Asymptotical Behavior
of Solutions of Nonlinear Elliptic Equations in $\mathbf{R}^N$

M. Grillot and P. Grillot

Abstract. In this paper we study the behavior near infinity of non-negative solutions $u \in C^2(\mathbb{R}^N)$ of the semi-linear elliptic equation

$$-\Delta u + u^q - u^p = 0$$

where $q \in (0,1)$, $p > q$ and $N \geq 2$. Especially, for a non-negative radial solution of this equation we prove the following alternative :

  either u has a compact support
  or u tends to one at infinity.

Moreover, we prove that if a general solution is sufficiently small in some sense, then it is compactly supported. To prove this result we use some inequalities between the solution and its spherical average at a shift point and consider a differential inequality. Finally, we prove the existence of non-trivial solutions which converge to one at infinity.

Keywords: *Laplacian, non-linearity, asymptotical behavior*

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1. Introduction

In this article we study non-negative solutions of the semi-linear elliptic equation with non-Lipschitzian non-linearity

$$-\Delta u + u^q - u^p = 0 \tag{1}$$

with $q \in (0,1)$ and $p > q$. This equation and, more generally, the equation

$$-\Delta u + f(u) = 0 \tag{2}$$

where $f : [0, +\infty) \to \mathbb{R}$ is a given continuous function, appears in models for many physical situations. On the one hand, equation (2) can be considered as non-linear

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Euclidean scalar field equation (see [2]). On the other hand, equation (2) can correspond to the stationary problem of the non-linear evolution equation $\frac{\partial u}{\partial t} = \Delta u + f(u)$. This equation occurs, for example, in population dynamics and chemical reactions (see [7, 8, 10]).

Our first question is the following:

Letting $u \in C^2(\mathbb{R}^N)$ be a non-negative solution of equation (1) in $\mathbb{R}^N$ with $N \geq 2$, can we describe the precise behavior of u near infinity? In the general case, this question is very difficult. For example, the sign of the Laplacian of u is not constant in the case where u oscillates around one. Another difficulty is that the non-linearity is non-Lipschitzian. Few people have tackled this question. The only results that we know are due to Cortazar, Elgueta and Felmer [6]. They consider the case $0 < q < 1 < p < \frac{N+2}{N-2}$ and $N \geq 3$ and prove that every $H^1(\mathbb{R}^N)$-function which satisfies equation (1) in the sense of distributions is a classical solution of this equation with compact support. Moreover, if this solution is positive, then it is radial. Therefore, it seems that the radial case is an important step in the study of equation (1).

Note that the function identically equal to one in $\mathbb{R}^N$ is a solution of equation (1) which is not in $H^1(\mathbb{R}^N)$.

Section 1 concerns the radial case. Our results complete those of [6]. We give a complete classification of solutions of equation (1) under the only restriction $q \in (0, 1)$, $p > q$ and $N \geq 2$. Our main result is given in

Theorem 1. *If $u \in C^2(\mathbb{R}^N)$ is a non-negative radial solution of equation (1) such that $u \not\equiv 0$ and $u \not\equiv 1$, then either u has compact support or u tends to one at infinity.*

The proof of this theorem consists of several steps. We begin to prove by energy arguments that the function u is bounded. We distinguish two cases according to the monotonicity of u. In the case where u is non-monotone, which is the more difficult one, we prove that u oscillates necessary around one. More precise, we give a complete analysis on the length of the oscillation of a solution u.

The second question is the existence of solutions of equation (1). Section 2 is devoted to this problem. If $0 < q < 1 < p < \frac{N+2}{N-2}$, then in [6] the existence of a non-trivial solution of equation (1) in $\mathbb{R}^N$ with compact support is proved. On the other hand, if we consider radial solutions, equation (1) is reduced to the ordinary differential equation

$$u'' + \tfrac{N-1}{r} u' = u^q - u^p \tag{3}$$

where u' denotes the derivative of u. Another result of [6] asserts that if $0 < q < 1 < p < \frac{N+2}{N-2}$, there exists a unique non-trivial solution of equation (3) such that

$$\left. \begin{array}{l} u(r) \geq 0 \ \ \text{in } [0, +\infty) \\ u(0) = 0 \end{array} \right\}. \tag{4}$$

We state a result in an exterior domain. Troughout the paper we denote by b^* the number

$$b^* = \left(\tfrac{p+1}{q+1} \right)^{\frac{1}{p-q}}. \tag{5}$$

Theorem 2. *Assume $q \in (0,1)$, $p > q$ and $N \geq 2$. Let $\gamma \in (0, b^*]$ and $r_0 > 0$. Then there exists a unique solution u of equation (3) in $(r_0, +\infty)$ such that $u(r_0) = \gamma$ and $u'(r_0) = 0$. Moreover, u is positive and converges to one at infinity.*

The uniqueness of solutions of equations (3) - (4) is proved in [6] whithout restrictions on p and q.

If we are not in the case $0 < q < 1 < p < \frac{N+2}{N-2}$, the existence of a non-negative solution of equation (3) in $(r_0, +\infty)$ with $u(r_0) = \gamma > b^*$ and $u'(r_0) = 0$ is an open problem.

The following notations are introduced in [6]. For $q \in (0,1)$ and $p > q$, let u be a solution of equation (3) in $(r_0, +\infty)$ such that $u(r_0) = \gamma > 0$ and $u'(r_0) = 0$. Define

$$R(\gamma) = \sup\left\{r > r_0 : u(s) > 0 \text{ and } u'(s) < 0 \text{ for all } s \in (r_0, r)\right\}.$$

It is proved in [6] that if $\gamma > 1$, then $R(\gamma) < \infty$. There are also introduced the sets

$$N = \left\{\gamma : \lim_{r \to R(\gamma)} u(r) = 0 \text{ and } \lim_{r \to R(\gamma)} u'(r) < 0\right\}$$

$$G = \left\{\gamma : \lim_{r \to R(\gamma)} u(r) = 0 \text{ and } \lim_{r \to R(\gamma)} u'(r) = 0\right\}$$

$$P = \left\{\gamma : \lim_{r \to R(\gamma)} u(r) > 0\right\}.$$

Note that these sets are mutually disjoint and that they form a partition of the interval $(1, +\infty)$. Also, it is proved that N and P are open subsets of $(1, +\infty)$. Theorem 2 implies $(1, b^*] \subset P$, and this result entails the existence of a real number $b > b^*$ such that Theorem 2 still holds for $\gamma \in (0, b)$. We do not know if $b = +\infty$ is possible.

In last, in Section 3 we prove that all small (in some sense) non-negative general solutions of equation (1) have necessarely a compact support. More precisely, we state the following

Theorem 3. *Let η be a positive real number such that $\max(A, B) \leq 1$ where*

$$A = \left(\left(\tfrac{2N}{N}\right)^{p+1-q} + \tfrac{\eta}{2}\right)^{-\frac{1}{p-q}} \qquad and \qquad B = \left(\left(\tfrac{2N}{N}\right)^{1-q} + \tfrac{\eta}{2}\right)^{-\frac{1}{p-q}}.$$

If $u \in C^2(\mathbb{R}^N)$ is a non-negative solution of equation (1) such that $u \not\equiv 0$ and $u(x) \leq \min(A, B)$ for large x, then u has compact support.

The proof of this theorem uses the spherical average of u. First we introduce the spherical coordinate (r, θ) of x in $\mathbb{R}^N$ with $r = |x|$ and $\theta \in S^{N-1}$. Next we denote by $\bar{u}$ the spherical average of u which is defined by

$$\bar{u}(r) = \tfrac{1}{|S^{N-1}|} \int_{S^{N-1}} u(r, \theta) \, d\theta \tag{6}$$

for all $r \geq 0$ and where $|S^{N-1}|$ denotes the Lebesgue measure of the unit sphere of $\mathbb{R}^N$. Now the idea of the proof of Theorem 3 is as follows. Since u satisfies equation (1) and

the inequality $u(x) \leq min(A, B) \leq 1$, u is subharmonic for large x. As in [5] we deduce an inequality between u and $\bar{u}$ of the type $u(x) \leq C\bar{u}(\frac{|x|}{2})$ and prove that $\bar{u}$ necessary tends to 0 at infinity. Finally, a maximum principle of the type of [6] leads us to the conclusion of the theorem.

Note that this property of solutions with compact support has been exhibited in [3, 4] where the problem is no longer in $\mathbb{R}^N$. There the authors consider equation (2) in a bounded domain, for example $B_1 \setminus \{0\}$, where B_1 denotes the unit ball in $\mathbb{R}^N$. In the case where $f(u) = u^q + \left(\frac{c}{|x|^2}\right)u$ with $q \in (0, 1)$ and $c \in \mathbb{R}$, it is proved in [3] that for $N \geq 3$ some solutions of equation (2) have compact support in $B_1 \setminus \{0\}$. The same is true [4] for $f(u) = u^q$ if $N = 2$. We also mention the papers [1, 7, 9] for general semi-linear or quasi-linear equations with monotonous non-linearities.

We finish our paper stating its last result as

Theorem 4. *Assume $q \in (0, 1)$ and $p \geq 1$. If $u \in C^2(\mathbb{R}^N)$ is a solution of equation* (1), *then* $\liminf_{|x| \to +\infty} u(x) \leq 1$. *Moreover, if $u(x) \geq c$ for large x with some $c \in (0, 1)$, then* $\limsup_{|x| \to +\infty} u(x) \geq 1$.

Corollary 1. *There does not exist solutions u of equation* (1) *such that $u(x) \geq c > 1$ for large x.*

Our work leads us to the following

Conjecture. If $u \in C^2(\mathbb{R}^N)$ is a solution of equation (1), then either u has compact support or $\lim_{|x| \to \infty} u(x) = 1$?

We now establish the notations that we will use troughout this paper. We introduce functions F and E defined in $(0, +\infty)$ by

$$F(v) = \frac{1}{q+1} v^{q+1} - \frac{1}{p+1} v^{p+1} \tag{7}$$
$$E(r) = \frac{1}{2} u'^2(r) - F(u(r)) \tag{8}$$

where u is a solution of equation (3). Observe that the function F is increasing in $(0, 1)$, decreasing in $(1, +\infty)$ and positive in $(0, b^*)$ where b^* is defined in (5). On the other hand, if u satisfies equation (3), then the derivative of E is

$$E'(r) = -\frac{N-1}{r} u'^2(r) \tag{9}$$

which implies that E is non-increasing. This will be used in several later comparison arguments.

2. Classification of non-negative radial solutions

Here we establish a classification of non-negative radial solutions of equation (1) starting with a result on their boundedness. We state it with an inequality which will be used later in Section 4.

Lemma 1. *Let $u = u(r)$, $u \in C^2(\alpha, +\infty)$ with $\alpha > 0$, be a solution of the differential inequality*

$$u'' + \frac{N-1}{r}\, u' \leq u^q - u^p \tag{10}$$

in $(\alpha, +\infty)$. Then u is bounded in $(\alpha, +\infty)$.

Proof. Assume that u is unbounded. Then $\limsup_{r \to +\infty} u(r) = +\infty$. If u is monotonous, then E is non-increasing and $\lim_{r \to +\infty} E(r) = +\infty$ which is a contradiction. If u is non-monotonous, then there exist sequences (r_n) and (μ_n) of maxima and minima of u, respectively, such that u is non-decreasing in (μ_n, r_n) and $\lim_{n \to +\infty} u(r_n) = +\infty$. Since u satisfies inequality (10), E is non-increasing in (μ_n, r_n). On the other hand, inequality (10) implies $u(\mu_n) \leq 1$. This entails $E(r_n) \leq E(\mu_n) \leq 0$ which is a contradiction to $\lim_{n \to +\infty} E(r_n) = +\infty$. Therefore u is bounded in $(\alpha, +\infty)$ ∎

Lemma 2. *Lut $u = u(r)$, $u \in C^2(\alpha, +\infty)$ with $\alpha > 0$, be a solution of equation (3). Then u and its derivatives u' and u'' are bounded.*

Proof. Lemma 1 implies that u is bounded. Since E is bounded from above, there exists a constant $C > 0$ such that $u'^2(r) \leq C + 2F(u(r))$. We deduce from the boundedness of u that u' is also bounded. Finally, we deduce from equation (3) that also u'' is bounded ∎

Note that there does not exist a local minimum or maximum point r such that $u(r)$ is strictly greater or less than 1, respectively. Because of Lemma 2 we can introduce some vocabulary:

Definition. We say that $u = u(r)$ *oscillates around one* if for all $R > 0$ there exist two points $r_1 > R$ and $r_2 > R$ such that $u(r_1) = u(r_2) = 1$ and $u(r) > 1$ in (r_1, r_2).

Note that if $u = u(r)$ oscillates around one, then for all $R > 0$ there exist two points $r_1' > R$ and $r_2' > R$ such that $u(r_1') = u(r_2') = 1$ and $u(r) < 1$ in (r_1', r_2'). This is a consequence of equation (3).

Lemma 3. *Let $u = u(r)$, $u \in C^2(0, +\infty)$, be a solution of equation (3) such that u oscillates around one and $\limsup_{r \to +\infty} u(r) = \bar{b} \in (1, b^*)$, with b^* from (5). Then we have the following property:*

(i) *If (σ_n) is a sequence such that $u(\sigma_n) = 1$ and $\lim_{n \to +\infty} \sigma_n = +\infty$, then there exists a real $\gamma > 0$ such that $|u'(\sigma_n)| > \gamma$ for all large n.*

(ii) *If (s_n) and (r_n) are two sequences such that $u(s_n) = 1$ and $u' > 0$ on (s_{2n}, r_n) or $u' < 0$ on (r_n, s_{2n+1}), then*

$$2(u(r_n) - 1) = \sqrt{2} \int_{s_{2n}}^{s_{2n+1}} \sqrt{\left|L + F(u(r)) + O(1)\right|}\, dr \tag{11}$$

as $r_n \to +\infty$, where F is the function defined in (7) and $L = \lim_{r \to +\infty} E(r)$.

Proof.

(i) Since u is a solution of equation (3), definition (8) implies

$$\tfrac{1}{2} u'^2(\sigma_n) = E(\sigma_n) + F(u(\sigma_n)) = E(\sigma_n) + \max_{v \in \mathbb{R}^+} F(v) = E(\sigma_n) + F(1) \qquad (12)$$

for all n. Since by (9) and Lemma 2 E is non-decreasing and bounded, $\lim_{r \to +\infty} E(r) = L$. We deduce from (12) that

$$\lim_{n \to +\infty} u'^2(\sigma_n) = 2(L + F(1)).$$

On the other hand, since u oscillates around one and $\limsup_{r \to +\infty} u(r) = \bar{b} \in (1, b^*)$, there exists a sequence (x_n) of strict maxima of u such that $\lim_{n \to +\infty} u(x_n) = \bar{b}$. Thus $\lim_{n \to +\infty} E(x_n) = -F(\bar{b}) < 0$ and therefore $L = -F(\bar{b}) < 0$. Since $\bar{b} \in (1, b^*)$, we deduce that $2(L + F(1)) = \tilde{\gamma}^2 > 0$ with $\tilde{\gamma} > 0$. This implies the first statement of Lemma 3 with $\gamma = \frac{\tilde{\gamma}}{2}$.

(ii) Because of (8) and since $\lim_{r \to +\infty} E(r) = L = -F(\bar{b})$, there exists a function Φ such that $\lim_{r \to \infty} \Phi(r) = 0$ and that Φ satisfies the equation

$$u'^2(r) = 2\big(L + F(u(r)) + \Phi(r)\big). \qquad (13)$$

Then

$$|u'(r)| = \sqrt{2}\sqrt{\big|L + F(u(r)) + \Phi(r)\big|}. \qquad (14)$$

Integrating this relation on $[s_{2n}, s_{2n+1}]$, we obtain

$$\int_{s_{2n}}^{s_{2n+1}} |u'(r)| \, dr = \int_{s_{2n}}^{r_n} u'(r) \, dr - \int_{r_n}^{s_{2n+1}} u'(r) \, dr = 2\big(u(r_n) - 1\big)$$

which implies (11) ∎

Lemma 4. *Let $v = v(r)$ be a non-negative uniformly continuous function on $(0, +\infty)$. If (σ_n) is a sequence such that $\lim_{n \to +\infty} \sigma_n = +\infty$, $\liminf_{n \to +\infty} v(\sigma_n) \geq \gamma$ and $\frac{n}{\sigma_n} > \beta$ for some reals $\gamma > 0$ and $\beta > 0$, then the integral $\int_1^{+\infty} \frac{v(r)}{r} \, dr$ is unbounded.*

Proof. Since v is a uniformly continous function and $\liminf_{n \to +\infty} v(\sigma_n) \geq \gamma$, there exists a real $\alpha > 0$ which does not depend on n and an integer N such that for all $n \geq N$

$$v(r) \geq \tfrac{\gamma}{2} \qquad \forall r \in (\sigma_n, \sigma_n + \alpha). \qquad (15)$$

Since $\lim_{n \to +\infty} \sigma_n = +\infty$, there exists a subsequence also denoted by (σ_n) such that $\sigma_n + \alpha < \sigma_{n+1}$ for all $n \geq N$. Therefore,

$$\sum_{n=N}^{N+m} \int_{\sigma_n}^{\sigma_{n+1}} \frac{v(r)}{r} \, dr \geq \tfrac{\gamma}{2} \sum_{n=N}^{N+m} \ln\left(1 + \tfrac{\alpha}{\sigma_n}\right) \qquad (16)$$

for all $m \in \mathbb{N}$. Since the sequence (σ_n) tends to infinity, we have $\ln(1 + \frac{\alpha}{\sigma_n}) \sim \frac{\alpha}{\sigma_n}$. But $\frac{1}{\sigma_n} > \frac{\beta}{n}$ for all n. Then (16) implies that the integral $\int_1^{+\infty} \frac{v(r)}{r} \, dr$ is unbounded ∎

Lemma 5. *Let $u = u(r)$, $u \in C^2(0, +\infty)$, be a solution of equation (3) oscillating around one and such that*

$$\limsup_{r \to +\infty} u(r) = \bar{b} \in (1, b^*) \qquad or \qquad \liminf_{r \to +\infty} u(r) = \bar{c} \in (0, 1)$$

with b^* defined by (5). Further, let (s_n) be a real sequence with $\lim_{n \to +\infty} s_n = +\infty$, $u(s_n) = 1$ $(n \in \mathbb{N})$ and

$$u(r) \geq 1 \;\; \forall\, r \in [s_{2n}, s_{2n+1}] \qquad or \qquad u(r) \leq 1 \;\; \forall\, r \in [s_{2n+1}, s_{2n+2}],$$

respectively. At last, let (r_n) or (μ_n) be a real sequence such that

$$\left.\begin{array}{l} r_n \in (s_{2n}, s_{2n+1}), \;\; u'(r_n) = 0 \\[4pt] u''(r_n) \leq 0, \;\; \lim_{n \to +\infty} u(r_n) = \bar{b} \end{array}\right\} \qquad or \qquad \left\{\begin{array}{l} \mu_n \in (s_{2n+1}, s_{2n+2}), \;\; u'(\mu_n) = 0 \\[4pt] u''(\mu_n) \geq 0, \;\; \lim_{n \to +\infty} u(\mu_n) = \bar{c}, \end{array}\right.$$

respectively. Then the sequence

$$(s_{2n+1} - s_{2n}) \qquad or \qquad (s_{2n+2} - s_{2n+1}), \tag{17}$$

respectively, is bounded.

Proof. We only consider the pair of sequences (s_n) and (r_n), the proof concerning the pair of sequences (s_n) and (μ_n) is similar. To prove that the sequence $(s_{2n+1} - s_{2n})$ is bounded, we assume the contrary. Then either the sequence $(r_n - s_{2n})$, or the sequence $(s_{2n+1} - r_n)$ or both of them are unbounded, with an extraction of a subsequence if necessary. Without loss of generality we can assume that $(r_n - s_{2n})$ is unbounded.

Denote by c and d the constants defined by $c = \frac{1+\bar{b}}{2}$ and $d = \frac{\bar{b}+b^*}{2}$. There exist an integer N and a sequence (γ_n) such that

$$s_{2n} \leq \gamma_n \leq r_n \;\; \text{and} \;\; u(\gamma_n) = c \qquad \forall\, n \geq N.$$

Since the sequence $(u(r_n))$ converges to $\bar{b}$ and is non-increasing because of the monotonicity of E, there exists an integer $n_1 \geq N$ such that

$$u(r) \in [c, d] \qquad \forall\, r \in [\gamma_n, r_n] \text{ and } n \geq n_1.$$

Now we distinguish the two cases where the sequence $(r_n - \gamma_n)$ is bounded or unbounded.

1. First we assume that $(r_n - \gamma_n)$ is unbounded and integrate equation (3) on $[\gamma_n, r_n]$ to get

$$-u'(\gamma_n) = u'(r_n) - u'(\gamma_n)$$
$$= -(N-1)\left(\frac{u(r)}{r}\right)\Big|_{\gamma_n}^{r_n} - (N-1)\int_{\gamma_n}^{r_n} \frac{u(s)}{s^2}\, ds + \int_{\gamma_n}^{r_n} (u^q - u^p)(s)\, ds. \tag{19}$$

Because of Lemma 2 both functions u and u' are bounded. Then we deduce that there exists a number $M > 0$ independent on n such that for all $n \geq n_1$

$$\left|\int_{\gamma_n}^{r_n} (u^q - u^p)(s)\, ds\right| < M. \tag{20}$$

Because of (18) and the monotonicity of the function $r \mapsto r^q - r^p$ on $[c, d]$ we obtain

$$0 > c^q - c^p \geq u^q(r) - u^p(r) \geq (d^q - d^p) \qquad \forall\, r \in (\gamma_n, r_n).$$

Then

$$(c^q - c^p)(r_n - \gamma_n) \geq \int_{\gamma_n}^{r_n} (u^q - u^p)(r)\, dr \geq (d^q - d^p)(r_n - \gamma_n)$$

which implies

$$\limsup_{n \to +\infty} \int_{\gamma_n}^{r_n} (u^q - u^p)(r)\, dr = -\infty \qquad \text{because} \quad \limsup_{n \to +\infty}(r_n - \gamma_n) = +\infty.$$

Therefore we obtain a contradiction to (20).

2. Now we consider the case where the sequence $(r_n - \gamma_n)$ is bounded. Recall that we assume $(r_n - s_{2n})$ to be unbounded. We deduce that $(\gamma_n - s_{2n})$ is unbounded. Let $\varepsilon > 0$ and $D > 0$ such that

$$F(c) - F(\bar{b}) - \varepsilon = D^2 > 0. \tag{21}$$

The assumptions of Lemma 3 are satisfied and we can use the function Φ introduced in (13). Let n_2 be an integer greater than n_1 such that for all $n \geq n_2$

$$\Phi(r) + \varepsilon \geq 0 \qquad \forall\, r \in [s_{2n}, \gamma_n]. \tag{22}$$

Since u is non-decreasing on $[s_{2n}, \gamma_n]$, we deduce that $F(u(r)) \geq F(c)$ for all $r \in [s_{2n}, \gamma_n]$ and equality (21) implies

$$F(u(r)) - F(\bar{b}) - \varepsilon \geq D^2 \qquad \forall\, r \in [s_{2n}, \gamma_n]. \tag{23}$$

Then, for all $n \geq n_2$ and for all $r \in [s_{2n}, \gamma_n]$, both relations (22) and (23) imply

$$\begin{aligned}
\sqrt{\left|F(u(r)) - F(\bar{b}) + \Phi(r)\right|} &= \sqrt{\left|F(u(r)) - F(\bar{b}) - \varepsilon + \Phi(r) + \varepsilon\right|} \\
&\geq \sqrt{F(u(r)) - F(\bar{b}) - \varepsilon} \\
&\geq D \\
&> 0.
\end{aligned} \tag{24}$$

Since $\gamma_n \leq s_{2n+1}$, we have

$$\int_{s_{2n}}^{s_{2n+1}} \sqrt{\left|F(u(r)) - F(\bar{b}) + \Phi(r)\right|}\, dr \geq \int_{s_{2n}}^{\gamma_n} \sqrt{\left|F(u(r)) - F(\bar{b}) + \Phi(r)\right|}\, dr \tag{25}$$

and we deduce from (11), (24) and (25)

$$2(u(r_n) - 1) \geq \sqrt{2}D(\gamma_n - s_{2n}) \tag{26}$$

for all $n \geq n_2$. Since $\lim_{n \to +\infty} u(r_n) = \bar{b}$ and the sequence $(\gamma_n - s_{2n})$ is unbounded we obtain a contradiction when n goes to infinity ∎

Now we prove Theorem 1. For that we proceed in several steps in which we denote by $C > 0$ a constant independent of r.

Proof of Theorem 1. Let $u \in C^2(\mathbb{R}^N)$ be a non-negative radial solution of equation (1) such that $u \not\equiv 0$ and $u \not\equiv 1$.

Step 1. Here we assume that u is monotone. Because of Lemma 2 we deduce the existence of a real $l \geq 0$ such that $\lim_{r \to +\infty} u(r) = l$. Assume $l \in (0,1)$. Then $l^q - l^p$ is positive. Therefore, there exists a real $r_0 > 0$ such that $(r^{N-1}u'(r))' \geq Dr^{N-1}$ for all $r \geq r_0$ with $D = \frac{l^q - l^p}{2}$. Integrating successivily two times this inequality on (r_0, r), we obtain $u(r) \geq Cr^2$ for large r where $C > 0$. But this is impossible because u is bounded.

Now if $l > 1$, there also exists another r_0 such that $(r^{N-1}u'(r))' \leq Dr^{N-1}$ for all $r \geq r_0$ which implies $u(r) \leq Dr^2$ for large r with $D < 0$. We still obtain a contradiction.

Then either $l = 1$ or $l = 0$. If $l = 0$, [6: Theorem 1] implies by comparison arguments that u has compact support.

Step 2. Here we assume u to be non-monotone for large r and prove $\lim_{r \to +\infty} u(r) = 1$. Recall that there does not exist a local minimum or maximum point r such that $u(r)$ is strictly greater or less than 1, respectively. This implies that there exist an integer N and sequences (r_n) and (μ_n) of strict maxima and minima of u, respetively, such that

$$\mu_n \leq r_n \leq \mu_{n+1} \leq r_{n+p} \tag{27}$$
$$0 \leq u(\mu_n) \leq 1 \leq u(r_n) \tag{28}$$

for all $n \geq N$ and $p \geq 1$. That is, u oscillates around one. Now we divide this step in several parts.

(i) We claim that there exists a real $\bar{b} \in [1, b^*]$ such that $(u(r_n))$ converges to $\bar{b}$. Indeed, since E is non-increasing, we have from inequality (27)

$$E(r_{n+p}) \leq E(\mu_{n+1}) \leq E(r_n) \leq E(\mu_n) \tag{29}$$

for all $n \geq N$ and $p \geq 1$. Moreover, the monotonicity of the function F defined in (7) and both inequalities (27) and (28) imply

$$0 \leq F(u(\mu_n)) \leq F(1). \tag{30}$$

We deduce with the help of (8) that $L \in [-F(1), 0]$, where L is given in Lemma 3, and that $(-F(u(r_n)))$ converges to L. Since F is one-to-one on $[1, +\infty)$, we conclude that there exists a real $\bar{b} \in [1, b^*]$ such that $\lim_{n \to +\infty} u(r_n) = \bar{b}$.

(ii) In the same way we can prove the existence of a real $\bar{c} \in [0, 1]$ such that $\lim_{n \to +\infty} u(\mu_n) = \bar{c}$.

(iii) We assume that $\bar{b} \in (1, b^*)$ and $\bar{c} \in (0, 1)$. Since u oscillates around one, we can define a sequence $(s_n)_{n \geq 0}$ by

$$s_0 = \inf\{r > r_N : u(r) = 1\}$$
$$s_{n+1} = \inf\{r > s_n : u(r) = 1\} \tag{31}$$

Without loss of generality we can assume

$$s_{2n} \leq r_n \leq s_{2n+1} \leq \mu_n \leq s_{2n+2}.$$

Then $u \geq 1$ on $[s_{2n}, s_{2n+1}]$ and $u \leq 1$ on $[s_{2n+1}, s_{2n+2}]$). Because of (29) the sequence $(u(r_n))$ is non-increasing and the sequence $(u(\mu_n))$ is non-decreasing. Therefore, (i) - (ii) and Lemma 5 imply that the sequences $(s_{2n+1} - s_{2n})$ and $(s_{2n+2} - s_{2n+1})$ are bounded. Thus the sequence $(s_{n+1} - s_n)$ is bounded. That is, there exists a constant $C > 0$ and an integer $\overline{N} \geq N$ such that $s_{n+1} - s_n \leq C$ for all $n \geq \overline{N}$. A straightfoward computation gives us for large n

$$\frac{1}{s_n} \geq \frac{1}{nC + s_0}. \tag{32}$$

On the other hand, we deduce after integration of (9) over $(1, +\infty)$

$$|L - E(1)| = (N - 1) \int_1^{+\infty} \frac{u'^2(r)}{r} \, dr \tag{33}$$

which implies that the integral $\int_1^{+\infty} \frac{u'^2(r)}{r} \, dr$ is bounded. Now we check the assumptions of Lemma 4 for $v = u'^2$ and $\sigma_n = s_n$. First, because of Lemma 2, u' is uniformly bounded. Next, Lemma 3 with $\sigma_n = s_n$ implies $\lim_{n \to +\infty} u'^2(s_n) \geq \gamma > 0$. Finally, inequality (32) holds and we obtain a contradiction.

(iv) If we assume $\bar{b} = b^*$, then $L = \lim_{n \to +\infty} E(r_n) = -F(b^*) = 0$. Therefore, we deduce from (29) when p tends to infinity that $E(\mu_n) \geq 0$ for all $n \geq N$. On the other hand, inequality (30) implies $E(\mu_n) \leq 0$. Thus E vanishes identical for large r and relation (9) entails that u is constant for large r. We deduce from (3) that $u \equiv 0$ or $u \equiv 1$ which contradicts the fact that $\lim_{n \to +\infty} u(r_n) = b^*$. Then $\bar{b} < b^*$.

(v) We deduce from substeps (iii) and (iv) that necessarily $\bar{b} = 1$. Then $L = -F(1)$ and we deduce from (29) that $-F(1) \leq -F(u(\mu_n)) \leq -F(u(r_n))$ for $n \geq N + 1$. Therefore, $\lim_{n \to +\infty} -F(u(\mu_n)) = -F(1)$ and inequality (28) implies $\lim_{n \to +\infty} u(\mu_n) = 1$. Thus $\lim_{r \to +\infty} u(r) = 1$ and the theorem is proved.

(vi) If $\bar{c} = 0$, we obtain a contradiction as in substep (iv). Then $\bar{c} = 1$ and as before we obtain the statement of the theorem ∎

3. Existence of non-trivial radial solutions

Here we prove the existence of solutions of equation (3) which tends to one. We start with the following

Lemma 6. *Let $\gamma \in (0, b^*]$ and $r_0 > 0$. Then there exists a unique solution u of equation (3) in $(r_0, +\infty)$ such that $u(r_0) = \gamma$ and $u'(r_0) = 0$. Moreover, u is positive in $(r_0, +\infty)$.*

Proof. Since $\gamma > 0$, there exists a solution u of equation (3) in some maximal interval $(r_0, r_0 + \delta)$ with $\delta \in (0, +\infty]$ and such that $u(r_0) = \gamma$ and $u'(r_0) = 0$. We

claim that u is positive in $(r_0, r_0 + \delta)$. Actually, $E(r_0) = -F(\gamma) \leq 0$, and if there exists $r_1 \in (r_0, r_0 + \delta)$ such that $u(r_1) = 0$, then $E(r_1) = \frac{(u'(r_1))^2}{2} \geq 0$. Since E is non-increasing, we have a contradiction if $\gamma < b^*$ which entails $-F(\gamma) < 0$. And if $\gamma = b^*$, then $E(r) = 0$ in $[r_0, r_1]$ and (9) implies $u'(r) = 0$ in (r_0, r_1). Thus $u(r) = b^*$ in $[r_0, r_1)$ which is not a solution of equation (3). This contradiction implies that u is positive in $(r_0, r_0 + \delta)$.

We claim now that $\delta = \infty$. Actually, assume $\delta < \infty$. Then there exists a number $m > 0$ such that $u(r) \geq m$ for all $r \in (r_0, r_0 + \delta)$. Moreover, for all $r \in (r_0, r_0 + \delta)$

$$-F(u(r)) \leq E(r) \leq E(r_0) = -F(u(r_0)) \leq 0.$$

We deduce that for all $r \in (r_0, r_0 + \delta)$

$$m \leq u(r) \leq b^*.$$

Recall that u satisfies equation (3) which is equivalent to $(r^{N-1}u')' = r^{N-1}(u^q - u^p)$. Then there exists a constant $M(\delta)$ such that for $r_0 < r < s < r_0 + \delta$

$$\left| s^{N-1}u'(s) - r^{N-1}u'(r) \right| = \left| \int_r^s t^{N-1}(u^q - u^p)(t)\,dt \right| \leq M(\delta)|s - r|.$$

That is, the function $r \mapsto r^{N-1}u'(r)$ is uniformly Lipschitz and there exists a real number l such that $\lim_{r \to r_0 + \delta} u'(r) = l$. Thus we can extend the solution which contradicts that the interval $(r_0, r_0 + \delta)$ is maximal. This ends the prove of the lemma ∎

Proof of Theorem 2. Because of Lemma 6 we only need to prove that the solution of equation (3) tends to one. Actually, if this is not the case, Theorem 1 implies that u has compact support. That is, there exists a real number $R > r_0$ such that $u \equiv 0$ on $[R, +\infty)$. Since E is non-increasing, we deduce $E(r_0) \geq 0$.

If $E(r_0) > 0$, then $-F(\gamma) > 0$, and then $\gamma > b^*$ which is a contradiction. If $E(r_0) = 0$, then $\gamma = b^*$ and $E(r) = 0$ for all $r \in (r_0, +\infty)$. Thus $E'(r) = 0$ for all $r \in (r_0, +\infty)$. That is, $u'(r) = 0$ for all $r \in (r_0, +\infty)$ and $u(r) = b^*$ for all $r \in (r_0, +\infty)$. The constant b^* is not a solution of equation (3). This is an other contradiction ∎

4. Solutions with compact support

In this section, we prove Theorems 3 and 4. For this we use a result of [5] which gives an estimate between the solution u and its spherical average $\overline{u}$. We recall this result as

Lemma 7. *Let $w \in C^2(\mathbb{R}^N)$ be a non-negative subharmonic function with $w \not\equiv 0$ near infinity. Then $\overline{w}$ is monotone for large $r = |x|$ and, for any $\varepsilon \in (0, 1)$,*

$$w(x) \leq C(N, \varepsilon)\, \varepsilon^{-N} \overline{w}(|x|(1 \pm \varepsilon)) \qquad \text{near infinity} \tag{34}$$

where $C(N, \varepsilon) = N^{-1}\big((1 + \varepsilon)^N - (1 - \varepsilon)^N\big)$, with sign $+\varepsilon$ if $\overline{w}$ is non-decreasing, and sign $-\varepsilon$ if $\overline{w}$ is non-increasing. Moreover, for any $Q > 1$ and large r,

$$\overline{w}^Q(r) \leq \overline{w^Q}(r) \leq (C(N, \varepsilon)\, \varepsilon^{-N})^Q \overline{w}^Q(r(1 \pm \varepsilon)) \tag{35}$$

and, for any $Q \in (0, 1)$,

$$\overline{w}^Q(r) \geq \overline{w^Q}(r) \geq (C(N, \varepsilon)\, \varepsilon^{-N})^{Q-1} \overline{w}^{Q-1}(r(1 \pm \varepsilon))\overline{w}(r). \tag{36}$$

Proof of Theorem 3. Since $N \geq 2$ and $\min(A, B) \leq 1$, the function u is subharmonic for large x. Consequently, the spherical average $\overline{u}$ of u is monotone in some interval $(r_0, +\infty)$ with $r_0 > 0$. Actually, $\overline{u}'' + \frac{N-1}{r}\overline{u}' \geq 0$ entails that there cannot exist the maximum of $\overline{u}$. On the other hand, Lemma 1 imlies that $\overline{u}$ is bounded. Therefore, there exists $\alpha \in [0, 1]$ such that $\lim_{r \to +\infty} \overline{u}(r) = \alpha$.

If $\alpha = 0$, Lemma 7 with $\varepsilon = \frac{1}{2}$ in (34) implies $\lim_{|x| \to \infty} u(x) = 0$. Hence using the comparison of [6] we establish the result in this case.

Now we assume $\alpha > 0$ and consider the following two cases.

Case 1: $p > 1$. Then, because of Lemma 7, we have for some $\varepsilon \in (0, 1)$

$$(\overline{u^q} - \overline{u^p})(r)$$
$$\geq (C(N, \varepsilon)\, \varepsilon^{-N})^{q-1} \tag{37}$$
$$\times \left[\overline{u}^{q-1}(r(1 \pm \varepsilon))\overline{u}(r) - C(N, \varepsilon)^{p+1-q}\varepsilon^{-N(p+1-q)}\overline{u}^p(r(1 \pm \varepsilon))\right]$$

with $C(N, \varepsilon) = N^{-1}((1 + \varepsilon)^N - (1 - \varepsilon)^N)$. Let ϕ be the function defined on $(0, 1]$ by

$$\phi(s) = \left[N^{-1}((1 + s)^N - (1 - s)^N)\right]^{p+1-q} s^{-N(p+1-q)}.$$

This function is decreasing on $(0, 1)$, satisfies $\phi(1) = \left(\frac{2^N}{N}\right)^{p+1-q}$ and $\lim_{s \to 0+} \phi(s) = +\infty$. Hence there exists $\varepsilon_0 > 0$ such that

$$\phi(\varepsilon_0) \leq \left(\frac{2^N}{N}\right)^{p+1-q} + \eta. \tag{38}$$

Because of (37), with ε replaced by ε_0, we find from here

$$(\overline{u^q} - \overline{u^p})(r)$$
$$\geq (C(N, \varepsilon_0)\, \varepsilon_0^{-N})^{q-1}\left[\overline{u}^{q-1}(r(1 \pm \varepsilon_0))\overline{u}(r) - \left(\left(\frac{2^N}{N}\right)^{p+1-q} + \eta\right)\overline{u}^p(r(1 \pm \varepsilon_0))\right].$$

On the other hand, since $\alpha \leq \min(A, B) \leq A$, we deduce

$$\alpha^q - \left(\left(\frac{2^N}{N}\right)^{p+1-q} + \eta\right)\alpha^p = \lambda > 0.$$

Consequently, since $\lim_{r \to +\infty} \overline{u}(r) = \alpha$, there exists a real $r_1 \geq r_0$ such that

$$(\overline{u^q} - \overline{u^p})(r) \geq \tfrac{\lambda}{2}\, (C(N, \varepsilon_0)\varepsilon_0^{-N})^{q-1}$$

which is equivalent to

$$(r^{N-1}\overline{u}')'(r) \geq \tfrac{\lambda}{2}\, (C(N, \varepsilon_0)\varepsilon_0^{-N})^{q-1} r^{N-1}$$

for all $r \geq r_1$. Two integrations over $[r_1, r]$ yield $\overline{u}(r) \geq Cr^2$, with $C > 0$, and we get a contradiction because $\overline{u}$ is bounded.

Case 2: $p \leq 1$. Then, because of Lemma 7, inequality (37) is replaced by

$$(\overline{u^q} - \overline{u^p})(r)$$
$$\geq (C(N, \varepsilon)\, \varepsilon^{-N})^{q-1}\left[\overline{u}^{q-1}(r(1 \pm \varepsilon))\overline{u}(r) - C(N, \varepsilon)^{1-q}\varepsilon^{-N(1-q)}\overline{u}^p(r)\right].$$

With similar arguments we obtain a new contradiction. Finally, in both cases we prove that u cannot have a limit different from zero ∎

Proof of Theorem 4. We devide the proof in steps (i) - (iii).

(i) Let $u \in C^2(\mathbb{R}^N)$ be a solution of equation (1). Recall that we introduced the spherical average in (6). Since $p \geq 1$, the Jensen inequality implies that $\overline{u}$ satisfies (10) in $(\alpha, +\infty)$, with $\alpha > 0$. Then Lemma 1 entails that $\overline{u}$ is bounded in $(\alpha, +\infty)$.

(ii) We claim that $\liminf_{|x| \to +\infty} u(x) \leq 1$. Actually, if $\liminf_{|x| \to +\infty} u(x) > 1$, then there exists a constant $l > 1$ such that $u(x) \geq l$ for large x. Thus $(u^q - u^p)(x) \leq -a$ for large x, with $a = l^p - l^q > 0$. Equation (1) implies $\overline{u}''(r) + \frac{N-1}{r}\overline{u}'(r) \leq -a$ for large r. We deduce that $(r^{N-1}\overline{u}')'(r) \leq -ar^{N-1}$ for large r. Indeed, integrating this inequality twice in some interval (α, r) we obtain $\overline{u}(r) \leq -ar^2 + d$ with $d \in \mathbb{R}$. This is a contradiction to the conclusion of step (i).

(iii) Assume that $u(x) \geq c > 0$ for large x, with some $c < 1$. If $\limsup_{|x| \to +\infty} u(x) < 1$, then there exists a constant $l < 1$ such that $c \leq u(x) \leq l$ for large x. Thus

$$(u^q - u^p)(x) \in \left[\min(c^q - c^p, l^q - l^p), \max(c^q - c^p, l^q - l^p) \right]$$

for large x. Equation (1) implies the existence of a constant $\lambda > 0$ such that $\Delta u(x) \geq \lambda$ for large x which entails $(r^{N-1}\overline{u}')'(r) \geq \lambda r^{N-1}$ for large r. Integrating this inequality twice in some interval (α, r) we get $\overline{u}(r) \geq C(\lambda, N)r^2$ which also contradicts the fact that $\overline{u}$ is bounded ∎

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