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On Bernis' Interpolation Inequalities in Multiple Space Dimensions

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Abstract. In spatial dimensions $d < 6$, we derive estimates of the form

$$\int_\Omega u^{n-4}|\nabla u|^6 + \int_\Omega u^{n-2}|\nabla u|^2|D^2u|^2 \leq C\int_\Omega u^n|\nabla\Delta u|^2$$

for functions $u \in H^2(\Omega)$ with vanishing normal derivatives on the boundary $\partial\Omega$. These inequalities imply that $\int_\Omega |\nabla\Delta u^{\frac{n+2}{2}}|^2$ can be controlled by $\int_\Omega u^n|\nabla\Delta u|^2$. This observation will be a key ingredient for the proof of certain qualitative results – e.g. finite speed of propagation or occurrence of a waiting time phenomenon – for solutions to fourth order degenerate parabolic equations like the thin film equation. Our result generalizes – in a slightly modified way – estimates in one space dimension which were obtained by F. Bernis.

Keywords: *Interpolation inequalities, fourth order degenerate parabolic equations, thin films*

AMS subject classification: 35K25, 35K55, 35K65, 46B70, 76D08

1. Introduction and statement of the result

In recent years, fourth order degenerate parabolic equations arising in hydrodynamics (cf. the *thin film equation*) or in materials sciences (cf. the *Cahn-Hilliard equation*) were the subject of a number of analytical and numerical studies (cf. [1 - 4, 6 - 12, 15 - 17, 19, 20] and the references therein). A model problem is given by

$$\left.\begin{aligned} u_t + \operatorname{div}(|u|^n\nabla\Delta u) &= 0 \qquad \text{in } \mathbb{R}^+\times\Omega\subset\mathbb{R}^{d+1} \\ \frac{\partial}{\partial\nu}u = \frac{\partial}{\partial\nu}\Delta u &= 0 \qquad \text{on } \mathbb{R}^+\times\partial\Omega \end{aligned}\right\} \tag{1.1}$$

and describes the surface tension driven evolution of the height u of a thin film of viscous liquid that spreads on a horizontal surface. The qualitative behavior of solutions strongly depends on the positive real exponent n. For $0 < n < 3$, there exist so called *strong solutions* (for more details, cf. – for instance – [11]) to initial data with compact support which converge for $t \to \infty$ to a solution constant in time given by the spatial mean of initial data (see [10]). For $n \geq 3$ it is conjectured (and in space dimension $d = 1$ proven for $n \geq 4$) that the solution's support is constant with respect to time.

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In space dimension $d = 1$, many results on the qualitative behavior of strong solutions could be obtained. For instance, Bernis proved in [3, 4] that strong solutions have the property of finite speed of propagation. If initial data are sufficiently smooth, a waiting time phenomenon occurs (see [11]). Moreover, it was possible to construct solutions to measure valued initial data (cf. [9]).

However, the techniques used in the parameter range $n \in (0,2)$ differ from those applied in the range $n \in [2,3)$. In the former case, the reasoning is based on the so called *α-entropy estimate* which reads as follows:

A positive constant $C = C(\alpha, n)$ exists such that for arbitrary $t > 0$ and arbitrary $\zeta \in C^2(\Omega)$ with $\mathrm{supp}(\zeta_x) \subset\subset \Omega$

$$\begin{aligned} &\tfrac{1}{\alpha(\alpha+1)} \int_\Omega \zeta^4 u^{\alpha+1}(t) + C^{-1} \int_0^t \int_\Omega \zeta^4 \left[|(u^{\frac{\alpha+n+1}{4}})_x|^4 + |(u^{\frac{\alpha+n+1}{2}})_{xx}|^2 \right] \\ &\quad \le \tfrac{1}{\alpha(\alpha+1)} \int_\Omega \zeta^4 u_0^{\alpha+1} + C \int_0^t \int_{\{\zeta>0\}} u^{\alpha+n+1} \left(|\zeta_x|^4 + \zeta^2 |\zeta_{xx}|^2 \right). \end{aligned} \tag{1.2}$$

In contrast, when $2 \le n < 3$, weighted versions of the basic energy estimate

$$\tfrac{1}{2} \sup_{t \in \mathbb{R}^+} \int_\Omega |u_x|^2 + \int_0^\infty \int_\Omega u^n u_{xxx}^2 \le \tfrac{1}{2} \int_\Omega |(u_0)_x|^2 \tag{1.3}$$

are used.

In the multi-dimensional case, Dal Passo, Garcke and the author of this paper succeeded in deriving an analogue of the entropy estimate (see [10]). As a consequence, for $0 < n < 2$ results on finite speed of propagation, on the occurrence of a waiting time phenomenon and on the existence of solutions of problem (1.1) to measure valued initial data could be established in higher space dimensions as well (see [8, 9, 11]).

However, in the parameter range $2 \le n < 3$ – which is probably more important with respect to applications (cf. [18]) – it turned out to be much more difficult to generalize results of one spatial dimension to higher dimensions. In fact, the questions whether strong solutions have the property of finite speed of propagation or whether a waiting time phenomenon occurs, are still open.

Let us briefly discuss the reason for this. The argumentation in the one-dimensional setting strongly relies on Bernis' inequalities (see [5])

$$\int_\Omega v^{n-4} v_x^6 \le C \int_\Omega v^{n-1} |v_{xx}|^3 \tag{1.4}$$

$$\int_\Omega v^{n-1} |v_{xx}|^3 \le C \int_\Omega v^n v_{xxx}^2. \tag{1.5}$$

These inequalities hold in the parameter range $\frac{1}{2} < n < 3$ for functions $v : \Omega \to \mathbb{R}^+$ such that $v_x|_{\partial\Omega} \equiv 0$. They imply in particular an estimate of the form

$$\int_\Omega \left(v^{\frac{n+2}{2}}\right)_{xxx}^2 \le C \int_\Omega v^n v_{xxx}^2. \tag{1.6}$$

This is the crucial estimate to apply Gagliardo-Nirenberg-type methods in the course of the proofs for the aforementioned qualitative results on problem (1.1).

The multi-dimensional analogue of the energy estimate (1.3) provides an $L^1(\Omega\times\mathbb{R}^+)$-estimate on $u^n|\nabla\Delta u|^2$. For this reason, higher dimensional versions of (1.6) with right-hand side given by $\int_\Omega v^n|\nabla\Delta v|^2$ would be desirable. But unfortunately, until now nobody succeeded in proving an appropriate multi-dimensional analogue of the Bernis inequalities (1.4) - (1.5). In fact, there are some doubts that an inequality of the form

$$\int_\Omega v^{n-1}|D^2 v|^3 \leq \int_\Omega v^n|\nabla\Delta v|^2$$

holds in general.

It is the purpose of this paper to overcome these difficulties and to demonstrate that the multi-dimensional equivalent of estimate (1.6) is true. The main idea is to substitute inequalities (1.4) - (1.5) by estimates of the form

$$\int_\Omega v^{n-4}|\nabla v|^6 \leq C\int_\Omega v^{n-2}|D^2 v|^2|\nabla v|^2 \tag{1.7}$$

$$\int_\Omega v^{n-2}|D^2 v|^2|\nabla v|^2 \leq C\int_\Omega v^n|\nabla\Delta v|^2. \tag{1.8}$$

More precisely, we will prove the following theorem.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^d$ $(d<6)$ be a bounded convex domain with a smooth boundary. Assume that $u\in H^2(\Omega)$ is strictly positive and satisfies*

- $\frac{\partial}{\partial\nu}u|_{\partial\Omega}\equiv 0$
- $\int_\Omega u^n|\nabla\Delta u|^2<\infty.$

Moreover, suppose that $2-\sqrt{1-\frac{d}{8+d}}<n<3$. Then, a positive constant C which only depends on d and n exists such that

$$\begin{aligned}\int_\Omega u^{n-4}|\nabla u|^6 dx+\int_\Omega u^{n-2}|D^2u|^2|\nabla u|^2 dx+\int_{\partial\Omega}u^{n-2}|\nabla u|^2 II(\nabla u,\nabla u)\,d\Gamma \\ \leq C(n,d)\int_\Omega u^n|\nabla\Delta u|^2.\end{aligned} \tag{1.9}$$

Here, $II(\cdot,\cdot)$ denotes the second fundamental form of $\partial\Omega$.

Remark.

1. Note that for Ω convex, $II(\cdot,\cdot)$ is positive semi-definite and symmetric.

2. If $2<n<3$, the constant C can be chosen independently of the dimension d.

3. In the physically relevant dimensions $d=2,3$, the lower bounds on n are given by real numbers which are approximately equal to 1.05 or 1.106, respectively. For $d=5$, we get a lower bound smaller than 1.216.

4. It is still an open problem whether the lower bound on n given in Theorem 1.1 is optimal. For the upper bound $n<3$, however, optimality follows by similar arguments as in the case of the one-dimensional Bernis inequalities (1.4) - (1.5) (cf. [5]).

Theorem 1.1 implies the following corollary.

Corollary 1.2. *Under the assumptions of Theorem* 1.1, *positive constants* C_1, C_2 *depending only on* d *and* n *exist such that*

$$\int_\Omega |\nabla\Delta u^{\frac{n+2}{2}}|^2 \le C_1(d,n)\int_\Omega u^n|\nabla\Delta u|^2 \tag{1.10}$$

$$\|u^{\frac{n+2}{6}}\|_{C^\alpha(\Omega)} \le C_2(d,n)\Big\{\Big(\int_\Omega u^n|\nabla\Delta u|^2\Big)^{\frac16} + \Big(⨍_\Omega u\Big)^{\frac{n+2}{6}}\Big\} \tag{1.11}$$

for $0 \le \alpha < (1-\frac{d}{6})$.

Remark. Also in this case, the constants can be chosen independently of the dimension d provided $2 < n < 3$.

In the subsequent section, we will prove Theorem 1.1 first for smooth functions. In fact, we will follow two different strategies – one for the case $2 < n < 3$, the other for the case $2 - \sqrt{1-\frac{d}{8+d}} < n \le 2$. In a forthcoming work, we intend to apply these new interpolation inequalities to prove finite speed of propagation, occurrence of a waiting time phenomenon and existence of solutions to measure valued initial data in multiple space dimensions in the parameter range $2 \le n < 3$. Not only for this purpose, we state a weighted version of the inequality proven in Theorem 1.1 at the end of Section 2.

Throughout the paper, we use the standard notation for Sobolev spaces. D^2u denotes the tensor of second order derivatives of a function $u \in H^2(\Omega)$, $\vec\nu$ stands for the outer normal vector to the domain Ω. Sometimes, we write $\langle u, v\rangle$ for the Euclidean scalar product of two vectors $u, v \in \mathbb{R}^d$. To avoid clumsy notation, $|\cdot|$ always denotes the Euclidean norm on spaces $\mathbb{R}$, $\mathbb{R}^d$ or $\mathbb{R}^{d\times d}$.

2. Proof of the Interpolation Inequalities

The first ingredient for the proof of Theorem 1.1 is the following lemma.

Lemma 2.1. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain with a smooth boundary. Assume that the function* $u \in C^\infty(\bar\Omega)$ *is positive and that its normal derivative vanishes on* $\partial\Omega$. *Then the identity*

$$\begin{aligned}\int_\Omega & u^{n-2}|D^2u\nabla u|^2 \\ &= -\tfrac12\int_\Omega u^{n-2}|\nabla u|^2|D^2u|^2 - \tfrac12\int_{\partial\Omega} u^{n-2}|\nabla u|^2 II(\nabla u,\nabla u)\,d\Gamma \\ &\quad - \tfrac{n-2}{2}\int_\Omega u^{n-3}|\nabla u|^2\langle\nabla u, D^2u\nabla u\rangle - \tfrac12\int_\Omega u^{n-2}|\nabla u|^2\nabla u\nabla\Delta u\end{aligned} \tag{2.1}$$

holds.

Proof. The formula follows by integration by parts:

$$\begin{aligned}\int_\Omega u^{n-2}|D^2u\nabla u|^2 &= \tfrac{1}{4}\int_\Omega u^{n-2}\langle \nabla|\nabla u|^2, \nabla|\nabla u|^2\rangle \\ &= -\tfrac{n-2}{4}\int_\Omega u^{n-3}|\nabla u|^2\langle \nabla u, \nabla|\nabla u|^2\rangle \\ &\quad -\tfrac{1}{4}\int_\Omega u^{n-2}|\nabla u|^2\Delta|\nabla u|^2 \\ &\quad +\tfrac{1}{4}\int_{\partial\Omega} u^{n-2}|\nabla u|^2 \underbrace{\langle \nabla|\nabla u|^2, \vec{\nu}\rangle}_{=2\langle \nabla u, D^2u\cdot\vec{\nu}\rangle}\, d\Gamma.\end{aligned}$$

Observe that $\nabla u|_{\partial\Omega}$ is a tangential vector field. This allows to apply the subsequent Lemma 2.2 to obtain $(D^2u\cdot\vec{\nu})_{\|} = -d\vec{\nu}\cdot\nabla u$. As a consequence (cf. [13]),

$$\langle \nabla u, D^2u\cdot\vec{\nu}\rangle = -II(\nabla u, \nabla u).$$

In addition, we have the identity

$$\Delta|\nabla u|^2 = 2(|D^2u|^2 + \nabla u\nabla\Delta u).$$

Together, this proves identity (2.1) ∎

Lemma 2.2. *Let $\Omega\subset\mathbb{R}^d$ be a domain with piecewise smooth boundary of class $C^{0,1}$. For every vector field $\eta\in H^2(\Omega;\mathbb{R}^N)$ which is tangential on $\partial\Omega$ we have*

$$(D\eta\cdot\vec{\nu})_{\|} = -d\vec{\nu}\cdot\eta$$

a.e. on $\partial\Omega$.

Proof. It can be found in [10: Lemma B.1] ∎

Let us now establish a result in the spirit of Theorem 1.1 for smooth functions in the parameter range $2<n<3$.

Lemma 2.3. *Let $\Omega\subset\mathbb{R}^d$ be a bounded convex domain with a smooth boundary. Assume that the function $u\in C^\infty(\bar{\Omega})$ is positive and that its normal derivative vanishes on $\partial\Omega$. If $2<n<3$, the estimates*

$$\int_\Omega u^{n-4}|\nabla u|^6 \le \left(\tfrac{5}{(n-3)(n-2)}\right)^2\int_\Omega u^n|\nabla\Delta u|^2 \tag{2.2}$$

and

$$\begin{aligned}\int_\Omega u^{n-2}|\nabla u|^2|\Delta u|^2 + 8\int_\Omega u^{n-2}|\nabla u|^2|D^2u|^2 + 12\int_\Omega u^{n-2}|D^2\nabla u|^2 \\ + 8\int_{\partial\Omega} u^{n-2}|\nabla u|^2 II(\nabla u,\nabla u)\, d\Gamma \\ \le \tfrac{25}{2(n-2)(3-n)}\int_\Omega u^n|\nabla\Delta u|^2\end{aligned} \tag{2.3}$$

hold true.

Remark. Note that this result does not depend on the dimension d.

Proof of Lemma 2.3. Integration by parts yields

$$\begin{aligned}\int_\Omega u^{n-4}|\nabla u|^6 &= -\tfrac{1}{n-3}\int_\Omega u^{n-3}|\nabla u|^4\Delta u \\ &\quad -\tfrac{4}{n-3}\int_\Omega u^{n-3}|\nabla u|^2\langle\nabla u, D^2u\nabla u\rangle \\ &\quad \underbrace{+\tfrac{1}{n-3}\int_{\partial\Omega} u^{n-3}|\nabla u|^4\langle\nabla u,\vec{\nu}\rangle\,d\Gamma}_{=0} \\ &= -\tfrac{4}{n-3}\int_\Omega u^{n-3}|\nabla u|^2\langle\nabla u, D^2u\nabla u\rangle \\ &\quad +\tfrac{1}{(n-3)(n-2)}\int_\Omega u^{n-2}|\nabla u|^2|\Delta u|^2 \\ &\quad +\tfrac{1}{(n-3)(n-2)}\int_\Omega u^{n-2}|\nabla u|^2\nabla u\nabla\Delta u \\ &\quad +\tfrac{2}{(n-3)(n-2)}\int_\Omega u^{n-2}\Delta u\langle\nabla u, D^2u\nabla u\rangle \\ &\quad \underbrace{-\tfrac{1}{(n-3)(n-2)}\int_{\partial\Omega} u^{n-2}|\nabla u|^2\Delta u\langle\nabla u,\vec{\nu}\rangle\,d\Gamma}_{=0} \\ &=: \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.\end{aligned} \tag{2.4}$$

We use identity (2.1) for $\mathcal{I}_1$ and obtain

$$\begin{aligned}\int_\Omega u^{n-4}|\nabla u|^6 &= \tfrac{5}{(n-3)(n-2)}\int_\Omega u^{n-2}|\nabla u|^2\nabla u\nabla\Delta u \\ &\quad +\tfrac{2}{(n-3)(n-2)}\int_\Omega u^{n-2}\Delta u\langle\nabla u, D^2u\nabla u\rangle \\ &\quad +\tfrac{1}{(n-3)(n-2)}\int_\Omega u^{n-2}|\nabla u|^2|\Delta u|^2 \\ &\quad +\tfrac{4}{(n-3)(n-2)}\int_\Omega u^{n-2}|\nabla u|^2|D^2u|^2 \\ &\quad +\tfrac{8}{(n-3)(n-2)}\int_\Omega u^{n-2}|D^2u\nabla u|^2 \\ &\quad +\tfrac{4}{(n-3)(n-2)}\int_{\partial\Omega} u^{n-2}|\nabla u|^2 II(\nabla u,\nabla u)\,d\Gamma.\end{aligned}$$

By use of Young's inequality, we may estimate

$$\begin{aligned}\int_\Omega u^{n-4}|\nabla u|^6 \leq \tfrac{5}{(n-3)(n-2)} &\int_\Omega u^{n-2}|\nabla u|^2 \nabla u \nabla \Delta u \\ &+ \tfrac{1}{2(n-3)(n-2)} \int_\Omega u^{n-2}|\nabla u|^2|\Delta u|^2 \\ &+ \tfrac{4}{(n-3)(n-2)} \int_\Omega u^{n-2}|\nabla u|^2|D^2 u|^2 \\ &+ \tfrac{6}{(n-3)(n-2)} \int_\Omega u^{n-2}|D^2 u \nabla u|^2 \\ &+ \tfrac{4}{(n-3)(n-2)} \int_{\partial\Omega} u^{n-2}|\nabla u|^2 II(\nabla u, \nabla u)\, d\Gamma.\end{aligned} \tag{2.5}$$

By convexity of Ω, $II(\cdot,\cdot)$ is positive semi-definite, and the fact $n \in (2,3)$ entails

$$\begin{aligned}\int_\Omega u^{n-4}|\nabla u|^6 &\leq \tfrac{5}{(n-3)(n-2)} \int_\Omega u^{n-2}|\nabla u|^2 \nabla u \nabla \Delta u \\ &\leq \tfrac{5}{|(n-3)(n-2)|} \left(\int_\Omega u^{n-4}|\nabla u|^6\right)^{\frac{1}{2}} \left(\int_\Omega u^n |\nabla \Delta u|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{2.6}$$

This proves (2.2). To establish (2.3), we rewrite (2.5) as

$$\begin{aligned}\int_\Omega u^{n-2}|\nabla u|^2|\Delta u|^2 + 8\int_\Omega u^{n-2}|\nabla u|^2|D^2 u|^2 + 12\int_\Omega u^{n-2}|D^2 u\nabla u|^2 & \\ + 2|n-2|\,|n-3| \int_\Omega u^{n-4}|\nabla u|^6 & \\ + 8\int_{\partial\Omega} u^{n-2}|\nabla u|^2 II(\nabla u, \nabla u)\, d\Gamma & \\ \leq 10\left|\int_\Omega u^{n-2}|\nabla u|^2 \nabla u \nabla \Delta u\right| & \\ \leq 5\varepsilon \int_\Omega u^{n-4}|\nabla u|^6 + 5\varepsilon^{-1}\int_\Omega u^n|\nabla\Delta u|^2. &\end{aligned} \tag{2.7}$$

The choice $\varepsilon = \frac{2|n-2|\,|n-3|}{5}$ gives (2.3) ∎

Observe that the constants occurring in Lemma 2.3 blow up when n approaches two from above. Moreover, the strategy of proof fails for values of $n < 2$. In that regime, we need another method which will be presented in the proof of the following lemma. That lemma extends results in the spirit of Lemma 2.3 to values of n slightly smaller two. It reads as follows:

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain with a smooth boundary. Assume that the function $u \in C^\infty(\bar\Omega)$ is positive and that its normal derivatives vanish on $\partial\Omega$. If n satisfies*

$$2 - \sqrt{1 - \tfrac{d}{8+d}} < n < 2 + \tfrac{2}{4+\sqrt{\frac{8+d}{2}}},$$

then a positive constant C which only depends on n and d exists such that

$$\int_\Omega u^{n-4}|\nabla u|^6 dx + \int_\Omega u^{n-2}|D^2u|^2|\nabla u|^2 dx + \int_{\partial\Omega} u^{n-2}|\nabla u|^2 II(\nabla u, \nabla u)\, d\Gamma \\ \leq C(n,d)\int_\Omega u^n|\nabla\Delta u|^2. \tag{2.8}$$

Remark. Note that we do not impose further restrictions on the dimension d.

Proof of Lemma 2.4. The argument essentially consists of two steps. Let us first prove the estimate

$$\left(\int_\Omega u^{n-4}|\nabla u|^6\right)^{\frac12} \\ \leq \tfrac{\sqrt{d}}{|n-3|}\left(\int_\Omega u^{n-2}|\nabla u|^2|D^2u|^2\right)^{\frac12} + \tfrac{4}{|n-3|}\left(\int_\Omega u^{n-2}|D^2u\nabla u|^2\right)^{\frac12}. \tag{2.9}$$

Indeed,

$$\begin{aligned}\int_\Omega u^{n-4}|\nabla u|^6 &= -\tfrac{1}{n-3}\int_\Omega u^{n-3}|\nabla u|^4\Delta u \\ &\quad - \tfrac{4}{n-3}\int_\Omega u^{n-3}|\nabla u|^2\langle\nabla u, D^2u\nabla u\rangle \\ &\quad + \tfrac{1}{n-3}\underbrace{\int_{\partial\Omega} u^{n-3}|\nabla u|^4\langle\nabla u,\vec\nu\rangle\, d\Gamma}_{=0} \\ &\leq \tfrac{1}{|n-3|}\left(\int_\Omega u^{n-4}|\nabla u|^6\right)^{\frac12}\left(\int_\Omega u^{n-2}|\nabla u|^2|\Delta u|^2\right)^{\frac12} \\ &\quad + \tfrac{4}{|n-3|}\left(\int_\Omega u^{n-4}|\nabla u|^6\right)^{\frac12}\left(\int_\Omega u^{n-2}|D^2u\nabla u|^2\right)^{\frac12}.\end{aligned}$$

Together with the trace-type estimate $|\Delta u|^2 \leq d\,|D^2u|^2$, the asserted estimate (2.9) follows.

The second ingredient in the proof is an estimate of terms involving products of first and second order derivatives. Starting point is the identity (2.1):

$$\begin{aligned}\int_\Omega u^{n-2}|D^2u\nabla u|^2 &= -\tfrac12\int_\Omega u^{n-2}|\nabla u|^2|D^2u|^2 \\ &\quad - \tfrac12\int_{\partial\Omega} u^{n-2}|\nabla u|^2 II(\nabla u,\nabla u)\, d\Gamma \\ &\quad - \tfrac{n-2}{2}\int_\Omega u^{n-3}|\nabla u|^2\langle\nabla u, D^2u\nabla u\rangle \\ &\quad - \tfrac12\int_\Omega u^{n-2}|\nabla u|^2\nabla u\nabla\Delta u.\end{aligned}$$

Introducing the abbreviations

$$J_1 = \int_\Omega u^{n-2}|D^2u\nabla u|^2 \tag{2.10}$$

$$J_2 = \int_\Omega u^{n-2}|\nabla u|^2|D^2u|^2 \tag{2.11}$$

$$B = \int_{\partial\Omega} u^{n-2}|\nabla u|^2 II(\nabla u,\nabla u)\, d\Gamma, \tag{2.12}$$

we may estimate by use of Schwarz inequality

$$\begin{aligned} &J_1 + \tfrac{1}{2}J_2 + \tfrac{1}{2}B \\ &\quad\le \tfrac{|n-2|}{2}\sqrt{J_1}\Big(\int_\Omega u^{n-4}|\nabla u|^6\Big)^{\frac{1}{2}} + \tfrac{1}{2}\Big(\int_\Omega u^{n-4}|\nabla u|^6\Big)^{\frac{1}{2}}\Big(\int_\Omega u^n|\nabla\Delta u|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Inserting estimate (2.9) and using Young's inequality yields

$$\begin{aligned} 2J_1 + J_2 + B &\le \tfrac{|n-2|}{|n-3|}\big(\sqrt{d}\sqrt{J_2} + 4\sqrt{J_1}\big)\sqrt{J_1} \\ &\qquad + \varepsilon_1\big(\sqrt{d}\sqrt{J_2} + 4\sqrt{J_1}\big)^2 + C_{\varepsilon_1}\int_\Omega u^n|\nabla\Delta u|^2 \\ &\le \tfrac{|n-2|}{|n-3|}\Big(\big(\tfrac{\sqrt{d}}{2\varepsilon_2} + 4\big)J_1 + \tfrac{\varepsilon_2\sqrt{d}}{2}J_2\Big) \\ &\qquad + \varepsilon_1\big(\sqrt{d}\sqrt{J_2} + 4\sqrt{J_1}\big)^2 + C_{\varepsilon_1}\int_\Omega u^n|\nabla\Delta u|^2. \end{aligned} \tag{2.13}$$

To absorb the terms on the right-hand side in an optimal way in the terms on the left-hand side, we choose ε_2 as the unique positive solution of the equation $8 + \sqrt{d}\varepsilon_2^{-1} = 2\sqrt{d}\varepsilon_2$. A straightforward calculation gives $\varepsilon_2 = \frac{2}{\sqrt{d}} + \sqrt{\frac{8+d}{2d}}$. If

$$\sqrt{d}\,\varepsilon_2\tfrac{|n-2|}{|n-3|} < 2, \tag{2.14}$$

the first term on the right-hand side can be absorbed on the left-hand side. An appropriate choice of ε_1 allows to absorb the second term on the right-hand side, too. Summing up, we obtain

$$J_1 + J_2 + B \le C(d,n)\int_\Omega u^n|\nabla\Delta u|^2,$$

and together with (2.9) the assertion of the lemma follows provided (d,n) satisfies condition (2.14). Another calculation shows that this is the case as long as $2 - \sqrt{1 - \frac{d}{8+d}} < n < 2 + \frac{2}{4+\sqrt{\frac{8+d}{2}}}$ ∎

Now, the proof of Theorem 1.1 follows by an approximation argument. It will be sketched in what follows.

Proof of Theorem 1.1. Let $u \in H^2(\Omega)$ be given satisfying the assumptions of the theorem. We may assume that $u \geq \delta > 0$ almost everywhere in Ω. Therefore, $\int_\Omega |\nabla\Delta u|^2 < \infty$. Let us construct a sequence $(u_k)_{k\in\mathbb{N}} \subset C^\infty(\bar{\Omega})$ with the following properties:

- $u_k \geq \frac{\delta}{2}$ for $k \in \mathbb{N}$, k sufficiently large
- $\frac{\partial}{\partial\nu}u_k = 0$ on $\partial\Omega$
- $u_k \to u$ strongly in $H^2(\Omega)$ for $k \to \infty$
- $\nabla\Delta u_k \to \nabla\Delta u$ strongly in $L^2(\Omega)$ for $k \to \infty$.

For this purpose, we consider a sequence $(f_k)_{k\in\mathbb{N}} \subset C^\infty(\bar{\Omega})$ (such a sequence exists due to the boundary regularity of Ω, cf. [14]) such that

- $\int_\Omega f_k = 0$ for all $k \in \mathbb{N}$
- $f_k \to -\Delta u$ strongly in $H^1(\Omega)$ for $k \to \infty$.

Associated with $(f_k)_{k\in\mathbb{N}}$ are functions $(u_k)_{k\in\mathbb{N}}$ which solve the Neumann problem

$$\left.\begin{aligned} -\Delta u_k &= f_k \quad \text{in } \Omega \\ \frac{\partial}{\partial\nu}u_k &= 0 \quad \text{on } \partial\Omega \\ \textstyle\int_\Omega u_k &= \textstyle\int_\Omega u \end{aligned}\right\}.$$

By elliptic regularity theory, it follows that $(u_k)_{k\in\mathbb{N}} \subset C^\infty(\bar{\Omega})$ strongly converges to u in $W^{2,p}(\Omega)$ for $1 \leq p < \frac{2N}{N-2}$. By Sobolev's imbedding theorem, $(u_k)_{k\in\mathbb{N}}$ strongly converges to u in $C^0(\Omega)$ for $N < 6$. Hence, the functions u_k have the aforementioned properties for indices $k \in \mathbb{N}$ sufficiently large. Moreover, they satisfy for k sufficiently large the inequalities (2.2), (2.3) or (2.8), respectively. From Vitali's theorem, we infer that

$$\lim_{k\to\infty}\int_\Omega u_k^n|\nabla\Delta u_k|^2 = \int_\Omega u^n|\nabla\Delta u|^2.$$

Together with Fatou's lemma, the assertion follows ∎

It remains to sketch the proof of Corollary 1.2.

Proof of Corollary 1.2. Inequality (1.10) is a direct consequence of inequality (1.9) and the identity

$$\begin{aligned}\nabla\Delta u^{s+1} = s(s-1)(s+1)u^{s-2}|\nabla u|^2\nabla u + 2s(s+1)u^{s-1}D^2u\nabla u \\ + s(s+1)u^{s-1}\nabla u\Delta u + (s+1)u^s\nabla\Delta u\end{aligned}$$

which holds for all $s \in \mathbb{R}$. Finally, inequality (1.11) follows by the estimate

$$\begin{aligned}\int_\Omega |\nabla u^{\frac{n+2}{6}}|^6 &= C(n)\int_\Omega u^{n-4}|\nabla u|^6 \\ &\leq C(d,n)\int_\Omega u^n|\nabla\Delta u|^2 \end{aligned}\tag{2.15}$$

and Sobolev's imbedding theorem ∎

Eventually, let us mention the following weighted version of Theorem 1.1 which will be of importance with respect to the applications mentioned in the introduction.

Theorem 2.5. *Suppose in addition to the assumptions of Theorem* 1.1 *that* $\phi \in C^1(\bar{\Omega})$ *is non-negative. Then, there exists a positive constant* $C = C(d,n)$ *such that for values of* n *contained in* $(2 - \sqrt{1 - \frac{d}{8+d}}, 3)$ *the estimate*

$$\int_\Omega \phi^6 u^{n-4} |\nabla u|^6 dx + \int_\Omega \phi^6 u^{n-2} |D^2 u|^2 |\nabla u|^2 dx + \int_{\partial\Omega} \phi^6 u^{n-2} |\nabla u|^2 II(\nabla u, \nabla u)\, d\Gamma \\ \leq C(n,d) \Big\{ \int_\Omega \phi^6 u^n |\nabla \Delta u|^2 + \int_{[\phi>0]} u^{n+2} |\nabla \phi|^6 \Big\}$$

holds.

The proof requires only minor modifications of the proof for Theorem 1.1. Therefore we omit it here.

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