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Automatic Control of the Temperature in Phase Change Problems with Memory

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Abstract. We study a parabolic two-phase system with memory occupying a bounded and smooth domain. The heat exchange at part of the boundary is controlled by a thermostat. Assuming on the phase variable either a relaxation dynamics or a Stefan condition, we prove existence and uniqueness results for feedback control problems corresponding to two different types of thermostat: the relay switch and the Preisach operator. These results are strictly related to the continuous dependence of the solution on the boundary datum, which is investigated in advance.

Keywords: *Parabolic Stefan problem, Heat conduction with memory, thermostats control, hysteresis operator of Preisach type, existence, uniqueness*

AMS subject classification: Primary 35R35, secondary 45K05, 80A22, 93B52

1. Introduction

We consider a bounded and smooth domain $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) occupied by a two-phase system whose state is described by a pair of state variables (θ, χ). Here θ is the relative temperature ($\theta = 0$ being the equilibrium temperature at which the two phases, for instance solid and liquid, can coexist) and χ is the concentration of the more energetic phase (i.e. water in a water-ice system).

On account of [8] (see also [5, 7, 17, 18]), we introduce the following constitutive laws for the internal energy $\mathcal{E}$ and the heat flux $\underline{q}$:

$$\mathcal{E}(x,t) = \varphi_0 \theta(x,t) + \psi_0 \chi(x,t) + \int_{-\infty}^{t} \varphi(t-s)\theta(x,s)\,ds + \int_{-\infty}^{t} \psi(t-s)\chi(x,s)\,ds$$

$$\underline{q}(x,t) = -k_0 \nabla\theta(x,t) - \int_{-\infty}^{t} k(t-s)\nabla\theta(x,s)\,ds$$

for $(x,t) \in \Omega \times \mathbb{R}$. Here φ_0, ψ_0, k_0 are positive constants and φ, ψ, k are time-dependent memory kernels.

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Assuming that the past histories of θ and χ are known up to $t = 0$ and using the energy balance

$$\partial_t \mathcal{E} + \nabla \cdot \underline{q} = g$$

where g is the heat supply, we deduce the integro-differential equation

$$\partial_t e - k_0 \Delta\theta - k * \Delta\theta = f \qquad \text{in } \Omega \times (0, \infty) \tag{1.1}$$

with

$$e = \varphi_0 \theta + \psi_0 \chi + \varphi * \theta + \psi * \chi \tag{1.2}$$

where $*$ stands for the usual time convolution product over $(0, t)$ and f is the sum of g with the contribution of the known term containing the past histories of θ and χ up to $t = 0$.

In order to describe the evolution of θ and χ, we need to couple (1.1) – (1.2) with a phase relationship. By considering the relaxation dynamics (cf., e.g., [7, 8, 20, 22, 23])

$$\alpha \partial_t \chi + \lambda(\chi) \ni \beta(\theta, \chi) \qquad \text{in } \Omega \times (0, \infty) \tag{1.3}$$

where α is a kinetic parameter, while λ is a maximal monotone graph and β a Lipschitz continuous function, we represent non-equilibrium phenomena like supercooling or superheating. Alternatively, we replace (1.3) with the standard equilibrium condition of the Stefan problem (see, e.g., [7, 8, 10, 11, 22])

$$\chi \in H(\theta) \qquad \text{in } \Omega \times (0, \infty), \tag{1.4}$$

where H is the Heaviside graph, i.e. $H(\eta) = 0$ if $\eta < 0$, $H(0) = [0, 1]$, and $H(\eta) = 1$ if $\eta > 0$.

We suppose that part of the boundary is at a given temperature (for instance, at the equilibrium temperature $\theta = 0$) and we consider the influence of a thermostat on the heat exchange at the remaining part of the boundary. Hence, we impose the following mixed boundary conditions. Letting $\{\Gamma_0, \Gamma_1\}$ be a partition of $\partial\Omega = \Gamma$ into two measurable subsets (Γ_1 of positive Lebesgue measure), we take

$$\left.\begin{aligned} \theta &= 0 && \text{on } \Gamma_0 \times (0, \infty) \\ -k_0 \frac{\partial \theta}{\partial \nu} - k * \frac{\partial \theta}{\partial \nu} &= \sigma(\theta - \theta_{\mathrm{e}}) && \text{on } \Gamma_1 \times (0, \infty) \end{aligned}\right\}.$$

Here σ is a positive constant, θ_{e} represents the external temperature and may depend on the past history of θ up to $t = 0$ as well. As usual, ν stands for the unit outward normal to $\partial\Omega$.

In addition, regarding the initial conditions, we associate with (1.1) – (1.3) the following ones

$$\left.\begin{aligned} \theta(\cdot, 0) &= \theta_0 \\ \chi(\cdot, 0) &= \chi_0 \end{aligned}\right\} \qquad \text{in } \Omega$$

while with (1.1), (1.2) and (1.4) we just need

$$(\varphi_0 \theta + \psi_0 \chi)(\cdot, 0) = e_0 \qquad \text{in } \Omega.$$

Now, we fix a finite time interval $(0, T)$ and we consider the following two problems.

Problem P1. Find a pair (θ, χ) such that

$$\left.\begin{aligned} e = \varphi_0\theta + \psi_0\chi + \varphi * \theta + \psi * \chi \quad & \text{in } \Omega \times (0,T) \\ \partial_t e - k_0\Delta\theta - k * \Delta\theta = f \quad & \text{in } \Omega \times (0,T) \\ \alpha\partial_t\chi + \lambda(\chi) \ni \beta(\theta,\chi) \quad & \text{in } \Omega \times (0,T) \\ \theta(\cdot,0) = \theta_0, \quad \chi(\cdot,0) = \chi_0 \quad & \text{in } \Omega \\ \theta = 0 \quad & \text{on } \Gamma_0 \times (0,T) \\ -k_0\frac{\partial\theta}{\partial\nu} - k * \frac{\partial\theta}{\partial\nu} = \sigma(\theta - \theta_{\mathrm{e}}) \quad & \text{on } \Gamma_1 \times (0,T) \end{aligned}\right\}. \tag{1.5}$$

Problem P2. Find a pair (θ, χ) such that

$$\left.\begin{aligned} e = \varphi_0\theta + \psi_0\chi + \varphi * \theta + \psi * \chi \quad & \text{in } \Omega \times (0,T) \\ \partial_t e - k_0\Delta\theta - k * \Delta\theta = f \quad & \text{in } \Omega \times (0,T) \\ \chi \in H(\theta) \quad & \text{in } \Omega \times (0,T) \\ (\varphi_0\theta + \psi_0\chi)(\cdot,0) = e_0 \quad & \text{in } \Omega \\ \theta = 0 \quad & \text{on } \Gamma_0 \times (0,T) \\ -k_0\frac{\partial\theta}{\partial\nu} - k * \frac{\partial\theta}{\partial\nu} = \sigma(\theta - \theta_{\mathrm{e}}) \quad & \text{on } \Gamma_1 \times (0,T) \end{aligned}\right\}. \tag{1.6}$$

Remark that both problems differ in equations $(1.5)_{3-4}$ and $(1.6)_{3-4}$ only. We will refer to Problems P1 and P2 as to the relaxed problem and the Stefan problem, respectively. Well-posedness was investigated for both cases in [7, 8]. Moreover, it was studied in [3] for the relaxed problem under weaker assumptions. In all these papers, a homogeneous Dirichlet boundary condition was considered. Since we are dealing with a non-homogeneous mixed (Dirichlet-Robin) boundary condition, the first aim is to obtain well-posedness for our relaxed and Stefan problems adapting the approach used in [3, 7, 8].

Moreover, when the data enjoy suitable regularity properties, we have a stronger regularity for the solution.

These well-posedness results are somewhat preliminaries to our feedback control problems. Actually, a thermostat device influences the evolution of the free boundary on account of suitable measurements of the temperature (cf., e.g., [9, 14]). First of all, we assume to detect the temperature θ by a real system of thermal sensors placed in the interior of the body and on its surface. Hence, according to [9], we suppose to know, for any $t \in [0,T]$,

$$\mathcal{M}(\theta)(t) = \fint_{\Omega_0} \theta(x,t)\omega_I(x)\,dx + \fint_{\Gamma_2} \big(1 * \theta\big)(y,t)\omega_S(y)\,d\Gamma.$$

Here $\Omega_0 \subset \Omega$ and $\Gamma_2 \subset \Gamma_1$ are of positive Lebesgue and surface measures, respectively, $\fint$ stands for the mean value, while $\omega_I : \Omega_0 \to [0,\infty)$ and $\omega_S : \Gamma_2 \to [0,\infty)$ are weight functions related to the characteristics of the sensors.

Now we consider the action of a thermostat. A heating/cooling device acts when the temperature $\mathcal{M}(\theta)$ detected by the sensors is 'too far' from the critical temperature.

This fact can be described by introducing a function $u(t)$ which represents the control input. According to [14] (see also [9, 19]), u modifies a fraction of θ_e; that is

$$\theta_e(y,t) \doteq \theta_A(y,t)u(t) + \theta_B(y,t) \qquad \forall\, (y,t) \in \Gamma_1 \times (0,T), \tag{1.7}$$

where u is the solution to the Cauchy problem

$$\left.\begin{aligned} bu' + u &= \mathcal{W}(\mathcal{M}(\theta)) + \theta_C \quad \text{on } [0,T] \\ u(0) &= u_0 \end{aligned}\right\}, \tag{1.8}$$

and $\theta_A, \theta_B : \Gamma_1 \times (0,T) \to \mathbb{R}$ and $\theta_C : [0,T] \to \mathbb{R}$ are given, b is a positive parameter, and $\mathcal{W}$ represents the action of the thermostat and $u_0 \in \mathbb{R}$.

Problems of this kind have been investigated, e.g., in [9, 14]. More precisely, the real-time control of a two-phase parabolic Stefan problem was introduced in [14]. On the basis of classical well-posedness results on parabolic equations, existence and uniqueness are obtained for two automatic control problems with mixed boundary conditions: in the former model, the characteristic function of the thermostat exhibits a simple jump discontinuity; in the latter, a hysteresis loop. The first analysis of some feedback control problems with memory, meaning that the memory effects are also included in the constitutive laws, is contained in [9]. Three types of thermostats are considered: the ideal switch, the relay switch and the Preisach operator. Existence and/or uniqueness are shown for the corresponding systems. We observe that [9] deals with a hyperbolic Stefan problem and that the notion of solution to the direct problem is weakened with respect to [14], but a Robin boundary condition is assumed on the whole boundary.

Here we extend the analysis of [9] to parabolic Stefan problems with memory with or without relaxation. Moreover, in this paper mixed boundary conditions will be considered: as we have pointed out above, this involves the proof of suitable well-posedness results.

We are going to consider a pair of thermostatic operators: $\mathcal{W}_1$, corresponding to the relay switch, and the Preisach operator $\mathcal{W}_2$. The former is characterized by two threshold functions $\rho_L, \rho_U \in C^0([0,T])$ such that, for some $\delta > 0$, $\rho_U(t) - \rho_L(t) \geq \delta$ for any $t \in [0,T]$. For any $r \in C^0([0,T])$, at time t_C, $\mathcal{W}_1(r)$ jumps up from -1 to $+1$ or, vice versa, down from $+1$ to -1 according to

$$\mathcal{W}_1(r)(t_c) = \begin{cases} +1 & \text{if } r(t_c) = \rho_L(t_c) \text{ and } \mathcal{W}_1(r)(t) = -1 \text{ just before} \\ -1 & \text{if } r(t_c) = \rho_U(t_c) \text{ and } \mathcal{W}_1(r)(t) = +1 \text{ just before} \end{cases}$$

The Preisach operator has been extensively studied by Visintin (see, e.g., [21, 24]) and is defined as follows. Let $\mathcal{P} = \{(\rho_1, \rho_2) \in \mathbb{R}^2 : \rho_1 < \rho_2\}$ be the so-called Preisach plane and $\zeta : \mathcal{P} \to \{+1, -1\}$ be a Borel measurable function. For any $(\rho_1, \rho_2) \in \mathcal{P}$ and any $r \in C^0([0,T])$ we define

$$\mathcal{H}_{(\rho_1,\rho_2)}(r,\zeta)(0) = \begin{cases} +1 & \text{if } r(0) \leq \rho_1 \\ \zeta(\rho_1,\rho_2) & \text{if } \rho_1 < r(0) < \rho_2 \\ -1 & \text{if } r(0) \geq \rho_2. \end{cases}$$

If $t \in (0,T]$, we set

$$\mathcal{T}_t = \{\tau \in (0,t] : r(\tau) = \rho_1 \text{ or } r(\tau) = \rho_2\}$$

and

$$\mathcal{H}_{(\rho_1,\rho_2)}(r,\zeta)(t) = \begin{cases} \mathcal{H}_{(\rho_1,\rho_2)}(r,\zeta)(0) & \text{if } \mathcal{T}_t = \emptyset \\ +1 & \text{if } \mathcal{T}_t \neq \emptyset \text{ and } r(\max \mathcal{T}_t) = \rho_1 \\ -1 & \text{if } \mathcal{T}_t \neq \emptyset \text{ and } r(\max \mathcal{T}_t) = \rho_2. \end{cases}$$

We point out that, for any pair $(\rho_1,\rho_2) \in \mathcal{P}$ and any ζ fixed,

$$\mathcal{H}_{(\rho_1,\rho_2)}(\cdot,\zeta) : C^0([0,T]) \to C^0_r([0,T)) \cap \mathrm{BV}(0,T).$$

Now, if μ is a non-negative Borel measure on $\mathcal{P}$, integrating $\mathcal{H}_{(\cdot,\cdot)}(r,\zeta)$ with respect to μ over the half plane $\mathcal{P}$ of the admissible thresholds, we obtain the Preisach operator

$$\mathcal{W}_2(r)(t) = \int_{\mathcal{P}} \mathcal{H}_{(\rho_1,\rho_2)}(r,\zeta)(t)\, d\mu(\rho_1,\rho_2).$$

The second part of the paper is devoted to study the following problems:

Problem (TPj). For a fixed thermostatic operator $\mathcal{W}_j$ $(j = 1,2)$, find a triplet (θ,χ,u) such that (θ,χ) is the solution to the relaxed, respectively the Stefan, problem with θ_{e} given by (1.7) and u obeing (1.8).

This rough formulation can be made more precise by incorporating the thermostat dynamics into the boundary conditions. Indeed, observe that regarding (1.8) as a Cauchy problem for u, we obtain

$$u(t) = \int_0^t e^{-\frac{t-\tau}{b}} \big(\mathcal{W}_j(\mathcal{M}(\theta))(\tau) + \theta_C(\tau)\big) d\tau + u_0 e^{-\frac{t}{b}} \tag{1.9}$$

for any $t \in (0,T)$, i.e. u is given by a Volterra operator. Inserting (1.9) into (1.7), for $j = 1,2$ we get

$$\theta_{\mathrm{e}} = \mathcal{F}\big(\mathcal{W}_j(\mathcal{M}(\theta))\big) \qquad \text{on } \Gamma_1 \times (0,T).$$

Here, for $r \in L^2(0,T)$,

$$\mathcal{F}(r)(y,t) = \int_0^t E(y,t,\tau) r(\tau)\, d\tau + E_0(y,t) \qquad \forall\, (y,t) \in \Gamma_1 \times (0,T)$$

wherein E and E_0 are easily recovered from (1.8) as

$$\begin{aligned} E(\cdot,t,\tau) &= e^{-\frac{t-\tau}{b}} \theta_A(\cdot,t) \\ E_0(\cdot,t) &= \left(\int_0^t e^{-\frac{t-\tau}{b}} \theta_C(\tau) d\tau + u_0 e^{-\frac{t}{b}} \right) \theta_A(\cdot,t) + \theta_B(\cdot,t) \end{aligned}$$

almost everywhere on Γ_1 and $t,\tau \in [0,T]$. However, we can consider a more general situation (see Section 4 below).

Now we can state our problems as follows, for $j = 1,2$ fixed.

Problem (TPj). Find (θ, χ) solution to the relaxed, respectively the Stefan, problem with

$$\theta_e = \mathcal{F}\big(\mathcal{W}_j(\mathcal{M}(\theta))\big) \qquad \text{on } \Gamma_1 \times (0,T).$$

We observe that we are dealing with nonlinear parabolic integro-differential problems with a nonlinear and non-local boundary condition.

Applying an inductive argument as in [14] (see also [9]), we will show that there exists a unique solution for our problems (relaxed and Stefan) corresponding to the relay switch. Concerning the Preisach operator, using the Schauder fixed point theorem, we can prove the existence of the solution. Uniqueness also holds, under suitable hypotheses on the measure μ, provided that $\omega_I = 0$.

The plan of the paper is the following:

Sections 2 and 3 are devoted to the well-posedness for the relaxed and the Stefan problems, respectively. In Section 4 we will establish our existence and uniqueness results for the two feedback control problems involving the relay switch, while in Section 5 we study the analogs in the case of the Preisach operator.

2. Relaxed problem: well-posedness

2.1 Variational formulation and main results. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega$ of class C^2 and $Q_T = \Omega \times (0,T)$; $\{\Gamma_0, \Gamma_1\}$ is a partition of $\partial\Omega$ into two measurable subsets (Γ_1 of positive Lebesgue measure). We introduce the Hilbert spaces $V = \{v \in H^1(\Omega) : \ v = 0 \text{ in } \Gamma_0\}$ and $H = L^2(\Omega)$. Since we allow Γ_0 to be a null set, we endow V with the norm $\|v\|_V = \{\int_\Omega \big[|\nabla v|^2 + |v|^2\big] dx\}^{1/2}$, while H is equipped with the usual norm denoted by $\|\cdot\|$. Further, V' represents the dual space of V and $\|\cdot\|_{V'}$ stands for its norm. Identifying H with its dual space H', it turns out that $V \subset H \subset V'$ with dense and compact injections. In addition, we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V' and V, by $(\cdot,\cdot), ((\cdot,\cdot))$ and $((\cdot,\cdot))_*$ the scalar products in H, V and V', respectively. Also, $J : V \to V'$ stands for the Riesz isomorphism $\langle Ju, v\rangle = ((u,v))$, for any $u, v \in V$. Let $(\cdot,\cdot)_{\Gamma_1}$ and $\|\cdot\|_{L^2(\Gamma_1)}$ be the the scalar product and the norm in $L^2(\Gamma_1)$. Henceforth $\varphi_0, \psi_0, k_0, \alpha$ and σ are positive constants.

Our assumptions on the data are the following:

- **(H1)** $\varphi \in L^2(0,T)$ and $\psi, k \in L^1(0,T)$.
- **(H2)** There exists $\gamma > 0$ such that $\gamma \int_0^t |v(s)|^2 ds \le \int_0^t \big(k_0 v(s) + (k * v)(s)\big) v(s)\, ds$ for any $v \in L^2(0,T)$ and any $t \in [0,T]$.
- **(H3)** $f \in L^1(0,T;H) + L^2(0,T;V')$.
- **(H4)** $\theta_e \in L^2(0,T;L^2(\Gamma_1))$.
- **(H5)** $\lambda : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone graph, particularly $\lambda = \partial\Lambda$.
- **(H6)** $\Lambda : \mathbb{R} \to [0,+\infty]$ is a proper, convex and lower-semicontinuous function such that $\Lambda(0) = 0 = \min \Lambda$.
- **(H7)** There exists $L > 0$ such that $|\beta(\theta_1,\chi_1) - \beta(\theta_2,\chi_2)| \le L\big\{|\theta_1 - \theta_2| + |\chi_1 - \chi_2|\big\}$ for any $(\theta_1,\chi_1), (\theta_2,\chi_2) \in \mathbb{R}^2$.
- **(H8)** $\theta_0 \in H$ and $\chi_0 \in K = \{\gamma \in H : \gamma \in \mathrm{D}(\lambda) \text{ a.e. in}\Omega\}$.

(H9) $\Lambda(\chi_0) \in L^1(\Omega)$.

(H10) h is a Carathéodory function depending on θ in a Lipschitz continuous way (being M its Lipschitz constant) and such that $h(0) = 0$.

Observe that assumption (H2) is due to the Second Principle of Thermodynamics (see [12]).

We shall deal with the following

Problem 1. Find (θ, e, χ, ξ) such that

$$\left.\begin{array}{l} \theta \in L^\infty(0,T;H) \cap L^2(0,T;V) \\ e \in W^{1,1}(0,T;H) + H^1(0,T;V') \\ \chi \in H^1(0,T;H),\ \xi \in L^2(0,T;H) \end{array}\right\} \tag{2.1}$$

and which, moreover, fulfills

$$\left.\begin{array}{rl} e = \varphi_0\theta + \psi_0\chi + \varphi * \theta + \psi * \chi & \text{a.e. in } Q_T \\ \langle \partial_t e + J(k_0\theta + k * \theta), v\rangle + \sigma(\theta_{\Gamma_1}, v)_{\Gamma_1} = & \\ \langle f, v\rangle + \big(h(\theta, k*\theta), v\big) + \sigma(\theta_{\mathrm{e}}, v)_{\Gamma_1} & \text{for all } v \in V,\ \text{a.e. in } (0,T) \\ \alpha\partial_t\chi + \xi = \beta(\theta, \chi) & \text{a.e. in } Q_T \\ \xi \in \lambda(\chi) & \text{a.e. in } Q_T \\ \theta(\cdot, 0) = \theta_0, \quad \chi(\cdot, 0) = \chi_0 & \text{a.e. in } \Omega \end{array}\right\}. \tag{2.2}$$

Remark 1. In our Problem P1, $h(\theta, k*\theta) = k_0\theta + k*\theta$, but we can consider a general function enjoying assumption (H10).

Remark 2. We point out that, because of $(2.1)_{1,3}$, $(2.1)_2$ follows from $(2.2)_{1-3}$. Moreover, $(2.1)_2$ implies $\partial_t\theta \in L^1(0,T;H) + L^2(0,T;V')$ which, due also to $(2.1)_1$, yields $\theta \in C^0([0,T];H)$ so that the first initial condition of $(2.2)_6$ makes sense as well.

Now we can state the existence and uniqueness of the solution to Problem 1 and the Lipschitz continuous dependence of θ on θ_{e} which we exploit in our feedback problems. Note that uniqueness is a straightforward consequence of estimate (2.3) below.

Theorem 2.1. *If assumptions $(H1)-(H10)$ hold, then the solution to Problem 1 exists. Moreover, if $k \in W^{1,1}(0,T)$, such solution is unique and there is a constant $C > 0$ such that for any pair $\theta_{\mathrm{e}\,i} \in L^2(0,T;L^2(\Gamma_1))$ $(i = 1,2)$, if we denote by (θ_i, e_i, χ_i) the solution to Problem 1 corresponding to $\theta_{\mathrm{e}\,i}$, then*

$$\begin{array}{l} \|\theta_1 - \theta_2\|_{L^2(0,T;H)} \\ +\|\nabla\big(1 * (\theta_1 - \theta_2)\big)\|_{L^\infty(0,T;H^N)} \\ +\|1 * (\theta_1 - \theta_2)\|_{L^\infty(0,T;L^2(\Gamma_1))} \\ +\|\chi_1 - \chi_2\|_{L^\infty(0,T;H)} \le C\|\theta_{\mathrm{e}\,1} - \theta_{\mathrm{e}\,2}\|_{L^2(0,T;L^2(\Gamma_1))} \end{array} \tag{2.3}$$

where C depends on

$$\Omega,\ \Gamma_1,\ T,\ \alpha,\ \sigma,\ \varphi_0,\ \psi_0,\ k_0,\ L,\ M$$

and on the norms

$$\|\varphi\|_{L^2(0,T)},\ \|k\|_{W^{1,1}(0,T)},\ \|\psi\|_{L^1(0,T)}.$$

Dealing with the Preisach operator we shall need a uniform bound for

$$\theta \in L^\infty(0,T;V) \cap H^1(0,T;H)$$

in terms of θ_e. This requires stronger regularity assumptions on the memory kernels and on the data θ_0, θ_e, f. Indeed, we have

Theorem 2.2. *Let assumptions* $(H2)$ *and* $(H5) - (H10)$ *hold and suppose* $\varphi, k \in W^{1,1}(0,T)$ *as well as* $\psi \in L^2(0,T)$. *Now, if* $\theta_0 \in V$, $\theta_e \in W^{1,1}(0,T;L^2(\Gamma_1))$ *and* $f \in L^2(0,T;H) + W^{1,1}(0,T;V')$, *then* $\theta \in L^\infty(0,T;V) \cap H^1(0,T;H)$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|\theta\|_{L^\infty(0,T;V)\cap H^1(0,T;H)} \le C\big\{1 + \|\theta_e\|_{W^{1,1}(0,T;L^2(\Gamma_1))}\big\} \tag{2.4}$$

holds where the constant C *depends on*

$$\Omega,\ \Gamma_1,\ T,\ M,\ \alpha,\ \sigma,\ \varphi_0,\ \psi_0,\ k_0,\ |\varphi(0)|,\ |k(0)|$$

and on the norms

$$\|\varphi\|_{W^{1,1}(0,T)},\ \|k\|_{W^{1,1}(0,T)},\ \|\psi\|_{L^2(0,T)}$$
$$\|\theta_0\|_V,\ \|\chi_0\|,\ \|\Lambda(\chi_0)\|_{L^1(\Omega)},\ \|f\|_{L^2(0,T;H)+W^{1,1}(0,T;V')}.$$

2.2 Relaxed problem: continuous dependence and uniqueness. As we have pointed out at the end of Section 1, we will deal with Problem 1 which is the variational formulation of a more general problem. We observe that it suffices to prove (2.3) since uniqueness is a direct consequence; let the quadruple $(\theta_i, e_i, \chi_i, \xi_i)$ be a solution to Problem 1 corresponding to $\theta_{e\,i}$ for $i = 1, 2$. Taking the difference of all equations in (2.2) written for the two solutions and denoting

$$\Theta = \theta_1 - \theta_2,\quad E = e_1 - e_2,\quad X = \chi_1 - \chi_2,\quad \Xi = \xi_1 - \xi_2,\quad \Theta_e = \theta_{e\,1} - \theta_{e\,2}$$

we infer that the quadruple (Θ, E, X, Ξ) solves the following problem with homogeneous initial conditions:

$$\left.\begin{aligned} &\langle \partial_t E + J(k_0\Theta + k*\Theta), v\rangle + \sigma(\Theta, v)_{\Gamma_1} \\ &\quad = \big(h(\theta_1, k*\theta_1) - h(\theta_2, k*\theta_2), v\big) + \sigma(\Theta_e, v)_{\Gamma_1} \quad \forall\, v \in V,\ \text{a.e. in } (0,T) \\ &\alpha\partial_t X + \Xi = \beta(\theta_1, \chi_1) - \beta(\theta_2, \chi_2) \quad \text{a.e. in } Q_T \\ &E = \varphi_0\Theta + \psi_0 X + \varphi*\Theta + \psi*X \quad \text{a.e. in} Q_T \end{aligned}\right\}. \tag{2.5}$$

In order to get an estimate for X, we multiply $(2.5)_3$ by X and integrate over Q_t. Assumptions (H5), (H7), (H8), the Hölder inequality and the standard Young inequality

$$ab \le \tfrac{\delta}{2}a^2 + \tfrac{1}{2\delta}b^2 \qquad (a, b \in \mathbb{R},\ \delta > 0) \tag{2.6}$$

yield, for any $t \in [0,T]$,

$$\tfrac{\alpha}{2}\|X(t)\|^2 \le \tfrac{\varphi_0\gamma}{4}\int_0^t \|\Theta(s)\|^2 ds + \big(\tfrac{L^2}{\varphi_0\gamma} + L\big)\int_0^t \|X(s)\|^2 ds. \tag{2.7}$$

Now we integrate $(2.5)_{1-2}$ over $(0,t)$ and choose as test function $v = \Theta$; integrating with respect to time, we get

$$\begin{aligned}
&\varphi_0\|\Theta\|^2_{L^2(0,t;H)} + \tfrac{k_0}{2}\|(1*\Theta)(t)\|^2_V + \tfrac{\sigma}{2}\|(1*\Theta)(t)\|^2_{L^2(\Gamma_1)}\\
&\quad = -\int_0^t (\varphi*\Theta,\Theta) - \psi_0\int_0^t (X,\Theta) - \int_0^t (\psi * X,\Theta) - \int_0^t \langle J(k*1*\Theta),\Theta\rangle\\
&\qquad + \int_0^t \Big(1*\big[h(\theta_1,k*\theta_1) - h(\theta_2,k*\theta_2)\big],\Theta\Big) + \sigma\int_0^t \big(1*\Theta,\Theta_{\rm e}\big)_{\Gamma_1}\\
&\quad = \sum_{i=1}^6 B_i(t).
\end{aligned}$$

By Hölder inequality we have

$$\begin{aligned}
|B_1(t)| &\le \|\varphi\|_{L^2(0,T)}\Big\{\int_0^t \|\Theta\|^2_{L^2(0,s;H)}ds\Big\}^{1/2}\|\Theta\|_{L^2(0,t;H)}\\
|B_2(t)| &\le \psi_0\|X\|_{L^2(0,t;H)}\|\Theta\|_{L^2(0,t;H)}.
\end{aligned}$$

By the Young inequality for convolution (see, e.g., [13]) we estimate the third integral as

$$|B_3(t)| \le \|\psi\|_{L^1(0,T)}\|X\|_{L^2(0,t;H)}\|\Theta\|_{L^2(0,t;H)}.$$

The same inequality, integration by parts and properties of convolution yield

$$\begin{aligned}
|B_4(t)| &\le \|k\|_{L^2(0,T)}\|1*\Theta\|_{L^2(0,t;V)}\|(1*\Theta)(t)\|_V\\
&\quad + \big[|k(0)| + \|k'\|_{W^{1,1}(0,T)}\big]\|1*\Theta\|^2_{L^2(0,t;V)}\\
|B_5(t)| &\le C\Big\{\int_0^t \|\Theta\|^2_{L^2(0,s;H)}ds\Big\}^{1/2}\|\Theta\|_{L^2(0,t;H)}\\
|B_6(t)| &\le \sigma\|(1*\Theta_{\rm e})(t)\|_{L^2(\Gamma_1)}\|(1*\Theta)(t)\|_{L^2(\Gamma_1)}\\
&\quad + \sigma\|\Theta_{\rm e}\|_{L^2(0,t;L^2(\Gamma_1))}\|1*\Theta\|_{L^2(0,t;L^2(\Gamma_1))}.
\end{aligned}$$

Collecting all estimates above for $B_1(t),\dots,B_6(t)$ and using (2.6), we get

$$\begin{aligned}
&\tfrac{\varphi_0}{2}\|\Theta\|^2_{L^2(0,t;H)} + \tfrac{k_0}{4}\|(1*\Theta)(t)\|^2_V + \tfrac{\sigma}{4}\|(1*\Theta)(t)\|^2_{L^2(\Gamma_1)}\\
&\quad \le C\Big\{\int_0^t \|\Theta\|^2_{L^2(0,s;H)}ds + \|1*\Theta\|^2_{L^2(0,t;V)} + \|1*\Theta\|^2_{L^2(0,t;L^2(\Gamma_1))}\\
&\qquad + \|X\|^2_{L^2(0,t;H)} + \|\Theta_{\rm e}\|^2_{L^2(0,T;L^2(\Gamma_1))}\Big\}.
\end{aligned}$$

Adding herein (2.7) multiplied by $\frac{1}{\gamma}$ and using a standard inequality, we obtain

$$\begin{aligned} &\tfrac{\varphi_0}{4}\|\Theta\|^2_{L^2(0,t;H)} + \tfrac{k_0}{4}\|(1*\Theta)(t)\|^2_V + \tfrac{\sigma}{4}\|(1*\Theta)(t)\|^2_{L^2(\Gamma_1)} + \tfrac{\alpha}{2\gamma}\|X(t)\|^2 \\ &\quad\le C\Big\{\int_0^t \|\Theta\|^2_{L^2(0,s;H)}ds + \|1*\Theta\|^2_{L^2(0,t;V)} + \|1*\Theta\|^2_{L^2(0,t;L^2(\Gamma_1))} \\ &\qquad + \|X\|^2_{L^2(0,t;H)} + \|\Theta_{\mathrm{e}}\|^2_{L^2(0,T;L^2(\Gamma_1))}\Big\}. \end{aligned}$$

Now the Gronwall lemma implies (2.3).

2.3 Relaxed problem: existence. In order to show that the solution to Problem 1 exists, we will study a sequence of approximating Problems P_n characterized by smoother kernels. By a fixed point argument, we infer that there exists a unique solution $(\theta_n, e_n, \chi_n, \xi_n)$ to Problem P_n. Moreover, we can prove that an *a priori* bound holds: then, at least for a subsequence, weak or weak star convergences are deduced, but due to the non-linearities due to λ, β and h, they do not allow to pass to the limit in the nonlinear terms. We overcome this obstacle by showing that the solutions to the approximating problems are Cauchy sequences with respect to suitable norms. Finally, an appropriate subsequence will converge to a solution to Problem 1. Indeed, due to uniqueness, the whole approximating sequence will converge to the solution.

First of all, we observe that, since $\varphi \in L^2(0,T)$ and $k \in L^1(0,T)$, it is possible to find two sequences $\{\varphi_n\}$ and $\{k_n\}$ in $W^{1,1}(0,T)$ such that

$$\begin{aligned} \varphi_n &\to \varphi \quad \text{in } L^2(0,T) \\ k_n &\to k \quad \text{in } L^1(0,T). \end{aligned} \tag{2.8}$$

By assumption (H2), $(2.8)_2$ and [3: Inequality (3.7)], without loss of generality, we can assume that, for any $n \in \mathbb{N}$, for any $v \in L^2(0,T)$ and any $t \in [0,T]$,

$$\tfrac{\gamma}{2}\int_0^t |v(s)|^2 ds \le \int_0^t \big(k_0 v(s) + (k_n * v)(s)\big)v(s)\,ds. \tag{2.9}$$

We will study the following approximating problems:

Problem P_n. Find $(\theta_n, e_n, \chi_n, \xi_n)$ such that

$$\left.\begin{aligned} \theta_n &\in C^0([0,T];H) \cap L^2(0,T;V) \\ \partial_t\theta_n &\in L^1(0,T;H) + L^2(0,T;V') \\ \chi_n &\in H^1(0,T;H), \quad \xi_n \in L^2(0,T;H) \end{aligned}\right\}$$

and, moreover,

$$\left.\begin{aligned} \varphi_0\theta_n + \psi_0\chi_n + \varphi_n * \theta_n + \psi * \chi_n &= e_n \quad \text{a.e. in } Q_T \\ \langle \partial_t e_n + J(k_0\theta_n + k_n * \theta_n), v\rangle + \sigma(\theta_n, v)_{\Gamma_1} & \\ = \langle f, v\rangle + \big(h(\theta_n, k*\theta_n), v\big) + \sigma(\theta_{\mathrm{e}}, v)_{\Gamma_1} &\quad \forall\, v \in V, \text{ a.e. in } (0,T) \\ \alpha\partial_t\chi_n + \xi_n &= \beta(\theta_n, \chi_n) \quad \text{a.e. in } Q_T \\ \xi_n &\in \lambda(\chi_n) \quad \text{a.e. in } Q_T \\ \theta_n(\cdot,0) = \theta_0, \quad \chi_n(\cdot,0) &= \chi_0 \quad \text{a.e. in } \Omega \end{aligned}\right\} \tag{2.10}$$

Lemma 2.1 *If assumptions* (H1) - (H10) *hold, let* $\varphi_n, k_n \in W^{1,1}(0,T)$ *be such that* (2.8) *is satisfied. Then, for any fixed* n*, there exists a unique solution to Problem* P_n.

Proof. We drop n for simplicity. Our argument is based on the Banach Fixed Point Theorem. Let us introduce the Hilbert spaces

$$X_T = \Big\{(u,\eta) \in L^2(0,T;H \times H) : 1 * u \in L^\infty(0,T;V)\Big\}$$

endowed with the norm

$$\|(u,\eta)\|_t^2 = \|u\|^2_{L^2(0,t;H)} + \|1 * u\|^2_{L^\infty(0,t;V)} + \|\eta\|^2_{L^2(0,t;H)}$$

and

$$Y_T = \Big\{(u,\eta) \in X_T : \eta(\cdot,t) \in K \text{ a.e. in } (0,T)\Big\}.$$

Note that Y_T is a closed and convex subset of X_T. Now we consider the following problem:

Problem (P). For any $(u,\eta) \in Y_T$ fixed, find a pair (θ,χ) such that

$$\left.\begin{array}{r}\theta \in C^0([0,T];H) \cap L^2(0,T;V)\\ \partial_t\theta \in L^1(0,T;H) + L^2(0,T;V')\\ \chi \in H^1(0,T;H),\ \chi(\cdot,t) \in K \quad \forall\, t \in [0,T]\end{array}\right\}$$

and which, moreover, fulfills

$$\left.\begin{array}{rl}\varphi_0\langle\partial_t\theta, v> + < J(k_0\theta), v> + \sigma(\theta,v)_{\Gamma_1} & \\ = \langle f,v\rangle + \big(h(u,k*u),v\big) + \sigma(\theta_{\mathrm{e}},v)_{\Gamma_1} & \\ -\langle\partial_t\big(\psi_0\chi + \varphi * u + \psi * \chi\big) + J(k*u), v\rangle & \forall\, v \in V, \text{ a.e. in } (0,T)\\ \alpha\langle\partial_t\chi, \chi-\gamma\rangle \le \langle\beta(u,\eta),\chi-\gamma\rangle & \forall\,\gamma \in K, \text{ a.e. in } (0,T)\\ \theta(\cdot,0) = \theta_0,\quad \chi(\cdot,0) = \chi_0 & \text{a.e. in } \Omega\end{array}\right\}.$$

By the same argument used in [8], we can easily see that there exists a unique solution to such problem; it turns out that a nonlinear and continuous operator $R : Y_T \to Y_T$ is defined as $R(u,\eta) = (\theta,\chi)$, where (θ,χ) solves the corresponding Problem (P). As in [8: Lemma 3.2], it is possible to prove that there exists a positive constant C such that, for any $(u_1,\eta_1),(u_2,\eta_2) \in Y_T$,

$$\big\|R(u_1,\eta_1) - R(u_2,\eta_2)\big\|_t^2 \le C\int_0^t \big\|(u_1,\eta_1) - (u_2,\eta_2)\big\|_\tau^2 d\tau.$$

Owing to this inequality we can deduce that

$$\big\|R(u_1,\eta_1) - R(u_2,\eta_2)\big\|_t^2 \le Ct\big\|(u_1,\eta_1) - (u_2,\eta_2)\big\|_t^2.$$

Then

$$\begin{aligned}\big\|R^2(u_1,\eta_1) - R^2(u_2,\eta_2)\big\|_t^2 &\le C\int_0^t \big\|R(u_1,\eta_1) - R(u_2,\eta_2)\big\|_\tau^2 d\tau\\ &\le \tfrac{C^2t^2}{2}\big\|(u_1,\eta_1) - (u_2,\eta_2)\big\|_t^2.\end{aligned}$$

By induction,

$$\big\|R^m(u_1,\eta_1) - R^m(u_2,\eta_2)\big\|_t^2 \le \tfrac{C^mT^m}{m!}\big\|(u_1,\eta_1) - (u_2,\eta_2)\big\|_t^2.$$

Then, if m is large enough, R^m is a contraction mapping in Y_T and the solution to Problem P_n exists and is unique ∎

We point out that $e_n \in W^{1,1}(0,T;H) + H^1(0,T;V')$. We can now prove that the following *a priori* estimates hold.

Lemma 2.2. *Under the assumptions of Lemma* 2.1*, if we write* $f = f_1 + f_2$*, where* $f_1 \in L^1(0,T;H)$ *and* $f_2 \in L^2(0,T;V')$*, there exist two constants* $C_1, C_2 > 0$ *both independent of* n *such that, for any* $n \in \mathbb{N}$*, the solution* $(\theta_n, e_n, \chi_n, \xi_n)$ *to Problem* P_n *satisfies*

$$\begin{aligned} \|\theta_n\|_{C^0([0,T];H)\cap L^2(0,T;V)} + \|\chi_n\|_{H^1(0,T;H)} &\le C_1 \\ \|e_n\|_{C^0([0,T];H)} + \|e_n - 1 * f_1\|_{H^1(0,T;V')} + \|\xi_n\|_{L^2(0,T;H)} &\le C_2. \end{aligned} \tag{2.11}$$

Proof. We multiply $(2.10)_4$ by χ_n and integrate over Q_t. Then, taking account of assumptions (H5), (H7), (H8) and $(2.10)_{5-6}$, we get

$$\tfrac{\alpha}{2}\big\{\|\chi_n(t)\|^2 - \|\chi_0\|^2\big\} \le C\bigg\{1 + \int_0^t \big[\|\chi_n(s)\|^2 + \|\theta_n(s)\|^2\big]ds\bigg\} \tag{2.12}$$

where $C > 0$ is independent of n. Let us recall that χ_n is a solution to $(2.10)_{4-6}$. Then assumptions (H5) - (H9) entail, by the third inequality in [4: p. 72/Theorem 3.6],

$$\alpha \int_0^t \|\partial_t\chi_n(s)\|^2 ds \le 2\|\Lambda(\chi_0)\|_{L^1(\Omega)} + C\bigg\{1 + \int_0^t \big[\|\theta_n(s)\|^2 + \|\chi_n(s)\|^2\big]ds\bigg\}. \tag{2.13}$$

Now we choose $v = \varphi_0\theta_n + \varphi_n * \theta_n$ in $(2.10)_{2-3}$: integrating with respect to time and taking account of (2.9), we deduce

$$\begin{aligned} \tfrac{1}{2}\big\|(\varphi_0\theta_n + \varphi_n * \theta_n)(t)\big\|^2 + \tfrac{\varphi_0\gamma}{2}\|\theta_n\|^2_{L^2(0,t;V)} + \sigma\varphi_0\|\theta_n\|^2_{L^2(0,t;L^2(\Gamma_1))} \\ \le \tfrac{\varphi_0^2}{2}\|\theta_0\|^2 + \sum_{i=1}^{7}|I_i(t)| \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} I_1(t) &= \psi_0 \int_0^t \langle \partial_t\chi_n, \varphi_0\theta_n + \varphi_n * \theta_n\rangle d\tau \\ I_2(t) &= \int_0^t \langle \partial_t(\psi * \chi_n), \varphi_0\theta_n + \varphi_n * \theta_n\rangle d\tau \\ I_3(t) &= \int_0^t \big(k_0\theta_n + k_n * \theta_n, \varphi_n * \theta_n\big) d\tau \\ I_4(t) &= \sigma \int_0^t (\theta_n, \varphi_n * \theta_n)_{\Gamma_1} d\tau \\ I_5(t) &= \int_0^t \langle f, \varphi_0\theta_n + \varphi_n * \theta_n\rangle d\tau \\ I_6(t) &= \int_0^t \big(h(\theta_n, k * \theta_n), \varphi_0\theta_n + \varphi_n * \theta_n\big) d\tau \\ I_7(t) &= \sigma \int_0^t \big(\theta_{\mathrm{e}}, \varphi_0\theta_n + \varphi_n * \theta_n\big)_{\Gamma_1} d\tau. \end{aligned}$$

We can estimate $I_i(t)$ for $i = 1, 2, 3$ as in [3: Formulas (4.22), (4.25), (4.26)] (henceforth $\delta > 0$ stands for a constant suitably chosen in any estimate). Namely,

$$|I_1(t)| \le \tfrac{\delta}{2}\|\partial_t\chi_n\|^2_{L^2(0,t;H)} + \tfrac{\psi_0^2}{2\delta}\int_0^t \|(\varphi_0\theta_n + \varphi_n * \theta_n)(s)\|^2 ds.$$

Let us recall that $\partial_t(\psi * \chi_n) = \chi_0\psi + \psi * \partial_t\chi_n$. Then we get

$$\begin{aligned}|I_2(t)| \le \|\chi_0\| &\int_0^t |\psi(s)|\,\|(\varphi_0\theta_n + \varphi_n * \theta_n)(s)\|ds \\ &+ \tfrac{\delta}{2}\|\psi\|^2_{L^1(0,T)}\|\partial_t\chi_n\|^2_{L^2(0,t;H)} \\ &+ \tfrac{1}{2\delta}\int_0^t \|(\varphi_0\theta_n + \varphi_n * \theta_n)(s)\|^2 ds.\end{aligned}$$

From (2.8) we infer

$$|I_3(t)| \le \tfrac{k_0+C}{2}\Big\{\delta\|\theta_n\|^2_{L^2(0,t;V)} + \tfrac{C}{\delta}\int_0^t \|\theta_n\|^2_{L^2(0,s;V)}ds\Big\}.$$

Now we consider the first term due to the boundary condition:

$$|I_4(t)| \le \tfrac{\sigma\delta}{2}\|\theta_n\|^2_{L^2(0,t;L^2(\Gamma_1))} + \tfrac{\sigma C}{2\delta}\int_0^t \|\theta_n\|^2_{L^2(0,s;L^2(\Gamma_1))}ds.$$

We write $f = f_1 + f_2$. Then we get

$$\begin{aligned}|I_5(t)| &\le \int_0^t \|f_1(s)\|\,\big\|(\varphi_0\theta_n + \varphi_n * \theta_n)(s)\big\|ds + \tilde{C}\tfrac{1}{\delta}\|f_2\|^2_{L^2(0,T;V')} + \tilde{C}\tfrac{\delta}{2}\|\theta_n\|^2_{L^2(0,t;V)} \\ |I_6(t)| &\le \tfrac{M}{2}\int_0^t \|\theta_n(s)\|^2 ds + \tfrac{M}{2}\int_0^t \|(\varphi_0\theta_n + \varphi_n * \theta_n)(s)\|^2 ds \\ |I_7(t)| &\le \tfrac{\sigma}{\delta}\|\theta_{\mathrm{e}}\|^2_{L^2(0,t;L^2(\Gamma_1))} + \tfrac{\sigma\delta}{2}\|\theta_n\|^2_{L^2(0,t;L^2(\Gamma_1))} + \tfrac{\sigma C\delta}{2}\int_0^t \|\theta_n\|^2_{L^2(0,s;L^2(\Gamma_1))}ds.\end{aligned}$$

It is easy to see that

$$\tfrac{\varphi_0^2}{8}\|\theta_n(t)\|^2 \le \tfrac{1}{4}\|(\varphi_0\theta_n + \varphi_n * \theta_n)(t)\|^2 + C\int_0^t \|\theta_n(s)\|^2 ds.$$

By this remark, inserting the above estimates for $I_1(t), \ldots, I_7(t)$ into (2.14) and taking advantage of (2.12) and (2.13), we obtain

$$\begin{aligned}&\tfrac{1}{4}\big\|(\varphi_0\theta_n + \varphi_n * \theta_n)(t)\big\|^2 + \tfrac{\varphi_0\gamma}{4}\|\theta_n\|^2_{L^2(0,t;V)} \\ &\quad + \tfrac{\sigma\varphi_0}{2}\|\theta_n\|^2_{L^2(0,t;L^2(\Gamma_1))} + \tfrac{\varphi_0^2}{8}\|\theta_n(t)\|^2 + \tfrac{\alpha}{2}\|\chi_n(t)\|^2 + \tfrac{\alpha}{2}\|\partial_t\chi_n\|^2_{L^2(0,t;H)} \\ &\le C\Big\{1 + \int_0^t \Big[\|\theta_n(s)\|^2 + \|\chi_n(s)\|^2 + \big\|(\varphi_0\theta_n + \varphi_n * \theta_n)(s)\big\|^2 \\ &\quad + \|\theta_n(s)\|^2_{L^2(0,s;L^2(\Gamma_1))}\Big]ds + \int_0^t \|\theta_n\|^2_{L^2(0,s;V)}ds \\ &\quad + \int_0^t \big(\|\chi_0\|\,|\psi(s)| + \|f_1(s)\|\big)\big\|(\varphi_0\theta_n + \varphi_n * \theta_n)(s)\big\|ds\Big\}.\end{aligned} \tag{2.15}$$

By a generalization of the Gronwall lemma [1: Teorema 2.1] we deduce $(2.11)_1$ while $(2.11)_2$ follows from (2.15) and $(2.10)_{1-3}$ ∎

From the *a priori* estimates we can conclude that, at least for a subsequence,

$$\left.\begin{aligned} \theta_n \to \theta &\quad \text{weakly star in } L^\infty(0,T;H)\cap L^2(0,T;V) \\ \chi_n \to \chi &\quad \text{weakly in } H^1(0,T;H) \\ e_n \to e &\quad \text{weakly star in } L^\infty(0,T;H) \\ e_n - 1*f_1 \to e - 1*f_1 &\quad \text{weakly in } H^1(0,T;V') \\ \xi_n \to \xi &\quad \text{weakly in } L^2(0,T;H) \end{aligned}\right\}. \tag{2.16}$$

As we have pointed out, these convergences are not enough in order to pass to the limit in (2.10); we also need some strong convergence.

By comparison in $(2.10)_{2-3}$, $(2.11)_1$ yields the existence of a constant $C_3 > 0$ such that for any $n \in \mathbb{N}$, $\|\partial_t\theta_n\|^2_{L^2(0,T;V')} \le C_3$. Then, by compactness,

$$\theta_n \to \theta \qquad \text{strongly in } L^2(0,T;H). \tag{2.17}$$

Writing $(2.10)_4$ for $n, m \in \mathbb{N}$ and taking the difference, we get

$$\alpha\partial_t(\chi_n - \chi_m) + \xi_n - \xi_m = \beta(\theta_n,\chi_n) - \beta(\theta_m,\chi_m) \qquad \text{a.e. in } Q_T.$$

Choose $v = \chi_n - \chi_m$ herein and integrate over $\Omega \times (0,t)$ to get

$$\begin{aligned} \tfrac{\alpha}{2}\|(\chi_n - \chi_m)(t)\|^2 + \int_0^t (\xi_n - \xi_m, \chi_n - \chi_m) \\ = \int_0^t \big(\big[\beta(\theta_n,\chi_n) - \beta(\theta_m,\chi_m)\big], \chi_n - \chi_m\big). \end{aligned}$$

Observe that assumptions (H5) and (H6) entail

$$\begin{aligned} &\tfrac{\alpha}{2}\big\|(\chi_n - \chi_m)(t)\big\|^2 \\ &\quad \le L\Big\{\|\theta_n - \theta_m\|_{L^2(0,t;H)} + \|\chi_n - \chi_m\|_{L^2(0,t;H)}\Big\}\|\chi_n - \chi_m\|_{L^2(0,t;H)}. \end{aligned}$$

By the Gronwall lemma and the strong convergence (2.17), we deduce that

$$\chi_n \to \chi \qquad \text{strongly in } L^\infty(0,T;H). \tag{2.18}$$

Now we proceed along the lines of [3]: we write a sketch of the proof for reader's convenience. Now (2.8), $(2.16)_{1-4}$ and (2.17) - (2.18) allow to pass to the limit in $(2.10)_{1-3}$ ending up with $(2.2)_{1-3}$. The pair (θ,χ) satisfies the initial conditions $(2.2)_6$ because of $(2.10)_6$. Relation $(2.2)_4$ is fulfilled since assumption (H7) and (2.17) - (2.18) yield $\beta(\theta_n,\chi_n) \to \beta(\theta,\chi)$ in $L^2(Q_T)$. Exploiting the monotonicity of λ, by [2: p. 42/Lemma 1.3], relation $(2.2)_5$ follows from

$$\limsup_{n\to\infty} \iint_Q \xi_n\chi_n \le \iint_Q \xi\chi.$$

This condition holds true because of $(2.10)_5$, $(2.16)_5$ and (2.18).

2.4 Relaxed problem: regularity. Let (θ,χ) be the solution to Problem 1. Since $\theta\in L^2(0,T;V)\cap C^0([0,T];H)$ and $\chi\in H^1(0,T;H)$, by Faedo-Galerkin method, we can easily prove that $\theta\in L^\infty(0,T;V)\cap H^1(0,T;H)$.

Now we will prove that estimate (2.4) holds as well. We can write $(2.2)_{1-3,6}$ as

$$\left.\begin{aligned}\varphi_0\langle\partial_t\theta,v\rangle+k_0\langle J(\theta),v\rangle+\sigma(\theta,v)_{\Gamma_1}+\langle G(\theta),v\rangle&\\ =\langle F,v\rangle+\sigma(\theta_{\mathrm{e}},v)_{\Gamma_1}&\quad\forall\ v\in V,\ \text{a.e. in }(0,T)\\ \theta(0)=\theta_0&\quad\text{a.e. in }\Omega\end{aligned}\right\}\tag{2.19}$$

where

$$\begin{aligned}\langle G(\theta),v\rangle&=\langle\partial_t(\varphi*\theta),v\rangle+\langle J(k*\theta),v\rangle\\ \langle F,v\rangle&=\langle f-\partial_t(\psi_0\chi+\psi*\chi),v\rangle+\big(h(\theta,k*\theta),v\big).\end{aligned}$$

Under the assumptions of Theorem 2.2, since $\chi\in H^1(0,T;H)$, it turns out that $F\in L^2(0,T;H)+W^{1,1}(0,T;V')$. Moreover, G is linear and continuous from $L^\infty(0,T;V)\cap H^1(0,T;H)$ to $L^2(0,T;H)+W^{1,1}(0,T;V')$.

First of all we point out that

$$\|\chi\|_{H^1(0,T;H)}\le C\big\{1+\|\theta_{\mathrm{e}}\|_{L^1(0,T;L^2(\Gamma_1))}\big\}\tag{2.20}$$

where the constant C depends on

$$\varphi_0,\ \psi_0,\ \sigma,\ M$$

and on the norms

$$\|\varphi\|_{L^2(0,T)},\ \|\psi\|_{L^1(0,T)},\ \|k\|_{W^{1,1}(0,T)},\ \|f\|_{L^1(0,T;H)+L^2(0,T;V')}\|\chi_0\|,\ \|\Lambda(\chi_0)\|_{L^1(\Omega)}.$$

In order to get (2.20), we infer as in (2.12) that

$$\|\chi(t)\|^2\le\|\chi_0\|^2+C\int_0^t\big(\|\chi(s)\|^2+\|\theta(s)\|^2\big)ds.\tag{2.21}$$

We need an estimate for $\|\theta\|_{L^2(0,t;H)}$ which can be obtained by integrating $(2.2)_{2-3}$ over $(0,t)$, choosing θ as test function and performing a further integration with respect to time. Then, owing to the properties of convolution and Gronwall Lemma, we get

$$\|\theta\|_{L^2(0,t;H)}\le C\Big\{1+\|\theta_{\mathrm{e}}\|_{L^1(0,t;L^2(\Gamma_1))}+\|\chi\|_{L^2(0,t;H)}\Big\}.$$

Substituting this inequality into (2.21), we derive

$$\begin{aligned}\|\chi(t)\|&\le C\big(1+\|\theta_{\mathrm{e}}\|_{L^1(0,t;L^2(\Gamma_1))}\big)\\ \|\theta\|_{L^2(0,t;H)}&\le C\big(1+\|\theta_{\mathrm{e}}\|_{L^1(0,t;L^2(\Gamma_1))}\big).\end{aligned}$$

Arguing as in (2.13), we obtain (2.20) by [4: p. 72/Theorem 3.6].

Now, in order to prove (2.4), we introduce the bilinear, continuous, symmetric and coercive form $a(u,v) = k_0\langle Ju, v\rangle$ on $V \times V$. Then if $J : V \to V'$ is the Riesz isomorphism and $I : V \to V'$ the injection operator, a satisfies the compatibility condition $a(Jv_1, v_2) = a(v_1, Jv_2)$ for all $v_1, v_2 \in J^{-1}(I(V))$. We recall that

$$F = F_1 + F_2 \in L^2(0,T;H) + W^{1,1}(0,T;V')$$

where

$$\begin{aligned} F_1 &= f_1 - \partial_t(\psi_0\chi + \psi * \chi) + h(\theta, k*\theta) \\ F_2 &= f_2. \end{aligned}$$

We are going to apply a regularization procedure introduced in [16] and whose main properties are summarized in [6: Appendix].

For $\varepsilon > 0$, we define $u_\varepsilon(t)$ as

$$u_\varepsilon(t) \in V \qquad \text{such that } (I + \varepsilon^2 J)u_\varepsilon(t) = \theta(t) \text{ a.e. in } (0,T).$$

Since $\theta \in L^\infty(0,T;V) \cap H^1(0,T;H)$, by [6: Proposition 6.1] we deduce that $u_\varepsilon \in H^1(0,T;V)$. Choosing $u'_\varepsilon(t)$ as test function in $(2.19)_{1-2}$ and integrating over $(0,t)$, we get

$$\begin{aligned} &\varphi_0 \int_0^t \langle \partial_t\theta, u'_\varepsilon\rangle ds + \int_0^t a(\theta, u'_\varepsilon)\, ds + \sigma \int_0^t (\theta, u'_\varepsilon)_{\Gamma_1} ds \\ &\qquad = \sigma \int_0^t (\theta_{\mathrm{e}}, u'_\varepsilon)_{\Gamma_1} ds + \int_0^t \langle F, u'_\varepsilon\rangle ds - \int_0^t \langle G(\theta), u'_\varepsilon\rangle ds. \end{aligned} \tag{2.22}$$

By definition of u_ε we have

$$\begin{aligned} \varphi_0 \int_0^t \langle \partial_t\theta, u'_\varepsilon\rangle\, ds &= \varphi_0 \int_0^t \langle \partial_t\theta - u'_\varepsilon, u'_\varepsilon \angle\, ds + \varphi_0 \int_0^t \langle u'_\varepsilon, u'_\varepsilon\rangle\, ds \\ &= \varphi_0\varepsilon^2 \int_0^t \langle J\, u'_\varepsilon, u'_\varepsilon\rangle ds + \varphi_0 \int_0^t \|\, u'_\varepsilon(s)\|_H^2 ds \\ &= \varphi_0\varepsilon^2 \|\, u'_\varepsilon\, \|^2_{L^2(0,t;V)} + \varphi_0 \|\, u'_\varepsilon\, \|^2_{L^2(0,t;H)}. \end{aligned} \tag{2.23}$$

As $\varepsilon \to 0$, by [6: Proposition 6.1/Formulas (6.7) and (6.8)], the right-hand side of (2.23) tends to $\varphi_0\|\partial_t\theta\|^2_{L^2(0,t;H)}$. Though $\theta \in L^\infty(0,T;V) \cap H^1(0,T;H)$, arguing as in [6: Proposition 6.3], we see that

$$\lim_{\varepsilon\to 0} \int_0^t a(\theta, u'_\varepsilon)\, ds = \tfrac{1}{2}\big\{a(\theta(t),\theta(t)) - a(\theta(0),\theta(0))\big\} = \tfrac{k_0}{2}\big\{\|\theta(t)\|_V^2 - \|\theta_0\|_V^2\big\}$$

a.e. in $(0,T)$. Integrating by parts, by [6: Formulas (6.11) and (6.7)] we obtain

$$\begin{aligned} \left| \int_0^t (\theta_{\mathrm{e}}, u'_\varepsilon)_{\Gamma_1} ds \right| &\le \|\theta_{\mathrm{e}}\|_{L^\infty(0,T;L^2(\Gamma_1))}\big\{\|\theta_0\|_V + \|\theta\|_{L^\infty(0,T;V)}\big\} \\ &\qquad + \|\partial_t\theta_{\mathrm{e}}\|_{L^1(0,T;L^2(\Gamma_1))}\|\theta\|_{L^\infty(0,T;V)} \\ &\le \|\theta_{\mathrm{e}}\|_{W^{1,1}(0,t;L^2(\Gamma_1))}\big\{\|\theta_0\|_V + \|\theta\|_{L^\infty(0,t;V)}\big\}. \end{aligned}$$

The last two terms on the right-hand side of (2.22) can be estimated as above, that is

$$\left| \int_0^t \langle F, u_\varepsilon' \rangle ds \right| \le \|F_1\|_{L^2(0,t;H)} \|\partial_t \theta\|_{L^2(0,t;H)} + \|F_2\|_{W^{1,1}(0,T;V')} \big\{ \|\theta\|_{L^\infty(0,t;V)} + \|\theta_0\|_V \big\}$$

and

$$\begin{aligned}\left| \int_0^t \langle G(\theta), u_\varepsilon' \rangle ds \right| \le & \big[|\varphi(0)| + \|\varphi'\|_{L^1(0,T)} \big] \|\theta\|_{L^2(0,t;H)} \|\partial_t \theta\|_{L^2(0,t;H)} \\ & + \|k\|_{L^2(0,T)} \|\theta\|_{L^2(0,t;V)} \|\nabla \theta(t)\|_{H^N} \\ & + \big[|k(0)| + \|k'\|_{L^1(0,T)} \big] \|\theta\|_{L^2(0,t;V)} \|\nabla \theta\|_{L^2(0,t;H^N)} \\ & + M \|\theta\|_{L^2(0,t;H)} \|\partial_t \theta\|_{L^2(0,t;H)}.\end{aligned}$$

Collecting the previous estimates, it turns out that

$$\begin{aligned}& \|\partial_t \theta\|_{L^2(0,t;H)} + \|\theta\|_{L^\infty(0,t;V)} + \|\theta\|_{L^\infty(0,t;L^2(\Gamma_1))} \\ & \quad \le C \Big\{ \|F\|_{L^2(0,t;H) + W^{1,1}(0,T;V')} + \|\theta_{\rm e}\|_{W^{1,1}(0,T;L^2(\Gamma_1))} + \|\theta_0\|_V \Big\}\end{aligned} \tag{2.24}$$

where the constant C depends on

$$\|\varphi\|_{W^{1,1}(0,T)},\ \|k\|_{W^{1,1}(0,T)},\ \|\psi\|_{L^1(0,T)},\ |\varphi(0)|,\ |k(0)|,\ \sigma,\ \alpha,\ \varphi_0,\ \psi_0\ k_0.$$

By the definition of F and $(2.19)_{1-2}$, it is easy to see that

$$\|F\|_{L^2(0,T;H) \cap W^{1,1}(0,T;V')} \le C \big\{ 1 + \|\theta_{\rm e}\|_{L^1(0,T;L^2(\Gamma_1))} \big\}$$

where the constant C depends on

$$\|f\|_{L^2(0,T;H) + W^{1,1}(0,T;V')},\ \psi_0,\ \|\psi\|_{L^2(0,T)},\ \|\chi_0\|,\ M,\ \|k\|_{W^{1,1}(0,T)},\ \|\Lambda(\chi_0)\|_{L^1(\Omega)}.$$

Inserting this inequality into (2.24), (2.4) follows.

3. Stefan problem: well-posedness

In this section we state our main results concerning the Stefan problem. First of all, we slightly modify the hypotheses on the memory kernels and the initial data:

(K1) $\varphi, \psi \in W^{1,1}(0,T)$ and $k \in L^2(0,T)$.

(K2) $e_0 \in H$.

Also, we suppose that assumptions (H2) - (H4) hold. With the same choice of V and H of Section 2, the variational formulation of Problem P2 reads

Problem 2. Find a triplet (θ, e, χ) such that

$$\theta \in L^\infty(0,T;H) \cap L^2(0,T;V)$$
$$e \in W^{1,1}(0,T;H) + H^1(0,T;V')$$
$$\chi \in L^\infty(Q_T)$$

and which, moreover, satisfies the equations

$$e = \varphi_0\theta + \psi_0\chi + \varphi * \theta + \psi * \chi \quad \text{in } V', \text{ a.e. in } (0,T) \tag{3.1}$$

$$\langle \partial_t e, v\rangle + \big(\nabla(k_0\theta + k * \theta), \nabla v\big) + \sigma(\theta, v)_{\Gamma_1}$$
$$= \langle f, v\rangle + \sigma(\theta_\mathrm{e}, v)_{\Gamma_1} \quad \forall\, v \in V, \text{ a.e. in } (0,T) \tag{3.2}$$

$$\chi \in H(\theta) \quad \text{a.e. in } Q_T \tag{3.3}$$

$$(\varphi_0\theta + \psi_0\chi)(\cdot, 0) = e_0 \quad \text{in } V'. \tag{3.4}$$

Theorem 3.1. *If assumptions* (K1) - (K2) *and* (H2) - (H4) *hold, then there exists at least one solution to Problem* 2.

Since we can not obtain any continuous dependence estimate for χ from (3.3), we apply as in [8] an inversion formula for Volterra integral equations (see [13]). It requires stronger regularity on memory kernels, but allows us to get rid of the convolution term in χ. Then, exploiting the monotonicity of H, it is possible to control this term. It remains to seek for an estimate for θ which is easy to derive.

Theorem 3.2. *If assumptions* (K2) *and* (H2) *hold and, moreover,*

$$\varphi, \psi, k \in W^{1,1}(0,T), \tag{3.5}$$

then there exists a constant $C > 0$ such that, denoting for any $\theta_{\mathrm{e}\,i} \in L^2(0,T;L^2(\Gamma_1))$ by (θ_i, χ_i) $(i = 1,2)$ the solution to Problem 2 corresponding to $\theta_{\mathrm{e}\,i}$, the estimate

$$\|\theta_1 - \theta_2\|_{L^2(0,T;H)} + \|1 * (\theta_1 - \theta_2)\|_{L^\infty(0,T;V)} \le C\|\theta_{\mathrm{e}\,1} - \theta_{\mathrm{e}\,2}\|_{L^2(0,T;L^2(\Gamma_1))} \tag{3.6}$$

holds where C depends on $\varphi_0, \psi_0, k_0, \sigma, \gamma, |k(0)|$ and on the norms of φ, ψ and k in $L^2(0,T), W^{1,1}(0,T)$ and $W^{1,1}(0,T)$, respectively.

Finally, we can prove the following regularity result.

Theorem 3.3. *If assumptions* (K2), (H2) *and* (3.5) *hold, let us suppose that*

$$\left.\begin{aligned} f &\in L^2(0,T;H) + W^{1,1}(0,T;V') \\ \theta_\mathrm{e} &\in W^{1,1}(0,T;L^2(\Gamma_1)) \\ \theta_0 &= (\varphi_0 I + \psi_0 H)^{-1}(e_0) \in V \end{aligned}\right\}. \tag{3.7}$$

Then $\theta \in L^\infty(0,T;V) \cap H^1(0,T;H)$ and there exists a constant $C > 0$ such that

$$\|\theta\|_{L^\infty(0,T;V)\cap H^1(0,T;H)} \le C\big\{1 + \|\theta_\mathrm{e}\|_{W^{1,1}(0,T;L^2(\Gamma_1))}\big\}. \tag{3.8}$$

The constant C depends on the norms $\|\varphi\|_{W^{1,1}(0,T)}, \|k\|_{W^{1,1}(0,T)}, \|\psi\|_{W^{1,1}(0,T)}, \|\theta_0\|_V$ and $\|f\|_{L^2(0,T;H)+W^{1,1}(0,T;V')}$.

3.1 Stefan problem: existence. We will show that there exists at least one solution to Problem 2 by the same procedure as in [8: Theorem 2.2], i.e. by obtaining as limit of a sequence of solutions of relaxed problems whose kinetic parameters tend to zero. Let $\theta_0 \in H$ and $\chi_0 \in K$, with K defined by

$$H = \Big\{\gamma \in H : 0 \le \gamma \le 1 \ \text{ a.e. in } \ \Omega\Big\},$$

be such that $\varphi_0\theta_0 + \psi_0\chi_0 = e_0$ a.e. in Ω. Assumption (K1) allows us to introduce a sequence $\{k_n\}$ in $C^1([0,T])$ such that $k_n \to k$ in $L^2(0,T)$.

Now we consider the following approximating problems.

Problem P1$_n$. Find a pair (θ_n, χ_n) such that

$$\left.\begin{aligned} &\theta_n \in L^\infty(0,T;H) \cap L^2(0,T;V) \\ &\chi_n \in H^1(0,T;H), \ \chi_n(\cdot,t) \in K \ \forall\, t \in [0,T] \end{aligned}\right\}$$

and which, moreover, satisfies

$$\begin{aligned} \langle \partial_t(\varphi_0\theta_n + \psi_0\chi_n), v\rangle + \langle \partial_t(\varphi * \theta_n + \psi * \chi_n), v\rangle & \\ + \big(\nabla(k_0\theta_n + k_n * \theta_n), \nabla v\big) + \sigma(\theta_n, v)_{\Gamma_1} & \\ = \langle f, v\rangle + \sigma(\theta_{\mathrm{e}}, v)_{\Gamma_1} & \quad \forall\, v \in V, \text{ a.e. in } (0,T) \qquad (3.9) \\ \tfrac{1}{n}\partial_t\chi_n + H^{-1}(\chi_n) \ni \theta_n & \quad \text{in } Q_T \qquad (3.10) \\ \theta_n(\cdot,0) = \theta_0, \ \chi_n(\cdot,0) = \chi_0 & \quad \text{in } \Omega. \qquad (3.11) \end{aligned}$$

We point out that (3.10) is a particular case of $(2.2)_4$, with $\lambda = H^{-1}$ and $\beta(\theta,\chi) = \theta$. Then by Theorem 2.1 it is easy to see that Problem P1$_n$ is well posed for any $n \in \mathbb{N}$. The following *a priori* estimate also holds:

Lemma 3.1. *If assumptions* (K1) - (K2) *and* (H2) - (H4) *hold, then there exists a constant $C > 0$ such that, for any $n \in \mathbb{N}$, the solution (θ_n, χ_n) to Problem* P1$_n$ *satisfies*

$$\|\theta_n\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\chi_n\|_{L^\infty(Q_T)} + \tfrac{1}{n}\|\partial_t\chi_n\|_{L^2(0,T;H)} \le C.$$

Proof. It is easy to recover such estimate because $\chi_n(\cdot,t) \in K$ for any $t \in [0,T]$ and any $n \in \mathbb{N}$. Moreover, the estimate for $\frac{1}{n}\|\partial_t\chi_n\|_{L^2(0,T;H)}$ follows from [4: p. 72/Theorem 3.6]. Finally, we take $v = \varphi_0\theta_n + \varphi_n * \theta_n$ in our variational equation (3.9) and, reasoning as in Lemma 2.2, we get the desired estimate ∎

From Lemma 3.1 we deduce that, at least for a subsequence,

$$\left.\begin{aligned} \theta_n \to \theta & \qquad \text{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V) \\ \chi_n \to \chi & \qquad \text{weakly star in } L^\infty(Q_T) \\ \tfrac{1}{n}\partial_t\chi_n \to 0 & \qquad \text{strongly in } L^2(0,T;H) \end{aligned}\right\} \qquad (3.12)$$

as $n \to \infty$. Now our aim is to prove that (θ, χ) solves Problem 2. We observe that, taking advantage of assumption (K1), we infer

$$\begin{aligned} \varphi_n * \theta_n &\to \varphi * \theta && \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \\ \psi * \chi_n &\to \psi * \chi && \text{weakly star in } W^{1,\infty}(0,T;H) \\ k_n * \theta_n &\to k * \theta && \text{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V). \end{aligned}$$

Taking now $\xi_n = \partial_t(\varphi_0\theta_n + \psi_0\chi_n) - f_1$, from (3.9) and the previous convergences we get that $\{\xi_n\}$ is bounded in $L^2(0,T;V')$ and, possibly taking a subsequence, $\xi_n \to \xi$ weakly in $L^2(0,T;V')$. Moreover, for any $v \in H_0^1(0,T;V)$ we have, integrating by parts with respect to time,

$$\begin{aligned} \int_0^T \langle \xi_n, v\rangle ds &= \int_0^T \big\langle \partial_t(\varphi_0\theta_n + \psi_0\chi_n), v\big\rangle ds - \int_0^T \langle f_1, v\rangle ds \\ &= -\int_0^T \big\langle (\varphi_0\theta_n + \psi_0\chi_n), \partial_t v\big\rangle ds - \int_0^T \langle f_1, v\rangle ds. \end{aligned}$$

Hence, because of $(3.12)_{1-2}$, we can pass to the limit and obtain that $\xi = \partial_t(\varphi_0\theta + \psi_0\chi) - f_1$, that is

$$\partial_t(\varphi_0\theta_n + \psi_0\chi_n) - f_1 \to \partial_t(\varphi_0\theta + \psi_0\chi) - f_1 \qquad \text{weakly in } L^2(0,T;V'). \tag{3.13}$$

On account of $(3.12)_{1-2}$ and (3.13), apply a compactness theorem [15: p. 58] to infer that

$$(\varphi_0\theta_n + \psi_0\chi_n) - 1 * f_1 \to (\varphi_0\theta + \psi_0\chi) - 1 * f_1$$

weakly star in $H^1(0,T;V') \cap L^\infty(0,T;H)$ and strongly in $L^2(0,T;V')$. Now it is allowed to pass to the limit in (3.9) and (3.11) ending up with (3.2) and (3.4).

It remains to show that (3.3) holds, too. Though the proof is the same as in [8: Theorem 2.2], we sketch it for the readers' convenience. Since $\chi_n(\cdot,t) \in K$, then $0 \le \chi \le 1$ a.e. in Q_T. We are going to show that

$$\int_0^T \big\langle \theta(\cdot,t), (\chi - \gamma)(\cdot,t)\big\rangle dt \ge 0 \qquad \forall\, \gamma \in \mathcal{K} \tag{3.14}$$

where

$$\mathcal{K} = \Big\{ \eta \in L^2(0,T,H) : \eta(\cdot,t) \in K \ \text{ for a.e. } \ t \in (0,T) \Big\}.$$

In the variational formulation of (3.10), we choose $\gamma \in \mathcal{K}$. Integrating over $(0,T)$, we get

$$\begin{aligned} &\tfrac{1}{n} \int_0^T \big\langle \partial_t\chi_n(\cdot,t), (\chi_n - \gamma)(\cdot,t)\big\rangle dt \\ &\quad \le \int_0^T \big\langle \chi_n(\cdot,t), \theta_n(\cdot,t)\big\rangle dt - \int_0^T \big\langle \theta_n(\cdot,t), \gamma(\cdot,t)\big\rangle dt \end{aligned} \tag{3.15}$$

for any $\gamma \in \mathcal{K}$. Due to (3.12), (3.14) holds true if

$$\limsup_{n\to\infty} \int_0^T \big\langle \chi_n(\cdot,t), \theta_n(\cdot,t)\big\rangle dt \le \int_0^T \big\langle \chi(\cdot,t), \theta(\cdot,t)\big\rangle dt. \tag{3.16}$$

Exploiting the previous compactness argument and the weak lower semicontinuity of the norms, from (3.15) we deduce

$$\begin{aligned}
&\limsup_{n\to\infty} \int_0^T \big\langle \chi_n(\cdot,t), \theta_n(\cdot,t)\big\rangle dt \\
&\quad = \limsup_{n\to\infty} \Big\{ \tfrac{1}{\psi_0} \int_0^T \big\langle (\varphi_0\theta_n + \psi_0\chi_n - 1 * f_1)(\cdot,t), \theta_n(\cdot,t)\big\rangle dt \\
&\qquad - \tfrac{\varphi_0}{\psi_0} \|\theta_n\|^2_{L^2(0,T,H)} + \tfrac{1}{\psi_0} \int_0^T \big\langle (1 * f_1)(\cdot,t), \theta_n(\cdot,t)\big\rangle dt \Big\} \\
&\quad \le \tfrac{1}{\psi_0} \int_0^T \big\langle (\varphi_0\theta + \psi_0\chi - 1 * f_1)(\cdot,t), \theta(\cdot,t)\big\rangle dt \\
&\qquad - \tfrac{\varphi_0}{\psi_0} \|\theta\|^2_{L^2(0,T,H)} + \tfrac{1}{\psi_0} \int_0^T \big\langle (1 * f_1)(\cdot,t), \theta(\cdot,t)\big\rangle dt,
\end{aligned}$$

that is (3.16).

3.2 Stefan problem: continuous dependence and uniqueness. As we wrote at the beginning of this section, we can not derive any estimate for χ from (3.3). Hence, assuming that the memory kernels are absolutely continuous and taking advantage of an inversion formula for Volterra integral equations (see [13]) and of the monotonicity of H, we can control only the variation of θ. Indeed, let (θ_i, e_i, χ_i) be the solution to Problem 2 corresponding to $\theta_{\mathrm{e}\,i}$ $(i = 1,2)$ and denote $\Theta = \theta_1 - \theta_2$, $E = e_1 - e_2$, $X = \chi_1 - \chi_2$ and $\Theta_{\mathrm{e}} = \theta_{\mathrm{e}\,1} - \theta_{\mathrm{e}\,2}$. Then (Θ, E, X) solves the problem

$$\begin{aligned}
E = \varphi_0\Theta + \psi_0 X + \varphi * \Theta + \psi * X \quad &\text{in } V' \\
\langle \partial_t E, v\rangle + \big(\nabla(k_0\Theta + k * \Theta), \nabla v\big) + \sigma(\Theta, v)_{\Gamma_1} = \sigma(\Theta_{\mathrm{e}}, v)_{\Gamma_1} \quad &\forall\, v \in V
\end{aligned}$$

a.e. in $(0,T)$. Integrating herein the second relation over $(0,t)$ and taking account the first relation we get

$$\begin{aligned}
\langle \psi_0 X + \psi * X, v\rangle &= -\langle \varphi_0\Theta + \varphi * \Theta, v\rangle \\
&\quad - \big(\nabla(k_0 1 * \Theta + k * 1 * \Theta), \nabla v\big) \\
&\quad - \sigma(1 * \Theta, v)_{\Gamma_1} + \sigma(1 * \Theta_{\mathrm{e}}, v)_{\Gamma_1} \\
&=: \langle \mathcal{F}, v\rangle
\end{aligned} \tag{3.17}$$

for any $v \in V$ and a.e. in $(0,t)$. We observe that $\mathcal{F} \in L^\infty(0,T;V')$. Since $\psi \in W^{1,1}(0,T)$, by [13: p. 42/Theorem 3.1] we know that there exists a unique $\Psi \in W^{1,1}(0,T)$ (named the *resolvent* of $\frac{\psi}{\psi_0}$) solution to $\psi_0\Psi + \psi * \Psi = \psi$ in $[0,T]$. By

[13: p. 44/Theorem 3.5], (3.17) is equivalent to $\psi_0 X = \mathcal{F} - \Psi * \mathcal{F}$ in V', a.e. in $(0,T)$. Then, for any $v \in V$ and a.e. in $(0,T)$,

$$\begin{aligned}\langle \psi_0 X, v\rangle = &-\langle \varphi_0\Theta + \varphi * \Theta, v\rangle - \big(k_0\nabla(1*\Theta) + k*\nabla(1*\Theta), \nabla v\big)\\ &- \sigma(1*\Theta, v)_{\Gamma_1} + \sigma(1*\Theta_{\mathrm{e}}, v)_{\Gamma_1} + \big\langle \varphi_0(\Psi*\Theta) + \Psi*\varphi*\Theta, v\big\rangle\\ &+ \big(k_0\Psi*\nabla(1*\Theta) + k*\Psi*\nabla(1*\Theta), \nabla v\big)\\ &+ \sigma(\Psi*1*\Theta, v)_{\Gamma_1} - \sigma(\Psi*1*\Theta_{\mathrm{e}}, v)_{\Gamma_1}.\end{aligned}$$

Choosing $v = \Theta$, integrating over $(0,t)$ and taking account of the monotonicity of H, we infer

$$\begin{aligned}\varphi_0\|\Theta\|^2_{L^2(0,t;H)} &+ \tfrac{k_0}{2}\|\nabla(1*\Theta)(t)\|^2_{H^N} + \tfrac{\sigma}{2}\|(1*\Theta)(t)\|^2_{L^2(\Gamma_1)}\\ &\le \left|\int_0^t (\varphi*\Theta,\Theta)\,ds\right| + \left|\int_0^t \big(\nabla(k*1*\Theta),\nabla\Theta\big)ds\right|\\ &\quad + \sigma\left|\int_0^t (1*\Theta_{\mathrm{e}},\Theta)_{\Gamma_1} ds\right| + \left|\int_0^t \big(\varphi_0(\Psi*\Theta) + \Psi*\varphi*\Theta,\Theta\big)ds\right|\\ &\quad + \left|\int_0^t \Big(k_0\Psi*\nabla(1*\Theta) + k*\Psi*\nabla(1*\Theta),\nabla\Theta\Big)ds\right|\\ &\quad + \sigma\left|\int_0^t (\Psi*1*\Theta,\Theta)_{\Gamma_1} ds\right| + \sigma\left|\int_0^t (\Psi*1*\Theta_{\mathrm{e}},\Theta)_{\Gamma_1} ds\right|\\ &=: \sum_1^7 |I_i(t)|.\end{aligned} \tag{3.18}$$

We estimate these integrals by the usual inequality (2.6), the Hölder inequality and properties of the convolution. Moreover, recall that, since $k \in W^{1,1}(0,T)$, then $k*\nabla\Theta = k(0)(1*\nabla\Theta) + k'*1*\nabla\Theta$. On account of that, we deduce

$$\begin{aligned}|I_1(t)| &\le \|\varphi*\Theta\|_{L^2(0,t;H)}\|\Theta\|_{L^2(0,t;H)}\\ |I_2(t)| &\le \big(|k(0)| + \|k'\|_{L^1(0,T)}\big)\|\nabla(1*\Theta)\|^2_{L^2(0,t;H^N)}\\ &\quad + \|\nabla(k*1*\Theta)(t)\|_{H^N}\|\nabla(1*\Theta)(t)\|_{H^N}\\ |I_3(t)| &\le \sigma\Big\{\|(1*\Theta)(t)\|_{L^2(\Gamma_1)}\|(1*\Theta_{\mathrm{e}})(t)\|_{L^2(\Gamma_1)}\\ &\quad + \|1*\Theta\|_{L^2(0,t;L^2(\Gamma_1))}\|\Theta_{\mathrm{e}}\|_{L^2(0,t;L^2(\Gamma_1))}\Big\}\\ |I_4(t)| &\le \Big\{\varphi_0\|\Psi*\Theta\|_{L^2(0,t;H)} + \|\varphi*\Psi*\Theta\|_{L^2(0,t;H)}\Big\}\|\Theta\|_{L^2(0,t;H)}\\ |I_5(t)| &\le \big(k_0 + \|k\|_{L^1(0,T)}\big)\|\Psi*\nabla(1*\Theta)\|_{L^\infty(0,t;H^N)}\|\nabla(1*\Theta)(t)\|_{H^N}\\ &\quad + \big(k_0|\Psi(0)| + k_0\|\Psi'\|_{L^1(0,T)} + \|(k*\Psi)'\|_{L^1(0,T)}\big)\\ &\quad \times \|\nabla(1*\Theta)\|^2_{L^2(0,t;H^N)}\\ |I_6(t)| &\le \sigma\big(|\Psi(0)| + \|\Psi'\|_{L^1(0,T)}\big)\|1*\Theta\|^2_{L^2(0,t;L^2(\Gamma_1))}\\ &\quad + \sigma\|(\Psi*1*\Theta)(t)\|_{L^2(\Gamma_1)}\|(1*\Theta)(t)\|_{L^2(\Gamma_1)}\\ |I_7(t)| &\le \sigma\Big\{\|(\Psi*1*\Theta_{\mathrm{e}})(t)\|_{L^2(\Gamma_1)}\|(1*\Theta)(t)\|_{L^2(\Gamma_1)}\\ &\quad + \|\Psi*\Theta_{\mathrm{e}}\|_{L^2(0,t;L^2(\Gamma_1))}\|1*\Theta\|_{L^2(0,t;L^2(\Gamma_1))}\Big\}.\end{aligned}$$

Taking account of these estimates in (3.18), we obtain that there exists a constant $C > 0$ such that

$$\begin{aligned} &\tfrac{\varphi_0}{2}\|\Theta\|^2_{L^2(0,t;H)} + \tfrac{k_0}{4}\|\nabla(1*\Theta)\|^2_{L^\infty(0,t,H^N)} + \tfrac{\sigma}{4}\|(1*\Theta)(t)\|^2_{L^2(\Gamma_1)} \\ &\quad \le C\Big\{\|\Theta_{\mathrm{e}}\|^2_{L^2(0,t;L^2(\Gamma_1))} + \|\nabla(1*\Theta)\|^2_{L^2(0,t;H^N)} \\ &\qquad + \|1*\Theta\|^2_{L^2(0,t;L^2(\Gamma_1))} + \int_0^t \|\Theta\|^2_{L^2(0,s;H)} ds\Big\} \end{aligned}$$

and by the Gronwall lemma we get the desired estimate (3.6). Of course, uniqueness is a straightforward consequence of the continuous dependence estimate.

3.3 Stefan problem: regularity. Integrating (3.2) over $(0,t)$, we obtain

$$\langle \psi_0\chi + \psi*\chi, v\rangle = \langle \mathcal{G}, v\rangle \qquad \forall\, v \in V, \text{ a.e. in } (0,T) \tag{3.19}$$

where

$$\begin{aligned} \langle \mathcal{G}, v\rangle = &-\langle \varphi_0\theta + \varphi*\theta, v\rangle - \big(k_0\nabla(1*\theta) + k*\nabla(1*\theta), \nabla v\big) \\ &- \sigma(1*\theta, v)_{\Gamma_1} + \sigma(1*\theta_{\mathrm{e}}, v)_{\Gamma_1} + \langle 1*f, v\rangle. \end{aligned}$$

Arguing as in Subsection 3.2, we write (3.19) as

$$\psi_0\chi = \mathcal{G} - \Psi*\mathcal{G} \qquad \text{in } V', \text{ a.e. in } (0,T)$$

where $\Psi \in W^{1,1}(0,T)$. Now, for $\varepsilon \in (0,1]$, we define a smooth approximation H_ε of H, for instance,

$$H_\varepsilon(s) = \begin{cases} 0 & \text{if } s \le 0 \\ \varepsilon^{-5}\big(10\varepsilon^2 s^3 - 15\varepsilon s^4 + 6s^5\big) & \text{if } 0 < s < \varepsilon \\ 1 & \text{if } s \ge \varepsilon. \end{cases}$$

We point out that $H_\varepsilon : \mathbb{R} \to [0,1]$ is a maximal monotone graph and $H_\varepsilon \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$. Since $f = f_1 + f_2 \in L^2(0,T;H) + W^{1,1}(0,T;V')$, we can find two sequences $\{f^1_\varepsilon\}$ and $\{f^2_\varepsilon\}$ such that

$$f^1_\varepsilon \in W^{1,1}(0,T;H), \quad f^2_\varepsilon \in L^2(0,T;H), \quad \partial_t f^2_\varepsilon \in L^2(0,T;V')$$

and

$$\left.\begin{aligned} f^1_\varepsilon &\to f_1 && \text{in } L^2(0,T;H) \\ f^2_\varepsilon &\to f_2 && \text{in } W^{1,1}(0,T;V') \end{aligned}\right\} \tag{3.20}$$

as $\varepsilon \to 0$. It turns out that $f^2_\varepsilon \in C^0([0,T];V')$ and $f^2_\varepsilon \to f_2$ in $L^\infty(0,T;V')$. We will denote by $f_\varepsilon = f^1_\varepsilon + f^2_\varepsilon$. Moreover, we introduce

$$\begin{aligned} \theta_{0\varepsilon} \in V &\quad \text{such that } \theta_{0\varepsilon} \to \theta_0 := (\varphi_0 I + \psi_0 H)^{-1}(e_0) \text{ in } V \\ \theta_{\mathrm{e}\,\varepsilon} \in H^1(0,T;L^2(\Gamma_1)) &\quad \text{such that } \theta_{\mathrm{e}\,\varepsilon} \to \theta_{\mathrm{e}} \text{ in } W^{1,1}(0,T;L^2(\Gamma_1)). \end{aligned}$$

We will deal with the approximating problems to find a pair $\theta_\varepsilon, \chi_\varepsilon \in H^1(0,T;H) \cap L^2(0,T;V)$ such that

$$\begin{aligned} &\langle \partial_t(\varphi_0\, \theta_\varepsilon + \psi_0\, \chi_\varepsilon + \varphi * \theta_\varepsilon), v\rangle \\ &\quad + k_0(\nabla\theta_\varepsilon, \nabla v) + \sigma(\theta_\varepsilon, v)_{\Gamma_1} \\ &= -(k * \nabla\theta_\varepsilon, \nabla v) + \sigma(\theta_{\mathrm{e}\,\varepsilon}, v)_{\Gamma_1} + \langle f_\varepsilon, v\rangle \\ &\quad + \big\langle \varphi_0 \partial_t(\Psi * \theta_\varepsilon) + \partial_t(\Psi * \varphi * \theta_\varepsilon), v\big\rangle \\ &\quad + \big(k_0 \Psi * \nabla\theta_\varepsilon + k * \Psi * \nabla\theta_\varepsilon, \nabla v\big) \\ &\quad + \sigma(\Psi * \theta_\varepsilon, v)_{\Gamma_1} - \sigma(\Psi * \theta_{\mathrm{e}\,\varepsilon}, v)_{\Gamma_1} - \langle \Psi * f_\varepsilon, v\rangle \quad \forall\, v \in V, \text{ a.e. in } (0,T) \end{aligned} \tag{3.21}$$

$$\chi_\varepsilon \in H_\varepsilon(\theta_\varepsilon) \quad \text{a.e. in } Q_T \tag{3.22}$$

$$\theta_\varepsilon(\cdot,0) = \theta_{0\varepsilon},\ \ \chi_\varepsilon(\cdot,0) = H_\varepsilon(\theta_{0\varepsilon}) \quad \text{a.e. in } \Omega. \tag{3.23}$$

Using the same argument as in [8: Theorems 2.2 and 2.3], it is easy to see that the solution to this problem exists and is unique. Arguing formally, choose $v = \partial_t\,\theta_\varepsilon$ as test function in (3.21) and integrate over $(0,t)$. Recalling that $\partial_t\,\chi_\varepsilon = H_\varepsilon'(\theta_\varepsilon)\partial_t\,\theta_\varepsilon$ and H_ε is increasing, we obtain

$$\begin{aligned} &\varphi_0 \|\partial_t\,\theta_\varepsilon\|^2_{L^2(0,t;H)} + \tfrac{k_0}{2}\|\nabla\theta_\varepsilon(t)\|^2_{H^N} + \tfrac{\sigma}{2}\|\theta_\varepsilon(t)\|^2_{L^2(\Gamma_1)} \\ &\le \Big(\tfrac{k_0}{2}\|\nabla\theta_{0\varepsilon}\|^2_{H^N} + \tfrac{\sigma}{2}\|\theta_{0\varepsilon}\|^2_{L^2(\Gamma_1)}\Big) \\ &\quad + \left|\int_0^t \langle \partial_t(\varphi * \theta_\varepsilon), \partial_t\,\theta_\varepsilon\rangle ds\right| + \left|\int_0^t (k * \nabla\theta_\varepsilon, \nabla\partial_t\,\theta_\varepsilon) ds\right| \\ &\quad + \sigma\left|\int_0^t (\theta_{\mathrm{e}\,\varepsilon}, \partial_t\,\theta_\varepsilon)_{\Gamma_1} ds\right| + \left|\int_0^t \langle f_\varepsilon, \partial_t\,\theta_\varepsilon\rangle ds\right| \\ &\quad + \left|\int_0^t \big\langle \varphi_0\partial_t(\Psi * \theta_\varepsilon) + \partial_t(\Psi * \varphi * \theta_\varepsilon), \partial_t\,\theta_\varepsilon\big\rangle ds\right| \\ &\quad + \left|\int_0^t \big(k_0\Psi * \nabla\theta_\varepsilon + k * \Psi * \nabla\theta_\varepsilon, \nabla\partial_t\,\theta_\varepsilon\big) ds\right| \\ &\quad + \sigma\left|\int_0^t (\Psi * \theta_\varepsilon, \partial_t\,\theta_\varepsilon)_{\Gamma_1} ds\right| + \sigma\left|\int_0^t (\Psi * \theta_{\mathrm{e}\,\varepsilon}, \partial_t\,\theta_\varepsilon)_{\Gamma_1} ds\right| \\ &\quad + \left|\int_0^t \langle \Psi * f_\varepsilon, \partial_t\,\theta_\varepsilon\rangle ds\right| \\ &= \Big(\tfrac{k_0}{2}\|\nabla\theta_{0\varepsilon}\|^2_{H^N} + \tfrac{\sigma}{2}\|\theta_{0\varepsilon}\|^2_{L^2(\Gamma_1)}\Big) + \sum_{i=0}^{9} S_i(t). \end{aligned}$$

Henceforth, C stands for a positive constant depending on $\varphi_0, \psi_0, k_0, \sigma, \Omega$ and on the norms of φ, ψ, k in $W^{1,1}(0,T)$. Exploiting the properties of convolution and integrating by parts, we derive the following estimates:

$$\begin{aligned} |S_1(t)| &\le \|\varphi\|_{L^2(0,T)}\|\theta_{0\varepsilon}\|_H \|\partial_t\,\theta_\varepsilon\|_{L^2(0,t;H)} \\ &\quad + \|\varphi * \partial_t\,\theta_\varepsilon\|_{L^2(0,t;H)}\|\partial_t\,\theta_\varepsilon\|_{L^2(0,t;H)} \\ |S_2(t)| &\le \|k * \nabla\theta_\varepsilon\|_{L^\infty(0,t;H^N)}\|\nabla\theta_\varepsilon\|_{L^\infty(0,t;H^N)} \\ &\quad + \big(|k(0)| + \|k'\|_{L^1(0,T)}\big)\|\nabla\theta_\varepsilon\|^2_{L^2(0,t;H^N)} \end{aligned}$$

$$
\begin{aligned}
|S_3(t)| &\le \sigma\Big\{\|\,\theta_{\mathrm{e}\,\varepsilon}\,\|_{L^\infty(0,T;L^2(\Gamma_1))}\|\,\theta_\varepsilon\,\|_{L^\infty(0,t;L^2(\Gamma_1))} \\
&\quad + \|\,\theta_{\mathrm{e}\,\varepsilon}(0)\|_{L^2(\Gamma_1)}\|\,\theta_{0\varepsilon}\,\|_{L^2(\Gamma_1)} \\
&\quad + \|\partial_t\,\theta_{\mathrm{e}\,\varepsilon}\,\|_{L^1(0,t;L^2(\Gamma_1))}\|\,\theta_\varepsilon\,\|_{L^\infty(0,t;L^2(\Gamma_1))}\Big\} \\
|S_4(t)| &\le \|f^1_\varepsilon\|_{L^2(0,T;H)}\|\partial_t\,\theta_\varepsilon\,\|_{L^2(0,t;H)} + \|f^2_\varepsilon\|_{L^\infty(0,T;V')}\|\,\theta_\varepsilon\,\|_{L^\infty(0,t;V)} \\
&\quad + \|f^2_\varepsilon\|_{L^\infty(0,T;V')}\|\,\theta_{0\varepsilon}\,\|_V + \|\partial_t f^2_\varepsilon\|_{L^1(0,T;V')}\|\,\theta_\varepsilon\,\|_{L^\infty(0,t;V)} \\
|S_5(t)| &\le \big(\varphi_0 + \|\varphi\|_{L^1(0,T)}\big) \\
&\quad \times \big\{\|\Psi\|_{L^2(0,T)}\|\,\theta_{0\varepsilon}\,\|_H + \|\Psi * \partial_t\,\theta_\varepsilon\,\|_{L^2(0,t;H)}\big\}\|\partial_t\,\theta_\varepsilon\,\|_{L^2(0,t;H)} \\
|S_6(t)| &\le \big(k_0 + \|k\|_{L^1(0,T)}\big)\|\Psi * \nabla\,\theta_\varepsilon\,\|_{L^\infty(0,t:H^N)}\|\nabla\,\theta_\varepsilon\,\|_{L^\infty(0,t;H^N)} \\
&\quad + \big\{k_0\big(|\Psi(0)| + \|\Psi'\|_{L^1(0,T)}\big) + \|(k*\Psi)'\|_{L^1(0,T)}\big\} \\
&\quad \times \|\nabla\,\theta_\varepsilon\,\|^2_{L^2(0;t;H^N)} \\
|S_7(t)| &\le \sigma\Big\{\|\Psi\|_{L^2(0,T)}\|\,\theta_\varepsilon\,\|_{L^2(0,t;L^2(\Gamma_1))}\|\,\theta_\varepsilon\,\|_{L^\infty(0,t;L^2(\Gamma_1))} \\
&\quad + \big(|\Psi(0)| + \|\Psi'\|_{L^1(0,T)}\big)\|\,\theta_\varepsilon\,\|^2_{L^2(0,t;L^2(\Gamma_1))}\Big\} \\
|S_8(t)| &\le \sigma\Big\{\|\Psi * \theta_{\mathrm{e}\,\varepsilon}\,\|_{L^\infty(0,t;L^2(\Gamma_1))}\|\,\theta_\varepsilon\,\|_{L^\infty(0,t;L^2(\Gamma_1))} \\
&\quad + \big(|\Psi(0)| + \|\Psi'\|_{L^1(0,T)}\big)\|\,\theta_{\mathrm{e}\,\varepsilon}\,\|_{L^2(0,t;L^2(\Gamma_1))}\|\,\theta_\varepsilon\,\|_{L^2(0,t;L^2(\Gamma_1))}\Big\} \\
|S_9(t)| &\le \|\Psi * f^1_\varepsilon\|_{L^2(0,T;H)}\|\partial_t\,\theta_\varepsilon\,\|_{L^2(0,t;H)} \\
&\quad + \|\Psi\|_{W^{1,1}(0,T)}\|f^2_\varepsilon\|_{W^{1,1}(0,T;V')}\|\,\theta_\varepsilon\,\|_{L^\infty(0,t;V)}.
\end{aligned}
$$

Collecting the estimates above, by the Gronwall lemma we see that there is a constant $C > 0$ independent of ε such that

$$
\begin{aligned}
&\|\partial_t\,\theta_\varepsilon\,\|^2_{L^2(0,t;H)} + \|\nabla\,\theta_\varepsilon\,\|^2_{L^\infty(0,t;H^N)} + \|\,\theta_\varepsilon\,\|^2_{L^\infty(0,t;L^2(\Gamma_1))} \\
&\quad \le C\Big\{\|\,\theta_{0\varepsilon}\,\|^2_V + \|\,\theta_{\mathrm{e}\,\varepsilon}\,\|^2_{W^{1,1}(0,T;L^2(\Gamma_1))} \\
&\qquad + \|f^1_\varepsilon\|^2_{L^2(0,t;H)} + \|f^2_\varepsilon\|_{W^{1,1}(0,T;V')}\|\,\theta_\varepsilon\,\|_{L^\infty(0,T;V)}\Big\}.
\end{aligned} \tag{3.24}
$$

Now we need to deduce an estimate for $\|\,\theta_\varepsilon\,\|_{L^\infty(0,t;H)}$; thus we choose as test function $v = \theta_\varepsilon$ in (3.21). Integrating (3.21) with respect to time and by definition of H_ε we see that there is a constant $C > 0$ such that

$$
\begin{aligned}
\|\,\theta_\varepsilon\,\|^2_{L^\infty(0,t;H)} &\le C\Big\{\sqrt{|\Omega|T}\,\|\partial_t\,\theta_\varepsilon\,\|_{L^2(0,t;H)} + \|\,\theta_{0\varepsilon}\,\|^2 + \|f^1_\varepsilon\|^2_{L^2(0,t;H)} \\
&\qquad + \|f^2_\varepsilon\|_{W^{1,1}(0,T;V')}\|\,\theta_\varepsilon\,\|_{L^\infty(0,t;V)}\Big\}.
\end{aligned} \tag{3.25}
$$

From (3.24) and (3.25), by (2.6) and (3.20), we see that there is a constant $C > 0$ independent of ε such that, for any $\varepsilon \in (0,1]$, the inequality

$$
\begin{aligned}
\|\,\theta_\varepsilon\,\|_{L^\infty(0,T;V)\cap H^1(0,T;H)} &\le C\Big\{1 + \|\theta_0\|_V + \|f^1\|_{L^2(0,T;H)} \\
&\qquad + \|f^2\|_{W^{1,1}(0,T;V')} + \|\theta_{\mathrm{e}}\|_{W^{1,1}(0,T;L^2(\Gamma_1))}\Big\}
\end{aligned}
$$

holds. Possibly taking a subsequence,

$$\theta_\varepsilon \to \theta \quad \text{weakly in } H^1(0,T;H) \text{ and weakly star in } L^\infty(0,T;V)$$
$$\chi_\varepsilon \to \chi \quad \text{weakly star in } L^\infty(Q_T)$$

as $\varepsilon \to 0$. But the first weak convergence therein yields $\theta_\varepsilon \to \theta$ strongly in $C^0([0,T];H)$. Now it is allowed to pass to the limit in (3.21) - (3.23), and (3.8) holds by the weak star lower semicontinuity of norms.

4. The relay switch

We can assume that the relay is initially switched on. Let us denote by (TP1) the feedback control problem corresponding to the relay switch, i.e. we shall deal with the following

Problem (TP1). Find a triplet (θ, χ, z) and a finite sequence of switching times $0 = t_0 < t_1 < ... < t_m = T$ such that $z \in L^\infty(0,T)$, (θ,χ) solves the relaxed (respectively, the Stefan) problem with $\theta_\mathrm{e} = \mathcal{F}(z)$,

$$z(t) = (-1)^h \quad \text{if } t \in [t_h, t_{h+1})$$
$$t_{h+1} = \inf\left\{\{T\} \cup K_{h+1}\right\}$$

where, for $h = 0, 1, ..., m-1$,

$$K_{h+1} = \left\{ t \in (t_h, T] : \mathcal{M}(\theta)(t) = \begin{cases} \rho_U(t) & \text{if } h \text{ is even} \\ \rho_L(t) & \text{if } h \text{ is odd} \end{cases} \right\}.$$

Let us recall that

$$\mathcal{M}(\theta)(t) = \fint_{\Omega_0} \theta(x,t)\omega_I(x)\,dx + \fint_{\Gamma_2} (1 * \theta)(y,t)\omega_S(y)\,d\Gamma$$

and, for $r \in L^2(0,T)$,

$$\mathcal{F}(r)(y,t) = \int_0^t E(y,t,\tau)r(\tau)\,d\tau + E_0(y,t) \qquad \forall\,(y,t) \in \Gamma_1 \times (0,T).$$

We assume the following:

(H11) $\omega_I \in L^2(\Omega_0; [0,+\infty))$ and $\omega_S \in L^2(\Gamma_2; [0,+\infty))$.

(H12) $E \in L^2(\Gamma_1 \times (0,T)^2)$, $E_t, E_\tau \in L^1(\Gamma_1 \times (0,T)^2)$, $E_0 \in W^{1,1}(0,T; L^2(\Gamma_1))$.

Now we can state our existence and uniqueness results.

Theorem 4.1. *If assumptions* (H2), (H5) - (H9), (H11) - (H12) *hold and*

$$\left.\begin{array}{r}\varphi,\psi,k\in W^{1,1}(0,T)\qquad\qquad\qquad\qquad\\ f\in L^2(0,T;H)+W^{1,1}(0,T;V'),\ \theta_0\in V\end{array}\right\},\tag{4.1}$$

then there exists a unique solution to problem (TP1) *in the relaxed problem case. Replacing assumption* (H8) *by assumption* (K2) *and adding* $(3.7)_3$*, the same result holds for problem* (TP1) *in the Stefan problem case.*

First of all we shall prove that for both problems the number of switching times is finite. In order to do that, we need some properties of the operators $\mathcal{F}$ and $\mathcal{M}$. If assumption (H12) holds, then observe that $\mathcal{F}$ is a continuous operator from $L^\infty(0,T)$ to $W^{1,1}(0,T;L^2(\Gamma_1))$ such that $\mathcal{F}(r)(\cdot,0)=E_0(\cdot,0)$ a.e. on Γ_1, for any $r\in L^\infty(0,T)$. Moreover, there exists a constant $\Lambda_1>0$ such that

$$\left.\begin{array}{r}\|\mathcal{F}(r)\|_{W^{1,1}(0,T;L^2(\Gamma_1))}\le\Lambda_1\big\{1+\|r\|_{L^\infty(0,T)}\big\}\\ \|(\mathcal{F}(r_1)-\mathcal{F}(r_2))(t)\|_{L^2(\Gamma_1)}\le\Lambda_1\|r_1-r_2\|_{L^2(0,t)}\end{array}\right\}\tag{4.2}$$

for any $t\in[0,T]$ and all $r,r_1,r_2\in L^\infty(0,T)$ (see [9: Proposition 2.2] for details).

On the other hand, if assumption (H11) holds, then $\mathcal{M}:\ L^\infty(0,T;V)\to L^\infty(0,T)$ turns out to be a linear and continuous operator such that, for any $v_1,v_2\in L^\infty(0,T;V)$,

$$\begin{aligned}&\big|\mathcal{M}(v_1)(t)-\mathcal{M}(v_2)(t)\big|\\ &\quad\le\big(\|\omega_I\|_{L^2(\Omega_0)}+\|\omega_S\|_{L^2(\Gamma_2)}\big)\\ &\qquad\times\big\{\|(v_1-v_2)(\cdot,t)\|_H+\|1*(v_1-v_2)(\cdot,t)\|_{L^2(\Gamma_1)}\big\}\end{aligned}\tag{4.3}$$

for almost any $t\in(0,T)$. Moreover, for any $v\in L^\infty(0,T;V)\cap H^1(0,T;H)$, $\mathcal{M}(v)\in C^{0,\frac12}([0,T])$ and

$$\begin{aligned}&\big|\mathcal{M}(v)(t)-\mathcal{M}(v)(\tau)\big|\\ &\le\Big\{\|\omega_I\|_{L^2(\Omega_0)}\|\partial_t v\|_{L^2(0,T;H)}+\|\omega_S\|_{L^2(\Gamma_2)}\|v\|_{L^2(0,T;L^2(\Gamma_1))}\Big\}|t-\tau|^{\frac12}\end{aligned}\tag{4.4}$$

for any $t,\tau\in[0,T]$. Particularly, if $\omega_I=0$, then

$$\|\mathcal{M}(v_1)-\mathcal{M}(v_2)\|_{C^0([0,T])}\le\Lambda_2\|1*(v_1-v_2)\|_{L^\infty(0,T;V)}\tag{4.5}$$

where $\Lambda_2=\|\omega_s\|_{L^2(\Gamma_2)}$.

Now we can prove

Lemma 4.1. *Let assumptions* (H2), (H5) - (H9), (H11) - (H12) *and* (4.1) *hold and* $\mathcal{M}(\theta(\cdot,0))\le\rho_L(0)$*. Then, for any*

$$w\in B=\big\{r\in L^\infty(0,T):\ \|r\|_{L^\infty(0,T)}\le 1\big\},$$

letting $(\theta(w), \chi(w))$ *be the solution to Problem* 1 *with* $\theta_{\mathrm{e}} = \mathcal{F}(w)$*, there exists* $\gamma > 0$ *such that, for any* $w \in B$ *and any pair* $t_L, t_U \in [0,T]$ *fulfilling*

$$\begin{aligned} \mathcal{M}(\theta(w))(t_L) &= \rho_L(t_L) \\ \mathcal{M}(\theta(w))(t_U) &= \rho_U(t_U) \end{aligned} \tag{4.6}$$

we have

$$|t_L - t_U| \geq \gamma.$$

In particular, $\mathcal{M}(\theta(w))$ *commutes at most* $\left[\frac{T}{\gamma}\right]$ *times between the threshold functions* ρ_L *and* ρ_U ($\left[\frac{T}{\gamma}\right]$ *being the integer part of* $\frac{T}{\gamma}$)*. The same result holds for Problem* 2*, provided assumption* (H8) *is replaced by assumption* (K2) *and* $(3.7)_3$ *holds.*

Proof. Taking into account (2.4), $(4.2)_1$ and (4.4) we have, for any $w \in B$,

$$\begin{aligned} &\big|\mathcal{M}(\theta(w))(t_L) - \mathcal{M}(\theta(w))(t_U)\big| \\ &\quad\leq \Big(\|\omega_I\|_{L^2(\Omega_0)}\|\partial_t\theta(w)\|_{L^2(0,T;H)} \\ &\qquad + \|\omega_S\|_{L^2(\Gamma_2)}\|\theta(w)\|_{L^2(0,T;L^2(\Gamma_1))}\Big)|t_L - t_U|^{\frac{1}{2}} \\ &\quad\leq C\big(1 + \|\mathcal{F}(w)\|_{W^{1,1}(0,T;L^2(\Gamma_1))}\big)|t_L - t_U|^{\frac{1}{2}} \\ &\quad\leq C\big(2 + \|w\|_{L^\infty(0,T)}\big)|t_L - t_U|^{\frac{1}{2}} \\ &\quad\leq \Lambda_3|t_L - t_U|^{\frac{1}{2}}. \end{aligned} \tag{4.7}$$

Because of the uniform continuity of ρ_L, there exists a constant $\gamma_1 > 0$ such that, for any $t, \tau \in [0,T]$ with $|t-\tau| \leq \gamma_1$, we have $|\rho_L(t) - \rho_L(\tau)| \leq \frac{\delta}{2}$. Then either $|t_L - t_U| \geq \gamma_1$ or $|t_L - t_U| < \gamma_1$. In the second case, by (4.6) and (4.7), we deduce

$$\begin{aligned} \Lambda_3|t_L - t_U|^{\frac{1}{2}} &\geq \big|\mathcal{M}(\theta(w))(t_L) - \mathcal{M}(\theta(w))(t_U)\big| \\ &= \big|\rho_L(t_L) - \rho_U(t_U)\big| \\ &\geq \rho_U(t_U) - \rho_L(t_U) - \big|\rho_L(t_L) - \rho_L(t_U)\big| \\ &\geq \tfrac{\delta}{2}. \end{aligned}$$

If $\gamma := \min\{\gamma_1, \frac{\delta^2}{(2\Lambda_3)^2}\}$, we have the first part of the thesis. The second follows from $\mathcal{M}(\theta(\cdot,0)) \leq \rho_L(0)$ ∎

In both cases, there exists a unique solution to Problem (TP1) because we can apply the inductive argument of [14] (see also [9: Theorem 4.1]).

4.1 Proof of Theorem 4.1. Since we have supposed that the relay is initially switched on, we define

$$w_0(t) = +1 \qquad \forall\, t \in [0,T].$$

Let us consider the triplet $(\theta(w_0), \chi(w_0), w_0)$, where $(\theta(w_0), \chi(w_0))$ solves the relaxed (respectively, the Stefan) problem with $\theta_{\mathrm{e}} = \mathcal{F}(w_0)$, and set

$$D_1 = \big\{t \in [0,T] : \mathcal{M}(\theta(w_0))(t) = \rho_U(t)\big\}.$$

If $D_1 = \emptyset$, then we are done because $(\theta(w_0), \chi(w_0), w_0)$ solves problem (TP1). Otherwise, we pick $t_1 = \inf D_1$ (it is a minimum because of the continuity of $\mathcal{M}$) and by Lemma 4.1 we see that $t_1 \geq \gamma$. Now we define

$$w_1(t) = \begin{cases} w_0(t) & \text{if } t \in [0, t_1) \\ -1 & \text{if } t \in [t_1, T]. \end{cases}$$

Then $(\theta(w_1), \chi(w_1))$ solves the relaxed (respectively the Stefan) problem with $\theta_e = \mathcal{F}(w_1)$ and

$$D_2 := \big\{ t \in (t_1, T] : \mathcal{M}(\theta(w_1))(t) = \rho_L(t) \big\}.$$

If $D_2 = \emptyset$, then the solution is $\big(\theta(w_1), \chi(w_1), w_1\big)$, otherwise we consider $t_2 = \inf D_2$ and we proceed as above, recalling that $t_2 \geq 2\gamma$. Finally, there exist $m \in \mathbb{N}$, a triplet $\big(\theta(w_m), \chi(w_m), w_m\big)$ and a sequence of switching times $\{t_h\}_{h=0}^m$ such that $m \leq \frac{T}{\gamma}$, $t_m = T$ and $\theta = \theta(w_m)$, $\chi = \chi(w_m)$, $z = w_m$ uniquely satisfies problem (TP1).

5. The Preisach operator

We will take advantage of some properties of the Preisach operator (see, e.g., [24: Chapter IV] for details). First of all, we are concerned with the continuity of this operator (see [24: Theorems 3.1 and 3.2]). We recall that if μ is a non-negative Borel measure on $\mathcal{P}$ with bounded density such that, for any $(\rho_1, \rho_2) \in \mathcal{P}$,

$$\mu(\{\rho_1\} \times R) = \mu(R \times \{\rho_2\}) = 0, \tag{5.1}$$

then

$$\left. \begin{array}{l} \|\mathcal{W}_2(r)\|_{L^\infty(0,T)} \leq \mu(\mathcal{P}) < \infty \quad \forall\, r \in C^0([0,T]) \\ \mathcal{W}_2 \ \text{ is strongly continuous from } C^0([0,T]) \text{ to } C^0([0,T]) \end{array} \right\}. \tag{5.2}$$

Moreover, as is shown in [24: Theorem 3.5], denoting by l the bidimensional Lebesgue measure, if the assumption

(K3) There exists a constant $\Lambda_\mu > 0$ such that $\mu(A) \leq \Lambda_\mu l(A)$ for all Lebesgue measurable sets $A \subset \mathcal{P}$

is fulfilled, then there exists a constant $\Lambda_4 > 0$ such that, for all $r_1, r_2 \in C^0([0,T])$ and for any $t \in [0,T]$,

$$\big|\big(\mathcal{W}_2(r_1) - \mathcal{W}_2(r_2)\big)(t)\big| \leq \Lambda_4 \|r_1 - r_2\|_{C^0([0,t])} \tag{5.3}$$

where the constant $\Lambda_4 > 0$ depends only on $\mu(\mathcal{P})$ and Λ_μ.

We now consider the following

Problem (TP2). Find (θ, χ) solution to the relaxed (respectively, the Stefan) problem with $\theta_e = \mathcal{F}(\mathcal{W}_2(\mathcal{M}(\theta)))$ on $\Gamma_1 \times (0,T)$.

Exploiting the results of Sections 2 and 3 and the properties of the operators $\mathcal{F}, \mathcal{M}, \mathcal{W}_2$, we are going to prove

Theorem 5.1. *Under assumptions* (H2), (H5) - (H9), (H11) - (H12) *and* (4.1) *and* (5.1)*, there exists a solution to problem* (TP2) *corresponding to the relaxed problem. Moreover, if* $\omega_I = 0$ *and assumption* (K3) *holds, then the solution is unique. Replacing assumption* $(H8)$ *by assumption* $(K2)$ *and assuming in addition* $(3.7)_3$*, then under the same hypotheses this result holds for problem* (TP2) *in the Stefan problem case.*

Proof. We consider the feedback control problem corresponding to the relaxed condition. As in [9: Theorem 5.2], we apply a fixed point argument to the operator

$$S : C^0([0,T]) \to C^0([0,T]), \qquad S(r) = \mathcal{M}(\theta(r))$$

where, as usual, $(\theta(r), \chi(r))$ is the solution to Problem 1 with $\theta_{\rm e} = \mathcal{F}(\mathcal{W}_2(r))$. More precisely, the continuity of $\mathcal{F}$ and (4.4) yield that S takes values in $C^{0,\frac{1}{2}}([0,T])$ and, by $(5.2)_1$, there exists a constant $\Lambda_5 > 0$ such that $\|S(r)\|_{C^{0,\frac{1}{2}}([0,T])} \le \Lambda_5$. Then S is compact provided we show that it is continuous.

If $\{r_j\}$ is a sequence converging in $C^0([0,T])$ to some r, then by $(5.2)_2$ we deduce that $\mathcal{W}_2(r_j) \to \mathcal{W}_2(r)$ in $C^0([0,T])$. Taking account of $(4.2)_2$ we infer $\mathcal{F}(\mathcal{W}_2(r_j)) \to \mathcal{F}(\mathcal{W}_2(r))$ in $C^0([0,T]; L^2(\Gamma_1))$. Now, from the continuous dependence of θ with respect to $\theta_{\rm e}$, we can deduce only that $\theta(r_j) \to \theta(r)$ in $L^2(0,T;H)$ and $(1*\theta)(r_j) \to (1*\theta)(r)$ in $L^2(0,T,L^2(\Gamma_1))$. But, by (4.3), it is enough to deduce that $S(r_j) \to S(r)$ in $L^2(0,T)$. Then, arguing as in [9], we can conclude that such convergence holds in $C^0([0,T])$ and, by the Schauder fixed point theorem, there exists at least one solution to problem (TP2).

We now prove that such solution is unique, provided $\omega_I = 0$ and assumption (K3) holds. Let us suppose that $(\theta_1,\chi_1),(\theta_2,\chi_2)$ are two different solutions to problem (TP2) and set $\theta_{{\rm e}\,i} = \mathcal{F}(\mathcal{W}_2(\mathcal{M}(\theta_i)))$ for $i = 1,2$. By the continuity of $\mathcal{F}$ and recalling $(5.2)_2, (4.2)_1$ and (2.3), we get

$$\begin{aligned}
\big\|(\theta_{{\rm e}\,1} - \theta_{{\rm e}\,2})(t)\big\|^2_{L^2(\Gamma_1)} &\le \Lambda_1^2 \big\|\mathcal{W}_2(\mathcal{M}(\theta_1)) - \mathcal{W}_2(\mathcal{M}(\theta_2))\big\|^2_{L^2(0,t)} \\
&\le (\Lambda_1\Lambda_4)^2 \int_0^t \big\|\mathcal{M}(\theta_1) - \mathcal{M}(\theta_2)\big\|^2_{C^0[0,s]} ds \\
&\le (\Lambda_1\Lambda_2\Lambda_4)^2 \int_0^t \big\|1*(\theta_1-\theta_2)\big\|^2_{L^\infty(0,s;V)} ds \\
&\le C \int_0^t \big\|\theta_{{\rm e}\,1} - \theta_{{\rm e}\,2}\big\|^2_{L^2(0,s;L^2(\Gamma_1))} ds \\
&\le C \int_0^t \big\|(\theta_{{\rm e}\,1} - \theta_{{\rm e}\,2})(s)\big\|^2_{L^2(\Gamma_1)} ds
\end{aligned}$$

and uniqueness follows from the Gronwall lemma. Arguing as above, we can prove the same result for Problem (TP2) with the Stefan condition.

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