Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 20 (2001), No. 4, 859–881

Free Boundary Problem for a One-Dimensional Transport Equation

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Abstract. For a linear transport equation in one space dimension with speeds in a compact interval and a general symmetric kernel for the change of velocity a problem with free boundary (Stefan problem) is stated. The case of constant speed corresponds to a Stefan problem for the damped wave equation (telegraph equation). Existence and uniqueness of the free boundary is shown, and the connection to the classical Stefan problem (parabolic limit) is exhibited.

Keywords: *Telegraph and transport equations, free boundary, correlated random walk*

AMS subject classification: 35R35, 35L50, 80A22

1. Introduction

The classical Stefan problem for the heat equation describes the melting of ice in a channel represented by a one-dimensional interval of variable length. At the fixed boundary a standard boundary condition prescribes the temperature or the heat flux, the free boundary is implicitly given by a Dirichlet condition (melting temperature) and a second condition connecting the displacement of the ice rim to the latent heat. The problem has been generalized in various directions (two-phase problem, semilinear equation, general boundary conditions) and there are many applications in science and technology. Of course, the heat equation can be also interpreted as a diffusion equation.

Even if the differential equation is linear, the Stefan problem is highly nonlinear. A standard approach to its solution is to transform the problem to an integral equation for the unknown boundary and the unknown temperature distribution and then to apply fixed point theorems. As a typical example we mention [3].

Although the diffusion equation (or Brownian motion) is the standard model for motion in space, it is only the limiting case of a class of correlated random walks, damped wave equations, and transport equations which are based on more detailed descriptions of many particles, i.e. on individual particle speed, and do not show the effect of infinitely fast propagation. These systems are hyperbolic in contrast to the parabolic limiting case.

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ISSN 0232-2064 / $ 2.50

Stefan problems for hyperbolic equations have been studied to some extent. Greenberg [8] proves existence of a free boundary for a damped wave equation (or telegraph equation) with a special boundary condition which describes in detail the underlying physical phenomena. Friedman and Bei Hu [4] prove existence for the telegraph equation and a more general class of boundary conditions. This work shows clearly that the hyperbolic problem requires novel techniques, e.g., the directions of characteristics must be taken into account, the compactness properties are much weaker than in the parabolic case.

Here a free boundary problem is formulated for a general linear transport equation in one space dimension. The particle speeds are strictly positive and the kernel governing change of velocities is symmetric. Particular attention is paid to the formulation of boundary conditions. At the fixed boundary a partially reflecting condition is imposed. Also at the free boundary particles are reflected whereby it is assumed that the motion of the free boundary can be neglected, when the reflection of an individual particle is considered. In other words, the speed of displacement of the free boundary is smaller than the minimal particle speed. This assumption is incorporated into the reflection law and into the law that connects the displacement to the number of arriving particles.

The free boundary value problem is transformed into an integral equation. For a fixed boundary a solution of the initial boundary value problem is shown to exist and its properties are investigated. This result is used as a tool to show existence for the free boundary value problem (Section 4). Uniqueness is shown by a Gronwall argument in Section 5. In Section 6 the limiting case of constant speed where the transport equation becomes the Goldstein-Kac random walk (see [7, 9]) is studied in its own right. Finally, in Section 7, it is shown that the formal parabolic limit of the hyperbolic free boundary problem is a classical Stefan problem. The strategy of proof resembles the approach in [4]. Due to the different interpretation of the variables as particle density (as opposed to linear combinations of temperature and heat flux in [4]), the positivity plays a different role and the boundary conditions are different.

1.1 The one-dimensional transport equation. The state of a particle is described by its position in space $x \in \mathbb{R}$ and its velocity $\gamma \in \mathbb{R}$. The particle moves with a fixed velocity, stops at a random time governed by a Poisson process with parameter μ and then selects a new velocity from a given set of velocities according to some distribution with density K. Given the previous velocity γ', the probability density of the new velocity is $K(\cdot, \gamma')$. We assume that velocities range in the set Γ defined by

$$\Gamma = \left\{\gamma \in \mathbb{R} : \gamma_0 \le |\gamma| \le \gamma_1\right\}$$

where $\gamma_0 > 0$ and $\gamma_1 < \infty$. Hence very fast and very slow particles are excluded.

The domain in the space-time continuum is

$$\Omega = \left\{(t, x) : 0 \le t \quad \text{and} \quad 0 \le x \le s(t)\right\}$$

where $x = 0$ is the fixed and $x = s(t)$ is the moving boundary. The function s starts from $s(0) = s_0 > 0$ and is positive as long as it exists. The particle density $u = u(t, x)$ is defined on Ω and satisfies the transport equation

$$u_t + \gamma \cdot u_x = -\mu u + \mu \int_\Gamma K(\gamma, \gamma')\, u(t, x, \gamma')\, d\gamma' \qquad \text{on } \Omega. \tag{1}$$

We assume that K is continuous in Γ^2, $K(\gamma,\gamma') \geq 0$ for $\gamma,\gamma' \in \Gamma$ and $\int_\Gamma K(\gamma,\gamma')\,d\gamma = 1$. Furthermore, K is assumed symmetric, i.e. $K(\gamma,\gamma') = K(\gamma',\gamma)$ for $\gamma,\gamma' \in \Gamma$. Of course, then also $\int_\Gamma K(\gamma,\gamma')\,d\gamma' = 1$. The initial condition has the form

$$u(0,x,\gamma) = u_0(x,\gamma) \qquad (x \in [0,s_0], \gamma \in \Gamma) \tag{2}$$

such that u_0 is at least continuous. Boundary conditions require special attention and will be discussed in the next subsection.

1.2 Boundary conditions. In the following we use the sets $\Gamma^+ = \Gamma \cap (0,\infty)$ and $\Gamma^- = \Gamma \setminus \Gamma^+$. Also, we write in short $u_\gamma(t,x) = u(t,x,\gamma)$. Boundary conditions can be prescribed only for ingoing particles, i.e. at $x = 0$ we can give data $u(t,0,\gamma)$ for $\gamma \in \Gamma^+$ and at $x = s(t)$ for $\gamma \in \Gamma^-$. For the moment assume that the free boundary is not there, i.e. the solution is defined for all $x \geq 0$ and $u_\gamma(t,\cdot)$ has compact support for each γ and t.

The total mass $U(t) = \int_0^\infty \int_\Gamma u(t,x,\gamma)\,d\gamma dx$ satisfies

$$\begin{aligned}
0 &= \frac{dU(t)}{dt} \\
&= \int_0^\infty \int_\Gamma \left[-\gamma u_x(t,x,\gamma) - \mu u(t,x,\gamma) + \mu \int_\Gamma K(\gamma,\gamma') u(t,x,\gamma')\,d\gamma' \right] d\gamma dx \\
&= -\int_\Gamma \gamma \int_0^\infty u_x(t,x,\gamma)\,dx d\gamma \\
&= \int_\Gamma \gamma u(t,0,\gamma)\,d\gamma.
\end{aligned}$$

In order to have total reflection at $x = 0$ and conservation of mass, the boundary condition at $x = 0$ must satisfy

$$\int_{\gamma_0}^{\gamma_1} \gamma u(t,0,\gamma)\,d\gamma = -\int_{-\gamma_1}^{-\gamma_0} u(t,0,\gamma)\gamma\,d\gamma.$$

This requirement is satisfied by any boundary condition of the form

$$u(t,0,\gamma) = \tfrac{1}{|\gamma|} \int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma') u(t,0,\gamma') |\gamma'|\,d\gamma' \qquad (\gamma \in \Gamma^+)$$

where L is continuous in $\Gamma^+ \times \Gamma^-$, $L(\gamma,\gamma') \geq 0$ and $\int_{\gamma_0}^{\gamma_1} L(\gamma,\gamma')\,d\gamma = 1$. Indeed,

$$\begin{aligned}
\int_{\gamma_0}^{\gamma_1} \gamma u(t,0,\gamma)\,d\gamma &= -\int_{\gamma_0}^{\gamma_1} \int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma') u(t,0,\gamma')\gamma'\,d\gamma' d\gamma \\
&= -\int_{-\gamma_1}^{-\gamma_0} u(t,0,\gamma')\gamma'\,d\gamma'.
\end{aligned}$$

Now we assume that only a share $r < 1$ of the particles arriving at the left boundary is reflected, and the remaining particles are absorbed. Furthermore, we add a source term at $x = 0$. Then we have the boundary condition

$$u(t,0,\gamma) = \tfrac{r}{|\gamma|} \int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')u(t,0,\gamma')|\gamma'|\,d\gamma' + f(t,\gamma) \qquad (\gamma \in \Gamma^+) \tag{3}$$

where $f(t,\gamma) \geq 0$ for all $\gamma \in \Gamma^+$.

The following propositions could also be shown for the limiting case $r = 1$ (total reflection), but the assumption $r < 1$ is justified in view of the the parabolic limit (see Section 7) and no further insight can be gained by treating the general case.

In order to derive boundary conditions at the moving boundary, we select an arbitrary, but fixed time t. Obviously, all particles $u(t,s(t),\gamma)$ with $\gamma \in \Gamma^-$ are ingoing and $u(t,s(t),\gamma)$ with $\gamma \in \Gamma^+$ are outgoing. We assume total reflection of the particles arriving at the free boundary $x = s(t)$ from the left. Let again $u_\gamma(t,\cdot)$ have a compact support in $\mathbb{R}$ for $t \geq 0$, $\gamma \in \Gamma$. Let $U(t) = \int_{-\infty}^{s(t)} \int_\Gamma u(t,x,\gamma)\,d\gamma dx$. Again we ask for conservation of mass:

$$\begin{aligned}
0 &= \frac{dU(t)}{dt} \\
&= \int_\Gamma u(t,s(t),\gamma)\,d\gamma\,\dot{s}(t) \\
&\quad + \int_{-\infty}^{s(t)} \int_\Gamma \Big[-\gamma u_x(t,x,\gamma) - \mu u(t,x,\gamma) + \mu \int_\Gamma K(\gamma,\gamma')u(t,x,\gamma')\,d\gamma' \Big] d\gamma dx \\
&= \int_\gamma u(t,s(t),\gamma)(\dot{s}(t) - \gamma)\,d\gamma.
\end{aligned}$$

Keeping the difference $\dot{s} - \gamma$ in all conditions and equations leads to extremely complicated expressions and arguments in proofs. Therefore we make the assumption that the speed $\dot{s}$ of the moving boundary is very small compared to the minimal particle speed, hence to any particle speed, and that the difference $\dot{s} - \gamma$ can be replaced by $-\gamma$. This simplification seems justified from a practical point of view (small fast particles fly against a heavy movable wall) but it makes us sacrifice conservation of mass. The resulting equation $0 = \int_\Gamma u(t,s(t),\gamma)(-\gamma)\,d\gamma$ can be rearranged to give

$$\int_{-\gamma_1}^{-\gamma_0} u(t,s(t),\gamma)(-\gamma)\,d\gamma = -\int_{\gamma_0}^{\gamma_1} u(t,s(t),\gamma)(-\gamma)\,d\gamma. \tag{4}$$

As before this condition is satisfied by boundary conditions

$$u(t,s(t),\gamma) = \tfrac{1}{|\gamma|} \int_{\gamma_0}^{\gamma_1} R(\gamma,\gamma')u(t,s(t),\gamma')|\gamma'|\,d\gamma' \qquad (\gamma \in \Gamma^-) \tag{5}$$

with continuous $R(\gamma,\gamma') \geq 0$ and $\int_{-\gamma_1}^{-\gamma_0} R(\gamma,\gamma')\,d\gamma = 1$. The reflection law (5) does not depend on $\dot{s}$ at all.

Again, the assumption that a share $(1-\kappa)$ of particles arriving at the free boundary is absorbed can be taken into account by multiplying the right-hand side of (5) by κ.

We define $R_{\max} = \sup_{\gamma\in\Gamma^-,\gamma'\in\Gamma^+}\{R(\gamma,\gamma')\}$. For the reflection operators $L(\gamma,\gamma')$ and $R(\gamma,\gamma')$ there are reciprocity conditions which guarantee the existence of a solution:

$$r\int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')|\gamma'|\,d\gamma' \le \hat{r}|\gamma| \qquad (\gamma\in\Gamma^+) \tag{6}$$

$$\kappa\int_{\gamma_0}^{\gamma_1} R(\gamma,\gamma')|\gamma'|\,d\gamma' \le \hat{\kappa}|\gamma| \qquad (\gamma\in\Gamma^-) \tag{7}$$

where $t\in[0,T]$ and $\hat{\kappa},\hat{r}<1$. These conditions say that particles do not become too slow while being reflected at the boundary. The normalization of R and the reciprocity condition (7) are not mutually exclusive. The function

$$R(\gamma,\gamma') = \tfrac{2}{\gamma_1^2-\gamma_0^2}\,|\gamma|$$

satisfies both conditions with $\hat{\kappa}=\kappa<1$. Similarly, one can show consistency of the normalization and reciprocity condition at the left boundary.

We further require conditions on the initial data and the source function

$$u_0(x,\gamma) > 0 \qquad (x\in[0,s_0],\ \gamma\in\Gamma) \tag{8}$$

$$f(t,\gamma) \ge 0 \qquad (t>0,\ \gamma\in\Gamma^+) \tag{9}$$

$$u_0\in C([0,s_0]\times\Gamma),\ u_0(\cdot,\gamma)\in C^{0,1}[0,s_0] \qquad (\gamma\in\Gamma) \tag{10}$$

$$f\in C([0,\infty)\times\Gamma),\ f(\cdot,\gamma)\in C^{0,1}[0,\infty) \qquad (\gamma\in\Gamma^+) \tag{11}$$

and compatibility conditions

$$u_0(0,\gamma) = \tfrac{r}{|\gamma|}\int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')u_0(0,\gamma')|\gamma'|\,d\gamma' + f(0,\gamma) \quad (\gamma\in\Gamma^+) \tag{12}$$

$$u_0(s_0,\gamma) = \tfrac{\kappa}{|\gamma|}\int_{\gamma_0}^{\gamma_1} R(\gamma,\gamma')u_0(s_0,\gamma')|\gamma'|\,d\gamma' \qquad (\gamma\in\Gamma^-). \tag{13}$$

1.3 Condition on the free boundary. Here we define a suitable Stefan condition connecting the movement of the free boundary to the particle flux. According to the boundary condition (5) the share κ of the particles arriving at the boundary is reflected and the share $(1-\kappa)$ is absorbed. We are guided by the idea that non-reflected particles push the boundary forward, independent of the velocity distribution of reflected particles. A simple law describing such behavior is

$$\dot{s}(t) = \frac{(1-\kappa)\int_{\gamma_0}^{\gamma_1}\gamma' u(t,s(t),\gamma')\,d\gamma'}{1+(1-\kappa)\int_{\gamma_0}^{\gamma_1} u(t,s(t),\gamma')\,d\gamma'}. \tag{14}$$

Obviously, this equation guarantees $\dot{s}(t)\le\gamma_1$. We choose (14) for the following reasons: The right-hand side is positive, monotone and bounded in u and is the simplest function with this property. The law respects the condition $\dot{s} \leq \gamma_1$. It reduces to the classical Stefan condition in the parabolic limit (see Section 7). The proofs can be extended to laws having these properties.

1.4 Derivation of the integral equations. With respect to a fixed velocity $\gamma \in \Gamma$, the domain

$$\Omega = \big\{(t,x) : 0 \leq t \;\text{ and }\; 0 \leq x \leq s(t)\big\}$$

is divided into three disjoint subdomains $I_\gamma, II_\gamma, III_\gamma$ related to a fixed particle velocity $\gamma \in \Gamma$, depending on initial or boundary data. A point $(t,x) \in \Omega$ belongs to

– domain I_γ, if its characteristic curve starts at a point of the initial manifold $(0,x), 0 \leq x \leq s_0$

– domain II_γ, if its characteristic curve starts at a point $(t,0)$ $(t>0)$ on the left boundary $x=0$

– domain III_γ if its characteristic curve starts at a point $(t,s(t))$ $(t>0)$ on the free boundary.

Obviously, the different domains have the form shown in Figure 1.

Figure 1: Subdomains for the integral equation

We set $u_{0,\gamma} = u_0(\cdot,\gamma)$ for $\gamma \in \Gamma$ and $f_\gamma = f(\cdot,\gamma)$ for $\gamma \in \Gamma^+$. Using the method of characteristics and the variation of constants formula system (1) - (5) can be carried into the form of an integral equation for which we get different expressions depending on the subdomain:

Domain I_γ:

$$\begin{aligned} u(t,x,\gamma) &= u_0(x-\gamma t,\gamma)e^{-\mu t} \\ &\qquad + \mu \int_0^t e^{-\mu(t-\eta)} \int_\Gamma K(\gamma,\gamma')u\big(\eta, x-\gamma t+\gamma\eta,\gamma'\big)\, d\gamma' d\eta. \end{aligned} \tag{15}$$

Domain II_γ:

$$\begin{aligned} u(t,x,\gamma) &= \left[\tfrac{r}{|\gamma|}\int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')u\big(t-\tfrac{x}{\gamma},0,\gamma'\big)|\gamma'|\, d\gamma' + f_\gamma\big(t-\tfrac{x}{\gamma}\big)\right] e^{-\frac{\mu x}{\gamma}} \\ &\qquad + \mu \int_{t-\frac{x}{\gamma}}^t e^{-\mu(t-\rho)} \int_\Gamma K(\gamma,\gamma')u\big(\rho,\gamma\big(\rho-t+\tfrac{x}{\gamma}\big),\gamma'\big)\, d\gamma' d\rho \end{aligned} \tag{16}$$

Domain III_γ: The starting point of the characteristic at the right boundary depends continuously on t, x and γ. If $\|\dot{s}\| < \gamma_0$ (using the supremum norm), then, using the implicit function theorem, the time coordinate can be expressed as a function $\Psi = \Psi_\gamma(t, x)$ which is obviously a solution of the equation $s(\Psi_\gamma(t,x)) + \gamma(t - \Psi_\gamma(t,x)) = x$. Then we get the integral equation

$$\begin{aligned} u(t,x,\gamma) = {} & \left[\tfrac{\kappa}{|\gamma|} \int_{\gamma_0}^{\gamma_1} R(\gamma,\gamma') u(\Psi, s(\Psi), \gamma') |\gamma'| \, d\gamma' \right] e^{-\mu(t-\Psi)} \\ & + \mu \int_\Psi^t e^{-\mu(t-\rho)} \int_\Gamma K(\gamma,\gamma') u\big(\rho, s(\Psi) + \gamma(\rho - \Psi), \gamma'\big) \, d\gamma' d\rho. \end{aligned} \tag{17}$$

2. Properties of the solution

2.1 Remark on positivity. Because of the interpretation of u as a particle density, the solution of problem (1) - (5) should remain non-negative. Theorem 2 in [1] can be used to show that solutions stay non-negative for any fixed boundary, actually for a more general class of problems.

2.2 Boundedness. The following lemma deals with the boundedness of the solution.

Lemma 1. *Let $\tilde{t} > 0$ and u be a continuous solution of integral equations* $(15)-(17)$ *on $\Omega_{\tilde{t}}$. Let s be a fixed boundary, where $\dot{s}(t) < \gamma_1$. Let reciprocity conditions* $(6)-(7)$ *hold. Then there exists a constant M depending only on $\tilde{t}$ and the upper bounds for initial and boundary data such that*

$$0 \le u(t,x,\gamma) \le M$$

for $0 \le x \le s(t), 0 \le t \le \tilde{t}$ and $\gamma \in \Gamma$.

Proof. Let

$$I(\tilde{t}) = \sup_{t\in[0,\tilde{t}], x\in[0,s(t)], \gamma\in\Gamma} |u(t,x,\gamma)|.$$

Let M_0 be an upper bound for the initial data and the source term $f_\gamma(t)$, where $\gamma \in \Gamma^+$. Choose $t \in [0,\tilde{t}]$, $x \in [0, s(t)]$ and $\gamma \in \Gamma$ such that $I(\tilde{t}) = u(t,x,\gamma)$. Then there are three possibilities:

1. $(t,x,\gamma) \in I_\gamma$. Then the integral equation yields

$$\begin{aligned} |u(t,x,\gamma)| &\le M_0 e^{-\mu t} + \mu \int_0^t e^{-\mu(t-\eta)} \int_\Gamma K(\gamma,\gamma') u(\eta, x - \gamma t + \gamma\eta, \gamma') \, d\gamma' d\eta \\ &\le M_0 + \mu \int_0^{\tilde{t}} I(\eta) \, d\eta. \end{aligned}$$

2. $(t,x,\gamma) \in II_\gamma$. Here we obtain

$$|u(t,x,\gamma)| \le \left[\tfrac{r}{|\gamma|}\int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')u\big(t-\tfrac{x}{\gamma},0,\gamma'\big)|\gamma'|\,d\gamma' + M_0\right] e^{-\frac{\mu x}{\gamma}}$$
$$+\,\mu\int_{t-\frac{x}{\gamma}}^{t} e^{-\mu(t-\rho)}\int_\Gamma K(\gamma,\gamma')u\big(\rho,\gamma\big(\rho-t+\tfrac{x}{\gamma}\big),\gamma'\big)\,d\gamma' d\rho$$
$$\le \hat{r}I(\tilde{t}) + M_0 + \mu\int_0^{\tilde{t}} I(\eta)\,d\eta.$$

3. $(t,x,\gamma) \in III_\gamma$. With the short notation $\Psi = \Psi_\gamma(t,x)$ we get

$$|u(t,x,\gamma)| \le \left[\tfrac{\kappa}{|\gamma|}\int_{\gamma_0}^{\gamma_1} R(\gamma,\gamma')u(\Psi,s(\Psi),\gamma)|\gamma'|\,d\gamma'\right] e^{-\mu(t-\Psi)}$$
$$+\,\mu\int_{\Psi}^{t} e^{-\mu(t-\rho)}\int_\Gamma K(\gamma,\gamma')u\big(\rho,s(\Psi)-\gamma(\rho-\Psi),\gamma'\big)\,d\gamma' d\rho$$
$$\le \hat{\kappa}I(\tilde{t}) + \mu\int_0^{\tilde{t}} I(\eta)\,d\eta.$$

Altogether we obtain

$$I(\tilde{t}) \le \max(\hat{\kappa},\hat{r})\; I(\tilde{t}) + M_0 + \mu\int_0^{\tilde{t}} I(\eta)\,d\eta.$$

Using the definition $M_2 = \max(\hat{\kappa},\hat{r})$ we get

$$I(\tilde{t}) \le \frac{M_0}{1-M_2} + \frac{\mu}{1-M_2}\int_0^{\tilde{t}} I(\eta)\,d\eta.$$

Gronwall's Lemma yields

$$I(\tilde{t}) \le M \qquad \text{where} \;\; M = \frac{M_0}{1-M_2}\,\exp\frac{\mu\tilde{t}}{1-M_2}.$$

Thus, we get $|u(t,x,\gamma)| \le M$ for all $(t,x,\gamma) \in \Omega_{\tilde{t}} \times \Gamma$ ∎

2.3 Properties of the free boundary. Here we get an upper bound for the free boundary.

Lemma 2. *For fixed* $\tilde{t} > 0$ *let*

$$M_0 = e^{-\frac{\mu\tilde{t}}{1-\max(\hat{\kappa},\hat{r})}} \cdot \frac{(1-\max(\hat{\kappa},\hat{r}))\gamma_0}{2(1-\kappa)(\gamma_1-\gamma_0)^2} \tag{18}$$

be an upper bound for the initial data $u_{0,\gamma}$ $(\gamma \in \Gamma)$ *and the source term* $f_\gamma(t)$ $(\gamma \in \Gamma^+, 0 \le t \le \tilde{t})$. *Let* (s,u) *be a solution of equations* $(15)-(17)$ *and* (14), *where* $s \in C^1[0,\tilde{t}]$ *and* $u \in C(\Omega_{\tilde{t}})$. *Then the free boundary satisfies the estimate* $\dot{s}(t) < \gamma_0$.

Proof. We define $M_2 = \max(\hat{\kappa}, \hat{r})$. Equation (18) yields

$$\left(\frac{M_0}{1-M_2}\right) e^{\frac{\mu\tilde{t}}{1-M_2}} \le \frac{\gamma_0}{2(1-\kappa)(\gamma_1-\gamma_0)^2}.$$

From Lemma 1 we know the estimate $|u(t,x,\gamma)| \le M$ for $0 \le x \le s(t)$, $0 \le t \le \tilde{t}$ and $\gamma \in \Gamma$, where $M = \frac{\gamma_0}{2(1-\kappa)(\gamma_1-\gamma_0)^2}$. Equation (14) can be rearranged to give

$$\dot{s}(t) = (1-\kappa) \int_{\gamma_0}^{\gamma_1} \big(\gamma' - \dot{s}(t)\big) u\big(t, s(t), \gamma'\big)\, d\gamma'$$

and we get

$$\dot{s}(t) \le (1-\kappa)(\gamma_1 - \dot{s}(t)) M (\gamma_1 - \gamma_0) \quad \Longleftrightarrow \quad \dot{s}(t) \le \frac{(1-\kappa)\gamma_1 M(\gamma_1-\gamma_0)}{1+(1-\kappa)M(\gamma_1-\gamma_0)}.$$

Inserting the definition of M gives

$$\dot{s}(t) \le \frac{\gamma_1(1-\kappa)(\gamma_1-\gamma_0)\,\frac{1}{2}\frac{\gamma_0}{(1-\kappa)(\gamma_1-\gamma_0)^2}}{1+(1-\kappa)(\gamma_1-\gamma_0)\,\frac{1}{2}\frac{\gamma_0}{(1-\kappa)(\gamma_1-\gamma_0)^2}} = \frac{\gamma_1\gamma_0}{2\gamma_1-\gamma_0} < \gamma_0$$

and the lemma is proved ∎

2.4 Continuity. The right-hand side of integral equations (15) - (17) is regarded as an integral operator F . For short we write $\tilde{u}$ instead of Fu. Let

$$\Omega_T = \Big\{(t,x) : \, 0 \le t \le T \text{ and } 0 \le x \le s(t)\Big\}.$$

Lemma 3. *Let assumptions* (8) – (11)*, compatibility conditions* (12) – (13) *and reciprocity conditions* (6) – (7) *hold for some* $T > 0$*. Let* $s \in C^{1,1}[0,T]$ *be a prescribed fixed boundary, where* $0 \le \dot{s}(t) < \gamma_0$ *for* $t \in [0,T]$*, and* $u \in C(\Omega_T \times \Gamma)$*. Then* $\tilde{u} \in C(\Omega_T \times \Gamma)$*.*

Proof. Inside the domains $I_\gamma - III_\gamma$ continuity follows from the definition of the operator. Now consider a point on the boundary between regions I_γ and II_γ. In order to show continuity in t and x it is sufficient to prove

$$\lim_{\substack{(t,x)\in I_\gamma \\ x\to\gamma t}} \tilde{u}_\gamma(t,x) = \lim_{\substack{(t,x)\in II_\gamma \\ x\to\gamma t}} \tilde{u}_\gamma(t,x).$$

Compatibility condition (12) yields

$$\begin{aligned}
&\lim_{\substack{(t,x)\in I_\gamma \\ x\to\gamma t}} \tilde{u}_\gamma(t,x) \\
&\quad = \lim_{\substack{(t,x)\in I_\gamma \\ x\to\gamma t}} \Big\{ u_0(x-\gamma t, \gamma) e^{-\mu t} \\
&\qquad + \mu \int_0^t e^{-\mu(t-\eta)} \int_\Gamma K(\gamma,\gamma') u\big(\eta, x-\gamma t+\gamma\eta, \gamma'\big)\, d\gamma' d\eta \Big\} \\
&\quad = u_0(0,\gamma) e^{-\mu t} + \mu \int_0^t e^{-\mu(t-\eta)} \int_\Gamma K(\gamma,\gamma') u(\eta,\gamma\eta,\gamma')\, d\gamma' d\eta
\end{aligned}$$

$$
\begin{aligned}
&= \left[\tfrac{r}{|\gamma|} \int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')u(0,0,\gamma')|\gamma'|\,d\gamma' + f_\gamma(0) \right] e^{-\mu t} \\
&\quad + \mu \int_0^t e^{-\mu(t-\rho)} \int_\Gamma K(\gamma,\gamma')u(\rho,\gamma\rho,\gamma')\,d\gamma' d\rho \\
&= \lim_{\substack{(t,x)\in II_\gamma \\ x\to\gamma t}} \Bigg\{ \left[\tfrac{r}{|\gamma|} \int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')u\big(t-\tfrac{x}{\gamma},0,\gamma'\big)|\gamma'|\,d\gamma' + f_\gamma\big(t-\tfrac{x}{\gamma}\big) \right] e^{-\frac{\mu x}{\gamma}} \\
&\quad + \mu \int_{t-\frac{x}{\gamma}}^0 e^{-\mu(t-\rho)} \int_\Gamma K(\gamma,\gamma')u\big(\rho,\gamma\big(\rho-t+\tfrac{x}{\gamma}\big),\gamma'\big)\,d\gamma' d\rho \Bigg\} \\
&= \lim_{\substack{(t,x)\in II_\gamma \\ x\to\gamma t}} \tilde{u}_\gamma(t,x).
\end{aligned}
$$

Now consider a point on the boundary between I_γ and III_γ. Again, it is sufficient to prove

$$
\lim_{\substack{(t,x)\in I_\gamma \\ x\to s_0+\gamma t}} \tilde{u}_\gamma(t,x) = \lim_{\substack{(t,x)\in III_\gamma \\ x\to s_0+\gamma t}} \tilde{u}_\gamma(t,x)
$$

which is guaranteed by compatibility condition (13). Similarly we obtain continuity in the variable γ from compatibility conditions (12) - (13) ∎

2.5 Lipschitz continuity. For the proof of the existence theorem for the free boundary value problem it is essential to have Lipschitz continuity of a solution. This property is shown in the following lemma.

Lemma 4. *Let $t_1 \le \frac{s_0}{\gamma_0}$. Let $s \in C^1$ be a fixed boundary with $\dot{s}(t) > 0$ and $\dot{s}(t) < \gamma_0$ for $t \in [0,t_1]$. Let assumptions $(8)-(11)$, compatibility conditions $(12)-(13)$ and reciprocity conditions $(6)-(7)$ hold. Let $u_\gamma, \gamma \in \Gamma$ be a solution of integral equations $(15)-(17)$. Then there exists a Lipschitz constant for u_γ, uniformly with respect to γ, which depends only on the initial and boundary conditions.*

Proof. For $h > 0$ define the functions

$$
J_\gamma(t) = \max_x \max_{|\xi|\le h, |\sigma|\le h} \big| u(t+\sigma, x+\xi, \gamma) - u(t,x,\gamma) \big|
$$

where $(t,x),(t+\sigma,x+\xi) \in \Omega_{t_1}$. For an arbitrary $t \in (0,t_1)$ we choose x^*,ξ^*,σ^* where $|\xi^*|,|\sigma^*| \le h$ such that

$$
J_\gamma(t) = \big| u(t+\sigma^*, x^*+\xi^*, \gamma) - u(t,x^*,\gamma) \big|.
$$

Further, we define

$$
J(t) = \sup_{\gamma\in\Gamma} J_\gamma(t).
$$

M is chosen in accordance with Lemma 1. Let $Lip(u_{0,\gamma}) \le M_1$ for all $\gamma \in \Gamma$ and $Lip(f_\gamma) \le M_1$ for all $\gamma \in \Gamma^+$.

Case 1: $(t,x^*),(t+\sigma^*,x^*+\xi^*)\in I_\gamma$ for an arbitrary fixed $\gamma\in\Gamma$. Without loss of generality we consider $\sigma^*\geq 0$. From integral equation (15) we get

$$\begin{aligned}
J_\gamma(t) &= \Big|u_0\big(x^*+\xi^*-\gamma(t+\sigma^*),\gamma\big)e^{-\mu(t+\sigma^*)}-u_0(x^*-\gamma t,\gamma)e^{-\mu t}\Big|\\
&\quad+\Big|\mu\int_0^{t+\sigma^*}e^{-\mu(t+\sigma^*-\eta)}\int_\Gamma K(\gamma,\gamma')u\big(\eta,x^*+\xi^*-\gamma(t+\sigma^*)+\gamma\eta,\gamma'\big)\,d\gamma' d\eta\\
&\quad-\mu\int_0^t e^{-\mu(t-\eta)}\int_\Gamma K(\gamma,\gamma')u\big(\eta,x^*-\gamma t+\gamma\eta,\gamma'\big)\,d\gamma' d\eta\Big|\\
&\leq \Big|u_0\big(x^*+\xi^*-\gamma(t+\sigma^*),\gamma\big)-u_0(x^*-\gamma t,\gamma)\Big|e^{-\mu(t+\sigma^*)}\\
&\quad+u_0(x^*-\gamma t,\gamma)\big|e^{-\mu(t+\sigma^*)}-e^{-\mu t}\big|+\mu M\int_t^{t+\sigma^*}e^{-\mu(t+\sigma^*-\eta)}\,d\eta\\
&\quad+\mu\Big|\int_0^t e^{-\mu(t+\sigma^*-\eta)}\\
&\quad\times\int_\Gamma K(\gamma,\gamma')\Big\{u_{\gamma'}\big(\eta,x^*+\xi^*-\gamma(t+\sigma^*+\eta)\big)-u_{\gamma'}\big(\eta,x^*-\gamma t+\gamma\eta\big)\Big\}d\gamma'\\
&\quad+\big(e^{-\mu(t+\sigma^*-\eta)}-e^{-\mu(t-\eta)}\big)\int_\Gamma K(\gamma,\gamma')u\big(\eta,x^*-\gamma t+\gamma\eta,\gamma'\big)\,d\gamma' d\eta\Big|\\
&\leq (1+\gamma)hM_1+M_0\mu h+M\big(e^{\mu(t+\sigma^*)}-e^{\mu t}\big)e^{-\mu(t+\sigma^*)}\\
&\quad+\mu(1+\gamma)\Big|\int_0^t e^{-\mu(t+\sigma^*-\eta)}J(\eta)\,d\eta\Big|+\mu M\Big|\int_0^t\big(e^{-\mu(t+\sigma^*-\eta)}-e^{-\mu(t-\eta)}\big)d\eta\Big|\\
&\leq H_1 h+\mu(1+\gamma_1)\int_0^t J(\eta)\,d\eta
\end{aligned}$$

where $H_1=2\mu M+(1+\gamma_1)M_1+M_0\mu$.

Case 2: $(t,x^*),(t+\sigma^*,x^*+\xi^*)\in II_\gamma$ for an arbitrarily chosen, fixed $\gamma\in\Gamma$. Without loss of generality we consider

$$t-\frac{x^*}{\gamma}\leq t+\sigma^*-\frac{x^*-\xi^*}{\gamma}\leq t\leq t+\sigma^*.$$

First, we need an auxiliary estimate. Remembering the choice of the time interval, we know that all the points on the left boundary belong to the domain $I_{\gamma'}$ for $\gamma'\in\Gamma^-$. Hence we get

$$\begin{aligned}
&\tfrac{r}{|\gamma|}\int_{-\gamma_1}^{-\gamma_0}L(\gamma,\gamma')\Big\{u\big(t+\sigma^*-\tfrac{x^*+\xi^*}{\gamma},0,\gamma'\big)-u\big(t-\tfrac{x^*}{\gamma},0,\gamma'\big)\Big\}|\gamma'|\,d\gamma' e^{-\frac{\mu(x^*+\xi^*)}{\gamma}}\\
&\leq \tfrac{r}{|\gamma|}\int_{-\gamma_1}^{-\gamma_0}L(\gamma,\gamma')\Big\{u_0\big(-\gamma'\big(t+\sigma^*-\tfrac{x^*+\xi^*}{\gamma}\big),\gamma'\big)e^{-\mu(t+\sigma^*-\frac{x^*+\xi^*}{\gamma})}\\
&\quad-u_0\big(-\gamma'\big(t-\tfrac{x^*}{\gamma}\big),\gamma'\big)e^{-\mu(t-\frac{x^*}{\gamma})}+\mu\int_0^{t+\sigma^*-\frac{x^*+\xi^*}{\gamma}}e^{-\mu(t+\sigma^*-\frac{x^*+\xi^*}{\gamma}-\eta)}\\
&\quad\times\int_\Gamma K(\gamma',\gamma'')u\big(\eta,-\gamma'(t+\sigma^*-\tfrac{x^*+\xi^*}{\gamma})+\gamma'\eta,\gamma''\big)\,d\gamma'' d\eta-\mu\int_0^{t-\frac{x^*}{\gamma}}e^{-\mu(t-\frac{x^*}{\gamma}-\eta)}
\end{aligned}$$

$$
\begin{aligned}
&\qquad \times \int_\Gamma K(\gamma',\gamma'')u\big(\eta,-\gamma'\big(t-\tfrac{x^*}{\gamma}\big)+\gamma'\eta,\gamma''\big)d\gamma'' d\eta\Big\}\,|\gamma'|\,d\gamma' e^{-\frac{\mu(x^*+\xi^*)}{\gamma}}\\
&\le \tfrac{r}{|\gamma|}\int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')\big(1+\tfrac{1}{\gamma}\big)\\
&\qquad \times \Big\{\big(\gamma_1 M_1+\mu M_0+3\mu M\big)h+\gamma_1\mu\int_0^{t-\frac{x^*}{\gamma}} J(\eta)\,d\eta\Big\}|\gamma'|\,d\gamma' e^{-\frac{\mu(x^*+\xi^*)}{\gamma}}\\
&\le \hat r\Big\{\big(\gamma_1 M_1+\mu M_0+3\mu M\big)\big(1+\tfrac{1}{\gamma}\big)h+\gamma_1\big(1+\tfrac{1}{\gamma}\big)\mu\int_0^{t-\frac{x^*}{\gamma}} J(\eta)\,d\eta\Big\}e^{-\frac{\mu(x^*+\xi^*)}{\gamma}}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&J_\gamma(t)\\
&=\Big|\Big[\tfrac{r}{|\gamma|}\int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')u\big(t+\sigma^*-\tfrac{x^*+\xi^*}{\gamma},0,\gamma'\big)|\gamma'|\,d\gamma'+f_\gamma\big(t+\sigma^*-\tfrac{x^*+\xi^*}{\gamma}\big)\Big]\,e^{-\frac{\mu(x^*+\xi^*)}{\gamma}}\\
&\quad -\Big[\tfrac{r}{|\gamma|}\int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')u\big(t-\tfrac{x^*}{\gamma},0,\gamma'\big)|\gamma'|\,d\gamma'+f_\gamma\big(t-\tfrac{x^*}{\gamma}\big)\Big]\,e^{-\frac{\mu x^*}{\gamma}}\Big|\\
&\quad +\Big|\mu\int_{t+\sigma^*-\frac{x^*+\xi^*}{\gamma}}^{t+\sigma^*} e^{-\mu(t+\sigma^*-\rho)}\int_\Gamma K(\gamma,\gamma')u\big(\rho,\gamma\big(\rho-(t+\sigma^*)+\tfrac{x^*+\xi^*}{\gamma}\big),\gamma'\big)d\gamma' d\rho\\
&\quad -\mu\int_{t-\frac{x^*}{\gamma}}^{t} e^{-\mu(t-\rho)}\int_\Gamma K(\gamma,\gamma')u\big(\rho,\gamma\big(\rho-t+\tfrac{x^*}{\gamma}\big),\gamma'\big)d\gamma' d\rho\Big|\\
&\le \hat r\Big\{\big(\gamma_1 M_1+\mu M_0+3\mu M\big)\big(1+\tfrac{1}{\gamma}\big)h+\gamma_1\big(1+\tfrac{1}{\gamma}\big)\mu\int_0^{t-\frac{x^*}{\gamma}} J(\eta)\,d\eta\Big\}e^{-\frac{\mu(x^*+\xi^*)}{\gamma}}\\
&\quad + M_0\frac{\mu}{\gamma}h+\mu M\Big|\int_{t+\sigma^*-\frac{x^*+\xi^*}{\gamma}}^{t-\frac{x^*}{\gamma}} e^{-\mu(t+\sigma^*-\rho)}d\rho\Big|\\
&\quad +\mu M\Big|\int_t^{t+\sigma^*} e^{-\mu(t+\sigma^*-\rho)}d\rho\Big|+M_1\big|\sigma^*-\tfrac{\xi^*}{\gamma}\big|e^{-\frac{\mu(x^*+\xi^*)}{\gamma}}\\
&\quad +\mu\Big|\int_{t-\frac{x^*}{\gamma}}^{t} e^{-\mu(t+\sigma^*-\rho)}\int_\Gamma K(\gamma,\gamma')\Big\{u_{\gamma'}\big(\rho,\gamma\big(\rho-t-\sigma^*+\tfrac{x^*+\xi^*}{\gamma}\big)\big)\\
&\quad -u_{\gamma'}\big(\rho,\gamma\big(\rho-t+\tfrac{x^*}{\gamma}\big)\big)\Big\}d\gamma'\\
&\quad +\big(e^{-\mu(t+\sigma^*-\rho)}-e^{-\mu(t-\rho)}\big)\int_\Gamma K(\gamma,\gamma')u\big(\rho,\gamma\big(\rho-t+\tfrac{x^*}{\gamma}\big),\gamma'\big)\,d\gamma' d\rho\Big|+\hat r M\frac{\mu}{\gamma}h\\
&\le H_2 h+\mu\big(1+\tfrac{1}{\gamma_0}\big)(\hat r\gamma_1+1)\int_0^t J(\rho)\,d\rho
\end{aligned}
$$

where

$$
\begin{aligned}
H_2 &= \hat r\big(\gamma_1 M_1+\mu M_0+3\mu M\big)\big(1+\tfrac{1}{\gamma_0}\big)\\
&\quad +\hat r M\frac{\mu}{\gamma}+M_1\big(1+\tfrac{1}{\gamma_0}\big)+M_0\frac{\mu}{\gamma_0}+\mu M\big(4+\tfrac{1}{\gamma_0}\big).
\end{aligned}
$$

Case 3: $(t,x^*),(t+\sigma^*,x^*+\xi^*)\in III_\gamma$ for an arbitrary, fixed $\gamma\in\Gamma$. In the sequel we use the short notation $\Psi_1=\Psi(t,x^*)$ and $\Psi_2=\Psi(t+\sigma^*,x^*+\xi^*)$. Without loss of generality we assume $\Psi_1 \le \Psi_2 \le t \le t+\sigma^*$. In accordance with the assumptions we have $|\Psi_1 - \Psi_2| \le h(1+\frac{1}{\gamma_0})$. The choice of t_1 guarantees that all points on the right boundary belong to the domain $I_{\gamma'}$ for $\gamma' \in \Gamma'$. Then we obtain

$$\begin{aligned}
&\max_{\gamma'\in\Gamma^+} \Big| u(\Psi_2, s(\Psi_2), \gamma') - u(\Psi_1, s(\Psi_1), \gamma') \Big| \\
&\quad \le \max_{\gamma'\in\Gamma^+} \Big\{ \big(2\mu M + (1+\gamma)M_1 + M_0\mu\big)h + \mu(1+\gamma) \int_0^t J(\eta)\, d\eta \Big\} \\
&\quad \times (1 + Lip(s)) \frac{|\Psi_1 - \Psi_2|}{h} \\
&\quad \le \Big\{ \big(2\mu M + (1+\gamma_1)M_1 + M_0\mu\big)h + \mu(1+\gamma_1) \int_0^t J(\eta)\, d\eta \Big\} \big(1+\tfrac{1}{\gamma_0}\big)(1+\gamma_0).
\end{aligned}$$

In accordance with the definition we get

$$\begin{aligned}
J_\gamma(t) &= \Bigg| \Big[\tfrac{\kappa}{|\gamma|} \int_{\gamma_0}^{\gamma_1} R(\gamma, \gamma') u(\Psi_2, s(\Psi_2), \gamma') |\gamma'|\, d\gamma' \Big] e^{-\mu(t-\Psi_2)} \\
&\quad - \Big[\tfrac{\kappa}{|\gamma|} \int_{\gamma_0}^{\gamma_1} R(\gamma, \gamma') u(\Psi_1, s(\Psi_1), \gamma') |\gamma'|\, d\gamma' \Big] e^{-\mu(t-\Psi_1)} \\
&\quad + \mu \int_{\Psi_2}^{t+\sigma^*} e^{-\mu(t+\sigma^*-\rho)} \int_\Gamma K(\gamma, \gamma') u\big(\rho, s(\Psi_2) - \gamma(\rho - \Psi_2), \gamma'\big)\, d\gamma' d\rho \\
&\quad - \mu \int_{\Psi_1}^{t} e^{-\mu(t+\sigma^*-\rho)} \int_\Gamma K(\gamma, \gamma') u\big(\rho, s(\Psi_1) - \gamma(\rho - \Psi_1), \gamma'\big)\, d\gamma' d\rho \Bigg| \\
&\le \hat\kappa \max_{\gamma'\in\Gamma^+} \big\{ u(\Psi_2, s(\Psi_2), \gamma') \big\} \mu |\Psi_2 - \Psi_1| \\
&\quad + \hat\kappa \max_{\gamma'\in\Gamma^+} \Big\{ u(\Psi_2, s(\Psi_2), \gamma') - u(\Psi_1, s(\Psi_1), \gamma') \Big\} e^{-\mu(t-\Psi_1)} \\
&\quad + \mu \left| \int_{\Psi_2}^t e^{-\mu(t+\sigma^*-\rho)} J(\rho) \big\{ \gamma_0\big(1+\tfrac{1}{\gamma_0}\big) + \gamma_1\big(1+\tfrac{1}{\gamma_0}\big) \big\} d\rho \right| \\
&\quad + \mu \left| \int_{\Psi_2}^t \mu h M\, d\rho \right| + \mu M |\Psi_1 - \Psi_2| + \mu M h \\
&\le \hat\kappa e^{-\mu(t-\Psi_1)} \Big\{ \big(2\mu M + (1+\gamma_1)M_1 + M_0\mu\big)h + \mu(1+\gamma_1) \int_0^t J(\eta)\, d\eta \Big\} \\
&\quad \times \big(1+\tfrac{1}{\gamma_0}\big)(1+\gamma_0) \\
&\quad + \hat\kappa M \mu \big(1+\tfrac{1}{\gamma_0}\big) h + \mu(\gamma_0+\gamma_1)\big(1+\tfrac{1}{\gamma_0}\big) \int_0^t J(\rho)\, d\rho + \mu^2 M T_1 h + 2\mu M \big(1+\tfrac{1}{\gamma_0}\big) h \\
&\le H_3 h + \mu\big(1+\tfrac{1}{\gamma_0}\big) \big\{ \hat\kappa(1+\gamma_0)(1+\gamma_1) + (\gamma_0+\gamma_1) \big\} \int_0^t J(\rho)\, d\rho
\end{aligned}$$

where

$$H_3 = \big(1+\tfrac{1}{\gamma_0}\big) \Big[(\hat\kappa+2)\mu + \hat\kappa(1+\gamma_0)\big(3\mu M + (1+\gamma_1)M_1\big) + \mu^2 M t_1 \Big].$$

In the sequel we use the fact that if $g: [a,c] \to \mathbb{R}$ is a continuous function, Lipschitz continuous on $[a,b]$ and on $[b,c]$, then it is Lipschitz continuous on $[a,c]$. Hence we need not investigate cases, where testing points belong to different domains.

Define

$$C_1 = \max\{H_i : i = 1,2,3\}$$
$$C_2 = \mu\big(1 + \tfrac{1}{\gamma_0}\big)\Big((\hat{r}+1)\gamma_1 + 1 + \gamma_0 + \hat{\kappa}(1+\gamma_0)(1+\gamma_1)\Big).$$

Altogether we obtain an estimate which holds for the whole domain Ω_{t_1}:

$$J(t) \le C_1 h + C_2 \int_0^t J(\rho)\, d\rho.$$

The Gronwall lemma yields $J(t) \le (C_1 h)\, e^{C_2 t}$ and we get $J(t) \le C_{ges}\, h$ with $C_{ges} = C_1 e^{C_2 t}$ ∎

The assertion of this lemma can be extended iteratively to arbitrary time intervals by an iteration process.

3. Existence and uniqueness for the fixed boundary problem

The following proposition yields existence and uniqueness for solutions of problem (15) - (17) with prescribed fixed right boundary s on the domain of definition

$$\Omega_T = \Big\{(t,x) : 0 \le t \le T \text{ and } 0 \le x \le s(t)\Big\}.$$

It will be used for the existence proof for the free boundary problem.

Proposition 1. *Let s be a fixed boundary, $s(t) > 0$ for $t \in [0,T]$, and s be continuously differentiable with $0 < \dot{s}(t) < \gamma_0$. Let assumptions (8) – (11), compatibility conditions (12) – (13) and reciprocity conditions (6) – (7) hold. Then there exists a unique continuous solution of problem (15) – (17).*

Proof. Lipschitz continuity of f_γ and $u_{0,\gamma}$ guarantees the existence of a constant $c > 0$ which satisfies

$$|u_{0,\gamma}(x_1) - u_{0,\gamma}(x_2)| \le c\,|x_1 - x_2| \qquad (x_1, x_2 \in [0, s_0])$$
$$|f_\gamma(t_1) - f_\gamma(t_2)| \le c\,|t_1 - t_2| \qquad (t_1, t_2 \ge 0,\ \gamma > 0).$$

We have already shown that every solution satisfies the integral equation

$$u_\gamma(t,x) =$$
$$\begin{cases} u_0(x-\gamma t, \gamma)e^{-\mu t} + \mu \int_0^t e^{-\mu(t-\eta)} \int_\Gamma K(\gamma,\gamma')u_{\gamma'}(\eta, x-\gamma t+\gamma\eta)\, d\gamma' d\eta & \text{in } I_\gamma \\ \Big[\frac{r}{|\gamma|}\int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')u(t-\frac{x}{\gamma},0,\gamma')|\gamma'|\,d\gamma' + f_\gamma(t-\frac{x}{\gamma})\Big]\, e^{-\frac{\mu x}{\gamma}} & \\ +\mu\int_{t-\frac{x}{\gamma}}^t e^{-\mu(t-\rho)}\int_\Gamma K(\gamma,\gamma')u(\rho,\gamma(\rho-t+\frac{x}{\gamma}),\gamma')\,d\gamma' d\rho & \text{in } II_\gamma \\ \frac{\kappa}{|\gamma|}\int_{\gamma_0}^{\gamma_1} R(\gamma,\gamma')u_{\gamma'}(\Psi,s(\Psi))|\gamma'|\,d\gamma' e^{-\mu(t-\Psi)} & \\ +\mu\int_\Psi^t e^{-\mu(t-\rho)}\int_\Gamma K(\gamma,\gamma')u\big(\rho, s(\Psi)+\gamma(\rho-\Psi),\gamma'\big)\,d\gamma' d\rho & \text{in } III_\gamma. \end{cases}$$

The right-hand side of this equation is produced by the operator F. We are looking for a fixed point of F. Recall that F maps u into $\tilde{u}$. The space $C(\Omega_T \times \Gamma)$ with norm $\|u\| = \max_{\gamma\in\Gamma} \|u_\gamma\|$ is a Banach space. For the solution u we define a box M as follows:

$$M = \Big\{u \in C(\Omega_T \times \Gamma) : \big|u_\gamma(t,x) - u_{0,\gamma}(x)\big| \le m \ \ \forall\,(t,x) \in \Omega_T, \gamma \in \Gamma\Big\}.$$

Claim. *There exist constants $T, m > 0$ such that $u \in M$ implies $\tilde{u} \in M$.*

Lemma 3 guarantees continuity of $\tilde{u}_\gamma$ as a function of (t,x) in Ω_T for $\gamma \in \Gamma$. First we choose $T > 0$ such that $|s(t) - s_0| < s_0$ for $0 \le t \le T$ and define

$$m_0 = \sup_{t\in[0,T]} |s(t) - s(0)|.$$

Then we choose $m > \frac{c\,m_0}{(1-\kappa)(s_0-m_0)}$ and T possibly smaller such that

$$T \le \min\left\{\frac{(1-\hat{\kappa})m - cm_0}{2\mu(\max\limits_{\gamma\in\Gamma}\|u_{0,\gamma}\| + m) + c\gamma_1},\ \frac{(1-\hat{r})m}{\mu(1+\hat{r})(\max\limits_{\gamma\in\Gamma}\|u_{0,\gamma}\| + m) + \mu\|f\| + c(1+\gamma_1)}\right\}.$$

Since $K(\gamma,\gamma') \ge 0$ and $\int_\Gamma K(\gamma,\gamma')\,d\gamma' = 1$, we get

$$\int_\Gamma K(\gamma,\gamma')u(t,x,\gamma')\,d\gamma' \le \max_{\gamma\in\Gamma}\{u(t,x,\gamma')\}.$$

Next we estimate $\tilde{u}_\gamma(t,x)$ in all three domains $I_\gamma - III_\gamma$:

Estimate for $\tilde{u}_\gamma(t,x)$ in I_γ:

$$\begin{aligned}
&\big|\tilde{u}_\gamma(t,x) - u_{0,\gamma}\big(\tfrac{xs_0}{s(t)}\big)\big| \\
&\le \|u_{0,\gamma}\|(1-e^{-\mu t}) + \gamma_1 cT + \Big(\max_{\gamma\in\Gamma}\|u_{0,\gamma}\| + m\Big)(1-e^{-\mu t}) + c\,\big|\tfrac{x(s(t)-s_0)}{s(t)}\big| \\
&\le T\Big(2\mu\max_{\gamma\in\Gamma}\|u_{0,\gamma}\| + \gamma_1 c + \mu m\Big) + cm_0 \\
&\le m.
\end{aligned}$$

Estimate for $\tilde{u}_\gamma(t,x)$ in II_γ: By the same technique, with $x \le \gamma t$ we get

$$\begin{aligned}
&\big|\tilde{u}_\gamma(t,x) - u_{0,\gamma}\big(\tfrac{xs_0}{s(t)}\big)\big| \\
&\quad\le \Bigg|\Bigg[\tfrac{r}{|\gamma|}\int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')u\big(t-\tfrac{x}{\gamma},0,\gamma'\big)|\gamma'|\,d\gamma' + f_\gamma\big(t-\tfrac{x}{\gamma}\big)\Bigg]e^{-\frac{\mu x}{\gamma}} - f_\gamma(0) \\
&\qquad - \tfrac{r}{|\gamma|}\int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')u(0,0,\gamma')|\gamma'|\,d\gamma' + u_\gamma(0,0) - u_{0,\gamma}\big(\tfrac{xs_0}{s(t)}\big) \\
&\qquad + \mu T\Big(\max_{\gamma\in\Gamma}\|u_{0,\gamma}\| + m\Big)\Bigg| \\
&\quad\le \Big|\Big(\hat{r}(\max_{\gamma\in\Gamma}\|u_{0,\gamma}\| + m\Big) + \|f\|\Big)(e^{-\frac{\mu x}{\gamma}} - 1)\Big| + \Big|\tfrac{r}{|\gamma|}\int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')|\gamma'|\,d\gamma'\,m\Big| \\
&\qquad + c\,\big|t - \tfrac{x}{\gamma}\big| + c\,\big|\tfrac{xs_0}{s(t)}\big| + \mu T\Big(\max_{\gamma\in\Gamma}\|u_{0,\gamma}\| + m\Big) \\
&\quad\le T\Big(\mu(1+\hat{r})\Big(\max_{\gamma\in\Gamma}\|u_{0,\gamma}\| + m\Big) + \mu\|f\| + c\,(1+\gamma_1)\Big) + \hat{r}m \\
&\quad\le m.
\end{aligned}$$

Estimate for $\tilde{u}_\gamma(t,x)$ in III_γ: Here we have $x > s_0 - \gamma_1 t$, using the short notation $\Psi = \Psi_\gamma(t,x)$ we obtain

$$\begin{aligned}
&\left|\tilde{u}_\gamma(t,x) - u_{0,\gamma}\big(\tfrac{xs_0}{s(t)}\big)\right| \\
&\quad \le \left|\hat{\kappa}\Big(\max_{\gamma\in\Gamma}\|u_{0,\gamma}\| + m\Big)\big(e^{-\mu(t-\Psi)} - 1\big)\right| + c\left|s_0 - \tfrac{xs_0}{s(t)}\right| \\
&\qquad + \mu T\Big(\max_{\gamma\in\Gamma}\|u_{0,\gamma}\| + m\Big) + \left|\tfrac{\kappa}{|\gamma|}\int_{\gamma_0}^{\gamma_1} R(\gamma,\gamma')|\gamma'|\,d\gamma'\,m\right| \\
&\quad \le T\Big(\mu(1+\hat{\kappa})\Big(\max_{\gamma\in\Gamma}\|u_{0,\gamma}\| + m\Big) + c\gamma_1\Big) + \hat{\kappa}m + cm_0 \\
&\quad \le m.
\end{aligned}$$

Hence the claim is proved.

Now we show the contraction property. We choose $u_1, u_2 \in M$ satisfying

$$\begin{aligned}
u_1(0,x,\gamma) &= u_0(x,\gamma) = u_2(0,x,\gamma) \\
u_i(t,0,\gamma) &= \tfrac{r}{|\gamma|}\int_{-\gamma_1}^{-\gamma_0} L(\gamma,\gamma')u_i(t,0,\gamma')|\gamma'|\,d\gamma' + f(t,\gamma) \quad (i=1,2).
\end{aligned}$$

In I_γ the following estimate is valid:

$$\begin{aligned}
&\|\tilde{u}_{1,\gamma} - \tilde{u}_{2,\gamma}\| \\
&\quad = \sup_{(t,x)\in I_\gamma}\Big|\mu\int_0^t e^{-\mu(t-\eta)}\int_\Gamma K(\gamma,\gamma')u_1\big(\eta, x-\gamma t+\gamma\eta,\gamma'\big)\,d\gamma' d\eta + u_0(x-\gamma t,\gamma)e^{-\mu t} \\
&\qquad - \mu\int_0^t e^{-\mu(t-\eta)}\int_\Gamma K(\gamma,\gamma')u_2\big(\eta, x-\gamma t+\gamma\eta,\gamma'\big)\,d\gamma' d\eta - u_0(x-\gamma t,\gamma)e^{-\mu t}\Big| \\
&\quad \le \sup_{(t,x)\in I_\gamma}\mu\int_0^t e^{-\mu(t-\eta)}d\eta\,\max_{\gamma\in\Gamma}\|u_{1,\gamma} - u_{2,\gamma}\| \\
&\quad \le \mu T\max_{\gamma\in\Gamma}\|u_{1,\gamma} - u_{2,\gamma}\|.
\end{aligned}$$

In II_γ we have $x \le \gamma t$, therefore we get

$$\|\tilde{u}_{1,\gamma} - \tilde{u}_{2,\gamma}\| \le \hat{r}\max_{\gamma\in\Gamma}\|u_{1,\gamma} - u_{2,\gamma}\| + \max_{\gamma\in\Gamma}\|u_1 - u_2\|\,\mu T.$$

Similarly, in III_γ we get

$$\|\tilde{u}_{1,\gamma} - \tilde{u}_{2,\gamma}\| \le \hat{\kappa}\max_{\gamma\in\Gamma}\|u_{1,\gamma} - u_{2,\gamma}\| + \mu T\max_{\gamma\in\Gamma}\|u_{1,\gamma} - u_{2,\gamma}\|.$$

We can decrease T once more if necessary, such that $\mu T + \max(\hat{\kappa},\hat{r}) < 1$. Then

$$\max_{\gamma\in\Gamma}\|\tilde{u}_{1,\gamma} - \tilde{u}_{2,\gamma}\| \le \big(\mu T + \max(\hat{\kappa},\hat{r})\big)\max_{\gamma\in\Gamma}\|u_{1,\gamma} - u_{2,\gamma}\|$$

is true for the whole domain $\tilde{\Omega}_T$. Hence we have a contraction constant strictly less than 1, and by Banach's theorem we get existence and uniqueness of a fixed point of F ∎

4. Existence for the free boundary problem

The existence for the free boundary problem is shown in the following proposition, using a fixed point argument.

Proposition 2. *Let assumptions* (8) – (11)*, compatibility conditions* (12) – (13) *and reciprocity conditions* (6) – (7) *hold for some* $\tilde{t} > 0$*. Let*

$$M_0 = e^{-\frac{\mu\tilde{t}}{1-\max(\hat{\kappa},\hat{r})}} \frac{(1-\max(\hat{\kappa},\hat{r}))\gamma_0}{2(1-\kappa)(\gamma_1-\gamma_0)^2}$$

be an upper bound for the initial data $u_{0,\gamma}$ $(\gamma \in \Gamma)$ *and for the source term on the left boundary* $f_\gamma(t)$ $(\gamma \in \Gamma^+,\ 0 \le t \le \tilde{t})$*. Then there exists a solution* (s,u) *of equations* (15) – (17) *and* (14) *such that* $s \in C^{1,1}([0,\tilde{t}])$ *and* $u_\gamma \in C^{0,1}(\Omega_{\tilde{t}})$ *for all* $\gamma \in \Gamma$*.*

Proof. We want to construct a solution for $t \le \tilde{t}$. Let s be a candidate for the free boundary. Positivity of u yields $\dot{s}(t) \ge 0$ and $s(t) \ge s_0 > 0$ for all $0 \le t \le \tilde{t}$. Obviously, $\dot{s}(t) < \gamma_1$ holds. Therefore, we know from Lemma 1 that

$$|u(t,x,\gamma)| \le \frac{\gamma_0}{2(1-\kappa)(\gamma_1-\gamma_0)^2}.$$

Define $c_1 = \frac{\gamma_1\gamma_0}{2\gamma_1-\gamma_0}$. By Lemma 2 each solution s of equation (14) satisfies $\dot{s}(t) \le c_1 < \gamma_0$ and we obtain

$$s(t) \le s_0 + \gamma_0\tilde{t} =: c_0.$$

Let $\sigma = \frac{s_0}{2\gamma_1}$. We consider the set of functions

$$\mathcal{B} = \left\{ s \in C^1[0,\sigma] :\ s(0) = s_0,\ \dot{s}(0) = s_1,\ \|\dot{s}\| \le c_1 \right\}$$

where

$$s_1 = \frac{(1-\kappa)\int_{\gamma_0}^{\gamma_1} \gamma' u_0(s_0,\gamma')\,d\gamma'}{1+(1-\kappa)\int_{\gamma_0}^{\gamma_1} u_0(s_0,\gamma')\,d\gamma'},$$

which guarantees compatibility with the initial data. For $t \in [0,\sigma]$ Proposition 1 yields the existence of a solution $u_\gamma \in C(\Omega_\sigma)$ $(\gamma \in \Gamma)$ for a fixed $s \in \mathcal{B}$ with $\dot{s}(t) > 0$. Now we define a map G by

$$G : \mathcal{B} \to C^1[0,\sigma], \qquad (Gs)(t) = s_0 + \int_0^t \frac{(1-\kappa)\int_{\gamma_0}^{\gamma_1} \gamma' u(\eta,s(\eta),\gamma')\,d\gamma'}{1+(1-\kappa)\int_{\gamma_0}^{\gamma_1} u(\eta,s(\eta),\gamma')\,d\gamma'}\,d\eta.$$

From the assumptions we get

$$\left.\begin{aligned} (Gs)(0) &= s_0 \\ \tfrac{d}{dt}(Gs)(0) &= s_1 \end{aligned}\right\}$$

and the estimate

$$\left|\frac{d}{dt}(Gs)(t)\right| = \left|\frac{(1-\kappa)\int_{\gamma_0}^{\gamma_1} \gamma' u(t,s(t),\gamma')\,d\gamma'}{1+(1-\kappa)\int_{\gamma_0}^{\gamma_1} u(t,s(t),\gamma')\,d\gamma'}\right| \le c_1,$$

hence $G(\mathcal{B}) \subset \mathcal{B}$.

Let $0 < L < \infty$ be a uniform Lipschitz constant for $u_\gamma \quad (\gamma \in \Gamma)$ (the existence is guaranteed by Lemma 4). Then the following estimate holds:

$$\begin{aligned}
&\left|\frac{d}{dt}Gs(t+h) - \frac{d}{dt}Gs(t)\right| \\
&\quad = \left|\frac{(1-\kappa)\int_{\gamma_0}^{\gamma_1}\gamma' u(t+h,s(t+h),\gamma')\,d\gamma'}{1+(1-\kappa)\int_{\gamma_0}^{\gamma_1}u(t+h,s(t+h),\gamma')\,d\gamma'} - \frac{(1+\kappa)\int_{\gamma_0}^{\gamma_1}\gamma' u(t,s(t),\gamma')\,d\gamma'}{1+(1-\kappa)\int_{\gamma_0}^{\gamma_1}u(t,s(t),\gamma')\,d\gamma'}\right| \\
&\quad \le \Bigg|(1-\kappa)\int_{\gamma_0}^{\gamma_1}\gamma' u(t+h,s(t+h),\gamma')\,d\gamma'\Bigg(1+(1-\kappa)\int_{\gamma_0}^{\gamma_1}u(t,s(t),\gamma')\,d\gamma' \\
&\qquad - (1-\kappa)\int_{\gamma_0}^{\gamma_1}\gamma' u(t,s(t),\gamma')\,d\gamma'\Bigg(1+(1-\kappa)\int_{\gamma_0}^{\gamma_1}u(t,s(t),\gamma')\,d\gamma'\Bigg) \\
&\qquad + (1-\kappa)\int_{\gamma_0}^{\gamma_1}\gamma' u(t,s(t),\gamma')\,d\gamma'\Bigg(1+(1-\kappa)\int_{\gamma_0}^{\gamma_1}u(t,s(t),\gamma')\,d\gamma'\Bigg) \\
&\qquad - (1-\kappa)\int_{\gamma_0}^{\gamma_1}\gamma' u(t,s(t),\gamma')\,d\gamma'\Bigg(1+(1-\kappa)\int_{\gamma_0}^{\gamma_1}u(t+h,s(t+h),\gamma')d\gamma'\Bigg)\Bigg| \\
&\quad \le \big(1+(1-\kappa)(\gamma_1-\gamma_0)M\big)(1-\kappa)L\int_{\gamma_0}^{\gamma_1}\gamma' d\gamma'(1+\gamma_0)h \\
&\qquad + (1-\kappa)^2 M\int_{\gamma_0}^{\gamma_1}\gamma'\,d\gamma'(\gamma_1-\gamma_0)L(1+\gamma_0)h \\
&\quad = (1-\kappa)L(1+\gamma_0)\big(1+2(1-\kappa)(\gamma_1-\gamma_0)M\big)\frac{1}{2}(\gamma_1^2-\gamma_0^2)h.
\end{aligned}$$

Obviously, $\mathcal{B}$ is closed and convex, and $G(\mathcal{B})$ is a bounded set in $C^{1,1}[0,\sigma]$. Therefore, [6: Lemma 6.36] yields that $G(\mathcal{B})$ is precompact in $C^{1,0}[0,\sigma]$. Schauder's fixed point theorem guarantees the existence of a fixed point $\tilde{s} \in \mathcal{B}$ of G, which is a solution for $t \in [0,\sigma]$. Iterating this argument, the solution can be extended to the whole time interval $[0,\tilde{t}]$ ∎

5. Uniqueness for the free boundary problem

Since Schauder's fixed point theorem does not yield uniqueness of the solution, this has to been shown separately.

Proposition 3. *Let the assumptions of Proposition* 2 *hold. Then for any* $t_1 > 0$ *there exists at most one solution* (s,u) *of equations* (15) – (17) *and* (14) *with* $s \in C^{1,1}([0,t_1])$ *and* $u_\gamma \in C^{0,1}(\Omega_{t_1})$ *for all* $\gamma \in \Gamma$.

Proof. First we want to show uniqueness for a $t \in [0,t_1^*]$, where $t_1^* \le \frac{s_0}{2\gamma_1}$. Assume there are two different solutions (s,u) and $(\tilde{s},\tilde{u})$. Then choose a constant M such that

$$|E|, Lip(E) \le M \qquad \text{for } E = u_\gamma, \tilde{u}_\gamma \ (\gamma \in \Gamma).$$

We set

$$
\begin{aligned}
s_0(t) &= \min\{s(t), \tilde{s}(t)\} \\
I(t) &= |\dot{s}(t) - \dot{\tilde{s}}(t)| \\
J_\gamma(t) &= \max\nolimits_{0 \le x \le s_0(t)} |u_\gamma(t,x) - \tilde{u}_\gamma(t,x)| \\
J(t) &= \max\nolimits_{\gamma \in \Gamma}\{J_\gamma(t)\}.
\end{aligned}
$$

Let I_γ be the domain corresponding to s and let $\tilde{I}_\gamma$ correspond to $\tilde{s}$. Then let $\hat{I}_\gamma = I_\gamma \cap \tilde{I}_\gamma$. Similarly $\widehat{III}_\gamma$ is defined. Let $t \in (0, t_1^*)$. Now we choose arbitrary but fixed $\gamma \in \Gamma$ and $x^* \in [0, s_0(t)]$ such that

$$
J_\gamma(t) = \big|u_\gamma(t, x^*) - \tilde{u}_\gamma(t, x^*)\big|.
$$

If $(t, x^*) \in \hat{I}_\gamma$, then the integral equations yield $J_\gamma(t) \le \mu \int_0^t J(\eta)\, d\eta$, similarly $J_\gamma(t) \le 2\mu \int_0^t J(\eta)\, d\eta$ for $(t, x^*) \in II_\gamma$. Further we need the estimate

$$
\max_{0 \le t' \le t} |s(t') - \tilde{s}(t')| \le \int_0^t |\dot{s}(\eta) - \dot{\tilde{s}}(\eta)|\, d\eta = \int_0^t I(\eta)\, d\eta.
$$

Now we consider $(t, x^*) \in \widehat{III}_\gamma$. Without loss of generality we assume $\Psi_\gamma(t, x^*) \ge \tilde{\Psi}_\gamma(t, x^*)$ and use the short notation $\Psi = \Psi_\gamma(t, x^*)$ and $\tilde{\Psi} = \tilde{\Psi}_\gamma(t, x^*)$. By definition we have

$$
\begin{aligned}
x^* - \gamma(t - \Psi) = s(\Psi) \quad &\Longleftrightarrow \quad \Psi = \tfrac{s(\Psi) - x^*}{\gamma} + t \quad \Longleftrightarrow \quad s(\Psi) - \gamma\Psi = x^* - \gamma t \\
x^* - \gamma(t - \tilde{\Psi}) = \tilde{s}(\tilde{\Psi}) \quad &\Longleftrightarrow \quad \tilde{\Psi} = \tfrac{\tilde{s}(\tilde{\Psi}) - x^*}{\gamma} + t \quad \Longleftrightarrow \quad \tilde{s}(\tilde{\Psi}) - \gamma\tilde{\Psi} = x^* - \gamma t
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
0 &\le \Psi - \tilde{\Psi} \\
&= \big|\tfrac{\tilde{s}(\tilde{\Psi})}{\gamma} - \tfrac{s(\Psi)}{\gamma}\big| \\
&\le \tfrac{1}{\gamma}|\tilde{s}(\tilde{\Psi}) - s(\tilde{\Psi})| + \tfrac{1}{\gamma}|s(\tilde{\Psi}) - s(\Psi)| \\
&\le \tfrac{1}{\gamma}\int_0^t I(\eta)\, d\eta + \tfrac{1}{\gamma}\tfrac{\gamma_1\gamma_0}{2\gamma_1 - \gamma_0}(\Psi - \tilde{\Psi}) \\
&\le \tfrac{1}{\gamma_0}\int_0^t I(\eta)\, d\eta + \tfrac{\gamma_1}{2\gamma_1 - \gamma_0}(\Psi - \tilde{\Psi})
\end{aligned}
$$

and therefore

$$
\Psi - \tilde{\Psi} \le \big(1 - \tfrac{\gamma_1}{2\gamma_1 - \gamma_0}\big)^{-1} \tfrac{1}{\gamma_0}\int_0^t I(\eta)\, d\eta = \tfrac{2\gamma_1 - \gamma_0}{\gamma_1 - \gamma_0}\ \tfrac{1}{\gamma_0}\int_0^t I(\eta)\, d\eta.
$$

Now we need some auxiliary estimates. In accordance with the definition, the estimate

$$
\begin{aligned}
&\Big|u\big(\rho, s(\Psi) + \gamma(\rho - \Psi), \gamma'\big) - \tilde{u}\big(\rho, \tilde{s}(\tilde{\Psi}) + \gamma(\rho - \tilde{\Psi}), \gamma'\big)\Big| \\
&\qquad = \Big|u\big(\rho, x_0 - \gamma t + \gamma\rho, \gamma'\big) - \tilde{u}\big(\rho, x_0 - \gamma t + \gamma\rho, \gamma'\big)\Big| \\
&\qquad \le J_{\gamma'}(\rho)
\end{aligned}
$$

holds where $\rho \in [\Psi, t]$. Similarly we obtain

$$\begin{aligned}
&\big|u(\Psi, s(\Psi), \gamma') - \tilde{u}(\tilde{\Psi}, \tilde{s}(\tilde{\Psi}), \gamma')\big| \\
&\qquad \le \Big|u(\Psi, s(\Psi), \gamma') - u(\alpha, \xi, \gamma') + u(\alpha, \xi, \gamma') \\
&\qquad\quad - \tilde{u}(\alpha, \xi, \gamma') + \tilde{u}(\alpha, \xi, \gamma') - \tilde{u}(\tilde{\Psi}, \tilde{s}(\tilde{\Psi}), \gamma')\Big| \\
&\qquad \le M\big(|\Psi - \alpha| + |s(\Psi) - \xi|\big) + J(\alpha) + M\big(|\tilde{\Psi} - \alpha| + |\tilde{s}(\tilde{\Psi}) - \xi|\big)
\end{aligned}$$

where $(\alpha, \xi) \in \hat{I}_{\gamma'}$ with $\tilde{\Psi} \le \alpha \le \Psi$ $(\gamma' \in \Gamma^+)$. Further, we compute

$$\begin{aligned}
I(t) &= \left| \frac{(1-\kappa)\int_{\gamma_0}^{\gamma_1} \gamma' u(t, s(t), \gamma')\, d\gamma'}{1 + (1-\kappa)\int_{\gamma_0}^{\gamma_1} u(t, s(t), \gamma')\, d\gamma'} - \frac{(1-\kappa)\int_{\gamma_0}^{\gamma_1} \gamma' \tilde{u}(t, \tilde{s}(t), \gamma')\, d\gamma'}{1 + (1-\kappa)\int_{\gamma_0}^{\gamma_1} \tilde{u}(t, \tilde{s}(t), \gamma')\, d\gamma'} \right| \\
&\le \left| (1-\kappa)\int_{\gamma_0}^{\gamma_1} \gamma' u(t, s(t), \gamma')\, d\gamma' \, (1-\kappa)\int_{\gamma_0}^{\gamma_1} \tilde{u}(t, \tilde{s}(t), \gamma') - u(t, s(t), \gamma')\, d\gamma' \right| \\
&\quad + \left(1 + (1-\kappa)\int_{\gamma_0}^{\gamma_1} u(t, s(t), \gamma')\, d\gamma'\right) \\
&\quad \times (1-\kappa)\int_{\gamma_0}^{\gamma_1} \gamma' \big\{u(t, s(t), \gamma') - \tilde{u}(t, \tilde{s}(t), \gamma')\big\}\, d\gamma' \Bigg| \\
&\le (1-\kappa)\big\{1 + 2(1-\kappa)M(\gamma_1 - \gamma_0)\big\}\big(\tfrac{\gamma_1^2}{2} - \tfrac{\gamma_0^2}{2}\big)\left\{M\int_0^t I(\eta)\, d\eta + \mu \int_0^t J(\eta)\, d\eta\right\}.
\end{aligned}$$

Hence the integral equations yield for $(t, x^*) \in \widehat{III}_\gamma$

$$\begin{aligned}
J_\gamma(t) &\le \Bigg| \left[\frac{\kappa}{|\gamma|}\int_{\gamma_0}^{\gamma_1} R(\gamma, \gamma') u(\Psi, s(\Psi), \gamma')|\gamma'|\, d\gamma'\right] e^{-\mu(t-\Psi)} \\
&\quad - \left[\frac{\kappa}{|\gamma|}\int_{\gamma_0}^{\gamma_1} R(\gamma, \gamma') \tilde{u}(\tilde{\Psi}, \tilde{s}(\tilde{\Psi}), \gamma')|\gamma'|\, d\gamma'\right] e^{-\mu(t-\tilde{\Psi})} \\
&\quad + \mu \int_{\Psi}^t e^{-\mu(t-\rho)} \int_\Gamma K(\gamma, \gamma') u\big(\rho, s(\Psi) + \gamma(\rho - \Psi), \gamma'\big)\, d\gamma' d\rho \\
&\quad - \mu \int_{\tilde{\Psi}}^t e^{-\mu(t-\rho)} \int_\Gamma K(\gamma, \gamma') \tilde{u}\big(\rho, \tilde{s}(\tilde{\Psi}) + \gamma(\rho - \tilde{\Psi}), \gamma'\big)\, d\gamma' d\rho \Bigg| \\
&\le \hat{\kappa} M \big|e^{-\mu(t-\Psi)} - e^{-\mu(t-\tilde{\Psi})}\big| + \hat{\kappa} e^{-\mu(t-\tilde{\Psi})}\big|u(\Psi, s(\Psi), \gamma') - \tilde{u}(\tilde{\Psi}, \tilde{s}(\tilde{\Psi}), \gamma')\big| \\
&\quad + \mu \left| \int_{\Psi}^t e^{-\mu(t-\rho)} \int_\Gamma K(\gamma, \gamma') J_{\gamma'}(\rho)\, d\gamma' d\rho \right| + \mu \left| \int_{\Psi}^{\tilde{\Psi}} e^{-\mu(t-\rho)} M\, d\rho \right| \\
&\le \hat{\kappa} M \mu |\Psi - \tilde{\Psi}| + \hat{\kappa}\Big\{M\Big(|\Psi - \alpha| + |\tilde{\Psi} - \alpha| + |s(\Psi) - \xi| + |\tilde{s}(\tilde{\Psi}) - \xi|\Big) + J(\alpha)\Big\} \\
&\quad + \mu \int_0^t J(\rho)\, d\rho + \mu M |\tilde{\Psi} - \Psi| \\
&\le \big\{\hat{\kappa} M \mu + \mu M + \hat{\kappa} M 2 + 2\hat{\kappa} M \gamma_1\big\} \tfrac{2\gamma_1 - \gamma_0}{\gamma_1 - \gamma_0} \tfrac{1}{\gamma_0} \int_0^t I(\eta)\, d\eta \\
&\quad + (1 + \hat{\kappa})\mu \int_0^t J(\eta)\, d\eta + 2\hat{\kappa} M \int_0^t I(\eta)\, d\eta.
\end{aligned}$$

Altogether we obtain

$$I(t)+J(t)$$
$$\leq \Big[(1-\kappa)\big\{1+2(1-\kappa)M(\gamma_1-\gamma_0)\big\}\big(\tfrac{\gamma_1^2}{2}-\tfrac{\gamma_0^2}{2}\big)(M+\mu)+(1-\hat{\kappa})\mu+2\hat{\kappa}M$$
$$+\big\{\hat{\kappa}M\mu+\mu M+2\hat{\kappa}M+2\hat{\kappa}M\gamma_1\big\}\,\tfrac{2\gamma_1-\gamma_0}{\gamma_1-\gamma_0}\,\tfrac{1}{\gamma_0}\Big]\int_0^t \big(I(\eta)+J(\eta)\big)\,d\eta.$$

From the Gronwall lemma we get $I(t)+J(t)\leq 0$, but from the definition we know $I(t),J(t)\geq 0$, therefore we have $I(t)=J(t)=0$ which implies uniqueness of solutions for $t\in[0,t_1^*]$. We iterate this principle until $t=t_1$ is reached ∎

6. Special case: constant speed

Now we want to investigate the special case of constant speed $\Gamma=\{\gamma\in\mathbb{R}:\,|\gamma|=\gamma^*\}$, i.e. $\gamma_0=\gamma^*=\gamma_1$. Then the transport equation assumes the form

$$u_t^+ + \gamma u_x^+ = -\tfrac{\mu}{2}\,(u^+-u^-) \tag{19}$$
$$u_t^- - \gamma u_x^- = -\tfrac{\mu}{2}\,(u^--u^+) \tag{20}$$

where u^+ denotes the particles moving to the right and, equivalently, u^- the particles moving to the left. This system is the well-known correlated random walk in one dimension.

Now we look for appropriate boundary conditions. Condition (14) for the free boundary can be brought into

$$\dot{s}(t)=\frac{\gamma^*(1-\kappa)u(t,s(t),\gamma^*)}{1+(1-\kappa)u(t,s(t),\gamma^*)}. \tag{21}$$

The right boundary condition has the general form

$$u(t,s(t),\gamma)=\tfrac{\kappa}{|\gamma|}\int_{\gamma_0}^{\gamma_1}R(\gamma,\gamma')u(t,s(t),\gamma')|\gamma'|\,d\gamma' \qquad (\gamma\in\Gamma^-)$$

corresponding to

$$u(t,s(t),-\gamma^*)=\tfrac{\kappa}{\gamma}R(-\gamma^*,\gamma^*)u(t,s(t),\gamma^*)\gamma^*=\kappa R(-\gamma^*,\gamma^*)u(t,s(t),\gamma^*)$$

in the special case. Thus we obtain

$$u(t,s(t),-\gamma^*)=\kappa u(t,s(t),\gamma^*) \tag{22}$$

as right boundary condition in the special case. Analogously we get

$$\begin{aligned} u(t,0,\gamma^*) &= \tfrac{r}{\gamma^*}L(\gamma^*,-\gamma^*)u(t,0,-\gamma^*)\gamma^*+f(t,\gamma^*)\\ &= rL(\gamma^*,-\gamma^*)u(t,0,-\gamma^*)+f(t,\gamma^*)\end{aligned}$$

for the left boundary condition so that it gets the form

$$u(t,0,\gamma^*)=ru(t,0,-\gamma^*). \tag{23}$$

Hence transport equation (1) together with initial and boundary conditions (2), (3), (5) and (14) reduces to system (19) - (23). For a detailed discussion of this special case see [10].

7. The parabolic limit

Now we want to consider the parabolic limit of system (19) - (23), which corresponds to the classical one-phase Stefan problem. If we let $\gamma \to \infty$ and $\mu \to \infty$ in the hyperbolic problem (19) - (23) such that $\frac{\gamma^2}{\mu} = D$, $0 < D < \infty$, we expect a formal transition to the classical parabolic Stefan problem (diffusion equation and appropriate initial and boundary conditions).

Therefore we introduce total mass $u = u^+ + u^-$ and flow $v = \gamma(u^+ - u^-)$. With the new variables system (19) - (20) assumes the form

$$u_t + v_x = 0 \tag{24}$$

$$\frac{1}{\mu} v_t + Du_x + v = 0. \tag{25}$$

Then, by eliminating mixed derivatives [9], we get the telegraph equation

$$\frac{1}{\mu} u_{tt} + u_t = Du_{xx}$$

from where the diffusion equation appears as the formal limit. In (u, v) coordinates boundary conditions (21) - (22) read

$$\left.\begin{aligned} u(t, s(t)) &= \frac{1+\kappa}{1-\kappa} \frac{v(t, s(t))}{\gamma} \\ \dot{s}(t) &= \frac{\gamma}{\gamma + v(t, s(t))} v(t, s(t)) \end{aligned}\right\}.$$

Using formally $\frac{v}{\gamma} \to 0$ and $v + Du_x \to 0$ (from (25)) we obtain

$$\left.\begin{aligned} u(t, s(t)) &= 0 \\ \dot{s}(t) &= -Du_x(t, s(t)) \end{aligned}\right\}.$$

The left boundary condition (23) reads

$$u(t, 0) = \frac{2f(t)}{1-r} + \frac{(r+1)}{(1-r)} \frac{v(t, 0)}{\gamma}$$

in (u, v) coordinates. Analogously we get

$$u(t, 0) = \frac{2f(t)}{1-r}$$

for $\gamma \to \infty$. Altogether we obtain the following parabolic system:

- Diffusion equation: $u_t = Du_{xx}$
- Initial conditions: $u(0, x) = u_0^+(x) + u_0^-(x)$
- Boundary conditions: $u(t, 0) = \frac{2f(t)}{1-r}$ and $u(t, s(t)) = 0$
- Stefan condition: $\dot{s}(t) = -Du_x(t, s(t))$

which corresponds to the classical parabolic one-phase Stefan problem (cf. [2, 5]).

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Received 27.11.2000