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L_1-Norms of Exponential Sums and the Corresponding Additive Problem

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Abstract. In this note, a new estimate of L_1-norm of certain exponential sum is obtained. At the same time, we establish a sharp lower bound for the cardinality of corresponding sumsets. In some cases this lower bound gives the true order of the cardinality.

Keywords: *L_1-norms, exponential sums, cardinality of sumsets*

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1. Introduction

Throughout the text the following notation will be used:

$A \ll B$ means $|A| \leq c\,B$ with some absolute constant c
$A \ll_{a,b,...} B$ means $|A| \leq c\,B$ with some constant c depending on $a, b, ...$ only
$B \gg A$ means the same as $A \ll B$.

The problem of obtaining lower bound estimations of the L_1-norm of exponential sums is of great interest in Functional Analysis, Analytic Number Theory and other topics of Mathematics. In this connection we would like to stress the Littlewood conjecture which reads as follows:

There exists an absolute constant $c > 0$ such that for any sequence of integers $f(1) < f(2) < ... < f(n)$ the inequality

$$\int_0^1 \left| \sum_{x=1}^{n} e^{2\pi i \beta f(x)} \right| d\beta > c \log n$$

holds.

This conjecture was proved in 1981 by S. V. Konyagin in [4] and by O. C. McGehee, L. Pigno and B. Smith in [6].

A. A. Karatsuba [3] noticed that the problem of lower bound estimations of a wide class of exponential sums is closely connected with the arithmetical problem of finding upper bounds for the number of solutions of the corresponding Diophantine Equations.

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Let $f(1) < f(2) < \ldots < f(n)$ be a sequence of integers, $J = J(n)$ – the number of solutions of the equation

$$f(x) + f(y) = f(u) + f(v) \qquad (1 \le x, y, u, v \le n).$$

Theorem [3]. *For any coefficients* $\gamma(x), |\gamma(x)| = 1$, *the inequality*

$$I = I(n) := \int_0^1 \left| \sum_{x=1}^{n} \gamma(x) e^{2\pi i \beta f(x)} \right| d\beta \ge n^{\frac{3}{2}} J^{-\frac{1}{2}} \tag{1}$$

is valid.

Note that

$$I \le \left(\int_0^1 \left| \sum_{x=1}^{n} e^{2\pi i \beta f(x)} \right|^2 d\beta \right)^{\frac{1}{2}} = n^{\frac{1}{2}}.$$

S. V. Konyagin [5], using methods of combinatorial geometry, proved that if $0 < f(2) - f(1) < f(3) - f(2) < \ldots < f(n) - f(n-1)$, then $J \ll n^{\frac{5}{2}}$. It means that for this sequence $I \gg n^{\frac{1}{4}}$. A new proof of Konyagin's theorem was given in our work [2], and in the case $f(x) = [Ax^{\alpha}]$ $(A > 0, \alpha > 2)$ we obtained the bound $J \ll n^{\frac{5}{2}} + n^{4-\alpha} \log^2 n$ where $n \ge n_1(\alpha, A) > 0$.

In the present paper we obtain the following results.

Theorem 1. *Let* $f(x) = [F(x)]$ *where the real-valued function* F *is three times continuously differentiable on the segment* $[1, n]$ *with* $F'(x) > 0, F''(x) > 0$ *and* $F'''(x) < 0$. *Then*

$$\frac{n^4}{F(n)+1} \ll J \ll (F'(1)^{-1} + 1) n^{\frac{5}{2}} + \frac{n^2 \log n}{F''(n)}.$$

Note that in Theorem 1 f is not necessarily strictly convex. The lower bound estimation is easy enough. Indeed,

$$\begin{aligned} n^2 &= \int_0^1 \left(\sum_{x=1}^{n} e^{2\pi i \beta f(x)} \right)^2 \left(\sum_{\lambda=0}^{2f(n)} e^{-2\pi i \beta \lambda} \right) d\beta \\ &\le \left(\int_0^1 \left| \sum_{x=1}^{n} e^{2\pi i \beta f(x)} \right|^4 d\beta \right)^{\frac{1}{2}} \left(\int_0^1 \left| \sum_{\lambda=0}^{2f(n)} e^{-2\pi i \beta \lambda} \right|^2 d\beta \right)^{\frac{1}{2}} \\ &= \big(2f(n) + 1\big)^{\frac{1}{2}} J^{\frac{1}{2}} \end{aligned}$$

from which the desired inequality follows. The upper bound we will obtain in Sections 3 - 4.

Corollary 1. *Under the assumption of Theorem* 1 *and* $F'(1) \ge 1$ *we have*

$$I \gg \min\left(n^{\frac{1}{4}}, (nF''(n))^{\frac{1}{2}} (\log n)^{-\frac{1}{2}}\right)$$

where I *is defined as in* (1).

In the case $F(x) = Ax^{\alpha}$ $(A > 0, 1 < \alpha \le \frac{3}{2})$ Corollary 1 improves the main theorem of [2] a bit by $(\log n)^{\frac{1}{2}}$ and Theorem 1 gives

$$n^{4-\alpha} \ll_{\alpha,A} J \ll_{\alpha,A} n^{4-\alpha} \log n.$$

In fact we also have established the following theorem.

Theorem 2. *Under the assumption of Theorem 1 and $F'(1) \geq 1$, let $|2S|$ denote the cardinality of the set of all integers of the form $[F(x)] + [F(y)]$ where x and y are integers with $1 \leq x, y \leq n$. Then*

$$n^2 \ll n|2S|^{\frac{2}{3}} + \frac{n}{F''(n)} + n\Big(\frac{|2S|}{F''(n)}\Big)^{\frac{1}{2}}.$$

Theorem 2 is a particular case of the direct additive problem. On this topic we refer readers to the work of G. Elekes, M. B. Nathanson and I. Z. Rusza [1].

Corollary 2. *For a fixed $A > 0$ and $1 < \alpha \leq \frac{3}{2}$ let $|2S|$ denote the cardinality of the set of all integers of the form $[Ax^\alpha] + [Ay^\alpha]$ where x and y are integers with $1 \leq x, y \leq n$. Then*

$$n^\alpha \ll_{\alpha,A} |2S| \ll_{\alpha,A} n^\alpha.$$

Corollary 2 establishes the exact order of the cardinality of $|2S|$ for this special case.

2. Preliminary remarks

For a given strictly increasing sequence of integers $f(1), ..., f(n)$ we denote by $J_0 = J_0(n)$ the number of solutions in the positive integers of the equation

$$f(x) + f(y) = f(u) + f(v) \qquad (1 \leq x \leq y \leq n, 1 \leq u \leq v \leq n).$$

Obviously, $J \leq 4J_0$. Therefore, in order to obtain an upper bound for J it is sufficient to obtain an upper bound for J_0.

Let $s_1, s_2, ..., s_\omega$ be distinct numbers of the form $f(x) + f(y)$ $(1 \leq x \leq y \leq n)$ and denote by m_j the number of solutions of the equation $s_j = f(x)+f(y)$ $(1 \leq x \leq y \leq n)$. Without loss of generality we may suppose that

$$m_1 \geq m_2 \geq ... \geq m_\omega. \tag{2}$$

Obviously,

$$J_0 = \sum_{j \leq \omega} m_j^2 \qquad \big(1 \leq m_j \leq n, \omega \leq n^2\big) \tag{3}$$

$$J \leq 4\sum_{j \leq \omega} m_j^2 \quad \big(\textstyle\sum_{j\leq\omega} m_j = \frac{1}{2}n(n+1)\big). \tag{4}$$

Let $s_1, s_2, ..., s_k$ be those of s_j, for which $m_1 \geq ... \geq m_k \geq 2$. Then $J \leq 4\sum_{j\leq k} m_j^2 + 4n^2$. Under the conditions of Theorem 1 and $F'(1) \geq 1$ our aim is to obtain the estimate

$$m_1 + ... + m_r \leq C_1\Big(nr^{\frac{2}{3}} + \frac{n}{F''(n)} + n\Big(\frac{r}{F''(n)}\Big)^{\frac{1}{2}}\Big) \tag{5}$$

for any r with $1 \leq r \leq \omega$ where C_1 is an absolute constant. Theorems 1 and 2 will follow from (2) - (5). Obviously, in order to establish (5) it is enough to consider only those r for which $r \le k$. Therefore we will suppose that $k \ge 1$. We need the condition $F'(1) \ge 1$ in order the sequence $[F(x)]$ $(1 \le x \le n)$ being strictly increasing. The case $F'(1) < 1$ will be reduced to the previous one.

For a given l with $1 \le l < n$ we denote by $J_l = J_l(n)$ the number of solutions of the Diophantine equation

$$f(x) + f(y) = f(x+l) + f(z) \qquad \big(1 \le x \le y \le n, x+l \le z \le n\big).$$

We need the following result from [2].

Lemma. *Suppose* $\Phi_l = \Phi_l(n)$ $(1 \le l < n)$ *is a sequence of real numbers such that* $J_l \le \Phi_l$ *for all* l. *Then for any positive integer* r *with* $1 \le r \le k$ *there exist a real number* a *and positive integers* q *and* $l_1, ..., l_q, l_{q+1}$ *such that either*

$$\Phi_{l_1} \ge \tfrac{1}{2}(m_1 + ... + m_r), \quad l_1 \le 2nr(m_1 + ... + m_r)^{-1}$$

or

$$\begin{aligned} &l_1 < ... < l_q < l_{q+1} < n, \quad 0 < a \le \Phi_{l_{q+1}} \\ &\Phi_{l_1} + ... + \Phi_{l_q} + a = (m_1 - 1) + ... + (m_r - 1) \\ &l_1\Phi_{l_1} + ... + l_q\Phi_{l_q} + l_{q+1}a \le nr \end{aligned}$$

hold.

3. The case $F'(1) \ge 1$

In this case $f(1) < f(2) < ... < f(n)$ and we may apply the Lemma. In order to apply it we estimate J_l which is the number of solutions of the equation

$$[F(x)] + [F(y)] = [F(x+l)] + [F(z)] \qquad (1 \le x < x+l \le z < y \le n).$$

Let us prove that $y - z \le 3\,l$. If $y = z + l + \delta_0$, then

$$\begin{aligned} 2 + F(x+l) - F(x) &\ge F(z+l+\delta_0) - F(z) \\ &= F(z+l) + \delta_0 F'(z+l+\theta\delta_0) - F(z) \\ &\ge F(x+l) - F(x) + \delta_0 \end{aligned}$$

whence $\delta_0 \le 2$, i.e. $y - z \le 3\,l$. We may fix $z = z_0$ such that $J_l \le nJ_l'$ where J_l' is the number of solutions of the equation

$$[F(x+l)] - [F(x)] = [F(z_0+\delta)] - [F(z_0)]$$

in the variables x and δ subject to $\delta \le 3\,l$ and $x + l \le n$. Then

$$J_l \le n \sum_{0<\delta\le 3l} J_l'(\delta) \tag{6}$$

where $J_l'(\delta)$ is the number of solutions of the equation

$$[F(x+l)]-[F(x)]=[F(z_0+\delta)]-[F(z_0)]$$

but now in one variable x subject to the condition $x+l\le n$. If x_0 is the least solution of this equation, then we have

$$\big(F(x+l)-F(x)\big)-\big(F(x_0+l)-F(x_0)\big)\le 2$$

whence using $F'''(x)<0$ we have

$$l(x-x_0)F''(n)\le\int_{x_0}^{x}\int_0^l F''(\phi+\psi)\,d\phi d\psi\le 2.$$

Thus $x-x_0\le\frac{2}{lF''(n)}+1$ and therefore $J_l'(\delta)\le\frac{2}{lF''(n)}+1$. From (6) it follows that

$$J_l\le\frac{6n}{F''(n)}+3ln.$$

We apply the Lemma with $\Phi_l=6(\frac{n}{F''(n)}+ln)$. The aim is to obtain inequality (5) for any r with $1\le r\le k$. According to the Lemma two cases are possible. In the first case we have

$$6\Big(l_1n+\frac{n}{F''(n)}\Big)\ge\frac{1}{2}(m_1+...+m_r),\quad l_1\le 2nr(m_1+...+m_r)^{-1}$$

from which inequality (5) follows. Now, assume the second case holds, i.e. for a given r there exists a real number a and positive integers q and $l_1,...,l_q$ such that

$$\begin{aligned}&l_1<...<l_q<l_{q+1}<n,\quad 0<a\le 6\Big(nl_{q+1}+\frac{n}{F''(n)}\Big)\\&6\Big(nl_1+\frac{n}{F''(n)}\Big)+...+6\Big(nl_q+\frac{n}{F''(n)}\Big)+a=(m_1-1)+...+(m_r-1)\\&6\,l_1\Big(nl_1+\frac{n}{F''(n)}\Big)+...+6\,l_q\Big(nl_q+\frac{n}{F''(n)}\Big)+l_{q+1}a\le nr.\end{aligned}$$

If $(m_1-1)+...+(m_r-1)\le\frac{60n}{F''(n)}$, then inequality (5) holds. Suppose now that $(m_1-1)+...+(m_r-1)>\frac{60n}{F''(n)}$. If $a<\frac{12n}{F''(n)}$, then $a<\frac{(m_1-1)+...+(m_r-1)}{5}$ and therefore we have the system

$$\left.\begin{aligned}&l_1<...<l_q<l_{q+1}<n\\&6\Big(nl_1+\frac{n}{F''(n)}\Big)+...+6\Big(nl_q+\frac{n}{F''(n)}\Big)>\frac{1}{3}(m_1+...+m_r)\\&6\,l_1\Big(nl_1+\frac{n}{F''(n)}\Big)+...+6\,l_q\Big(nl_q+\frac{n}{F''(n)}\Big)\le nr\\&q\le(m_1+m_2+...+m_r)^{\frac12}n^{-\frac12}\end{aligned}\right\}.\qquad(7)$$

If $a > \frac{12n}{F''(n)}$, then $l_{q+1} \geq \frac{a}{12n}$ and we would get the system

$$\left.\begin{array}{l} l_1 < ... < l_q < l_{q+1} < n \\ 6\Big(nl_1 + \dfrac{n}{F''(n)}\Big) + ... + 6\Big(nl_q + \dfrac{n}{F''(n)}\Big) + a > \dfrac{1}{3}(m_1 + ... + m_r) \\ 6\,l_1\Big(nl_1 + \dfrac{n}{F''(n)}\Big) + ... + 6\,l_q\Big(nl_q + \dfrac{n}{F''(n)}\Big) + \dfrac{a^2}{12n} \leq nr \\ q \leq (m_1 + m_2 + ... + m_r)^{\frac{1}{2}} n^{-\frac{1}{2}} \end{array}\right\}. \tag{8}$$

Now note that if in (8) we take $a = 0$, then we would get system (7). Therefore, it is sufficient to obtain (5) by using system (8) for $a \geq 0$. So, let (8) be true for some $a \geq 0$. Among $l_1, ..., l_q$ some may be less than $\frac{1}{F''(n)}$. Let

$$l_1 < ... < l_t < \frac{1}{F''(n)} \leq l_{t+1} \leq ... \leq l_q < n$$

(if such t does not exist or $t = q$, then the proof is analogous). From system (8) we have

$$\left.\begin{array}{l} \dfrac{12tn}{F''(n)} + 12n(l_{t+1} + ... + l_q) + a > \dfrac{1}{3}(m_1 + ... + m_r) \\ \dfrac{6(l_1 + ... + l_t)n}{F''(n)} + 6n(l_{t+1}^2 + ... + l_q^2) + \dfrac{a^2}{12n} \leq nr \\ t < q \leq (m_1 + m_2 + ... + m_r)^{\frac{1}{2}} n^{-\frac{1}{2}} \end{array}\right\}. \tag{9}$$

Therefore

$$\frac{t}{F''(n)} + l_{t+1} + ... + l_q + \frac{a}{12n} > \frac{m_1 + ... + m_r}{36n} \tag{10}$$

$$\frac{t^2}{F''(n)} + l_{t+1}^2 + ... + l_q^2 + \frac{a^2}{72n^2} \leq \frac{r}{6}. \tag{11}$$

If $\frac{t}{F''(n)} > \frac{m_1+...+m_r}{72n}$, then from (11)

$$\frac{r}{6} \geq \frac{t^2}{F''(n)} > (m_1 + ... + m_r)^2 (72n)^{-2} F''(n)$$

follows from which we derive (5). Let now $\frac{t}{F''(n)} < \frac{m_1+...+m_r}{72n}$. Then from (10) and (11) we have

$$l_{t+1} + ... + l_q + \frac{a}{12n} > \frac{m_1 + ... + m_r}{72n}$$

$$l_{t+1}^2 + ... + l_q^2 + \Big(\frac{a}{12n}\Big)^2 \leq \frac{r}{6}.$$

Taking into account (9) we have

$$\frac{r}{6} \geq \frac{1}{q}\Big(l_{t+1} + ... + l_q + \frac{a}{12n}\Big)^2 > (m_1 + ... + m_r)^{\frac{3}{2}} n^{-\frac{3}{2}} 72^{-2}$$

and therefore $m_1 + ... + m_r \le 100nr^{\frac{2}{3}}$. Thus, estimation (5) is proved for all r with $1 \le r \le k$ and therefore for all $r \le \omega$.

Now, Theorem 2 follows from (5) by using (4) if we take $r = \omega$. In order to prove Theorem 1 we note that for any r

$$\text{either} \quad m_1 + ... + m_r \le 3C_1 nr^{\frac{2}{3}} \tag{12}$$

$$\text{or} \quad m_1 + ... + m_r \le \frac{3C_1 n}{F''(n)} \tag{13}$$

$$\text{or} \quad m_1 + ... + m_r \le 3C_1 n\Big(\frac{r}{F''(n)}\Big)^{\frac{1}{2}}. \tag{14}$$

Let B_1 be the set of those r for which (12) takes place. Then for $r \in B_1$ we have $m_r \le 3C_1 nr^{-\frac{1}{3}}$. Therefore

$$\sum_{r\in B_1} m_r^2 \le \sum_{r\in B_1, r\le n^{\frac{3}{2}}} (3C_1 nr^{-\frac{1}{3}})^2 + \sum_{r\in B_1, r> n^{\frac{3}{2}}} 3C_1 n(n^{\frac{3}{2}})^{-\frac{1}{3}} m_r \ll n^{\frac{5}{2}}.$$

Let B_2 be the set of those r for which (13) takes place. Then using $m_r \le n$ we easily get

$$\sum_{r\in B_2} m_r^2 \le \frac{3C_1 n^2}{F''(n)}.$$

At last, let B_3 be the set of those r for which (14) takes place. Then $m_r \le n(rF''(n))^{-\frac{1}{2}}$ and therefore

$$\sum_{r\in B_3} m_r^2 \le \sum_{r\le n^2} \frac{n^2}{F''(n)} r^{-1} \ll \frac{n^2 \log n}{F''(n)}.$$

Now Theorem 1 follows from (4) ∎

4. The case $F'(1) < 1$

This case we easily reduce to the previous one. Let J be the number of solutions of the equation

$$[F(x)] + [F(y)] = [F(u)] + [F(v)] \qquad (1 \le x, y, u, v \le n).$$

Then

$$F(x) + F(y) = F(u) + F(v) + 2\theta$$

where θ is some function subject to $|\theta| \le 1$. Taking $g(x) = \frac{F(x)}{F'(1)}$ we have

$$g(x) + g(y) = g(u) + g(v) + \frac{2\theta}{F'(1)}.$$

Therefore

$$-\frac{2}{F'(1)} \le \big([g(x)] + [g(y)]\big) - \big([g(u)] + [g(v)]\big) \le \frac{2}{F'(1)}.$$

If we denote by $E(k)$ the number of solutions of the equation

$$[g(x)] + [g(y)] = [g(u)] + [g(v)] + k \qquad (1 \le x, y, u, v \le n),$$

then

$$J \le \sum_{|k| \le \frac{2}{F'(1)}} E(k).$$

But $E(k) \le E(0)$. Therefore

$$J \le \Big(1 + \frac{2}{F'(1)}\Big) E(0).$$

Taking this into account and that $g'(1) \ge 1$ we obtain the desired estimation of J by estimating $E(0)$ using the previous result in Section 3.

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