy\right)\\
&= \frac{(-1)^k\varepsilon^{-1-k}}{\Gamma(a+k+1)}\int_0^\infty y^{a+k}\varphi^{(k)}\left(\frac{y-x}{\varepsilon}\right)dy\\
&= \frac{(-1)^k\varepsilon^{-k}}{\Gamma(a+k+1)}\int_{-x/\varepsilon}^{l}(x+\varepsilon u)^{a+k}\varphi^{(k)}(u)\,du.
\end{aligned}$$

It is assumed here (without lost of generality) that $\operatorname{supp}\varphi \subseteq [-l,l]$ for some $l \in \mathbb{R}_+$ and the substitution $u = \frac{y-x}{\varepsilon}$ has been made. Similarly, taking into account the equation $\partial_x^k\,\nu_-^{b+k} = (-1)^k\nu_-^b$ we get

$$\begin{aligned}
\widetilde{\nu_-^b}(\varphi_\varepsilon,x) &= \frac{\varepsilon^{-1-k}}{\Gamma(b+k+1)}\int_{-\infty}^0(-y)^{b+k}\varphi^{(k)}\left(\frac{y-x}{\varepsilon}\right)dy\\
&= \frac{\varepsilon^{-k}}{\Gamma(b+k+1)}\int_{-l}^{-x/\varepsilon}(-x-\varepsilon v)^{b+k}\varphi^{(k)}(v)\,dv.
\end{aligned}$$

For any $\psi \in \mathcal{D}(\mathbb{R})$, denoting the functional value $F_1(\varepsilon) = \langle\widetilde{\nu_+^a}(\varphi_\varepsilon,x)\,\widetilde{\nu_-^b}(\varphi_\varepsilon,x),\,\psi(x)\rangle$ and also $G = \Gamma(a+k+1)\Gamma(b+k+1)F_1(\varepsilon)$, we have

$$G = \frac{(-1)^k}{\varepsilon^{2k}}\int_{-l\varepsilon}^{l\varepsilon}\psi(x)\int_{-x/\varepsilon}^{l}\varphi^{(k)}(u)\int_{-l}^{-x/\varepsilon}(x+\varepsilon u)^{a+k}(-x-\varepsilon v)^{b+k}\varphi^{(k)}(v)\,dv\,du\,dx.$$

We have taken into account that $-l \le -\frac{x}{\varepsilon} \le l$, that is $-l\varepsilon \le x \le l\varepsilon$. Further, on making the substitution $w = -\frac{x}{\varepsilon}$ we obtain

$$\begin{aligned}
G &= (-1)^k\varepsilon^{a+b+1}\\
&\quad\times\int_{-l}^{l}\psi(-\varepsilon w)\int_{w}^{l}\varphi^{(k)}(u)\int_{-l}^{w}(u-w)^{a+k}(w-v)^{b+k}\varphi^{(k)}(v)\,dv\,du\,dw.
\end{aligned}$$

Now, by the Taylor theorem, $\psi(-\varepsilon w) = \psi(0) - (\varepsilon w)\psi'(\eta w) \quad (\eta \in (0,1))$. Employing this and changing twice the order of integration – which is permissible here – we get

$$G = (-1)^k \psi(0)\, \varepsilon^{a+b+1} \\ \times \int_{-l}^{l} \varphi^{(k)}(u) \int_{-l}^{u} \varphi^{(k)}(v) \int_{v}^{u} (u-w)^{a+k}(w-v)^{b+k}\, dwdvdu + o(1).$$

Note that to obtain the last term we have taken into account the requirement $a + b > -2$ and that the multiplying expression amounts to definite integrals majorizable by constants. Further, the substitution $w \to t = \frac{w-v}{u-v}$ together with the relations $w - v = (u-v)t$ and $u - w = (u-v)(1-t)$ gives

$$G = (-1)^k\, \psi(0)\, \varepsilon^{a+b+1} \\ \times \int_{-l}^{l} \varphi^{(k)}(u) \int_{-l}^{u} (u-v)^{a+b+2k+1} \varphi^{(k)}(v)\, dvdu \int_0^1 (1-t)^{a+k}\, t^{b+k} dt + o(1).$$

By the definition of the first-order Euler integral [6: Section 22.4.4],

$$\int_0^1 (1-t)^{a+k}\, t^{b+k} dt = \frac{\Gamma(a+k+1)\Gamma(b+k+1)}{\Gamma(a+b+2k+2)} \qquad (a+k, b+k > -1).$$

Taking account of this and integrating k-times by parts with respect to the variable v – the integrated part being zero each time – we obtain

$$\begin{aligned} F_1(\varepsilon) &= \frac{(-1)^k \psi(0)\, \varepsilon^{a+b+1}}{\Gamma(a+b+2k+2)} \int_{-l}^{l} \varphi^{(k)}(u) \int_{-l}^{u} (u-v)^{a+b+2k+1} \varphi^{(k)}(v)\, dvdu + o(1) \\ &= \frac{(-1)^k \psi(0)\, \varepsilon^{a+b+1}}{\Gamma(a+b+k+2)} \int_{-l}^{l} \varphi^{(k)}(u) \int_{-l}^{u} (u-v)^{a+b+k+1} \varphi(v)\, dvdu + o(1). \end{aligned}$$

Recall that the formula for differentiation of integrals

$$\partial_u \left(\int_{-l}^{u} T(u,v)\, dv \right) = \int_{-l}^{u} \partial_u T(u,v)\, dv + T(u,v)|_{v=u} \tag{7}$$

holds [6: Section 4.6.1]. This equation, the fact that $T(u,v)|_{v=u} = 0$ in the particular case under calculation, as well as the requirement $a + b > -2$ will all yield

$$\begin{aligned} F_1(\varepsilon) &= \frac{\psi(0)\, \varepsilon^{a+b+1}}{\Gamma(a+b+k+2)} \int_{-l}^{l} \varphi(u)\, \partial_u^k \left(\int_{-l}^{u} (u-v)^{a+b+k+1} \varphi(v)\, dv \right) du + o(1) \\ &= \frac{\psi(0)\, \varepsilon^{a+b+1}}{\Gamma(a+b+2)} \int_{-l}^{l} \varphi(u) \int_{-l}^{u} (u-v)^{a+b+1} \varphi(v)\, dvdu + o(1). \end{aligned} \tag{8}$$

Consider next the functional value

$$F_2(\varepsilon) = \left\langle \widetilde{\nu_-^{a+b+1}}(\varphi_\varepsilon, x)\, \widetilde{\delta}(\varphi_\varepsilon, x), \psi(x) \right\rangle$$

where $a + b > -2$ and $\psi \in \mathcal{D}(\mathbb{R})$. On changing the variables and applying the Taylor theorem, we obtain

$$\begin{aligned} F_2(\varepsilon) &= \frac{\varepsilon^{-2}}{\Gamma(a+b+2)} \int_{-l\varepsilon}^{l\varepsilon} \psi(x)\varphi\left(-\frac{x}{\varepsilon}\right) \int_{-l\varepsilon+x}^{0} (-y)^{a+b+1}\varphi\left(\frac{y-x}{\varepsilon}\right) dydx \\ &= \frac{\varepsilon^{a+b+1}}{\Gamma(a+b+2)} \int_{-l}^{l} \psi(-\varepsilon u)\varphi(u) \int_{-l}^{u} (u-v)^{a+b+1}\varphi(v)\, dvdu \\ &= \frac{\psi(0)\,\varepsilon^{a+b+1}}{\Gamma(a+b+2)} \int_{-l}^{l} \varphi(u) \int_{-l}^{u} (u-v)^{a+b+1}\varphi(v)\, dvdu + o(1). \end{aligned} \tag{9}$$

We now distinguish the following subcases:

(a) $a+b+1 > 0$. Then $\lim_{\varepsilon\to 0} F_1(\varepsilon) = \lim_{\varepsilon\to 0} F_2(\varepsilon) = 0$.

(b) $a+b+1 = 0$, or else, $b = -a-1$. In this particular case, equations (3) and (4) for $m = 1$ yield

$$\widetilde{\nu_+^a} \cdot \widetilde{\nu_-^b} - \widetilde{\nu_-^{a+b+1}} \cdot \widetilde{\delta}(x) = \widetilde{\nu_+^a} \cdot \widetilde{\nu_-^{-a-1}} - \widetilde{H} \cdot \widetilde{\delta}(x) \approx \tfrac{1}{2}\delta(x) - \tfrac{1}{2}\delta(x) = 0.$$

(c) $-2 < a+b < -1$. Equations (8) and (9) now give $F_1(\varepsilon) - F_2(\varepsilon) = o(1)$. Thus

$$\lim_{\varepsilon\to 0} \langle \widetilde{\nu_+^a}(\varphi_\varepsilon, x)\, \widetilde{\nu_+^b}(\varphi_\varepsilon, x) - \widetilde{\nu_-^{a+b+1}} \cdot \widetilde{\delta}(x),\, \psi(x)\rangle = 0.$$

By Definition 1.3, this proves equation (6) for $x \in \mathbb{R}$. Allowing for Notation 1.4, the result in the case of several variables follows immediately from Theorem 2 ∎

Replacing further x by $-x$ and interchanging a and b, we obtain

Corollary 4. *For any $a, b \in \Omega^m$ such that $a + b > -2$, the M-type product*

$$\widetilde{\nu_+^a} \cdot \widetilde{\nu_-^b} - \widetilde{\nu_+^{a+b+1}} \cdot \widetilde{\delta}(x) \approx 0 \qquad (x \in \mathbb{R}^m) \tag{10}$$

holds in $\mathcal{G}(\mathbb{R}^m)$.

The next assertion specifies further the results given by equations (6) and (10).

Theorem 5. *If $a, b \in \Omega^m$ are such that $a + b > -2$, then for each $p \in \mathbb{N}_0^m$ the M-type products*

$$\widetilde{\nu_+^a} \cdot \widetilde{\nu_-^b} - \widetilde{\nu_-^{a+b+p+1}} \cdot \widetilde{\delta^{(p)}}(x) \approx 0 \tag{11}$$

$$\widetilde{\nu_+^a} \cdot \widetilde{\nu-^b} - (-1)^{|p|}\widetilde{\nu_+^{a+b+p+1}} \cdot \widetilde{\delta^{(p)}}(x) \approx 0 \tag{12}$$

hold in $\mathcal{G}(\mathbb{R}^m)$.

Proof. In the one-variable case, for arbitrary $\psi \in \mathcal{D}(\mathbb{R})$ and $p \in \mathbb{N}_0$ consider the functional value

$$F(\varepsilon) = \left\langle \widetilde{\nu_-^{a+b+p+1}}(\varphi_\varepsilon, x)\, \widetilde{\delta^{(p)}}(\varphi_\varepsilon, x),\, \psi(x) \right\rangle.$$

On changing the variables and applying the Taylor theorem, we obtain

$$
\begin{aligned}
F(\varepsilon) &= \frac{(-1)^p\,\varepsilon^{-p-2}}{\Gamma(a+b+p+2)} \int_{-l\varepsilon}^{l\varepsilon} \psi(x)\varphi^{(p)}\left(-\frac{x}{\varepsilon}\right) \int_{-l\varepsilon+x}^{0} (-y)^{a+b+p+1} \varphi\left(\frac{y-x}{\varepsilon}\right) dydx \\
&= \frac{(-1)^p\,\varepsilon^{-p}}{\Gamma(a+b+p+2)} \int_{-l}^{l} \psi(-\varepsilon u)\varphi^{(p)}(u) \int_{-l}^{u} (u-v)^{a+b+p+1} \varphi(v)\, dvdu \\
&= \frac{(-1)^p\,\psi(0)\,\varepsilon^{a+b+1}}{\Gamma(a+b+p+2)} \int_{-l}^{l} \varphi^{(p)}(u) \int_{-l}^{u} (u-v)^{a+b+p+1}\, \varphi(v)\, dvdu + o(1).
\end{aligned}
$$

Taking further into account equation (7), with $T(u,v)|_{v=u} = 0$ in this particular case, as well as the requirement $a+b > -2$, we get

$$
\begin{aligned}
F(\varepsilon) &= \frac{\psi(0)\,\varepsilon^{a+b+1}}{\Gamma(a+b+p+2)} \int_{-l}^{l} \varphi(u)\, \partial_u^p \left(\int_{-l}^{u} (u-v)^{a+b+p+1} \varphi(v)\, dv \right) du + o(1) \\
&= \frac{\psi(0)\,\varepsilon^{a+b+1}}{\Gamma(a+b+2)} \int_{-l}^{l} \varphi(u) \int_{-l}^{u} (u-v)^{a+b+1} \varphi(v)\, dvdu + o(1).
\end{aligned}
\tag{13}
$$

On the other hand, following exactly the calculations in the proof of Theorem 3, for the functional value

$$
F_1(\varepsilon) = \left\langle \widetilde{\nu_+^a}(\varphi_\varepsilon, x)\, \widetilde{\nu_-^b}(\varphi_\varepsilon, x),\, \psi(x) \right\rangle
$$

we get

$$
F_1(\varepsilon) = \frac{\psi(0)\,\varepsilon^{a+b+1}}{\Gamma(a+b+2)} \int_{-l}^{l} \varphi(u) \int_{-l}^{u} (u-v)^{a+b+1} \varphi(v)\, dvdu + o(1). \tag{14}
$$

Now equations (13) and (14) give $F_1(\varepsilon) - F(\varepsilon) = o(1)$, and therefore

$$
\lim_{\varepsilon \to 0} \left\langle \widetilde{\nu_+^a}(\varphi_\varepsilon, x)\, \widetilde{\nu_-^b}(\varphi_\varepsilon, x) - \widetilde{\nu_-^{a+b+p+1}}(\varphi_\varepsilon, x)\, \widetilde{\delta^{(p)}}(\varphi_\varepsilon, x),\, \psi(x) \right\rangle = 0.
$$

By Definition 1.3 this proves equation (11) for $x \in \mathbb{R}$. Taking account of Notation 1.4, the result in the case of several variables follows directly from Theorem 2. If we replace x by $-x$, interchange a and b, and apply the formula $\delta^{(p)}(-x) = (-1)^{|p|}\partial^p\delta(x)$ for $x \in \mathbb{R}^m$, we obtain equation (12) ∎

Combining now the results given by equations (6), (11) and (10), (12), respectively, we obtain

Corollary 6. *If $-1 < c \in \mathbb{R}^m$, then for each $p \in \mathbb{N}^m$ the M-type products*

$$
\begin{aligned}
\widetilde{\nu_-^{c+p}} \cdot \widetilde{\delta^{(p)}}(x) - \widetilde{\nu_-^c} \cdot \widetilde{\delta}(x) \approx 0 \\
\widetilde{\nu_+^{c+p}} \cdot \widetilde{\delta^{(p)}}(x) - (-1)^{|p|} \widetilde{\nu_+^c} \cdot \widetilde{\delta}(x) \approx 0
\end{aligned}
$$

hold in $\mathcal{G}(\mathbb{R}^m)$.

We note that the particular case $c = 0$ is in consistency with 'ordinary' Colombeau products given by equation (3) above.

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