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A Priori Gradient Bounds and Local $C^{1,\alpha}$-Estimates for (Double) Obstacle Problems under Non-Standard Growth Conditions

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Abstract. We prove local gradient bounds and interior Hölder estimates for the first derivatives of functions $u \in W^1_{1,loc}(\Omega)$ which locally minimize the variational integral $I(u) = \int_\Omega f(\nabla u)\,dx$ subject to the side condition $\Psi_1 \le u \le \Psi_2$. We establish these results for various classes of integrands f with non-standard growth. For example, in the case of smooth f the (s,μ,q)-condition is sufficient. A second class consists of all convex functions f with (p,q)-growth.

Keywords: *Non-standard growth, (double) obstacle problems, a priori estimates, regularity of minimizers*

AMS subject classification: 49N60, 35J85, 49J40

1. Introduction

In this paper we discuss the regularity properties of functions $u \in W^1_{1,loc}$ which locally minimize the variational integral

$$I(u) = \int_\Omega f(\nabla u)\,dx$$

subject to the constraint $\Psi_1 \le u \le \Psi_2$ almost everywhere on Ω (double obstacle problems). Here Ω denotes a bounded domain in $\mathbb{R}^n$ $(n \ge 2)$ and $f : \mathbb{R}^n \to [0,\infty)$ is a given strictly convex function such that $f(Z)$ grows faster than $|Z|$ as $|Z| \to \infty$.

To be precise we briefly summmarize our setting. We consider locally Lipschitz functions Ψ_1 and Ψ_2 such that $\Psi_2 - \Psi_1 \ge m$ holds for a number $m > 0$, and we say that $u \in W^1_{1,loc}(\Omega)$ is locally *I-minimizing* with respect to the (double) side condition

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$\Psi_1 \le u \le \Psi_2$ if and only if $f(\nabla u)$ is in $L^1_{loc}(\Omega)$ and

$$\begin{aligned} I\big(u, \operatorname{spt}(u-v)\big) &= \int_{\operatorname{spt}(u-v)} f(\nabla u)\, dx \\ &\le \int_{\operatorname{spt}(u-v)} f(\nabla v)\, dx = I\big(v, \operatorname{spt}(u-v)\big) \end{aligned} \tag{1.1}$$

holds for any $v \in W^1_{1,loc}(\Omega)$ such that $\operatorname{spt}(u-v) \Subset \Omega$ and $\Psi_1 \le v \le \Psi_2$ almost everywhere in Ω. Constrained problems of this type have been recently faced by few authors under different assumptions, covering degenerate energy densities f (see [5, 31, 32]), possibly with non-standard growth conditions of a particular type (see [23]). Here we investigate the smoothness of local minimizers under growth and differentiability assumptions on f which are quite different from the standard hypotheses usually considered, proving new regularity results recovering and substantially extending most of the previous ones available in the literature.

Before going into details let us briefly outline the history of the regularity results for the single and double obstacle problem. The most common case is the so-called *p-growth behaviour* of f which means that f is of class C^2 satisfying

$$|Z|^p \le f(Z) \le L(1+|Z|^p) \tag{1.2}$$

$$\nu(1+|Z|^2)^{\frac{p-2}{2}}|Y|^2 \le D^2 f(Z)(Y,Y) \le L(1+|Z|^2)^{\frac{p-2}{2}}|Y|^2 \tag{1.3}$$

for all $Z, Y \in \mathbb{R}^n$ with constants $\nu, L > 0$ and with a fixed exponent $p > 1$ (we refer the reader to the papers [6, 10, 11, 22, 28, 31] and the references quoted therein). Of course, it should be mentioned that the classical case $p = 2$ is extensively treated in the monographs [14, 21]. Under assumptions (1.2) - (1-3) optimal smoothness of local minimizers (depending on the structure of Ψ_1 and Ψ_2) has been established. For example, it is shown in [31] that any solution u of (1.1) has Hölder continuous first derivatives provided $\nabla\Psi_1$ and $\nabla\Psi_2$ are Hölder continuous functions.

In recent years integrands f with non-standard growth became object of intensive investigation. Ten years ago Marcellini (see [24 - 27]) replaced assumption (1.2) by the so-called (p,q)-growth condition

$$|Z|^p \le f(Z) \le L(1+|Z|^q) \tag{1.4}$$

with $1 < p \le q$ and proved (using also an appropriate version of assumption (1.3)) $C^{1,\alpha}$-regularity of unconstrained local minimizers provided $q < \frac{np}{n-2}$ in the case $n > 2$ (see comment (v) below for details). For related results also in the vectorial setting we refer to the papers [2, 27].

On the other hand, many problems in Mathematical Physics (see, for example, [16] or [17]) motivate the study of functionals of nearly linear growth like

$$I_1(u) = \int_\Omega |\nabla u| \ln(1+|\nabla u|)\, dx \tag{1.5}$$

or its iterated version

$$I_k(u) = \int_\Omega |\nabla u| \ln\Big(1 + \ln\Big(1 + \ldots \ln(1 + |\nabla u|) \ldots\Big)\Big) dx \tag{1.6}$$

which are obviously not of (p,q)-growth for any $1 < p \le q$. Partial $C^{1,\alpha}$-regularity results for free minimizers of energies given by (1.5) or (1.6) covering also the vector-valued case were presented first in [15, 16, 18]; later on these results were completed in [8] and full regularity was proved in [29] (see also [13]).

The first result in our paper addresses the double obstacle problem for functionals given by (1.5) and (1.6) but also covers the case of integrands like $f(Z) = |Z|^p \ln(1+|Z|)$, its iterated versions and in addition includes integrands of (p,q)-growth as studied by Marcellini. We can even consider integrands f of (s,μ,q)-growth which means that f has to satisfy the following set of hypotheses: let $F : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ denote a continuous function, fix some real number $s \ge 1$ and assume

$$\lim_{t\to\infty} \frac{F(t)}{t} = \infty \qquad \text{and} \qquad F(t) \ge c_0 t^s \text{ for large values of } t. \tag{1.7}$$

The integrand f is required to be a non-negative function of class $C^2(\mathbb{R}^n)$ such that, for all $Z, Y \in \mathbb{R}^n$,

$$c_1 F(|Z|) \le f(Z) \tag{1.8}$$

$$|D^2 f(Z)|\,|Z|^2 \le c_2(1 + f(Z)) \tag{1.9}$$

$$\lambda(1 + |Z|^2)^{-\frac{\mu}{2}} |Y|^2 \le D^2 f(Z)(Y,Y) \le \Lambda(1 + |Z|^2)^{\frac{q-2}{2}} |Y|^2 \tag{1.10}$$

where $\mu \in \mathbb{R}$, $q > 1$ and $c_0, c_1, c_2, \lambda, \Lambda$ denote positive constants. If $n \ge 3$, we assume in addition that

$$q < (2 - \mu)\,\tfrac{n}{n-2} \tag{1.11}$$

is satisfied. Note that, on account of $q > 1$, (1.11) gives the upper bound

$$\mu < 1 + \tfrac{2}{n} \tag{1.12}$$

which we also assume in the case $n = 2$. Under these hypotheses, our results are summarized in the following

Theorem 1.1.

(a) *Assume that f satisfies* (1.7) – (1.12). *Then any solution u of* (1.1) *is locally Lipschitz continuous if so are the two obstacles Ψ_1 and Ψ_2. If we assume the obstacles to have Hölder continuous gradients, then the solution of* (1.1) *is of class $C^{1,\alpha}_{loc}(\Omega)$ for some $0 < \alpha < 1$.*

(b) *If condition* (1.9) *is dropped and if we replace condition* (1.11) *by the stronger condition*

$$q < (2 - \mu) + s\,\tfrac{2}{n}, \tag{1.13}$$

then we also obtain the conclusion of statement (a).

Let us briefly comment on our conditions:

(i) Condition (1.7) together with the second part of condition (1.10) implies $s \le q$ (compare [1: Lemma 2.1] if $q < 2$). For this reason (1.13) is more restrictive than (1.11), and (1.13) reduces to (1.11) if s reaches the optimal value q.

(ii) The case of integrands with nearly linear growth like (1.5) and (1.6) is covered choosing $s = 1$.

(iii) We may assume that $2 - \mu \le s$ (which is obvious if $\mu \ge 1$ or $\mu \le 0$; in the case $0 < \mu < 1$ again compare [1: Lemma 2.1]). The lower bound $q \ge 2 - \mu$ follows from (1.10).

(iv) In Section 3 we will construct an example of an integrand $f_{\mu,q}$ satisfying (1.10) precisely with exponents μ and q for a given range of values for μ and q. The balancing condition (1.9) is also satisfied. Moreover, the growth of $f_{\mu,q}$ is exactly q. Thus we obtain regularity under the condition $q < \frac{(2-\mu)n}{n-2}$.

For the unbalanced case described in Theorem 1.1/(b) we give an example of an integrand f depending also on the parameter s by the way demonstrating the importance of condition (1.13).

(v) Suppose that we are given numbers $q > p > 1$ and that (1.10) holds with $\mu = 2 - p$. This case corresponds to the version of (p, q)-growth introduced by Marcellini in the paper [25] where the growth behaviour is formulated in terms of the second derivatives. Marcellini then proved regularity of unconstrained local solutions u assuming (1.11) but without any balancing condition. Instead of this he requires u to be of class $W^1_{q,loc}(\Omega)$, hence in our setting we can choose $s = q$ and get regularity under the same condition on q and p as in [25]. Thus we recover Marcellini's regularity result and extend it to the constrained case.

(vi) Now let us assume that just (1.10) is true with $q > p > 1$ and $\mu = 2 - p$. Then we have (1.7) with $s = p$, and part (b) of Theorem 1.1 implies regularity in the case that $q < \frac{p(n+2)}{n}$. The latter condition also occurs in the second part of the paper [25]; it turns out to be sufficient to obtain existence for the kind of equations considered by Marcellini.

There exist some preliminary versions of Theorem 1.1: in [12] there is considered the case of a single obstacle Ψ of class $W^2_\infty(\Omega)$ for the logarithmic energy introduced in (1.5) and partial C^1-regularity was proved provided $n \le 4$. Assuming condition (1.13) with $s = 1$ the nearly linear setting was studied in [13], and singular points where excluded for any dimension n still dealing with a single obstacle Ψ and also under stronger hypotheses on Ψ than stated in Theorem 1.1 above.

Observe that a modification of Moser's iteration argument was applied in [13]. Here we use De Giorgi's technique which turned out to be useful in the case of linear growth studied in [4, 20] and which now is seen to cover any of the above mentioned growth conditions.

Next, we turn our attention to the double obstacle problem in the context of energies with (p, q)-growth as stated in (1.4) (thus exluding integrals as in (1.5) or (1.6)): we now move in a different direction by weakening, with respect to the cases considered in the literature, not only the growth assumptions but also those regarding the smoothness of the integrand f. Very recently (see [7, 9]) some surprising regularity properties like Lipschtiz continuity were proved without any differentiability assumption on f. Here we want to prove similar results in the constrained situation. More precisely, we obtain Lipschitz regularity of solutions without assuming any differentiability property for f, in particular, any ellipticity condition involving $D^2 f$ is dropped. In place of this a "qualified" form of convexity (see (1.15)) is assumed while the obstacles Ψ_1 and Ψ_2 have to satisfy a local Lipschitz condition.

Theorem 1.2. *Let $f \in C^0(\mathbb{R}^n)$ be such that:*

$$(\sigma^2 + |Z|^2)^{\frac{p}{2}} \le f(Z) \le L(\sigma^2 + |Z|^2)^{\frac{p}{2}} + L(\sigma^2 + |Z|^2)^{\frac{q}{2}} \tag{1.14}$$

$$\int_{[0,1]^n} \big(f(Z + D\varphi) - f(Z)\big)\, dx \ge \nu \int_{[0,1]^n} \big(\sigma^2 + |Z|^2 + |D\varphi|^2\big)^{\frac{p-2}{2}} |D\varphi|^2 dx \tag{1.15}$$

for any $Z \in \mathbb{R}^n$ and $\varphi \in C_0^\infty\big((0,1)^n\big)$, where $1 < p \le q < p\frac{n+1}{n}$, $\nu > 0$, $L \ge 1$ and $\sigma \in [0,1]$. Then any solution $u \in W^1_{1,loc}(\Omega)$ to (1.1) is locally Lipschitz continuous if so are the two obstacles Ψ_1 and Ψ_2.

In addition to (1.14) and (1.15) suppose that

(i) *$f \in C^2(\mathbb{R}^n)$ if $p \ge 2$ or $\sigma > 0$*

or

(ii) *$f \in C^2(\mathbb{R}^n \sim \{0\}) \cap C^{1,p-1}(\mathbb{R}^n)$ when $1 < p < 2$ and $\sigma = 0$.*

Moreover, we assume that for $\sigma = 0$ we have

$$\limsup_{|z| \to 0} \frac{|D^2 f(z)|}{|z|^{p-2}} \le L < +\infty. \tag{1.16}$$

Then u is in the space $C^{1,\alpha}_{loc}(\Omega)$ provided Ψ_1 and Ψ_2 have locally Hölder continuous gradients.

We remark that the result of Theorem 1.2, which is obtained using an appropriate modification of the approximation and (Moser-) iteration technique presented in [7], is completely new even in the standard case $p = q$ and also includes the degenerate p-case treated in [31] (that follows choosing $\sigma = 0$). Actually, the degenerate case in [31] is extended not only because the functional has (p,q)-growth but also since no hypotheses has been made on the growth of the second derivatives of f: (1.16) only controls the kind of degeneration of $D^2 f$.

Let us give some further comments on the hypotheses of Theorem 1.2: condition (1.15) requires a kind of uniform (quasi-) convexity of our integrand f, and in [9] it is shown that under suitable hypotheses on f inequality (1.15) is equivalent to the usual pointwise condition. We will comment on this during the proof of the second part of Theorem 1.2 (see Lemma 4.3). So, comparing the assumptions of Theorems 1.1 and 1.2 we remark: according to [1: Lemma 2.1] the right-hand side of (1.10) implies the right-hand side of (1.14) whereas the left-hand side of (1.10) gives (1.15). So, if f is smooth, then Theorem 1.2 is a consequence of Theorem 1.1, which holds under even weaker assumptions relating p and q. On the other hand, despite its apparently involved formulation the convexity condition (1.15) is very general. For example, all integrands f of the form $f(Z) = |Z|^p + h(Z)$ are included where h is a general convex function satisfying nothing but a q-growth assumption of the type $0 \le h(z) \le L(1+|z|^q)$. However, according to this generality, the relation between p and q is more restrictive than the one stated in Theorem 1.1.

With obvious changes in notation (see Theorem 1.1) it should of course be possible to give a variant of Theorem 1.2 also for non-smooth convex integrands f of nearly linear growth. Since the iteration technique requires some technical modifications, we did not include this aspect for the sake of clearness and brevity.

2. Proof of Theorem 1.1

In the following ε and δ will denote two sequences of positive real numbers such that $\varepsilon \to 0$ and $\delta \to 0$. From time to time we shall pass to any subsequence that will still be denoted by ε and δ, respectively. Moreover, c will denote a finite, positive constant, not necessarily the same in any two occurencies, while only the relevant dependences will be highlighted. The proof of Theorem 1.1 is organized in the following five steps:

- approximation
- linearization
- a priori L^q-estimates
- a priori L^∞-estimates
- conclusion.

Step 1 (*Approximation*). Let $\{\varphi_t\}_{t>0}$ be a family of smooth mollifiers. We denote by $u_\varepsilon, \Psi_{1,\varepsilon}$ and $\Psi_{2,\varepsilon}$ the ε-mollification with kernel φ_ε of u, Ψ_1 and Ψ_2, respectively. Furthermore, let $m > 0$ be such that $\Psi_2 - \Psi_1 \ge m$ and and fix $\bar\varepsilon > 0$ such that $\Psi_{2,\varepsilon} - \Psi_{1,\varepsilon} \ge \frac{m}{2}$ whenever $0 < \varepsilon < \bar\varepsilon$. We fix $R > 0$ and $x_0 \in \Omega$ with the property $B_{2R} \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ $(\varepsilon < \bar\varepsilon)$ where $B_r = B_r(x_0)$. Then we define

$$\mathbb{K}'_\varepsilon = \Big\{ w \in u_\varepsilon + \mathring{W}^1_q(B_{2R}) : \Psi_{1,\varepsilon} \le w \le \Psi_{2,\varepsilon} \Big\}$$

and $v_{\varepsilon,\delta} \in \mathbb{K}'_\varepsilon$ as the unique solution of the Dirichlet problem

$$J_\delta(w) = \int_{B_{2R}} f_\delta(\nabla w)\, dx \to \min \qquad \text{in } \mathbb{K}'_\varepsilon \tag{2.1}$$

where for any $\delta > 0$

$$f_\delta(Z) = f(Z) + \delta(1+|Z|^2)^{\frac{q}{2}}. \tag{2.2}$$

Observe that we have by standard results (see, e.g., [6, 31] and the references given at the end of the proof of Lemma 2.1)

$$v_{\varepsilon,\delta} \in C^{1,\alpha}(B_{2R}) \cap W^2_{q,loc}(B_{2R})$$

for some $0 < \alpha < 1$.

From now on we shall drop the subscripts ε and δ just denoting

$$v_{\varepsilon,\delta} \equiv v, \qquad f_\delta \equiv f, \qquad \Psi_{i,\varepsilon} \equiv \Psi_i \ (i \in \{1,2\}), \qquad \mathbb{K}'_\varepsilon = \mathbb{K}'.$$

The full notation will be recovered later, in Step 5.

Step 2 (*Linearization*). Here we are going to prove the following

Lemma 2.1. *Under the assumptions of Theorem 1.1, v is of class $W_t^2(B_{2R})$ for any $t<\infty$ and*

$$Df(\nabla v)\in W^1_{t,loc}(B_{2R}). \tag{2.3}$$

Moreover, the equation

$$\int_{B_{2R}} Df(\nabla v)\cdot\nabla\varphi\,dx=\int_{B_{2R}}\varphi g\,dx \tag{2.4}$$

is valid for any $\varphi\in C_0^1(B_{2R})$, where

$$g=\mathbf{1}_{S_1}\big(-\operatorname{div}[Df(\nabla\Psi_1)]\big)+\mathbf{1}_{S_2}\big(-\operatorname{div}[Df(\nabla\Psi_2)]\big)$$

and $S_i=\big\{x\in B_{2R}:\,v=\Psi_i\big\}\quad(i\in\{1,2\})$.

Proof. Following the lines of [10 - 13] or [4] we fix $0<s<\frac{m}{10}$ and consider a function $h_s:[0,+\infty)\to[0,1]$ of class C^1 such that $h_s=1$ on $[0,s]$, $h_s=0$ on $[2s,+\infty)$ and $h_s'\le 0$. Given $\eta\in C_0^1(B_{2R})$, $\eta\ge 0$, we let

$$w_t=v+t\eta h_s\circ(v-\Psi_1)$$

which belongs to the class $\mathbb{K}'$ if the positive number t satisfies $t\sup_\Omega\eta\le\frac{m}{10}$. From the minimum property of v we deduce

$$\int_{B_{2R}} Df(\nabla v)\cdot\nabla\big(\eta h_s\circ(v-\Psi_1)\big)\,dx\ge 0$$

hence there is a Radon measure $\lambda_1=\lambda_1(s)$ such that

$$\int_{B_{2R}} Df(\nabla v)\cdot\nabla\big(\eta h_s\circ(v-\Psi_1)\big)=\int_{B_{2R}}\eta\,d\lambda_1(s). \tag{2.5}$$

Actually, $\lambda_1(s)$ does not depend on s (use the comparison function $w_t=v+t\eta[h_s\circ(v-\Psi_1)-h_{s'}\circ(v-\Psi_1)]$ with $s<s'$, $\eta\in C_0^1(B_{2R})$, $\eta\ge 0$ and $|t|>0$ small enough). Hence we may write λ_1 in equation (2.5). In order to estimate λ_1, we fix $\eta\in C_0^1(B_{2R})$, $\eta\ge 0$, and observe by (2.5)

$$\begin{aligned}
\int_{B_{2R}}\eta\,d\lambda_1&=\int_{B_{2R}} Df(\nabla v)\cdot\nabla\eta h_s\circ(v-\Psi_1)\,dx\\
&\quad+\int_{B_{2R}} Df(\nabla\Psi_1)\cdot\eta\nabla\big(h_s\circ(v-\Psi_1)\big)\,dx\\
&\quad+\int_{B_{2R}}\big(Df(\nabla v)-Df(\nabla\Psi_1)\big)\cdot(\nabla v-\nabla\Psi_1)\eta h_s'\circ(v-\Psi_1)\,dx\\
&\le\int_{B_{2R}} Df(\nabla v)\cdot\nabla\eta h_s\circ(v-\Psi_1)\,dx\\
&\quad-\int_{B_{2R}}\operatorname{div}\big(Df(\nabla\Psi_1)\big)\eta h_s\circ(v-\Psi_1)\,dx\\
&\quad-\int_{B_{2R}} Df(\nabla\Psi_1)\cdot\nabla\eta h_s\circ(v-\Psi_1)\,dx\\
&\to\int_{B_{2R}\cap[v=\Psi_1]}\eta\big(-\operatorname{div}(Df(\nabla\Psi_1))\big)dx\quad\text{as } s\downarrow 0.
\end{aligned}$$

Therefore λ_1 is of the form

$$\lambda_1 = \mathbf{1}_{[v=\Psi_1]}\Theta_1\big(-\operatorname{div}(Df(\nabla\Psi_1))\big) \times \text{Lebesgue measure} \tag{2.6}$$

for a density function $\Theta_1 : \Omega \to [0,1]$. In a similar way, using $w_t = v - t\eta h_s \circ (\Psi_2 - v)$ with s, t, η as stated before (2.5), we get the equation

$$-\int_{B_{2R}} Df(\nabla v)\cdot\nabla\big(\eta h_s \circ (\Psi_2 - v)\big)\,dx = \int_{B_{2R}} \eta\, d\lambda_2 \tag{2.7}$$

for another Radon measure λ_2 indepedent of s. In place of (2.6) we get

$$\lambda_2 = \mathbf{1}_{[v=\Psi_2]}\Theta_2\big(\operatorname{div}(Df(\nabla\Psi_2))\big) \times \text{Lebesgue measure.} \tag{2.8}$$

Putting together (2.5) - (2.8) we arrive at

$$\begin{aligned}&\int_{B_{2R}} Df(\nabla v)\cdot\nabla\Big\{\varphi\big[h_s\circ(v-\Psi_1)+h_s\circ(\Psi_2-v)\big]\Big\}dx\\ &\quad=\int_{B_{2R}}\varphi\Big\{\Theta_1\mathbf{1}_{S_1}\big(-\operatorname{div}(Df(\nabla\Psi_1))\big)+\Theta_2\mathbf{1}_{S_2}\big(-\operatorname{div}(Df(\nabla\Psi_2))\big)\Big\}dx\end{aligned} \tag{2.9}$$

being valid for all $\varphi \in C_0^1(B_{2R})$ and any $s \in (0, \frac{m}{10})$. Let us fix s and φ as above. Then, for $t \in \mathbb{R}$ such that $|t|\sup_\Omega|\varphi| < s$, the function

$$w_t = v + t\varphi\big\{1 - \big[h_s\circ(v-\Psi_1)+h_s\circ(\Psi_2-v)\big]\big\}$$

is in the class $\mathbb{K}'$, the minimality of v implies

$$\int_{B_{2R}} Df(\nabla v)\cdot\nabla\Big\{\varphi\Big(1-\big[h_s\circ(v-\Psi_1)+h_s\circ(\Psi_2-v)\big]\Big)\Big\}dx = 0.$$

Thus, from (2.9), $v \in \overset{\circ}{W}{}_2^1(\Omega)$ is a weak solution of the equation $-\operatorname{div}(Df(\nabla v)) = g$ with $g \in L^\infty(B_{2R})$. Recalling the growth condition (1.10) for $D^2 f$ we see (compare [13] or [21] for details) that $v \in W_t^2(B_{2R})$ for any finite t. Hence we may integrate by parts in (2.5) and (2.7) to get (2.6) and (2.8) with densities $\equiv 1$ which finally proves the lemma ∎

Remark 2.2. Of course, Lemma 2.1 is valid under weaker assumptions as stated in Theorem 1.1.

Step 3 (*A priori L^q-estimates*). To obtain uniform L^q-estimates for ∇v we fix

$$M > 1 + \|\nabla\Psi_1\|^2_{L^\infty(B_{2R})} + \|\nabla\Psi_2\|^2_{L^\infty(B_{2R})} \tag{2.10}$$

and for $0 < \rho \le R$ we set

$$U_\kappa^\rho = \big\{x \in B_{R+\rho} :\, 1 + |\nabla v|^2 > \kappa\big\}.$$

Lemma 2.3. *There is a constant $c = c(R)$ independent of ε and δ such that, for any $\kappa > 2M$ and $\eta \in C_0^1(B_{R+\rho})$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on B_R and $|\nabla\eta| \le \frac{2}{\rho}$,*

$$\int_{U_\kappa^\rho} \eta^2 (1+|\nabla v|^2)^{-\frac{\mu}{2}} |\nabla^2 v|^2 dx \le \tfrac{c}{\rho^2} \int_{B_{R+\rho}\sim B_R} |D^2 f(\nabla v)|\,|\nabla v|^2 dx.$$

Proof. Fix $\kappa > 2M$ and set for all $t \in \mathbb{R}$

$$\tilde{h}(t) = \min\big\{\max[t-1,0],1\big\}$$
$$h(t) = h_M(t) = \tilde{h}(M^{-1}t),$$

hence $h(t) = 0$ if $t < M$ and $h(t) = 1$ if $t > 2M$. In (2.4) we may replace φ by $\partial_s\varphi$ where $s \in \{1,\dots,n\}$. Integrating by parts and using (2.3) we obtain

$$\int_{B_{2R}} D^2 f(\nabla v)\,(\partial_s \nabla v, \nabla\varphi)\,dx = -\int_{B_{2R}} g\,\partial_s\varphi\,dx \tag{2.12}$$

remaining valid for any $\varphi \in \mathring{W}_2^1(\Omega)$. Then we introduce the quantity:

$$\Gamma = \Gamma(\nabla v) = 1 + |\nabla v|^2. \tag{2.13}$$

By Lemma 2.1 we may pick $\varphi = \eta^2 \partial_s v\, h(\Gamma)$ as test function in (2.12). Since $\nabla v = \nabla\Psi_i$ almost everywhere on S_i it is seen by (2.11) that $h(\Gamma) = 0$ almost everywhere on S_i. Thus the right-hand side of (2.12) vanishes and we obtain (from now on summation with respect to $s = 1,\dots,n$)

$$\begin{aligned} 0 &= \int_{B_{R+\rho}} D^2 f(\nabla v)\big(\partial_s\nabla v, \nabla\{\eta^2 \partial_s v\, h(\Gamma)\}\big)\,dx \\ &= \int_{B_{R+\rho}} D^2 f(\nabla v)\,(\partial_s\nabla v, \partial_s\nabla v)\eta^2 h(\Gamma)\,dx \\ &\quad + \int_{B_{R+\rho}} D^2 f(\nabla v)\,(\partial_s\nabla v, \nabla h(\Gamma))\eta^2\partial_s v\,dx \\ &\quad + \int_{B_{R+\rho}} D^2 f(\nabla v)\,(\partial_s\nabla v, \nabla\eta^2)\partial_s v\, h(\Gamma)\,dx \\ &=: A_1 + A_2 + A_3. \end{aligned} \tag{2.14}$$

Since $\partial_j h(\Gamma) = 2h'(\Gamma)\nabla v\,\partial_j\nabla v$ and $h' \ge 0$ we see that A_2 is positive on account of

$$A_2 = \int_{B_{R+\rho}} D^2 f(\nabla v)\big(\nabla|\nabla v|^2, \nabla|\nabla v|^2\big) h'(\Gamma)\eta^2 dx \ge 0.$$

Now use Young's inequality to handle A_3 and observe that

$$\int_{B_{R+\rho}} D^2 f(\nabla v)(\nabla\eta,\nabla\eta)\partial_s v\,\partial_s v\, h(\Gamma)\,dx \le \tfrac{c}{\rho^2}\int_{B_{R+\rho}\sim B_R} |D^2 f(\nabla v)|\,|\nabla v|^2 dx.$$

Finally, (2.14), (2.10) and $\kappa > 2M$ imply the assertion (ignoring the "δ-part" on the left-hand side) ∎

As an application we get

Lemma 2.4. *Let the assumptions of Theorem* 1.1 *hold and set* $\chi = \frac{n}{n-2}$ *if* $n \geq 3$. *In the case* $n = 2$ *define a number* $\chi > 1$ *through the condition*

$$\chi \begin{cases} > \frac{q}{2-\mu} & \text{in the case (a) of Theorem 1.1} \\ > \frac{2s}{s+2-\mu-q} & \text{in the case (b) of Theorem 1.1.} \end{cases}$$

Then there are local constants $c = c(R)$ *and* $\beta = \beta(n, s, q, \mu)$ *independent of* ε *and* δ *such that*

$$\int_{B_R} (1 + |\nabla v|^2)^{\frac{(2-\mu)\chi}{2}} dx \leq c \left\{ \int_{B_{2R}} (1 + f(\nabla v))\, dx \right\}^{\beta}.$$

Note that our assumptions imply $q < (2 - \mu)\chi$. The proof given below in fact will show that in the case $n = 2$ we can choose for χ any finite number. Of course, the constants will depend on the quantity χ.

Proof of Lemma 2.4. (a) Let $\rho = R$, fix $\kappa > 2M$ and define $h(t) = h_\kappa(t)$ and Γ according to (2.11) and (2.13). Then we have with η as in Lemma 2.3 and using Sobolev's inequality

$$\begin{aligned} \int_{B_R} (1 + |\nabla v|^2)^{\frac{(2-\mu)\chi}{2}} dx &\leq c \int_{B_{2R}} \big(\eta h(\Gamma)[1 + |\nabla v|^2]^{\frac{(2-\mu)}{4}}\big)^{2\chi} dx + c(\kappa) \\ &\leq c \left[\int_{B_{2R}} \big|\nabla\big(\eta h(\Gamma)[1 + |\nabla v|^2]^{\frac{2-\mu}{4}}\big)^2 \big| dx \right]^{\chi} + c(\kappa) \\ &\leq c(\kappa)\big(1 + T_1 + T_2 + T_3\big)^{\chi} \end{aligned}$$

where we abbreviated

$$\begin{aligned} T_1 &= \int_{B_{2R}} |\nabla\eta|^2 h^2(\Gamma)[1 + |\nabla v|^2]^{\frac{2-\mu}{2}} dx \\ T_2 &= \int_{B_{2R}} \eta^2 |\nabla h(\Gamma)|^2 [1 + |\nabla v|^2]^{\frac{2-\mu}{2}} dx \\ T_3 &= \int_{B_{2R}} (\eta h(\Gamma))^2 \big|\nabla[1 + |\nabla v|^2]^{\frac{2-\mu}{4}}\big|^2 dx. \end{aligned}$$

The bound for T_1 follows from $\frac{2-\mu}{2} \leq \frac{s}{2}$ and (1.7) - (1.8). Since $\nabla h(\Gamma) = 0$ on the complement of $U_\kappa^R \sim U_{2\kappa}^R$ we may estimate T_2 by

$$\begin{aligned} T_2 &\leq c(\kappa) \int_{U_\kappa^R \sim U_{2\kappa}^R} \eta^2 |\nabla h(\Gamma)|^2 dx \\ &\leq c(\kappa) \int_{U_\kappa^R \sim U_{2\kappa}^R} \eta^2 (1 + |\nabla v|^2)^{-\frac{\mu}{2}} |\nabla^2 v|^2 dx \\ &\leq c(\kappa, R) \int_{U_\kappa^R} (1 + f(\nabla v))\, dx \end{aligned}$$

since $1+|\nabla v|^2$ is bounded on $U^R_\kappa \sim U^R_{2\kappa}$ where we used Lemma 2.3 and the balancing condition (1.9). For T_3 observe

$$\left|\nabla[1+|\nabla v|^2]^{\frac{2-\mu}{4}}\right|^2 \le C[1+|\nabla v|^2]^{-\frac{\mu}{2}}|\nabla^2 v|^2.$$

Hence Lemma 2.3) and (1.9) also give the bound for T_3, i.e. the first part of the lemma.

(b) We fix $R<r<\frac{3}{2}R$ and $0<\rho<\frac{1}{2}R$ and consider $\tilde\eta \in C^1_0(B_{r+\frac{\rho}{2}})$ with $\tilde\eta\equiv 1$ on B_r and $|\nabla\tilde\eta|\le\frac{4}{\rho}$. As above we obtain

$$\begin{aligned}&\int_{B_r}(1+|\nabla v|^2)^{\frac{(2-\mu)\chi}{2}}dx\\ &\quad\le c\Big\{1+\frac{1}{\rho^2}\int_{B_{2R}}(1+|\nabla v|^2)^{\frac{2-\mu}{2}}dx+c\int_{B_{r+\frac{\rho}{2}}\cap U^R_\kappa}(1+|\nabla v|^2)^{-\frac{\mu}{2}}|\nabla^2 v|^2dx\Big\}^\chi.\end{aligned}$$

Now we apply Lemma 2.3, where we replace R by $r+\frac{\rho}{2}$ and ρ by $\frac{\rho}{2}$. Observing the growth condition for $D^2 f$ we arrive at

$$\begin{aligned}&\int_{B_r}(1+|\nabla v|^2)^{\frac{(2-\mu)\chi}{2}}dx\\ &\quad\le c\Big\{1+\tfrac{1}{\rho^2}\int_{B_{2R}}(1+|\nabla v|^2)^{\frac{2-\mu}{2}}dx+\frac{c}{\rho^2}\int_{B_{r+\rho}\sim B_r}(1+|\nabla v|^2)^{\frac{q}{2}}dx\Big\}^\chi.\end{aligned}\tag{2.15}$$

This corresponds to the inequality given in [7: after (4.6)], where we now can choose $t=q$. With this choice, the following interpolation procedure of [7] reads as $\|\nabla u\|_q \le \|\nabla u\|_s^\theta \|\nabla u\|^{1-\theta}_{(2-\mu)\chi}$ where $\theta\in(0,1)$ is such that $\frac{1}{q}=\frac{\theta}{s}+\frac{1-\theta}{(2-\mu)\chi}$. Note that the arguments of [7] require the bound $\frac{q}{2-\mu}(1-\theta)<1$ which for $n\ge 3$ is equivalent to (1.13). If $n=2$, then the above inequality reads as $\chi>\frac{s}{s+2-\mu-q}$ which clearly holds according to our choice of χ. Thus we may follow the lines of [7] again to get the claim of the lemma ∎

Step 4 (A priori L^∞-estimates). Now let us introduce the notation

$$\begin{aligned}\omega=\omega_{\varepsilon,\delta}&=\ln(1+|\nabla v|^2)\\ A(h,r)=A_{\varepsilon,\delta}(h,r)&=\{x\in B_r : \omega\ge h\}\quad(h\ge 0)\end{aligned}$$

where we assume in the following that the balls B_{2r} are compactly contained in Ω.

Lemma 2.5. *Consider $\eta\in C^1_0(B_R)$ with $0\le\eta\le 1$. Then for any $k\ge k_0(M)$*

$$\begin{aligned}\int_{A(k,R)}(1+|\nabla v|^2)^{1-\frac{\mu}{2}}|\nabla\omega|^2\eta^2dx+\int_{A(k,R)}(1+|\nabla v|^2)^{-\frac{\mu}{2}}(\omega-k)^2\eta^2|\nabla^2 v|^2dx\\ \le C\int_{A(k,R)}(1+|\nabla v|^2)^{\frac{q}{2}}(\omega-k)^2|\nabla\eta|^2dx.\end{aligned}\tag{2.16}$$

Here $C<+\infty$ only depends on the data and is independent of δ and k, $k_0(M)$ denotes a constant depending only on Ψ_1 and Ψ_2 through the quantity M appearing in (2.10).

Proof. (i) In (2.12) we pick $\varphi = \eta^2 \partial_s v \max[\omega - k, 0]$. On $[v = \Psi_i]$ we have

$$\max[\omega - k, 0] = \max[\ln(1 + |\nabla \Psi_i|^2) - k, 0] = 0$$

provided $k \geq k_0(M) := \sup_{B_R} \max_{i=1,2} \ln(1 + |\nabla \Psi_i|^2)$. Thus (2.12) reduces to

$$\int_{A(k,R)} D^2 f(\nabla v)\big(\partial_s \nabla v, \nabla\{\eta^2 \partial_s v(\omega - k)\}\big) dx = 0 \qquad (k \geq k_0(M)). \tag{2.17}$$

Next observe

$$\begin{aligned}
&\int_{A(k,R)} D^2 f(\nabla v)(\partial_s \nabla v, \partial_s v \nabla \omega) \eta^2 dx \\
&\qquad = \tfrac{1}{2} \int_{A(k,R)} D^2 f(\nabla v)(\nabla \omega, \nabla \omega)(1 + |\nabla v|^2) \eta^2 dx \\
&\qquad \geq \tfrac{\lambda}{2} \int_{A(k,R)} (1 + |\nabla v|^2)^{1 - \frac{\mu}{2}} |\nabla \omega|^2 \eta^2 dx
\end{aligned} \tag{2.18}$$

where we neglected the δ-part of f on the right-hand side. It remains to estimate

$$\int_{A(k,R)} D^2 f(\nabla v)(\partial_s \nabla v, \nabla \partial_s v) \eta^2 (\omega - k)\, dx \geq 0$$

and

$$\begin{aligned}
&\left| \int_{A(k,R)} D^2 f(\nabla v)(\partial_s \nabla v \partial_s v, \nabla \eta^2)(\omega - k)\, dx \right| \\
&\qquad = \left| \int_{A(k,R)} D^2 f(\nabla v)(\nabla \omega, \nabla \eta)(1 + |\nabla v|^2) \eta (\omega - k)\, dx \right| \\
&\qquad \leq \int_{A(k,R)} \big(D^2 f(\nabla v)(\nabla \omega, \nabla \omega)\big)^{\frac{1}{2}} \eta (1 + |\nabla v|^2)^{\frac{1}{2}} \\
&\qquad\quad \times \big(D^2 f(\nabla v)(\nabla \eta, \nabla \eta)\big)^{\frac{1}{2}} (\omega - k)(1 + |\nabla v|^2)^{\frac{1}{2}} dx \\
&\qquad \leq \varepsilon \int_{A(k,R)} D^2 f(\nabla v)(\nabla \omega, \nabla \omega) \eta^2 (1 + |\nabla v|^2)\, dx \\
&\qquad\quad + \tfrac{1}{\varepsilon} \int_{A(k,R)} D^2 f(\nabla v)(\nabla \eta, \nabla \eta)(\omega - k)^2 (1 + |\nabla v|^2)\, dx.
\end{aligned}$$

For ε small enough we get from (2.17)

$$\begin{aligned}
&\int_{A(k,R)} D^2 f(\nabla v)(\nabla \omega, \nabla \omega) \eta^2 (1 + |\nabla v|^2)\, dx \\
&\qquad \leq C \int_{A(k,R)} D^2 f(\nabla v)(\nabla \eta, \nabla \eta)(1 + |\nabla v|^2)(\omega - k)^2 dx
\end{aligned}$$

and by (1.10) and (2.18) we have bounded the first integral on the right-hand side of (2.16).

(ii) This time we pick $\varphi = \eta^2\partial_s v \max[\omega - k, 0]^2$ in (2.12). As in (i) we get for $k \geq k_0(M)$

$$\begin{aligned}
&\int_{A(k,R)} D^2 f(\nabla v)(\partial_s\nabla v, \partial_s\nabla v)(\omega - k)^2\eta^2 dx \\
&\quad + \int_{A(k,R)} D^2 f(\nabla v)(\partial_s v\, \partial_s\nabla v, \nabla\omega)2(\omega - k)\eta^2 dx \\
&= -\int_{A(k,R)} D^2 f(\nabla v)(\partial_s\nabla v, \nabla\eta)2\eta(\omega - k)^2\partial_s v dx.
\end{aligned} \tag{2.19}$$

As for (2.18), by ellipticity the second integral on the left-hand side is ≥ 0. The right-hand side of (2.19) is bounded by

$$\begin{aligned}
&C\int_{A(k,R)} \big(D^2 f(\nabla v)(\partial_s\nabla v, \partial_s\nabla v)\big)^{\frac{1}{2}}\eta\big(D^2 f(\nabla v)(\nabla\eta, \nabla\eta)\big)^{\frac{1}{2}}|\nabla v|(\omega - k)^2 dx \\
&\quad \leq C\Big\{\varepsilon\int_{A(k,R)} D^2 f(\nabla v)(\partial_s\nabla v, \partial_s\nabla v)\eta^2(\omega - k)^2 dx \\
&\quad\quad + \tfrac{1}{\varepsilon}\int_{A(k,R)} |\nabla v|^2 D^2 f(\nabla v)(\nabla\eta, \nabla\eta)(\omega - k)^2 dx\Big\}
\end{aligned}$$

and by choosing ε properly we get

$$\begin{aligned}
&\int_{A(k,R)} D^2 f(\nabla v)(\partial_s\nabla v, \partial_s\nabla v)\eta^2(\omega - k)^2 dx \\
&\quad \leq C\int_{A(k,R)} |\nabla v|^2 D^2 f(\nabla v)(\nabla\eta, \nabla\eta)(\omega - k)^2 dx
\end{aligned}$$

which completes the proof of Lemma 2.5 ∎

Finally, we introduce the notation

$$\begin{aligned}
a(h, r) = a_{\varepsilon,\delta}(h, r) &= \int_{A(h,r)} (1 + |\nabla v|^2)^{\frac{q}{2}} dx \\
\tau(h, r) = \tau_{\varepsilon,\delta}(h, r) &= \int_{A(h,r)} (1 + |\nabla v|^2)^{\frac{q}{2}}(\omega - h)^2 dx
\end{aligned}$$

to obtain

Lemma 2.6. *Let $\chi > 1$ as defined in Lemma 2.4, and $h \geq k_0(M)$ and $0 < r < R$. Then:*

(i) $\tau(h, r) \leq C\, a(h, r)^{\frac{\chi-1}{\chi}}(R - r)^{-2}\tau(h, R)$.

(ii) $a(h, r) \leq (h - k)^{-2}\tau(k, r)$ *for $h \geq k \geq k_0(M)$.*

Proof. Statement (ii) is immediate. As to statement (i), we consider $\eta \in C_0^1(B_R)$ such that $\eta \equiv 1$ on B_r, $0 \leq \eta \leq 1$ and $|\nabla\eta| \leq C(R - r)^{-1}$. Again we let $\Gamma = \Gamma(\nabla v) =$

$1+|\nabla v|^2$ and select $\beta\in[0,\frac{q}{2})$ to be fixed later. Then by Sobolev's inequality (for simplicity we let $n\geq 3$)

$$\begin{aligned}
&\int_{A(h,r)} \Gamma^{\frac{q}{2}}(\omega-h)^2 dx \\
&\quad = \int_{A(h,r)} \Gamma^{\frac{q}{2}-\beta}(\omega-h)^2\Gamma^{\beta} dx \\
&\quad \leq \Big(\int_{A(h,r)} \Gamma^{(\frac{q}{2}-\beta)\chi}(\omega-h)^{2\chi} dx\Big)^{\frac{1}{\chi}} \underbrace{\Big(\int_{A(h,r)} \Gamma^{\frac{\chi}{\chi-1}\beta} dx\Big)^{\frac{\chi-1}{\chi}}}_{=:X} \\
&\quad \leq X\Big(\int_{A(h,R)} \big\{\eta\,\Gamma^{\frac{1}{2}(\frac{q}{2}-\beta)}(\omega-h)\big\}^{2\chi} dx\Big)^{\frac{1}{\chi}} \\
&\quad \leq CX\int_{A(h,R)} \big|\nabla\big(\eta(\omega-h)\Gamma^{\frac{1}{2}(\frac{q}{2}-\beta)}\big)\big|^2 dx
\end{aligned}$$

and the remaining integral splits into the sum of the following terms:

$$\int_{A(h,R)} |\nabla\eta|^2(\omega-h)^2\Gamma^{\frac{q}{2}-\beta} dx \leq C(R-r)^{-2}\tau(h,R)$$

$$\int_{A(h,R)} \eta^2|\nabla\omega|^2\Gamma^{\frac{q}{2}-\beta} dx \leq \text{r.-h. side of (2.16) if } \tfrac{q}{2}-\beta\leq 1-\tfrac{\mu}{2}$$

$$\begin{aligned}
\int_{A(h,R)} \eta^2(\omega-h)^2\Gamma^{\frac{q}{2}-\beta-2}|\nabla\Gamma|^2 dx &\leq C\int_{A(h,R)} \eta^2(\omega-h)^2\Gamma^{\frac{q}{2}-\beta-1}|\nabla^2 v|^2 dx \\
&\leq \text{ r.-h. side of (2.16)}
\end{aligned}$$

if again the above inequality holds for β. So let us define $\beta=\frac{1}{2}(q+\mu)-1\geq 0$. Finally $X\leq a(h,r)^{\frac{\chi-1}{\chi}}$ follows from assumption (1.11) and altogether we have proved Lemma 2.6 ∎

From Lemma 2.6 we deduce as in [20: Lemma 3.7] (compare [19: Proposition 5.1]) the existence of a positive number $d\geq k_0(M)$ such that $a(d,\frac{R}{2})\tau(d,\frac{R}{2})=0$, hence $|A(d,\frac{R}{2})|=0$ and in conclusion $A(d,\frac{R}{2})=\emptyset$. This implies

$$|\nabla v|^2\leq e^d \qquad \text{on } B_{\frac{R}{2}}. \tag{2.20}$$

By construction d is bounded in terms of the quantities $\tau(0,R)$ and $a(0,R)$, thus on account of Lemma 2.4 and (2.20) we have proved the gradient bounds

$$\|\nabla v_{\varepsilon,\delta}\|_{L^\infty(B_{\frac{R}{2}})}\leq C \tag{2.21}$$

for $v=v_{\varepsilon,\delta}$ where $C=C\big(\int_{B_R} f_\delta(\nabla v_{\varepsilon,\delta})\,dx\big)$.

Step 5 (*Conclusion*). Recovering the full notation we finally choose

$$\delta=\delta(\varepsilon):=\big(1+\varepsilon^{-1}+\|\nabla u_\varepsilon\|^{2q}_{L^q(B_{2R})}\big)^{-1}$$

and set $v_\varepsilon = v_{\varepsilon,\delta(\varepsilon)}$ and $f_\varepsilon = f_{\delta(\varepsilon)}$. Using the minimality of v_ε and Jensen's inequality we have

$$\begin{aligned}\int_{B_{2R}} F(|\nabla v_\varepsilon|)\,dx &\le \int_{B_{2R}} f(\nabla v_\varepsilon)\,dx\\ &\le \int_{B_{2R}} f_\varepsilon(\nabla u_\varepsilon)\,dx \qquad (2.22)\\ &\le \int_{B_{2R}} f(\nabla u)\,dx + o(\varepsilon).\end{aligned}$$

On one hand, (2.22) proves together with (2.21) uniform gradient bounds on $B_{\frac{R}{2}}$, on the other hand we may suppose on account of (2.22) that $v_\varepsilon \rightharpoonup v$ weakly in $W^1_1(B_{2R})$ and almost everywhere in B_R, thus $\Psi_1 \le v \le \Psi_2$. Letting $\varepsilon \to 0$ and using lower semicontinuity we get

$$\int_{B_{2R}} f(\nabla v)\,dx \le \liminf_{\varepsilon\to 0} \int_{B_{2R}} f(\nabla v_\varepsilon)\,dx \le \int_{B_{2R}} f(\nabla u)\,dx$$

and the minimality of u gives

$$\int_{B_{2R}} f(\nabla u)\,dx = \int_{B_{2R}} f(\nabla v)\,dx,$$

i.e. $v = u$ by the uniqueness of minimizers. So far it is proved, via a standard covering argument, that the solution u is locally Lipschitz if so are the obstacles. Once ∇u is known to be bounded the type of growth of f becomes irrelevant and the whole theorem follows (compare again [13, 31]).

3. Examples

Starting with the nearly linear case we construct an example satisfying (1.8) - (1.10) with optimal exponents in (1.10): for $\mu > 1$ set

$$\begin{aligned}\varphi(r) &= \int_0^r\int_0^s (1+t^2)^{-\frac{\mu}{2}}\,dtds \quad (r \in \mathbb{R}_0^+)\\ \Phi(Z) &= \int_0^{|Z|}\int_0^s (1+t^2)^{-\frac{\mu}{2}}\,dtds = \varphi(|Z|) \quad (Z \in \mathbb{R}^n).\end{aligned}$$

Lemma 3.1. *The function* Φ *satisfies*

(i) $D\Phi(Z) = Z\int_0^1 (1+t^2|Z|^2)^{-\frac{\mu}{2}}\,dt$

(ii) $\frac{\partial^2\Phi}{\partial Z_\alpha \partial Z_\beta}(Z) = [\delta_{\alpha\beta} - |Z|^{-2}Z_\alpha Z_\beta]\int_0^1 (1+t^2|Z|^2)^{-\frac{\mu}{2}}\,dt + |Z|^{-2}Z_\alpha Z_\beta (1+|Z|^2)^{-\frac{\mu}{2}}$

(iii) $D^2\Phi(Z)(Y,Y) \ge \frac{1}{4}|Y|^2(1+|Z|^2)^{-\frac{\mu}{2}}$

(iv) $|D^2\Phi(Z)|\,|Z|^2 \le C\,|Z|$

for all $Z, Y \in \mathbb{R}^n$ *with a suitable constant* $C > 0$.

Proof. Using a linear transformation, the proof of statements (i) and (ii) is obvious. Moreover, statement (iii) is a consequence of statement (ii) and follows by considering the cases $|Y \cdot Z| \leq \frac{1}{2}|Y|\,|Z|$ and $|Y \cdot Z| > \frac{1}{2}|Y|\,|Z|$, respectively. We like to remark that the exponent $-\frac{\mu}{2}$ occuring on the right-hand side of statement (iii) is the best possible which can be seen by considering Y parallel to Z.

Next we are going to prove statement (iv). Observing

$$|D^2\Phi(Z)| = \sup_{|Y|=1} D^2\Phi(Z)(Y,Y) \leq 2\int_0^1 (1+t^2|Z|^2)^{-\frac{\mu}{2}}\,dt$$

we get

$$|Z|^2|D^2\Phi(Z)| \leq 2|Z|\int_0^{|Z|}(1+s^2)^{-\frac{\mu}{2}}\,ds \leq 2|Z|\int_0^{\infty}(1+s^2)^{-\frac{\mu}{2}}\,ds$$

the last integral being finite on account of $\mu > 1$ ∎

From $\varphi'(r) \leq \int_0^\infty (1+t^2)^{-\frac{\mu}{2}}\,dt < \infty$ it follows that φ is at most of linear growth, thus we have to modify our construction.

Let $q > 1$ and define $\rho(t) = (1+t^2)^{\frac{q}{2}}$. The function $\tilde{\rho}$ is given for all $n \in \mathbb{N}_0$ and $t \in [2n, 2n+2)$ by

$$\tilde{\rho}(t) = \begin{cases} \rho(t) & \text{if } 2n \leq t < 2n+1 \\ \rho(2n+1) + (t - [2n+1])\big(\rho(2n+2) - \rho(2n+1)\big) & \text{if } 2n+1 \leq t < 2n+2. \end{cases}$$

We extend $\tilde{\rho}$ to the whole line by setting $\tilde{\rho}(-t) = \tilde{\rho}(t)$ $(t \geq 0)$ and consider a mollification $(\tilde{\rho})_\varepsilon$ with some small $\varepsilon > 0$.

Lemma 3.2.

(i) $(\tilde{\rho})_\varepsilon$ *is an* N*-function, i.e. convex and additionally* $\lim_{t\to+\infty} \frac{1}{t}(\tilde{\rho})_\varepsilon(t) = +\infty$.

(ii) *Let* $g(Z) = (\tilde{\rho})_\varepsilon(|Z|)$ $(Z \in \mathbb{R}^n)$. *Then* $0 \leq D^2g(Z)(Y,Y) \leq c(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2$ *for all* $Z, Y \in \mathbb{R}^n$.

(iii) g *satisfies* $|Z|^2|D^2g(Z)| \leq c(g(Z)+1)$ *for any* $Z \in \mathbb{R}^n$ *where* $c > 0$ *denotes a constant.*

Proof. By construction we have statement (i). Now fix $\varepsilon = \frac{1}{10}$ and consider the mollification

$$(\tilde{\rho})_\varepsilon(s) = \tfrac{1}{\varepsilon}\int_{-\infty}^{+\infty} k\big(\tfrac{s-t}{\varepsilon}\big)\tilde{\rho}(t)\,dt.$$

We fix $n_0 \in \mathbb{N}$ and sketch the proofs of statements (ii) and (iii) for a given $s \in U(t_0)$, where $U(t_0)$ is some small neighbourhood of $t_0 = 2n_0+1$. To this purpose we let $a = \frac{s-t_0}{\varepsilon}$ and compute

$$(\tilde{\rho})''_\varepsilon(s) = \int_a^\infty k(y)\rho''(s-\varepsilon y)\,dy + \tfrac{k(a)}{\varepsilon}\Big(\lim_{t\downarrow t_0}\tilde{\rho}'(t) - \lim_{t\uparrow t_0}\tilde{\rho}'(t)\Big). \tag{3.1}$$

Now ρ is strictly convex implying

$$\lim_{t\downarrow t_0}\tilde{\rho}'(t) \leq \rho'(2n+2) \qquad \text{and} \qquad \lim_{t\uparrow t_0}\tilde{\rho}'(t) \geq \rho'(2n),$$

thus by (3.1) there is a constant (depending on ε) such that

$$(\tilde{\rho})''_\varepsilon(s) \leq (\rho)''_\varepsilon(s) + c\big(\rho'(2n+2) - \rho'(2n)\big) = (\rho)''_\varepsilon(s) + c\rho''(\xi) \tag{3.2}$$

for $\xi \in (2n, 2n+2)$. With (3.2) the lemma is proved by direct computations ∎

Given Lemma 3.2, we finally set

$$f(Z) = g(Z) + \Phi(Z) \qquad (Z \in \mathbb{R}^n)$$
$$F(t) = (\tilde{\rho})_\varepsilon(t) + \varphi(t) \qquad (t \in \mathbb{R}).$$

If we choose $\mu, q \in (1,2)$, then f satisfies (1.8) - (1.10), (1.7) with $s = q$, and due to the degeneracy of $D^2 g$ the lower bound in (1.10) can not be improved. Thus, if we also impose (1.12), then f is admissible in Theorem 1.1.

Suppose we are given numbers $q > p > 1$, then we replace μ by $2 - p$ and obtain completly analogous results with balancing condition $|Z|^2 |D^2\Phi(Z)| \le c(1 + |Z|^p)$. The function g remains unchanged. In particular, if we now choose p and q to satisfy $q < \frac{pn}{n-2}$, then regularity of local minimizers follows from Theorem 1.1 but can not be deduced by Theorem 1.2.

Finally, we modify our example in order to demonstrate the flexibility of condition (1.13). Suppose that we are given numbers $2 \le s \le q$ and $\mu \in \mathbb{R}$. Let $p = 2 - \mu$ and assume for simplicity that $n = 3$. Suppose further that $p < s$. We let

$$\tilde{f}(Z) = (\tilde{\rho}_s)_\varepsilon(|z_1|) + (\tilde{\rho}_q)_\varepsilon \sqrt{z_2^2 + z_3^2}$$

where $(\tilde{\rho}_s)_\varepsilon$ and $(\tilde{\rho}_q)_\varepsilon$ are defined as before Lemma 3.2 with respect to the exponents s and q. We have

$$0 \le D^2 \tilde{f}(Z)(Y,Y) \le C(1 + |Z|^2)^{\frac{q-2}{2}} |Y|^2 \tag{3.3}$$

and

$$c(1 + |Z|^2)^{\frac{s}{2}} \le \tilde{f}(Z) \le C(1 + |Z|^2)^{\frac{q}{2}} \tag{3.4}$$

for all $Z, Y \in \mathbb{R}^n$ with constants $c > 0$ and $C > 0$. Note that the exponents in (3.3) and (3.4) can not be improved. Moreover, due to the degeneracy of $D^2 \tilde{f}$, the lower bound in (3.3) is the best possible.

In the case $1 < s < 2$ the right-hand side inequality of (3.3) fails to be true. Here we modify the example by letting

$$\tilde{f}(Z) = (\tilde{\rho}_s)_\varepsilon(\Gamma), \qquad \Gamma = \left(z_1^2 + z_2^2 + |z_3|^{2(1+\gamma)}\right)^{\frac{1}{2}}$$

for some appropriate $\gamma > 0$. Then (3.3) (for some $q > s$) and the first inequality of (3.4) are valid, the second one of (3.4) holds for some $\tilde{q}$ with $s < \tilde{q} < q$.

In both examples we then set $f(Z) = \Phi(Z) + \tilde{f}(Z)$ which only in the limit case $s = q$ is of balanced type. In the case $\mu \ge 1$, Φ is of lower growth than any power $|Z|^{1+\vartheta}$ $(\vartheta > 0)$ for $\mu < 1$ we get $\Phi(Z) \le C(1 + |Z|^2)^{\frac{p}{2}}$, the exponent p being optimal. Moreover, we have inequality (iii) from Lemma 3.1, and regularity of local solutions follows if

$$q < p + s\,\tfrac{2}{3}. \tag{3.5}$$

From (3.5) it is evident in which way the parameter s improves regularity. The quantities μ and q in the above example describe the behaviour of the second derivative $D^2 f$, and as a matter of fact the upper bound for $D^2 f$ implies the corresponding upper bound for f itself. In contrast to this the lower growth order s of f is quite strong and can not be deduced from the lower bound on $D^2 f$. By incorporating s as an additional quantity in the condition for regularity we obtain better results as, for example, in [13] where regularity for the above example would follow provided that $q < p + \frac{2}{3}$, and the latter condition does not take care of the choice of s.

4. Proof of Theorem 1.2

We start by making some preliminary reductions. Let us observe that, since both Ψ_1 and Ψ_2 are of class $C^{0,1}_{loc}(\Omega)$ and since the argumentation is purely local, we may suppose without loss of generality that $\Psi_1, \Psi_2 \in W^1_\infty(\Omega)$ and that after translation $0 \le \Psi_1 \le \Psi_2$. Moreover, again without loss of generality we may suppose that

$$X := \|D\Psi_1\|^2_{L^\infty(\Omega)} + \|D\Psi_2\|^2_{L^\infty(\Omega)} < \tfrac{1}{10}. \tag{4.1}$$

This last point (4.1) may be needs some comments. Suppose that $X > 0$ (otherwise we are trivially done) and pick $\lambda := [10X]^{-1}$. We observe that u is a solution to the original problem if and only if the function $\tilde{u} = \lambda u$ is a solution to a similar obstacle problem with $f(Z), \Psi_1$ and Ψ_2 replaced by $\tilde{f}(Z) = f(\frac{Z}{\lambda}), \tilde{\Psi}_1 = \lambda\Psi_1$ and $\tilde{\Psi}_2 = \lambda\Psi_2$, respectively. Moreover, we observe that $\tilde{f}$ satisfies hypotheses (1.14) and (1.15) with different constants of ellipiticity and growth

$$\tilde{\nu} = \tilde{\nu}(\nu, X) \qquad \text{and} \qquad \tilde{L} = \tilde{L}(L, X). \tag{4.2}$$

Therefore, up to passing to $\tilde{u}$ proving our theorem for $\tilde{u}$ and going back to u, we may assume (4.1). Of course, an explicit dependence on the quantity X will not appear, the dependence will only appear through (4.2).

Adjusting the constants L and ν we finally suppose that

$$\sigma \le \tfrac{1}{10}. \tag{4.3}$$

Now we really start proving Theorem 1.2, again organizing the proof in several steps:

- approximation
- linearization
- a priori estimates and
- conclusion.

We shall keep the same notation as introduced in the proof of Theorem 1.1.

Steps 1 - 2 (Approximation and Linearization). The approximation procedure has to be refined, so let us recall the following approximation result taken from [7].

Lemma 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying* (1.14) *and* (1.15). *Then there is a family $\{f_\delta\}_{0<\delta<1}$ of C^2 functions $f_\delta : \mathbb{R}^n \to \mathbb{R}$ such that $f_\delta \to f$ uniformly on compact subsets of $\mathbb{R}^n$. Moreover, we have*

$$f_\delta(Z) \ge \Lambda^{-1}(\sigma^2 + \delta^2 + |Z|^2)^{\frac{q}{2}} + C_*^{-1}(\sigma^2 + \delta^2 + |Z|^2)^{\frac{p}{2}} \tag{4.4}$$

$$f_\delta(Z) \le C_*(\sigma^2 + \delta^2 + |Z|^2)^{\frac{q}{2}} + C_*(\sigma^2 + \delta^2 + |Z|^2)^{\frac{p}{2}} \tag{4.5}$$

$$|Df_\delta(Z)| \le C_*(\sigma^2 + \delta^2 + |Z|^2)^{\frac{q-1}{2}} + C_*(\sigma^2 + \delta^2 + |Z|^2)^{\frac{p-1}{2}} \tag{4.6}$$

$$|D^2 f_\delta(Z)| \le \Lambda(\sigma^2 + \delta^2 + |Z|^2)^{\frac{q-2}{2}} + \Lambda \tag{4.7}$$

$$D^2 f_\delta(Z)(Y,Y) \ge C_*^{-1}(\sigma^2 + \delta^2 + |Z|^2)^{\frac{p-2}{2}}|Y|^2 + \Lambda^{-1}(\sigma^2 + \delta^2 + |Z|^2)^{\frac{q-2}{2}}|Y|^2 \tag{4.8}$$

for any $Z, Y \in \mathbb{R}^n$. Here we have

$$\begin{aligned} C_* &= C_*(n,q,p,L,\nu) \quad \textit{independent of } \sigma,\delta \\ \Lambda &= \Lambda(n,p,q,L,\nu,\delta) \quad \textit{independent of } \sigma. \end{aligned}$$

Now, the approximation we are going to use follows the same ideas as in the proof of Theorem 1.1: $B_R = B_R(x_0) \Subset \Omega, u_\varepsilon, \Psi_{1,\varepsilon}, \Psi_{2,\varepsilon}$ and $v_{\varepsilon,\delta}$ have the same meaning but this time, when defininig $v_{\varepsilon,\delta}$, in (2.1) we shall use the approximating sequence $\{f_\delta\}_{0<\delta<1}$ provided by Lemma 4.1 above to regularize the energy density f, instead of the functions in (2.2). Note that, with a slight change of notation, we now consider balls B_R instead of B_{2R} as done in the previous sections. With these definitions also the linearization procedure works and we come up with the statement of Lemma 2.1. In the same manner as outlined in Section 2, it is seen that $v_{\varepsilon,\delta} \in C^{1,\alpha}(B_R) \cap W^2_{2,loc}(B_R)$. Again we drop the indexes ε and δ for a moment.

Step 3 (A priori estimates). We have the following variants of Lemma 2.4 and estimate (2.21).

Lemma 4.2. *Assume that $\delta < \frac{1}{10}$ and that*

$$\chi \begin{cases} = \frac{n}{n-2} & \textit{if } n > 2 \\ > \max\left\{\frac{p}{3p-2q}, \frac{4q-2p}{p}\right\} & \textit{if } n = 2. \end{cases}$$

Then there is a constant $\beta = \beta(n,p,q)$ and a local constant $c = c(n,p,q,L,\nu)$, both being independent of ε and δ such that for all $0 < \rho < R$

$$\sup_{B_{\frac{R}{2}}} |\nabla v| \le c \left\{ \int_{B_R} f(\nabla v)\, dx + 1 \right\}^\beta \tag{4.9}$$

$$\int_{B_\rho} |\nabla v|^{p\chi} dx \le c\, c(\rho) \left\{ \int_{B_R} f(\nabla v)\, dx + 1 \right\}^\beta . \tag{4.10}$$

Proof. Testing the linearized equation (2.4) with $\varphi = \eta^2 \partial_s \psi$ where $\eta \in C_0^\infty(B_R)$ and $\psi \in C^\infty(B_R)$, we obtain by partial integration

$$\begin{aligned} &\int_{B_R} D^2 f(\nabla v)(\partial_s \nabla v, \nabla\psi)\eta^2 dx = \\ &2\int_{B_R} Df(\nabla v)\cdot\nabla\eta\,\eta\,\partial_s\psi\, dx - \int_{B_R} g\,\eta^2\partial_s\psi\, dx - 2\int_{B_R} Df(\nabla v)\cdot\nabla\psi\,\eta\,\partial_s\eta\, dx \end{aligned} \tag{4.11}$$

and via an approximation argument this is also true for all $\psi \in W_2^1(B_R)$. Now choose $\frac{1}{10} \le \kappa \le \frac{1}{5}$ such that (recall (4.1))

$$\|\nabla\Psi_1\|^2_{L^\infty(B_R)} + \|\nabla\Psi_2\|^2_{L^\infty(B_R)} \le \frac{\kappa}{2},$$

set (compare (2.11))

$$\begin{aligned} \tilde{h}(t) &= \min\{\max[t-1,0],1\} \\ h(t) &= h_\kappa(t) = \tilde{h}(\kappa^{-1}t) \end{aligned}$$

and, finally, define (compare 2.13))

$$\Gamma = \Gamma(\nabla v) = \sigma^2 + \delta^2 + |\nabla v|^2 .$$

Now we fix $\gamma > 0$ and choose $\psi = \partial_s v \Gamma^\gamma h(\Gamma)$ in (4.11). As in Section 2, the integral on the right-hand side of (4.11) which is generated by the obstacles vanishes and we obtain (using summation with respect to $s = 1, \dots, n$)

$$\begin{aligned} I + II + III :=& \int_{B_R} D^2 f(\nabla v)(\partial_s \nabla v, \partial_s \nabla v)\Gamma^\gamma h(\Gamma)\, \eta^2 dx \\ &+ \gamma \int_{B_R} D^2 f(\nabla v)(\partial_s \nabla v, \nabla |\nabla v|^2)\Gamma^{\gamma-1} h(\Gamma)\, \partial_s v\, \eta^2 dx \\ &+ \int_{B_R} D^2 f(\nabla v)(\partial_s \nabla v, \nabla |\nabla v|^2)\Gamma^\gamma h'(\Gamma)\, \partial_s v\, \eta^2 dx \\ \le& \, c \int_{B_R} |Df(\nabla v)|\, |\nabla \eta|\, \eta\, |\nabla \psi|\, dx \\ \le& \, c \int_{B_R} \eta\, |\nabla \eta|\, |Df(\nabla v)| \Big[|\nabla^2 v|\, \Gamma^\gamma h(\Gamma) \\ &+ \gamma\, \Gamma^{\gamma-1}\, |\nabla v|\, \big|\nabla |\nabla v|^2\big| h(\Gamma) + |\nabla v|\, \big|\nabla |\nabla v|^2\big| \Gamma^\gamma h'(\Gamma) \Big] dx \\ =:& \, IV. \end{aligned} \tag{4.12}$$

Now we use the ellipticity and growth properties (4.4) - (4.8) stated in Lemma 4.1 to get

$$\begin{aligned} I \ge& \, C_*^{-1} \int_{B_R} \Gamma^{\frac{p-2}{2}+\gamma} |\nabla^2 v|^2 h(\Gamma)\, \eta^2 dx \\ II =& \, \tfrac{\gamma}{2} \int_{B_R} D^2 f(\nabla v)\big(\nabla |\nabla v|^2, \nabla |\nabla v|^2\big)\Gamma^{\gamma-1} h(\Gamma)\, \eta^2 dx \\ \ge& \, \tfrac{\gamma}{2}\, C_*^{-1} \int_{B_R} \Gamma^{\frac{p-2}{2}+\gamma-1} \big|\nabla |\nabla v|^2\big|\, h(\Gamma)\, \eta^2 dx \\ III \ge& \, \tfrac{1}{2}\, C_*^{-1} \int_{B_R} \Gamma^{\frac{p-2}{2}+\gamma} \big|\nabla |\nabla v|^2\big|\, h'(\Gamma)\, \eta^2 dx \\ IV \le& \, c\, C_* \int_{B_R} \eta\, |\nabla \eta|\, [\Gamma^{\frac{q-1}{2}} + \Gamma^{\frac{p-1}{2}}] \Big[|\nabla^2 v|\, \Gamma^\gamma h(\Gamma) \\ &+ \gamma\, \Gamma^{\gamma-1} |\nabla v|\, \big|\nabla |\nabla v|^2\big|\, h(\Gamma) + |\nabla v|\, \big|\nabla |\nabla v|^2\big|\, \Gamma^\gamma h'(\Gamma) \Big] dx. \end{aligned} \tag{4.13}$$

Thus, (4.12) and (4.13) prove the existence of a real number c, independent of ε, δ and σ such that

$$\begin{aligned} \sum_{i=1}^{3} A_i :=& \int_{B_R} \Gamma^{\frac{p-2}{2}+\gamma} |\nabla^2 v|^2 h(\Gamma)\, \eta^2 dx \\ &+ \gamma \int_{B_R} \Gamma^{\frac{p-2}{2}+\gamma-1} \big|\nabla |\nabla v|^2\big|\, h(\Gamma)\, \eta^2 dx \\ &+ \int_{B_R} \Gamma^{\frac{p-2}{2}+\gamma} \big|\nabla |\nabla v|^2\big|\, h'(\Gamma)\, \eta^2 dx \end{aligned}$$

$$
\begin{aligned}
&\leq c \int_{B_R} \eta\,|\nabla\eta|\,[\Gamma^{\frac{q-1}{2}} + \Gamma^{\frac{p-1}{2}}]\Big[|\nabla^2 v|\,\Gamma^\gamma h(\Gamma) \\
&\quad + \gamma\,\Gamma^{\gamma-1}|\nabla v|\,\big|\nabla|\nabla v|^2\big|\,h(\Gamma) + |\nabla v|\,\big|\nabla|\nabla v|^2\big|\,\Gamma^\gamma h'(\Gamma)\Big]dx \qquad (4.14)\\
&=: \sum_{j=1}^{2}\sum_{i=1}^{3} B_i^j.
\end{aligned}
$$

We start estimating B_1^j using Young's inequality and setting $\tau = q - p$:

$$
\begin{aligned}
B_1^1 &\leq \tfrac{1}{4}\int_{B_R} \Gamma^{\frac{p-2}{2}+\gamma}|\nabla^2 v|^2\,h(\Gamma)\,\eta^2\,dx + 4\int_{B_R}\Gamma^{\frac{p}{2}+\tau+\gamma}h(\Gamma)\,|\nabla\eta|^2 dx\\
&\leq \tfrac{1}{4}\,A_1 + 4\int_{B_R}\Gamma^{\frac{p}{2}+\tau+\gamma}\,|\nabla\eta|^2 dx\\
B_1^2 &\leq \tfrac{1}{4}\,A_1 + 4\int_{B_R}\Gamma^{\frac{p}{2}+\gamma}|\nabla\eta|^2 dx.
\end{aligned}
$$

Clearly, B_2^j can be handled in the same way. For B_3^j we observe

$$
\begin{aligned}
B_3^1 &\leq \int_{B_R}\Gamma^{\frac{q-1}{2}}\Gamma^\gamma\Gamma^{\frac{1}{2}}\big|\nabla|\nabla v|^2\big|\,h'(\Gamma)\,\eta\,|\nabla\eta|\,dx\\
&\leq \tfrac{1}{4}\int_{B_R}\Gamma^{\frac{p-2}{2}+\gamma}\big|\nabla|\nabla v|^2\big|^2 h'(\Gamma)\,\eta^2\,dx + 4\int_{B_R}\Gamma^{\frac{p}{2}+\tau+\gamma+1}h'(\Gamma)\,|\nabla\eta|^2 dx\\
&\leq \tfrac{1}{4}\,A_3 + 4\int_{B_R}\Gamma^{\frac{p}{2}+\tau+\gamma+1}h'(\Gamma)\,|\nabla\eta|^2 dx\\
B_3^2 &\leq \tfrac{1}{4}\,A_3 + 4\int_{B_R}\Gamma^{\frac{p}{2}+\gamma+1}h'(\Gamma)\,|\nabla\eta|^2 dx.
\end{aligned}
$$

Subtracting $\frac{1}{2}\sum A_i$ in (4.14) and then neglecting A_3 we have proved the existence of a constant $c = c(n,p,q,L,\nu)$ such that

$$
\begin{aligned}
&\int_{B_R}\Gamma^{\frac{p-2}{2}+\gamma}|\nabla^2 v|^2 h(\Gamma)\,\eta^2 dx + \gamma\int_{B_R}\Gamma^{\frac{p-2}{2}+\gamma-1}\big|\nabla|\nabla v|^2\big|\,h(\Gamma)\,\eta^2 dx \leq\\
&c(\gamma+1)\int_{B_R}\big[\Gamma^{\frac{p}{2}+\gamma}+\Gamma^{\frac{p}{2}+\tau+\gamma}\big]\,|\nabla\eta|^2 dx + c\int_{B_R}\Big[\Gamma^{\frac{p}{2}+\gamma+1}+\Gamma^{\frac{p}{2}+\tau+\gamma+1}\Big]h'(\Gamma)\,|\nabla\eta|^2 dx.
\end{aligned}
\tag{4.15}
$$

As in Section 2 the integrand of the second term on the right hand-side of (4.15) is supported on $\kappa \leq \Gamma \leq 2\kappa$, on the left-hand side we observe

$$
\Gamma^{\frac{p-2}{2}+\gamma-1}\big|\nabla|\nabla v|^2\big|^2 \leq c\Gamma^{\frac{p-2}{2}+\gamma}\,|\nabla^2 v|^2,
$$

hence there is a constant $c = c(n,p,q,L,\nu)$ such that

$$
\int_{B_R}\Gamma^{\frac{p-2}{2}+\gamma-1}\big|\nabla|\nabla v|^2\big|^2 h(\Gamma)\,\eta^2 dx \leq c\int_{B_R}\Big[\Gamma^{\frac{p}{2}+\gamma}+\Gamma^{\frac{p}{2}+\tau+\gamma}\Big]|\nabla\eta|^2 dx. \tag{4.16}
$$

Now we set

$$G(s) = 1 + \int_0^s \sqrt{t^{\frac{p-2}{2}+\gamma-1} h(t)}\, dt$$

and claim the existence of a real number $c = c(p, \kappa)$ such that for all $s \geq 0$

$$\frac{1 + s^{\frac{p}{4}+\frac{\gamma}{2}}}{c(\gamma+1)} \leq G(s) \leq c(1 + s^{\frac{p}{4}+\frac{\gamma}{2}}). \tag{4.17}$$

In fact, the second inequality follows by the elementary calculation

$$G(s) \leq 1 + \int_0^s t^{\frac{p-2}{4}+\frac{\gamma-1}{2}}\, dt = 1 + \frac{s^{\frac{p}{4}+\frac{\gamma}{2}}}{\frac{p}{4}+\frac{\gamma}{2}}.$$

For the first inequality we first consider the case $0 < s < 3\kappa$ recalling that $\kappa \leq \frac{1}{5}$:

$$G(s) \geq 1 \geq 1 + s^{\frac{p}{4}+\frac{\gamma}{2}} - \left(\tfrac{3}{5}\right)^{\frac{p}{4}} \geq \left(1 - \sqrt[4]{\tfrac{3}{5}}\right)(1 + s^{\frac{p}{4}+\frac{\gamma}{2}}).$$

For $s \geq 3\kappa$ observe that

$$\begin{aligned} G(s) &\geq 1 + \int_{2\kappa}^s t^{\frac{p-2}{4}+\frac{\gamma-1}{2}}\, dt \\ &\geq 1 + \frac{s^{\frac{p}{4}+\frac{\gamma}{2}} - (2\kappa)^{\frac{p}{4}+\frac{\gamma}{2}}}{\frac{p}{4}+\frac{\gamma}{2}} \\ &\geq 1 + \frac{s^{\frac{p}{4}+\frac{\gamma}{2}} - (\frac{2}{3}s)^{\frac{p}{4}+\frac{\gamma}{2}}}{\frac{p}{4}+\frac{\gamma}{2}} \\ &\geq \frac{1 - \sqrt[4]{\frac{2}{3}}}{\frac{p}{4}+\frac{\gamma}{2}} [\tfrac{1}{4} + s^{\frac{p}{4}+\frac{\gamma}{2}}] \\ &\geq \tfrac{c(p)}{\gamma+1} [1 + s^{\frac{p}{4}+\frac{\gamma}{2}}] \end{aligned}$$

and (4.17) is established. The left-hand side of (4.17) implies

$$c\,(\gamma+1)^{-2\chi}(1 + \Gamma^{(\frac{p}{2}+\gamma)\chi}) \leq c\,(\gamma+1)^{-2\chi}(1 + \Gamma^{\frac{p}{4}+\frac{\gamma}{2}})^{2\chi} \leq G(\Gamma)^{2\chi},$$

the right-hand side of (4.17) gives with (4.16) and Sobolev's inequality in the case $n \geq 3$

$$\begin{aligned} &\left(\int_{B_R} \eta^{2\chi} G(\Gamma)^{2\chi} dx\right)^{\frac{1}{\chi}} \\ &\leq c \int_{B_R} |\nabla(\eta\, G(\Gamma))|^2 dx \\ &\leq c \int_{B_R} |\nabla\eta|^2 G(\Gamma)^2 dx + c \int_{B_R} \eta^2 |\nabla G(\Gamma)|^2 dx \\ &\leq c \int_{B_R} |\nabla\eta^2| (1 + \Gamma^{\frac{p}{4}+\frac{\gamma}{2}})^2 dx + c \int_{B_R} \eta^2 \Gamma^{\frac{p-2}{2}+\gamma-1} h(\Gamma) \left|\nabla|\nabla v|^2\right| dx \\ &\leq c \int_{B_R} |\nabla\eta|^2 (1 + \Gamma^{\frac{p}{2}+\tau+\gamma})\, dx. \end{aligned}$$

Thus we have proved the existence of a real number $c = c(n,p,q,L,\nu)$ such that

$$\left(\int_{B_R} \eta^{2\chi}(1+\Gamma^{(\frac{p}{2}+\gamma)\chi})\,dx\right)^{\frac{1}{\chi}} \le c\,(\gamma+1)^2 \int_{B_R} |\nabla\eta|^2\Gamma^{\frac{p}{2}+\tau+\gamma}dx. \tag{4.18}$$

We observe that this is exactly inequality (4.11) of [7]. What is more, the case $\gamma = 0$ corresponds to the equation after (4.6) in [7] and directly gives (4.10) (compare [7: Proposition 4.1]). Once this is known we proceed from (4.18) with the iteration given in [7: Proposition 4.2/Step 2] to prove (4.9) and the whole lemma ∎

Step 4 (*Conclusion*). Again recovering the full notation and using the minimality of $v_{\varepsilon,\delta}$ it is known so far

$$\int_{B_R} |\nabla v_{\varepsilon,\delta}|^p dx \le C_* \int_{B_R} f_\delta(\nabla u_\varepsilon)\,dx \tag{4.19}$$

$$\sup_{B_{\frac{R}{2}}} |\nabla v_{\varepsilon,\delta}| \le c\left(1+\int_{B_R} f_\delta(\nabla u_\varepsilon)\,dx\right)^{\beta} \tag{4.20}$$

$$\int_{B_\rho} |\nabla v_{\varepsilon,\delta}|^{p\chi} dx \le c(\rho,R)\left(1+\int_{B_R} f_\delta(\nabla u_\varepsilon)\,dx\right)^{\beta}. \tag{4.21}$$

For fixed $\varepsilon > 0$ we have by construction $f_\delta \to f$ uniformly on compact sets as $\delta \to 0$, thus $f_\delta(\nabla u_\varepsilon) \to f(\nabla u_\varepsilon)$ in $L^1(B_R)$ as $\delta \to 0$. Then (4.19) and (4.21) yield a suitable subsequence such that for $\delta \to 0$

$$v_{\varepsilon,\delta} \begin{cases} \rightharpoondown w_\varepsilon & \text{in } W^1_p(B_R) \cap W^1_{p\chi,loc}(B_R) \\ \to w_\varepsilon & \text{a.e. on } B_R \end{cases} \tag{4.22}$$

where the latter convergence immediately proves that the limit w_ε respects the mollified obstacles $\Psi_{i,\varepsilon}$ $(i = 1,2)$. Now we proceed exactly as in the conclusion of Theorem 1.1 (compare (2.22)) to obtain as $\varepsilon \to 0$

$$\begin{aligned} &w_\varepsilon \rightharpoondown w \ \text{ in } W^1_p(B_R) \text{ and a.e. on } B_R \\ &\Psi_1 \le w \le \Psi_2 \ \text{ a.e. on } B_R \\ &\sup\nolimits_{B_{\frac{R}{2}}} |\nabla w| \le c\big(1+\textstyle\int_{B_R} f(\nabla u)\,dx\big)^{\beta}. \end{aligned} \tag{4.23}$$

We finally claim that for all $\varepsilon > 0$

$$\int_{B_R} f(\nabla w_\varepsilon)\,dx \le \liminf_{\delta\downarrow 0} \int_{B_R} f_\delta(\nabla v_{\varepsilon,\delta})\,dx. \tag{4.24}$$

We first note that lower semicontinuity and (4.22) give for fixed $\rho < R$

$$\int_{B_\rho} f(\nabla w_\varepsilon)\,dx \le \liminf_{\delta\downarrow 0} \int_{B_\rho} f(\nabla v_{\varepsilon,\delta})\,dx. \tag{4.25}$$

On the other hand we have

$$\begin{aligned}\int_{B_R} f_\delta(\nabla v_{\varepsilon,\delta})\,dx &\ge \int_{B_\rho} f_\delta(\nabla v_{\varepsilon,\delta})\,dx \\ &= \int_{B_\rho} f(\nabla v_{\varepsilon,\delta})\,dx + \int_{B_\rho}\big[f_\delta(\nabla v_{\varepsilon,\delta}) - f(\nabla v_{\varepsilon,\delta})\big]dx.\end{aligned} \tag{4.26}$$

Given $M > 0$, the second integral I on the right-hand side of (4.26) is estimated by

$$|I| \le \int_{B_\rho\cap[|\nabla v_{\varepsilon,\delta}|\le M]} \big|f_\delta(\nabla v_{\varepsilon,\delta}) - f(\nabla v_{\varepsilon,\delta})\big|\,dx + c\int_{B_\rho\cap[|\nabla v_{\varepsilon,\delta}|>M]} (1 + |\nabla v_{\varepsilon,\delta}|^q)\,dx,$$

where the second part II is handled in the following way: by equiintegrability (since we have (4.10), $\varepsilon > 0$ is fixed and $q < p\chi$) fix $t > 0$ and choose $M(t)$ large enough such that $II \le t$ for all $\delta > 0$. Thus $\limsup_{\delta\downarrow 0} |I| \le t$ holds true on account of uniform convergence of f_δ on compact sets. With (4.25) and (4.26) we obtain

$$\liminf_{\delta\downarrow 0} \int_{B_R} f_\delta(\nabla v_{\varepsilon,\delta})\,dx + t \ge \int_{B_\rho} f(\nabla w_\varepsilon)\,dx$$

and letting first $t \downarrow 0$ and then $\rho \uparrow R$, (4.24) is proved. At this point, arguing as for Theorem 1.1 it turns out that $u \equiv w$ so that the local boundedness of ∇u and hence the local Lipschtiz continuity of u follows.

Next we prove local Hölder continuity of ∇u. Since our arguments are purely local, we may assume that $|\nabla u| \le M < +\infty$ a.e. on Ω for some number M.

Lemma 4.3. *Under the hypotheses imposed on f stated in the second part of Theorem 1.2 the convexity condition (1.15) implies*

$$D^2 f(Z)(Y,Y) \ge 2\nu(\sigma^2 + |Z|^2)^{\frac{p-2}{2}}|Y|^2 \tag{4.27}$$

for all $Y, Z \in \mathbb{R}^n$, where in the case $p < 2$ together with $\sigma = 0$ it has to be assumed $z \ne 0$.

Proof. The proof is elementary, for example, we may follow the arguments used by Morrey [30: Proof of Theorem 4.4.3]. We briefly sketch the ideas: set

$$\Theta(t) = \int_\Omega \big(f(Z + t\nabla\varphi) - f(Z)\big)\,dx - \nu\int_\Omega h(t,x)\,t^2|\nabla\varphi|^2 dx$$

$$h(t,x) = \big(\sigma^2 + |Z|^2 + t^2|\nabla\varphi|^2\big)^{\frac{p-2}{2}}.$$

By condition (1.15), Θ reaches its minimum at $t = 0$, hence $\Theta''(0) \ge 0$ which means that

$$\int_\Omega D^2 f(Z)(\nabla\varphi,\nabla\varphi)\,dx \ge 2\nu\int_\Omega (\sigma^2 + |Z|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2 dx. \tag{4.28}$$

Next, consider $\psi \in C_0^1(\Omega)$ with $\psi \ge 0$ and $\eta \in C^1(\mathbb{R})$ such that η and η' are of class L^∞. For $\xi \in \mathbb{R}^n$ set $\varphi(x) = \eta(s\,x\cdot\xi)\,\psi(x)$ $(s > 0)$. We then have

$$\nabla\varphi(x) = \eta'(s\,x\cdot\xi)\,s\,\xi\,\psi(x) + \eta(s\,x\cdot\xi)\,\nabla\psi(x).$$

Inserting this into (4.28), dividing by s^2 and then letting $s \to \infty$ we get

$$\Big(D^2 f(Z)(\xi,\xi) - 2\nu(\sigma^2 + |Z|^2)^{\frac{p-2}{2}}|\xi|^2\Big)\liminf_{s\to\infty}\int_\Omega \psi^2(x)(\eta'(s\,x\cdot\xi))^2 dx \ge 0.$$

Using this for $\eta = \sin$ and $\eta = cos$ and adding the results we deduce (4.27) from the arbitrariness of ψ ∎

To proceed further let us first consider the case $\sigma > 0$. Setting $a(Z) = Df(Z)$ we quote [21: p. 97/Lemma 4.3] noting that $a(Z)$ is locally coercive on account of (4.27): there exists a strongly coercive vector field [21: p. 94/Definition 4.1] $\tilde{a}$ such that $\tilde{a}(Z) = a(Z)$ for $|Z| \le M$. This field is of class $C^1(\mathbb{R}^n)$ and it is easy to check (using the formula for $\tilde{a}$) that $\tilde{a}$ satisfies hypotheses (1.4) - (1.6) of [31] with $p = 2$. Observing that $\int_\Omega \tilde{a}(\nabla u) \cdot \nabla \varphi \, dx \ge 0$ for any φ with compact support such that $\Psi_1 \le u + \varphi \le \Psi_2$, Hölder continuity of ∇u follows from [31: Theorem 2.8].

In the case $\sigma = 0$ we set

$$\tilde{a}(Z) = \psi(|Z|)\, a(Z) + k\, g(|Z|)\, |Z|^{p-2}\, Z$$

with ψ, k, g exactly as in [21: p. 97]. With the help of (1.16) we deduce [31: Formula (1.5)]. Condition [31: (1.6)] trivially holds for $\tilde{a}$. Using $g' \ge 0$ together with (4.27) we get

$$\begin{aligned} \nabla \tilde{a}(Z)\, Y \cdot Y \ge 2\nu\, \psi(|Z|)\, |Z|^{p-2}\, |Y|^2 + \psi'(|Z|)\, \frac{(Z \cdot Y)}{|Z|}\, (a(Z) \cdot Y) \\ + k\, g(|Z|) \Big[|Z|^{p-2}\, |Y|^2 + (p-2)\, |Z|^{p-4}\, (Z \cdot Y)^2 \Big] \end{aligned}$$

with

$$|Z|^{p-2}\, |Y|^2 + (p-2)\, |Z|^{p-4}\, (Z \cdot Y)^2 \ge \begin{cases} |Z|^{p-2}\, |Y|^2 & \text{if } p \ge 2 \\ (p-1)|Z|^{p-2}\, |Y|^2 & \text{if } 1 < p < 2. \end{cases}$$

In the case $\psi'(|Z|) \ne 0$ we have $|Z| \in [2M, 3M]$, hence

$$\left| \psi'(|Z|)\, \frac{(Z \cdot Y)}{|Z|}\, (a(Z) \cdot Y) \right| \le c(p, M)\, |Y|^2 |Z|^{p-2}$$

and (observe that $g \ge c_0 > 0$ on $[2M, \infty]$)

$$k\, g(|Z|) \Big[|Z|^{p-2}\, |Y|^2 + (p-2)\, |Z|^{p-4}\, (Z \cdot Y)^2 \Big] \ge k\, c_0\, c(p)\, |Z|^{p-2} |Y|^2.$$

So, if we choose k large enough, we get

$$\nabla \tilde{a}(Z)\, Y \cdot Y \ge \big[\alpha\, \psi(|Z|) + \beta\, g(|Z|) \big] |Z|^{p-2}\, |Y|^2$$

with positive numbers α and β. This implies [31: (1.4)], and the proof can be finished as before ∎

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