

Boundary Layer Correctors for the Solution of Laplace Equation in a Domain with Oscillating Boundary

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Abstract. We study the asymptotic behaviour of the solution of Laplace equation in a domain with very rapidly oscillating boundary. The motivation comes from the study of a longitudinal flow in an infinite horizontal domain bounded at the bottom by a plane wall and at the top by a rugose wall. The rugose wall is a plane covered with periodic asperities which size depends on a small parameter $\varepsilon > 0$. The assumption of sharp asperities is made, that is the height of the asperities does not vanish as $\varepsilon \rightarrow 0$. We prove that, up to an exponentially decreasing error, the solution of Laplace equation can be approximated, outside a layer of width 2ε , by a non-oscillating explicit function.

Keywords: *Asymptotic behaviour, oscillating boundary, boundary layers*

AMS subject classification: 35B40, 35C20

1. Motivation

Let us consider a viscous fluid in an infinite horizontal domain limited at the bottom by a plane wall \mathcal{P} and at the top by a rough wall \mathcal{R}_ε . We assume that \mathcal{P} moves at a constant horizontal velocity $\gamma = (\gamma', 0)$ ($\gamma' \in \mathbb{R}^2$) and that \mathcal{R}_ε is at rest. The latter is assumed to consist of a plane wall covered with periodic asperities which size depends on a small parameter $\varepsilon > 0$. Let $0 < a_i < b_i < l_i$ ($i = 1, 2$). We denote

$$\begin{aligned} S &= (0, l_1) \times (0, l_2) \\ \tilde{S} &= (a_1, b_1) \times (a_2, b_2) \\ S_\varepsilon &= \varepsilon S \\ \tilde{S}_\varepsilon &= \varepsilon \tilde{S} \end{aligned}$$

Let η be a non-negative Lipschitz-continuous function on \mathbb{R}^2 , S -periodic, and let η_ε be the S_ε -periodic function defined by

$$\eta_\varepsilon(x') = \begin{cases} l_3 \left(1 + \varepsilon \eta\left(\frac{x'}{\varepsilon}\right)\right) & \text{if } x' \in S_\varepsilon \setminus \tilde{S}_\varepsilon \\ l'_3 \left(1 + \varepsilon \eta\left(\frac{x'}{\varepsilon}\right)\right) & \text{if } x' \in \tilde{S}_\varepsilon \end{cases}$$

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with $l'_3 > l_3 > 0$ and $x' = (x_1, x_2)$. Observe that η_ε has a jump on the boundary $\partial\tilde{S}_\varepsilon$ of \tilde{S}_ε ; this jump tends to $l'_3 - l_3$ as $\varepsilon \rightarrow 0$, and then the amplitude of the oscillations of η_ε is large. The domain of the flow is

$$\mathcal{O}_\varepsilon = \left\{ x = (x', x_3) \in \mathbb{R}^3 \mid x' \in \mathbb{R}^2 \text{ and } 0 < x_3 < \eta_\varepsilon(x') \right\}.$$

It is bounded at the bottom by

$$\mathcal{P} = \left\{ x = (x', x_3) \in \mathbb{R}^3 \mid x' \in \mathbb{R}^2 \text{ and } x_3 = 0 \right\}$$

and at the top by $\mathcal{R}_\varepsilon = \partial\mathcal{O}_\varepsilon \setminus \mathcal{P}$, where $\partial\mathcal{O}_\varepsilon$ denotes the boundary of \mathcal{O}_ε . The profile of the asperities is then assumed to be the graph of the function η_ε on \mathbb{R}^2 . The mean height of the asperities does not vanish as $\varepsilon \rightarrow 0$. In the case where η is identically 0 the plate \mathcal{R}_ε is comb shaped.

The velocity $v_\varepsilon = (v_{\varepsilon 1}, v_{\varepsilon 2}, v_{\varepsilon 3})$ and the pressure p_ε of the fluid satisfy the stationary Navier-Stokes equations

$$\left. \begin{aligned} -\nu\Delta v_\varepsilon + (v_\varepsilon \cdot \nabla)v_\varepsilon + \nabla p_\varepsilon &= 0 && \text{in } \mathcal{O}_\varepsilon \\ \nabla v_\varepsilon &= 0 && \text{in } \mathcal{O}_\varepsilon \\ v_\varepsilon &= 0 && \text{on } \mathcal{R}_\varepsilon \\ v_\varepsilon &= \gamma && \text{on } \mathcal{P} \end{aligned} \right\} \tag{1}$$

and they are assumed to be periodic with respect to x_1 and x_2 , with periods εl_1 and εl_2 . Here $\nu > 0$ is the viscosity. We assume that $\frac{1}{\varepsilon}$ is an integer so that η_ε , v_ε and p_ε are also periodic with respect to x_1 and x_2 , with periods l_1 and l_2 . Then \mathcal{O}_ε can be viewed as generated by periodic translations of the bounded domain

$$\tilde{\Omega}_\varepsilon = \left\{ x \in \mathbb{R}^3 \mid x' \in S \text{ and } 0 < x_3 < \eta_\varepsilon(x') \right\}.$$

We denote

$$\begin{aligned} \tilde{P} &= \left\{ x = (x', x_3) \in \mathbb{R}^3 \mid x' \in S \text{ and } x_3 = 0 \right\} \\ \tilde{L}_\varepsilon &= \left\{ x = (x', x_3) \in \mathbb{R}^3 \mid x' \in \partial S \text{ and } 0 < x_3 < \eta_\varepsilon(x') \right\} \end{aligned}$$

and $\tilde{R}_\varepsilon = \partial\tilde{\Omega}_\varepsilon \setminus (\tilde{L}_\varepsilon \cup \tilde{P})$, where ∂S is the boundary of S .

Suppose now that $\gamma = (0, g, 0)$ and that the function η is independent of x_2 , i.e. η is a Lipschitz-continuous function on $(0, l_1)$ and $\eta_\varepsilon = \eta_\varepsilon(x_1)$ with

$$\eta_\varepsilon(x_1) = \begin{cases} l_3 \left(1 + \varepsilon \eta \left(\frac{x_1}{\varepsilon} \right) \right) & \text{if } x_1 \in (0, \varepsilon l_1) \setminus (\varepsilon a_1, \varepsilon b_1) \\ l'_3 \left(1 + \varepsilon \eta \left(\frac{x_1}{\varepsilon} \right) \right) & \text{if } x_1 \in (\varepsilon a_1, \varepsilon b_1). \end{cases}$$

Then, a particular solution $(v_\varepsilon, p_\varepsilon)$ of problem (1) is in the form $v_\varepsilon = (0, u_\varepsilon, 0)$, $p_\varepsilon = 0$, provided u_ε satisfies the Laplace equation in the bi-dimensional section

$$\Omega_\varepsilon = \left\{ x = (x_1, x_3) \in \mathbb{R}^2 \mid 0 < x_1 < l_1 \text{ and } 0 < x_3 < \eta_\varepsilon(x_1) \right\},$$

i.e. $u_\varepsilon \in H^1_\varepsilon(\Omega_\varepsilon)$ and

$$\left. \begin{aligned} \Delta u_\varepsilon &= 0 && \text{in } \Omega_\varepsilon \\ u_\varepsilon &= 0 && \text{on } R_\varepsilon \\ u_\varepsilon &= g && \text{on } P \end{aligned} \right\}. \tag{2}$$

Here

$$\begin{aligned} P &= \left\{ x = (x_1, x_3) \in \mathbb{R}^2 \mid 0 < x_1 < l_1 \text{ and } x_3 = 0 \right\} \\ L &= \left\{ x = (x_1, x_3) \in \mathbb{R}^2 \mid x_1 = 0 \text{ and } 0 < x_3 < \eta(0) \right\} \\ &\cup \left\{ x = (x_1, x_3) \in \mathbb{R}^2 \mid x_1 = l_1 \text{ and } 0 < x_3 < \eta(l_1) \right\} \end{aligned}$$

and $R_\varepsilon = \partial\Omega_\varepsilon \setminus (P \cup L)$, $\partial\Omega_\varepsilon$ being the boundary of Ω_ε . Setting

$$D_\varepsilon = \left\{ x = (x_1, x_3) \in \mathbb{R}^2 \mid x_1 \in \mathbb{R} \text{ and } 0 < x_3 < \eta_\varepsilon(x_1) \right\}$$

we denote, for each $m \geq 0$,

$$H^m_{\text{per}}(\Omega_\varepsilon) = \left\{ f \in H^m_{\text{loc}}(D_\varepsilon) \mid f \in H^m(\Omega_\varepsilon), f(x_1 + l_1, x_3) = f(x_1, x_3) \text{ a.e. in } D_\varepsilon \right\}$$

endowed with the norm of $H^m(\Omega_\varepsilon)$.

Our aim is to study the asymptotic behaviour, as $\varepsilon \rightarrow 0$, of the solution u_ε of problem (2). The main difficulty is due to the fact that the amplitude of the oscillations of the boundary is large.

Problems involving rough boundaries, in the case where the frequency and the amplitude of the oscillations of the boundary are of the same order ε , have been addressed by many authors. In [2], Y. Achdou, O. Pironneau and F. Valentin consider a laminar flow over a rough wall with periodic roughness elements. Using asymptotic expansions and corresponding boundary layer correctors, the authors derive first and second order effective boundary conditions. In [1], an approximation at $O(\varepsilon^{\frac{3}{2}})$ order for the H^1 -norm is derived and analyzed for Laplace equation, using a domain decomposition argument. In [3], G. Allaire and M. Amar give a non-oscillating approximation at $O(\varepsilon^{\frac{3}{2}})$ order for the H^1 -norm for Laplace equation. In [10], W. Jäger and A. Mikelić consider a laminar viscous channel flow, with the lateral surface of the channel containing surface irregularities. The fluid satisfies a no-slip boundary condition on the rugose surface and it is supposed that a uniform pressure gradient is maintained in the longitudinal direction in the channel. So the limit flow is a Hagen-Poiseuille flow. Using the corresponding boundary layers, the authors derive a wall law which gives an approximation of the tangential drag force at order $O(\varepsilon^{\frac{3}{2}})$. For a flow governed by problem (1) in the domain \mathcal{O}_ε corresponding to $l'_3 = l_3$, it is proved that, outside a neighbourhood of the rugose zone, the flow behaves asymptotically as a Couette flow, up to an exponentially small error (see [4]). Laplace equation in a domain with very rapidly oscillating locally periodic boundary, the amplitude of the oscillations being ε and the frequency ε^α ($\alpha > 1$) is considered by G. A. Chechkin, A. Friedman, and A. L. Piatniski [7]. In this paper, the authors analyze a first order approximation in the H^1 -norm. Asymptotic limits of

boundary-value problems in oscillating domains, in the case where the amplitude of the oscillations does not vanish as $\varepsilon \rightarrow 0$, are studied in [6, 18]. Problems in domains with fragmented boundaries are treated in [11, 15]. For boundary-value problems in thick periodic junctions, see [16, 17].

In the present paper, we study the asymptotic behaviour, as $\varepsilon \rightarrow 0$, of the solution u_ε of problem (2) in the case of large amplitude. We assume that

$$\eta(y_1) > 0 \quad \forall y_1 \in (0, l_1).$$

This assumption is made for the sake of technical simplicity; the case where η vanishes over a subinterval of S can be treated with slight modifications. The Hausdorff limit of the sequence $(\overline{\Omega_\varepsilon})_{\varepsilon>0}$ is the closed set $\overline{\Omega_0}$, with $\Omega_0 = (0, l_1) \times (0, l'_3)$. We prove that, up to an exponentially decreasing error, u_ε can be approximated by a non-oscillating (that is independent of $\frac{x_1}{\varepsilon}$) explicit function for $x_3 < l_3 - \varepsilon$, and that $u_\varepsilon \approx 0$ for $x_3 > l_3 + \varepsilon$. The approximation is derived from asymptotic expansions of u_ε (for $x_3 < l_3$ and $x_3 > l_3$) connected at the interface $\{x \in \mathbb{R}^2 \mid x_3 = l_3\}$. The proof relies on a kind of de Saint-Venant estimates, that is decay properties for the solution of Laplace equation in semi-infinite domains. This result generalizes the one given in [5] which deals with the case $l'_3 = l_3$ in which the amplitude of the oscillations of η_ε vanishes as $\varepsilon \rightarrow 0$.

The paper is organized as follows. In Section 2 we establish a convergence result for the solutions of problem (2). Section 3 is devoted to the asymptotic approximation result. We first state the main result. Starting from formal asymptotic expansions of u_ε , for $x_3 < l_3$ and $x_3 > l_3$, we derive a non-oscillating explicit approximation. We then give decay estimates at infinity for the solution of Laplace equations in semi-infinite domains. We finally prove the main result.

2. Convergence result

Let u_ε be the unique solution in $H^1_{\text{per}}(\Omega_\varepsilon)$ of (2). We denote

$$\begin{aligned} \tilde{\Omega} &= (0, l_1) \times (0, l'_3(1 + \bar{\eta})) \\ \bar{\eta} &= \sup_{x_1} \eta(x_1) \\ \Omega^- &= (0, l_1) \times (0, l_3) \\ \Omega^+ &= \tilde{\Omega} \setminus \Omega^-. \end{aligned}$$

Clearly, $\Omega_\varepsilon \subset \tilde{\Omega}$ for any $\varepsilon \in (0, 1)$. Let then \tilde{u}_ε denote the extension of u_ε into $\tilde{\Omega}$ by 0 and let \tilde{u}_0 be the function defined on $\tilde{\Omega}$ by

$$\tilde{u}_0(x) = \begin{cases} (1 - \frac{x_3}{l_3})g & \text{if } x \in \Omega^- \\ 0 & \text{elsewhere.} \end{cases} \tag{3}$$

We have the following result.

Proposition 1. *The sequence $(\tilde{u}_\varepsilon)_{\varepsilon>0}$ converges in $H^1(\tilde{\Omega})$ towards \tilde{u}_0 as $\varepsilon \rightarrow 0$.*

Proof. Let $h = s(x_3)g$, with $s \in C^2(\mathbb{R})$, $s(0) = 1$ and $s(x_3) = 0$ for $x_3 > \frac{l_3}{2}$. We have

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot (\nabla u_\varepsilon - \nabla h) = 0 \tag{4}$$

whence $\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \leq C$ where C is a number independent of ε . Then, by the Poincaré-Friedrichs inequality, the sequence (\tilde{u}_ε) is bounded in $H^1(\tilde{\Omega})$. Therefore, up to the extraction of a subsequence, (\tilde{u}_ε) converges weakly in $H^1(\tilde{\Omega})$ and strongly in $L^2(\tilde{\Omega})$ to some function $v \in H^1(\tilde{\Omega})$. By the continuity of the trace from $H^1(\tilde{\Omega})$ onto $H^{\frac{1}{2}}(P)$ and the compact imbedding of $H^{\frac{1}{2}}(P)$ into $L^2(P)$, $(\tilde{u}_\varepsilon|_P)$ converges in $L^2(P)$ to $v|_P$ so that

$$v = g \quad \text{on } P. \tag{5}$$

For any $\varphi \in \mathcal{D}(\Omega^-)$ we have

$$0 = \int_{\Omega^-} \nabla \tilde{u}_\varepsilon \cdot \nabla \varphi \longrightarrow \int_{\Omega^-} \nabla v \cdot \nabla \varphi = 0,$$

hence

$$\Delta v = 0 \quad \text{in } \Omega^-. \tag{6}$$

For any $x = (x_1, x_3) \in \Omega_\varepsilon \cap \Omega^+$ we can write

$$\tilde{u}_\varepsilon(x_1, x_3) = u_\varepsilon(x) = u_\varepsilon(\tilde{x}_1, x_3) + \int_{\tilde{x}_1}^{x_1} \partial_{x_1} u_\varepsilon(t, x_3) dt = \int_{\tilde{x}_1}^{x_1} \partial_{x_1} u_\varepsilon(t, x_3) dt$$

where \tilde{x}_1 is such that $(\tilde{x}_1, x_3) \in R_\varepsilon$. Using this expression, the Cauchy-Schwarz inequality and then integrating over Ω^+ , we obtain

$$\int_{\Omega^+} |\tilde{u}_\varepsilon|^2 \leq C\varepsilon^2$$

where C is a number independent of ε . Letting $\varepsilon \rightarrow 0$ it follows that v vanishes in Ω^+ and, in particular,

$$v = 0 \quad \text{for } x_3 = l_3. \tag{7}$$

Hence, from (5) - (7) and the fact that $v \in H^1_{\text{per}}(\Omega^-)$, $v = \tilde{u}_0$ in Ω^- and then $v = \tilde{u}_0$ in $\tilde{\Omega}$.

Let us now prove the strong convergence in $H^1(\tilde{\Omega})$ which will be done by proving that

$$\|\nabla \tilde{u}_\varepsilon\|_{(L^2(\tilde{\Omega}))^3} \longrightarrow \|\nabla \tilde{u}_0\|_{(L^2(\tilde{\Omega}))^3}. \tag{8}$$

First we have from (4) and integration by parts

$$\int_{\tilde{\Omega}} |\nabla \tilde{u}_\varepsilon|^2 = - \int_{\tilde{\Omega}} \tilde{u}_\varepsilon \Delta h \longrightarrow - \int_{\tilde{\Omega}} v \Delta h.$$

Multiplying (6) by v and integrating by parts we get

$$\int_{\tilde{\Omega}} \nabla v \cdot (\nabla v - \nabla h) = 0$$

from which

$$\int_{\tilde{\Omega}} |\nabla v|^2 = - \int_{\tilde{\Omega}} v \Delta h$$

follows. Hence (8) is established ■

3. Asymptotic approximation

In the sequel our purpose will be to construct an approximation of u_ε with an exponentially decreasing error with respect to ε . Let Λ be the infinite vertical domain in \mathbb{R}^2 defined by

$$\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Gamma$$

with

$$\begin{aligned} \Lambda_1 &= \left\{ y = (y_1, y_3) \in \mathbb{R}^2 \mid 0 < y_1 < l_1 \text{ and } y_3 < 0 \right\} \\ \Lambda_2 &= \left\{ y = (y_1, y_3) \in \mathbb{R}^2 \mid a_1 < y_1 < b_1 \text{ and } y_3 > 0 \right\} \\ &\quad \cup \left\{ y = (y_1, y_3) \in \mathbb{R}^2 \mid 0 < y_1 < l_1 \text{ and } 0 < y_3 < l_3 \eta(y_1) \right\} \\ \Gamma &= \left\{ y = (y_1, y_3) \in \mathbb{R}^2 \mid 0 < y_1 < l_1 \text{ and } y_3 = 0 \right\}. \end{aligned}$$

We denote

$$\begin{aligned} \tilde{\Lambda} &= \left\{ y = (y_1, y_3) \in \mathbb{R}^2 \mid a_1 < y_1 < b_1 \text{ and } y_3 > 0 \right\} \\ \tilde{\Gamma} &= \left\{ y = (y_1, y_3) \in \mathbb{R}^2 \mid a_1 < y_1 < b_1 \text{ and } y_3 = 0 \right\}. \end{aligned}$$

Let ψ_1 and ψ_2 be the functions defined by

$$\left. \begin{aligned} \psi_1 &\in H^1_{\text{loc,per}}(\Lambda_1), \int_{\Lambda_1} |\nabla \psi_1|^2 < +\infty \\ \psi_2 &\in H^1(\Lambda_2) \end{aligned} \right\} \tag{9}$$

and

$$\left. \begin{aligned} \Delta \psi_i &= 0 && \text{in } \Lambda_i \ (i = 1, 2) \\ \psi_2 &= 0 && \text{on } \partial\Lambda_2 \setminus \Gamma \\ \psi_1 &= \psi_2 && \text{on } \Gamma \\ \partial_{y_3} \psi_1 &= 1 + \partial_{y_3} \psi_2 && \text{on } \Gamma \end{aligned} \right\}. \tag{10}$$

The existence of ψ_1 and ψ_2 can be proved by passing to the limit, as $m \rightarrow +\infty$, on the solutions of problem (10) posed in the bounded domains

$$\begin{aligned} \Lambda_1^m &= \{ y = (y_1, y_3) \in \Lambda_1 \mid y_3 > -m \} \\ \Lambda_2^m &= \{ y = (y_1, y_3) \in \Lambda_2 \mid y_3 < m \} \end{aligned}$$

and which vanish for $y_3 = -m$ and $y_3 = m$, respectively. Let then ψ be the function defined in Λ as

$$\psi|_{\Lambda_i} = \psi_i \quad (i = 1, 2) \tag{11}$$

where ψ_1 and ψ_2 are defined by (9) and (10). We call β the mean of ψ over an horizontal section of Λ_1 , i.e.

$$\beta = \beta(\delta) = \frac{1}{l_1} \int_0^{l_1} \psi_1(y_1, -\delta) dy_1 \quad (\delta > 0). \tag{12}$$

Indeed, we will see later that β does not depend on δ . We now define the function $u_{0\varepsilon}$ in Ω_ε by

$$u_{0\varepsilon} = \begin{cases} \left(1 - \frac{x_3}{l_3 + \varepsilon\beta}\right)g & \text{for } 0 < x_3 < l_3 \\ 0 & \text{for } x_3 > l_3. \end{cases} \tag{13}$$

The main result of the paper is the following one.

Theorem 1. *There exists $\theta_\varepsilon \in H^1_{\text{per}}(\Omega_\varepsilon)$ such that*

$$u_\varepsilon = u_{0\varepsilon} + \theta_\varepsilon \tag{14}$$

with, for any $\alpha \in \mathbb{N}^2$ and any x such that $|x_3 - l_3| \geq \varepsilon$,

$$|\partial^\alpha \theta_\varepsilon(x)| \leq C_{l,\alpha} \exp\left(-c_l \frac{|x_3 - l_3|}{\varepsilon}\right) \tag{15}$$

where $c_l > 0$.

The correcting term θ_ε is defined in Ω_ε by

$$\theta_\varepsilon(x) = \begin{cases} \frac{\varepsilon g}{l_3 + \varepsilon \beta} \left(\psi\left(\frac{x_1}{\varepsilon}, \frac{x_3 - l_3}{\varepsilon}\right) - \beta \right) + \zeta_\varepsilon(x) & \text{for } 0 < x_3 < l_3 \\ \frac{\varepsilon g}{l_3 + \varepsilon \beta} \psi\left(\frac{x_1}{\varepsilon}, \frac{x_3 - l_3}{\varepsilon}\right) + \zeta_\varepsilon(x) & \text{for } x_3 > l_3 \end{cases}$$

where ψ is given by (9) - (11) and β is given by (12). The residue ζ_ε is the solution in $H^1_{\text{per}}(\Omega_\varepsilon)$ of the problem

$$\left. \begin{aligned} \Delta \zeta_\varepsilon &= 0 && \text{in } \Omega_\varepsilon \\ \zeta_\varepsilon &= -\frac{\varepsilon g}{l_3 + \varepsilon \beta} \left(\psi\left(\frac{x_1}{\varepsilon}, \frac{-l_3}{\varepsilon}\right) - \beta \right) && \text{on } P \\ \zeta_\varepsilon &= -\frac{\varepsilon g}{l_3 + \varepsilon \beta} \psi\left(\frac{x_1}{\varepsilon}, \frac{l'_3 - l_3}{\varepsilon} + l'_3 \eta\left(\frac{x_1}{\varepsilon}\right)\right) && \text{for } x \in \tilde{R}_\varepsilon \\ \zeta_\varepsilon &= 0 && \text{for } x \in R_\varepsilon \setminus \tilde{R}_\varepsilon \end{aligned} \right\} \tag{17}$$

where

$$\tilde{R}_\varepsilon = \left\{ x \in \mathbb{R}^2 \mid a_1 < x_1 < b_1 \text{ and } x_3 = \eta_\varepsilon(x_1) \right\}.$$

Let us emphasize that $u_{0\varepsilon}$ and θ_ε do not belong to $H^1(\Omega_\varepsilon)$ but, due to the interface conditions in (10), $u_{0\varepsilon} + \theta_\varepsilon \in H^1_{\text{per}}(\Omega_\varepsilon)$ and $\Delta(u_{0\varepsilon} + \theta_\varepsilon) = 0$ in Ω_ε .

In the theorem and in the sequel we denote $l = (l_1, l_3)$ and the constants are indexed with parameters which they depend on. The proof of Theorem 1 will be given in the final subsection, and relies on a property of exponential decay of the function ψ . Let us remark that this result generalizes to the Laplace equation in higher space dimension.

3.1 Formal asymptotic expansion. The expression of u_ε given by (14) is derived from a formal asymptotic expansion. Let us seek an expansion of u_ε in Ω_ε in the form

$$u_\varepsilon(x_1, x_3) = \begin{cases} \left(1 - \frac{x_3}{l_3} d_\varepsilon\right) g + \frac{g}{l_3} \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \left(\psi_k\left(\frac{x_1}{\varepsilon}, \frac{x_3 - l_3}{\varepsilon}\right) - \beta_k \right) + \zeta_\varepsilon(x_1, x_3) & \text{if } 0 < x_3 < l_3 \\ \left(1 - d_\varepsilon\right) g + \frac{g}{l_3} \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \left(\psi_k\left(\frac{x_1}{\varepsilon}, \frac{x_3 - l_3}{\varepsilon}\right) - \beta_k \right) + \zeta_\varepsilon(x_1, x_3) & \text{if } x_3 > l_3 \end{cases} \tag{18}$$

with

$$d_\varepsilon = \sum_{k=0}^{\infty} \alpha_k \varepsilon^k \quad (\alpha_k \in \mathbb{R})$$

where $(\psi_k)_{k \geq 0}$ is a sequence of functions defined in the infinite domain Λ , and β_k is the mean of ψ_k over an horizontal section of Λ_1 . We are going to identify the functions ψ_k , the sequence $(\alpha_k)_{k \geq 0}$ and the residual term ζ_ε in order to obtain (14). First we impose $\Delta \zeta_\varepsilon = 0$ in Ω_ε . Thus the difference $u_\varepsilon - \zeta_\varepsilon$ given by (18) has to be harmonic, too. Its derivative with respect to x_3 must have no jump at the points where $x_3 = l_3$. As $x_3 \rightarrow l_3$, the value of the derivative is

$$\begin{aligned} -\frac{g}{l_3}d_\varepsilon + \frac{g}{l_3} \sum_{k \geq 0} \varepsilon^k \partial_{y_3} \psi_k \left(\frac{x_1}{\varepsilon}, 0 \right) & \quad \text{as } x_3 < l_3 \\ \frac{g}{l_3} \sum_{k \geq 0} \varepsilon^k \partial_{y_3} \psi_k \left(\frac{x_1}{\varepsilon}, 0 \right) & \quad \text{as } x_3 > l_3. \end{aligned}$$

We then impose

$$\partial_{y_3} \psi_k|_{y_3=0^+} = \partial_{y_3} \psi_k|_{y_3=0^-} + \alpha_k \quad (k \geq 0).$$

Therefore, the coefficients α_k temporarily assumed to be known, it suffices to choose

$$\psi_k = \alpha_k \psi \quad (k \geq 0)$$

where ψ is defined by (9) - (11). Thus $\beta_k = \alpha_k \beta$. Then, for $x_3 > l_3$, we impose the constant term of expansion (18) to vanish. Thus

$$(1 - d_\varepsilon)g - \frac{g}{l_3} \varepsilon \sum_{k=0}^{\infty} \beta_k \varepsilon^k = 0,$$

that is $(1 - d_\varepsilon)g - \frac{g}{l_3} \varepsilon \beta d_\varepsilon = 0$ from which

$$d_\varepsilon = \sum_{k=0}^{\infty} \alpha_k \varepsilon^k = \frac{l_3}{l_3 + \beta \varepsilon} \quad \text{for } \beta \varepsilon < l_3$$

and

$$\sum_{k=0}^{\infty} \varepsilon^k (\psi_k - \beta_k) = \sum_{k=0}^{\infty} \varepsilon^k (\alpha_k \psi - \alpha_k \beta) = \frac{l_3}{l_3 + \beta \varepsilon} (\psi - \beta)$$

follow. Putting these expressions together with (18) we get formally the asymptotic expansion (14).

3.2 Decay estimates. Here we give a property of exponential decay of ψ as $|y_3| \rightarrow +\infty$. We begin with the following result.

Lemma 1. *The mean β defined by (12) does not depend on δ .*

Proof. Integrating the equation $\Delta\psi_1 = 0$ with respect to y_1 ($0 < y_1 < l_1$) and taking the periodicity into account we get, for $y_3 < 0$,

$$\frac{d^2}{dy_3^2} \int_0^{l_1} \psi_1(y_1, y_3) dy_1 = 0,$$

hence

$$\int_0^{l_1} \psi_1(y_1, y_3) dy_1 = ay_3 + b$$

where a and b are constants. Since $\nabla\psi_1 \in (L^2(\Lambda_1))^3$, we have $a = 0$ which proves the result ■

Proposition 2. *Let ψ be defined by (9) – (11) and β be given by (12) with $\delta > 0$. Then:*

(i) For any $\alpha \in \mathbb{N}^2$, $0 < y_1 < l_1$ and $y_3 \leq -\delta$,

$$|\partial^\alpha(\psi - \beta)(y_1, y_3)| \leq C_{l,\delta,\alpha} \exp(c_l y_3). \tag{19}$$

(ii) For any $\alpha \in \mathbb{N}^2$, $a_1 < y_1 < b_1$ and $y_3 \geq (1 + \bar{\eta})$,

$$|\partial^\alpha\psi(y_1, y_3)| \leq C_{l,\alpha} \exp(-c_l y_3). \tag{20}$$

Here $c_l > 0$.

The proof will follow from the next two lemmas.

Lemma 2. *Let $\varphi \in H^1_{loc,per}(\Lambda_1)$ be such that $\nabla\varphi \in (L^2(\Lambda_1))^3$ and*

$$\left. \begin{aligned} \Delta\varphi &= 0 && \text{in } \Lambda_1 \\ \int_\Gamma \varphi &= 0 \end{aligned} \right\}.$$

Then, for any $\alpha \in \mathbb{N}^2$, for any $0 < y_1 < l_1$ and $y_3 \leq t < 0$,

$$|\partial^\alpha\varphi(y_1, y_3)| \leq C_{l,t,\alpha} \|\varphi\|_{L^2(\Gamma)} \exp(c_l y_3) \tag{21}$$

with $c_l > 0$.

Lemma 3. *Let $\varphi \in H^1(\tilde{\Lambda})$ be such that*

$$\left. \begin{aligned} \Delta\varphi &= 0 && \text{in } \tilde{\Lambda} \\ \varphi &= 0 && \text{on } \partial\tilde{\Lambda} \setminus \tilde{\Gamma} \end{aligned} \right\}.$$

Then, for any $\alpha \in \mathbb{N}^2$, for any $a_1 < y_1 < b_1$ and $y_3 \geq t > 0$,

$$|\partial^\alpha\varphi(y_1, y_3)| \leq C_{l,t,\alpha} \|\varphi\|_{L^2(\tilde{\Gamma})} \exp(-c_l y_3) \tag{22}$$

with $c_l > 0$.

Estimates (21) and (22) are of the so-called de Saint-Venant type. Such estimates occur in many problems (see [19: pp. 67 – 97] for elasticity and [9: pp. 260 – 262] for fluids). Lemma 2 is proved in [14: pp. 49 – 58] by means of a Tartar’s lemma. Other proofs are given in [3, 12, 13]. The proof of Lemma 3 can be done by adapting that of Lemma 2. One can also adapt the proof given for the case of Stokes equations in [8: pp. 319 – 320].

Proof of Proposition 2. Applying Lemma 2 to $\varphi(y_1, y_3) = \psi(y_1, y_3 - \frac{\delta}{2}) - \beta$ with $t = -\frac{\delta}{2}$ in Λ_1 , and taking Lemma 1 into account yields

$$|\partial^\alpha(\psi - \beta)(y_1, y_3)| \leq C_{l,\delta,\alpha} \|\psi - \beta\|_{L^2(\Gamma_1^\delta)} \exp(c_l y_3) \quad (y_3 \leq -\delta)$$

where $\Gamma_1^\delta = \{y \in \Lambda_1 : y_3 = -\frac{\delta}{2}\}$. Moreover, we have

$$|\beta| \leq \frac{1}{l_1} \int_0^{l_1} \left| \psi\left(y_1, -\frac{\delta}{2}\right) \right| dy_1 \leq C_l \|\psi\|_{L^2(\Gamma_1^\delta)}.$$

To prove (19), it then suffices to prove that

$$\|\psi\|_{L^2(\Gamma_1^\delta)} \leq C_{l,\delta}. \tag{23}$$

From the variational formulation of (10) we deduce that

$$\int_\Lambda |\nabla \psi|^2 = \int_\Gamma \psi \leq C_l \|\psi\|_{L^2(\Gamma)}.$$

Then, using the Poincaré inequality and the trace theorem we get $\|\psi\|_{L^2(\Gamma)} \leq C_l$ and $\|\nabla \psi\|_{(L^2(\Lambda))^3} \leq C_l$. Since

$$\|\psi\|_{L^2(\Gamma_1^\delta)} \leq C_{l,\delta} \|\psi\|_{H^1(\Lambda_1^\delta)} \leq C_{l,\delta} (\|\psi\|_{L^2(\Gamma)} + \|\nabla \psi\|_{(L^2(\Lambda))^3})$$

where $\Lambda_1^\delta = \{y \in \Lambda_1 : -\frac{\delta}{2} < y_3 < 0\}$ we obtain therefore (23). Employing Lemma 3 in $\tilde{\Lambda}$ to $\varphi(y_1, y_3) = \psi(y_1, y_3 + l_3 \bar{\eta})$ with $t = l_3$ and proceeding to similar computations we prove (20) ■

3.3 Proof of Theorem 1. The functions $\psi, u_{0\varepsilon}, \zeta_\varepsilon, \theta_\varepsilon$ and the real β being defined respectively by (9) - (11), (13), (17), (16) and (12), we denote

$$u_\varepsilon^* = u_{0\varepsilon} + \theta_\varepsilon.$$

By construction, and in particular due to the jump condition in (10), the function u_ε^* belongs to $H_{\text{per}}^1(\Omega_\varepsilon)$ and is harmonic in Ω_ε ; it satisfies the boundary conditions

$$u_\varepsilon^* = \begin{cases} 0 & \text{on } R_\varepsilon \\ g & \text{on } P \end{cases}$$

since $\theta_\varepsilon = 0$ on $P \cup R_\varepsilon$. Then u_ε^* is the solution in $H_{\text{per}}^1(\Omega_\varepsilon)$ of problem (2) and therefore $u_\varepsilon^* = u_\varepsilon$ which proves (14).

Let us now prove (15). From (17) and Proposition 2 we have, for any $\alpha_1 \in \mathbb{N}$,

$$|\partial^{\alpha_1} \zeta_\varepsilon(x)| \leq C_{l,\alpha_1} \exp\left(-\frac{c_l}{\varepsilon}\right) \quad \text{on } P \cup \tilde{R}_\varepsilon \tag{24}$$

with $c_l > 0$. The maximum principle then implies

$$|\zeta_\varepsilon(x)| \leq C_{l,\alpha_1} \exp\left(-\frac{c_l}{\varepsilon}\right) \quad \text{in } \Omega_\varepsilon. \quad (25)$$

Now let $\alpha = (\alpha_1, \alpha_3) \in \mathbb{N}^2$ be fixed. Letting $\delta = 1$ in Proposition 2 we have, for $x = (x_1, x_3) \in \Omega_\varepsilon$ such that $x_3 \leq l_3 - \varepsilon$,

$$\left| \partial^\alpha (\psi - \beta) \left(\frac{x_1}{\varepsilon}, \frac{x_3 - l_3}{\varepsilon} \right) \right| \leq C_{l,\alpha} \exp\left(-c_l \frac{l_3 - x_3}{\varepsilon}\right) \quad (26)$$

and for $x = (x_1, x_3) \in \Omega_\varepsilon$ such that $x_3 \geq l_3 + \varepsilon$,

$$\left| \partial^\alpha \psi \left(\frac{x_1}{\varepsilon}, \frac{x_3 - l_3}{\varepsilon} \right) \right| \leq C_{l,\alpha} \exp\left(-c_l \frac{x_3 - l_3}{\varepsilon}\right). \quad (27)$$

Then estimate (15) for $|\alpha| = 0$ follows readily from (14) and (25) - (27). Now, from the local regularizing properties of the Laplace equation and the Sobolev imbedding theorem, we have for any $x = (x_1, x_3) \in \Omega_\varepsilon$ such that $x_3 \leq l_3 - \varepsilon$,

$$|\partial^\alpha \zeta_\varepsilon(x)| \leq C_{l,\varepsilon,\alpha} \left(\|\zeta_\varepsilon\|_{L^2(S \times (0, l_3 - \frac{\varepsilon}{2}))} + \|\partial^{\alpha_1} \zeta_\varepsilon\|_{L^2(P)} \right).$$

Hence, from (24) and (25), $|\partial^\alpha \zeta_\varepsilon(x)| \leq C_{l,\varepsilon,\alpha} \exp\left(-c_l \frac{l_3 - x_3}{\varepsilon}\right)$. It can be checked that the dependence of $C_{l,\varepsilon,\alpha}$ on ε is of order $\frac{1}{\varepsilon^{|\alpha|}}$. Then we can choose new constants independent of ε to get $|\partial^\alpha \zeta_\varepsilon(x)| \leq C_{l,\alpha} \exp\left(-c_l \frac{l_3 - x_3}{\varepsilon}\right)$. Using this inequality and (26), we obtain estimate (15) for $x = (x_1, x_3) \in \Omega_\varepsilon$ such that $x_3 \leq l_3 - \varepsilon$. Arguing as before and using (27) we get estimate (15) for $x = (x_1, x_3) \in \Omega_\varepsilon$ such that $x_3 \geq l_3 + \varepsilon$ ■

Remark. In industrial applications the walls are not plane. For a flow in a domain limited by a smooth (and not plane) wall and a rugose wall, with asperities of large height, the velocity and the pressure of the fluid are not S_ε -periodic, and the study of the asymptotic behaviour is more complicated. For a longitudinal flow in a such domain, we are now studying a first order approximation for the H^1 -norm. Our result in the present paper may be considered as a first step in this direction.

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