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Iteration Procedures of Shuttle Iteration Type in Continuous Non-Monotone Problems

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Abstract. We suggest and study iteration procedures converging from below and above to robust stable solutions and to robust stable continuous branches of solutions for quasilinear boundary-value problems with continuous non-monotone non-linearities. The iterations are constructed by modifications of the shuttle iteration method, which is used in problems with monotone operators leaving invariant a cone interval.

Keywords: *Cone semiordering, monotone operator, robust stable solution, continuous branch of solutions, second order elliptic operator*

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1. Introduction

Consider the equation $x = Tx$ in a Banach space with cone semiordering [4, 11]. If the operator T is monotone and maps some cone interval into itself, then in general situations this equation has a solution in the cone interval. Moreover, under natural conditions, there are solutions possessing important additional properties: robust stable solutions of equations with continuous operators, regular solutions of equations with discontinuous operators (a solution is called regular if it is a robust stable continuity point of the discontinuous operator T), robust stable continuous curves and continuous branches of solutions. The existence of such solutions is proved in [5, 8, 9, 14, 15] (see also references therein). Some other methods and results can be found, e.g., in [13].

In [8], the authors suggested a special iteration procedure called the *shuttle iteration method* to construct robust stable and regular solutions of equations with monotone continuous and discontinuous operators. The iteration procedure converges to such solutions and contains their upper and lower estimates.

In this paper, we consider problems with continuous, but non-monotone operators. The problem can be reduced to the Hammerstein type equation $x = B\hat{f}x$ with linear positive compact operator B and nonlinear non-monotone superposition operator $\hat{f}$. By modification of the shuttle iteration method, we construct iteration procedures that

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converge from below and above either to a robust stable solution of the problem, or to a robust stable continuum of solutions.

We use the iterations

$$x_{n+1} = T_n x_n := (I + \alpha_n B)^{-1} B(\hat{f}_n + \alpha_n I) x_n,$$

where $\hat{f}_n$ are Lipschitz continuous approximations of the non-linearity $\hat{f}$. The numbers α_n are sufficiently large (generally speaking, $\alpha_n \to \infty$), the operators T_n are monotone for all n.

2. Problem statement

Consider a bounded closed domain Ω in the space $\mathbb{R}^N$ $(N \geq 1)$ and the differential expression

$$Lx = \sum_{i,j=1}^{N} a_{ij}(t) \frac{\partial^2 x}{\partial t_i \partial t_j} + \sum_{i=1}^{N} b_i(t) \frac{\partial x}{\partial t_i} + c(t)x \tag{2.1}$$

where $t = \{t_1, \dots, t_N\} \in \Omega$ and the coefficients a_{ij} satisfy the ellipticity condition

$$-\sum_{i,j=1}^{N} a_{ij}(t) \xi_i \xi_j \geq a \sum_{i=1}^{N} \xi_i^2 \qquad (a > 0,\, t \in \Omega).$$

Let us use the same notation L for the linear differential operator defined by differential expression (2.1) and the zero boundary condition

$$x(t) = 0 \qquad (t \in \partial\Omega). \tag{2.2}$$

This operator is considered in the space $C(\Omega)$. The coefficients a_{ij}, b_i, c and the boundary $\partial\Omega$ of the domain Ω are supposed to be sufficiently smooth, which guarantees the following classical properties of the operator L (see, e.g., [1, 12]).

(i) *The resolvent set of the operator L contains an interval $(-\infty, -\alpha_0)$. Every operator $A[\alpha] = (L + \alpha I)^{-1}$ with $\alpha > \alpha_0$ (here I is the identity) is completely continuous in the space $C(\Omega)$ and positive with respect to the semiordering generated by the cone* [1)]

$$K_+ = \Big\{ x \in C(\Omega) : x(t) \geq 0 \textit{ for all } t \in \Omega \Big\}$$

of non-negative functions, i.e. $A[\alpha]x \in K_+$ for every $x \in K_+$.

In this paper, we study the quasilinear boundary-value problem

$$Lx = f(t, x) \tag{2.3}$$

[1)] The general theory of Banach spaces with cone semiordering can be found in [4, 11]. We only use some notation and simple facts.

where the nonlinear function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to the set of its arguments. By property (i), problem (2.3) is equivalent to the equation

$$x = F[\alpha]x := A[\alpha](\hat{f}x + \alpha x) \qquad (x \in C(\Omega)) \tag{2.4}$$

for every $\alpha > \alpha_0$, where $\hat{f} : C(\Omega) \to C(\Omega)$ is the superposition operator $(\hat{f}x)(t) = f(t, x(t))$ generated by the function f. Everywhere it is assumed that the non-linearity f satisfies the estimates

$$h(t,x) < f(t,x) < g(t,x) \qquad (t \in \Omega,\ x \in \mathbb{R}) \tag{2.5}$$

and the problems

$$\begin{aligned} Lx &= h(t,x) \\ Lx &= g(t,x) \end{aligned} \tag{2.6}$$

have solutions $x_-, x_+ \in C(\Omega)$ such that

$$x_-(t) \le x_+(t) \qquad (t \in \Omega). \tag{2.7}$$

The functions h and g are supposed to be continuous with respect to the set of their arguments; the existence of solutions (2.7) is discussed in Section 6.

Under the assumptions above, property (i) implies that for every sufficiently large $\alpha > \alpha_0$ the operator $F[\alpha]$ maps the convex closed set [2)]

$$\langle x_-, x_+ \rangle = \Big\{ x \in C(\Omega) : x_-(t) \le x(t) \le x_+(t) \text{ for all } t \in \Omega \Big\}$$

into itself (the details are in the next section). By the Schauder principle, there is at least one solution of equation (2.4) or, which is the same, of problem (2.3) in $\langle x_-, x_+ \rangle$.

We are interested in solutions possessing additional properties.

A solution $x_* = x_*(t)$ of (2.3) is called *robust stable* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that any problem

$$Lx = f_1(t,x) \tag{2.8}$$

with a continuous non-linearity f_1 satisfying

$$|f_1(t,x) - f(t,x)| < \delta \qquad (t \in \Omega,\ x \in \mathbb{R}) \tag{2.9}$$

has at least one solution $\tilde{x}_*$ satisfying $\|\tilde{x}_* - x_*\|_C < \varepsilon$. In other words, problem (2.8) has a solution in arbitrary small vicinity of the robust stable solution of problem (2.3) whenever perturbation of the nonlinearity is uniformly sufficiently small [3)]. In particular, any isolated solution of (2.3) with a non-zero topological index [10] is robust stable.

In the Sections 3 - 5 we construct and study a number of iteration procedures that converge either a to a robust stable solution of problem (2.3), or to a robust stable continuum of solutions (see the exact formulations below). Section 6 contains some discussion of the results and examples. Further, Sections 7 - 8 contain the proofs.

[2)] This set is called a *cone interval.*

[3)] We consider perturbations of the non-linearity only. Also, it is possible to consider perturbations both of the linear and nonlinear parts of the problem.

3. Existence of robust stable solutions

Set $\rho = \max\{\|x_-\|_C, \|x_+\|_C\} + 1$. We use a fixed increasing sequence of functions h_n,

$$h(t,x) \le h_1(t,x) < h_2(t,x) < \ldots < h_n(t,x) < \ldots \qquad (t \in \Omega,\ |x| \le \rho) \tag{3.1}$$

and a decreasing sequence of functions g_n,

$$g(t,x) \ge g_1(t,x) > g_2(t,x) > \ldots > g_n(t,x) > \ldots \qquad (t \in \Omega,\ |x| \le \rho) \tag{3.2}$$

that converge uniformly to the function f from below and above:

$$\lim_{n\to\infty} \max_{t\in\Omega,|x|\le\rho} |h_n(t,x) - f(t,x)| = \lim_{n\to\infty} \max_{t\in\Omega,|x|\le\rho} |g_n(t,x) - f(t,x)| = 0. \tag{3.3}$$

All the functions g_n, h_n are continuous in t and Lipschitz continuous in x:

$$\begin{aligned} |h_n(t,x) - h_n(t,y)| &\le \alpha_n |x-y| \\ |g_n(t,x) - g_n(t,y)| &\le \beta_n |x-y| \end{aligned} \qquad (t \in \Omega;\ |x|, |y| \le \rho). \tag{3.4}$$

Evidently, sequences (3.1) and (3.2) can be constructed for any estimates (2.5) (possibly, it should be that $\alpha_n, \beta_n \to \infty$). The functions g, g_n, h, h_n generate continuous bounded superposition operators $\hat g, \hat g_n, \hat h, \hat h_n$ in the space $C(\Omega)$.

Without loss of generality, suppose that the sequences of the Lipschitz coefficients α_n, β_n are non-decreasing and $\alpha_1, \beta_1 > \alpha_0$, therefore the operators $A[\alpha_n] = (L + \alpha_n I)^{-1}$ and $A[\beta_n] = (L + \beta_n I)^{-1}$ are defined and positive for all n. Set

$$\begin{aligned} H_n &= A[\alpha_n](\hat h_n + \alpha_n I) \\ G_n &= A[\beta_n](\hat g_n + \beta_n I). \end{aligned}$$

Since the functions $h_n(t,x) + \alpha_n x$ and $g_n(t,x) + \beta_n x$ increase in x for $t \in \Omega$ and $|x| \le \rho$, it follows that each of the operators H_n, G_n is monotone on the cone interval $\langle x_-, x_+\rangle$, i.e.

$$x \ll y, \quad x, y \in \langle x_-, x_+\rangle \quad \Longrightarrow \quad H_n x \ll H_n y, \quad G_n x \ll G_n y; \tag{3.5}$$

here and henceforth we write [4] $x \ll y$, or $y \gg x$, whenever $y - x \in K_+$.

The estimates $h(t,x) \le h_n(t,x) \le g_n(t,x) \le g(t,x)$ imply the relations

$$\begin{aligned} x_- &= A[\alpha_n](\hat h x_- + \alpha_n x_-) \ll H_n x_- \ll H_n x_+ \ll A[\alpha_n](\hat g x_+ + \alpha_n x_+) = x_+ \\ x_- &= A[\beta_n](\hat h x_- + \beta_n x_-) \ll G_n x_- \ll G_n x_+ \ll A[\beta_n](\hat g x_+ + \beta_n) x_+ = x_+, \end{aligned}$$

hence each of the operators H_n and G_n maps the cone interval $\langle x_-, x_+\rangle$ into itself [5]. Therefore for every $x_0, \tilde x_0 \in \langle x_-, x_+\rangle$ and every n the sequences

$$y_0^{(n)} = x_0, \quad y_k^{(n)} = H_n y_{k-1}^{(n)} \quad (k \ge 1) \tag{3.6}$$

$$z_0^{(n)} = \tilde x_0, \quad z_k^{(n)} = G_n z_{k-1}^{(n)} \quad (k \ge 1) \tag{3.7}$$

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[4] The relation $x \ll y$ is the semiordering generated by the cone K_+ in the space $C(\Omega)$.

[5] This implies that the operator $F[\alpha]$ also maps the set $\langle x_-, x_+\rangle$ into itself whenever α is sufficiently large. Indeed, one can take $\alpha_1 = \beta_1 = \alpha$. Then the estimates $h_1(t,x) \le f(t,x) \le g_1(t,x)$ imply $H_1 x \ll F[\alpha]x \ll G_1 x$ and therefore the inclusions $H_1 x, G_1 x \in \langle x_-, x_+\rangle$ imply $F[\alpha]x \in \langle x_-, x_+\rangle$ for all $x \in \langle x_-, x_+\rangle$.

are contained in $\langle x_-, x_+\rangle$.

We use sequences (3.6), (3.7) with initial values $x_0, \tilde{x}_0 \in \langle x_-, x_+\rangle$ such that

$$\begin{aligned} x_0 &\gg H_n x_0 \\ \tilde{x}_0 &\ll G_n \tilde{x}_0. \end{aligned} \tag{3.8}$$

Relations (3.5) and (3.8) imply that sequences (3.6) and (3.7) decreases and increases, respectively:

$$y_0^{(n)} \gg y_1^{(n)} \gg \ldots \gg y_k^{(n)} \gg \ldots \tag{3.9}$$

$$z_0^{(n)} \ll z_1^{(n)} \ll \ldots \ll z_k^{(n)} \ll \ldots. \tag{3.10}$$

Since the operators H_n, G_n are completely continuous, it follows that monotone bounded sequences (3.9), (3.10) are compact and converge to solutions $y_*, z_* \in \langle x_-, x_+\rangle$ of the equations $x = H_n x, x = G_n x$, which are equivalent to the problems $Lx = h_n(t,x), Lx = g_n(t,x)$.

For every n denote by u_n the limit of the decreasing sequence

$$y_0^{(n)} = x_+, \quad y_k^{(n)} = H_n y_{k-1}^{(n)} \quad (k \geq 1). \tag{3.11}$$

Lemma 1. *The sequence of solutions $u_n \in \langle x_-, x_+\rangle$ of the problems $Lx = h_n(t,x)$ increases and converges uniformly to a solution $u_* \in \langle x_-, x_+\rangle$ of problem* (2.3).

By Lemma 1,

$$u_*(t) = \lim_{n\to\infty} \lim_{k\to\infty} ((H_n)^k x_+)(t) = \sup_n \inf_k ((H_n)^k x_+)(t). \tag{3.12}$$

Now, for every n consider the sequence

$$z_0^{(n)} = u_*, \quad z_k^{(n)} = G_n z_{k-1}^{(n)} \quad (k \geq 1) \tag{3.13}$$

with initial value (3.12). Since $u_* = A[\beta_n](\hat{f} u_* + \beta_n u_*) \ll G_n u_*$, sequence (3.13) increases and converges uniformly. Denote its limit by v_n.

Lemma 2. *The sequence of solutions $v_n \in \langle x_-, x_+\rangle$ of the problems $Lx = g_n(t,x)$ decreases and converges uniformly to a solution $v_* \in \langle x_-, x_+\rangle$ of problem* (2.3).

By construction,

$$v_*(t) = \lim_{n\to\infty} \lim_{k\to\infty} ((G_n)^k u_*)(t) = \inf_n \sup_k ((G_n)^k u_*)(t) \tag{3.14}$$

and $u_* \ll v_*$, the cone interval

$$\langle u_*, v_*\rangle = \left\{ x \in C(\Omega) : \, u_* \ll x \ll v_* \right\}$$

is included in $\langle x_-, x_+\rangle$. Simple examples show that the points u_* and v_* can be different (see Section 6), or they can coincide.

A set $M \subset C(\Omega)$ is called a *continuous branch connecting the points u_* and v_** if $u_*, v_* \in M$ and $M \cap \partial U \neq \emptyset$ for any bounded open domain $U \subset C(\Omega)$ such that either $u_* \in U, v_* \notin U$ or $u_* \notin U, v_* \in U$; here ∂U is the boundary of U.

Denote by Π_0 the set of all the solutions of problem (2.3). A non-empty subset Π of the set Π_0 is said to be *robust stable* if any ε-vicinity

$$\Pi^\varepsilon = \left\{ x \in C(\Omega) : \, \inf_{y\in\Pi} \|x - y\|_C < \varepsilon \right\} \qquad (\varepsilon > 0)$$

of Π contains at least one solution of every problem (2.8) such that estimate (2.9) holds for a sufficiently small $\delta = \delta(\varepsilon) > 0$.

Theorem 1. *Let u_*, v_* be solutions* (3.12), (3.14) *of problem* (2.3). *The following statements are valid:*

1. *If $u_* = v_*$, then u_* is a robust stable solution of problem* (2.3).

2. *If u_* is an isolated solution, then $u_* = v_*$ and for every $\alpha > \alpha_0$ the function u_* is a singular point of the topological index* 1 *for the completely continuous vector field*

$$x - A[\alpha](\hat{f} + \alpha I)x, \qquad x \in C(\Omega). \tag{3.15}$$

3. *If $u_* \neq v_*$, then the set of all solutions of problem* (2.3) *that lie in the cone interval $\langle u_*, v_* \rangle$ is a robust stable continuous branch connecting the points u_* and v_*.*

By statement 3 of Theorem 1, problem (2.3) has a continuum of solutions in $\langle u_*, v_* \rangle$ if $u_* \neq v_*$.

If problem (2.3) has at most a countable number of solutions, then by statement 1, at least one of them is robust stable. If the number of the solutions is finite, then statement 2 is applicable, hence there is a robust stable solution of the topological index 1.

4. Approximation of solutions

In this section, we construct some iteration procedures converging to solution (3.12) of problem (2.3). At every iteration, the linear problem

$$Ly + \alpha y = \varphi \tag{4.1}$$

should be solved for given $\varphi \in C(\Omega)$ and $\alpha > \alpha_0$.

Lemma 3. *Suppose that for some n sequences* (3.6), (3.7) *are contained in the cone interval $\langle x_-, x_+ \rangle$ and relations* (3.9), (3.10) *hold. Suppose*

$$h_n(t,x) < \phi(t,x) < g_n(t,x) \qquad (t \in \Omega,\ |x| \leq \rho) \tag{4.2}$$

where the function ϕ is continuous in t and Lipschitz continuous in x:

$$|\phi(t,x) - \phi(t,y)| \leq c\,|x-y| \qquad (t \in \Omega;\ |x|, |y| \leq \rho). \tag{4.3}$$

Set $(\hat{\phi}x)(t) = \phi(t, x(t))$ $(x \in C(\Omega))$. Then for any fixed $\alpha > \alpha_0$ the estimates

$$\begin{aligned} A[\alpha](\hat{\phi} y_k^{(n)} + \alpha y_k^{(n)}) &\gg y_k^{(n)} \\ A[\alpha](\hat{\phi} z_k^{(n)} + \alpha z_k^{(n)}) &\ll z_k^{(n)} \end{aligned} \tag{4.4}$$

are valid for all sufficiently large k.

By Lemma 3, the elements of sequence (3.11) satisfy

$$H_{n+1} y_k^{(n)} \gg y_k^{(n)} \tag{4.5}$$

for all sufficiently large k. Recall that $y_k^{(n)} \to u_n$.

Theorem 2. *Let $\{s(n)\}$ be an arbitrary sequence of indices such that for every n the element $y_{s(n)}^{(n)}$ of sequence* (3.11) *satisfies* (4.5), *i.e.* $H_{n+1}y_{s(n)}^{(n)} \gg y_{s(n)}^{(n)}$. *Then the relations*

$$u_1 \ll y_{s(1)}^{(1)} \ll u_2 \ll y_{s(2)}^{(2)} \ll \ldots \ll u_n \ll y_{s(n)}^{(n)} \ll u_{n+1} \ll \ldots$$

are valid. Therefore the sequence $\{\bar{y}_n\} = \{y_{s(n)}^{(n)}\}$ $(n \geq 1)$ *converges uniformly from below to solution* (3.12) *of problem* (2.3).

Let $\bar{y}_n$ be the first element satisfying (4.5) in sequence (3.11), i.e. $\bar{y}_n = y_{s(n)}^{(n)}$ with

$$s(n) = \min\{k :\ H_{n+1}y_k^{(n)} \gg y_k^{(n)}\}. \tag{4.6}$$

By Theorem 2, the functions $\bar{y}_n$ monotonically and uniformly approximate the function u_* from below. To find the approximation $\bar{y}_n$ of order n, one needs to construct a finite number of elements $y_k^{(n)}$ of sequence (3.11), the initial element is always x_+. The required number (4.6) of iterations (3.11) is not known *a priori*, therefore the function $H_{n+1}y_k^{(n)}$ should be also constructed and compared with $y_k^{(n)}$ for each $k \in \mathbb{N}$. The calculations for a given n are complete as soon as relation (4.5) is valid, this means that $k = s(n)$ and $y_k^{(n)} = \bar{y}_n$. Then, one can proceed to the construction of the approximation of next order.

Now consider the situation when the functions x_-, x_+ are not known. Instead, suppose we are given the function $\tilde{x}_0 = \tilde{x}_0(t) \in \langle x_-, x_+\rangle$ such that $G_1\tilde{x}_0 \gg \tilde{x}_0$. By Lemma 3, the increasing sequence

$$z_0 = \tilde{x}_0, \quad z_m = G_1 z_{m-1} \quad (m \geq 1) \tag{4.7}$$

contains a subsequence $\{z_{m_n}\}$ such that $H_n z_{m_n} \ll z_{m_n}$ $(n \geq 1)$; it is natural to select this subsequence by the rule

$$m_0 = 0, \quad m_n = \min\{m > m_{n-1} :\ H_n z_m \ll z_m\} \quad (n \geq 1).$$

For every n one can construct the elements $y_1^{(n)}, \ldots, y_{s(n)}^{(n)}$ of the decreasing sequence

$$y_0^{(n)} = z_{m_n}, \quad y_k^{(n)} = H_n y_{k-1}^{(n)} \quad (k \geq 1) \tag{4.8}$$

where $s(n)$ is given by (4.6).

Theorem 3. *The sequence* $\{d_n\} = \{y_{s(n)}^{(n)}\}$ *increases. If*

$$g = g_1, \qquad x_+ = z_* := \lim_{m\to\infty} z_m, \tag{4.9}$$

then the sequence $\{d_n\}$ converges uniformly from below to solution (3.12) *of problem* (2.3).

Relations (4.9) are not an additional restriction. Indeed, if they are not satisfied originally, we can consider them as a new definition of the functions g and x_+. This definition is correct, since the limit z_* of sequence (4.7) is a solution of the problem $Lx = g_1(t,x)$ and the inequalities

$$h(t,x) < f(t,x) < g_1(t,x), \qquad x_- \ll z_*$$

are valid, i.e. both our main hypotheses (2.5) and (2.7) are satisfied for the new functions (4.9) and the original functions h, x_-. Also note that $\tilde{x}_0 \in \langle x_-, z_*\rangle$.

5. Two-sided approximations

Here we construct iteration procedures that converge to a robust stable solution [6] of problem (2.3) (or to a robust stable continuum of solutions) and contain both upper and lower estimates of the solution (the continuum of solutions, respectively). Everywhere $\tilde{x}_0 \in \langle x_-, x_+ \rangle$ is a given function such that $\tilde{x}_0 \ll G_1 \tilde{x}_0$.

The first iteration procedure is a simple modification of the shuttle iteration method of [8]. Set $\tilde{u}_0 = \tilde{x}_0$ and define the sequences $\tilde{u}_n$ and $\tilde{v}_n$ by induction as follows. Suppose the function $\tilde{u}_{n-1}$ is already constructed; then $\tilde{v}_n$ is an element of the sequence

$$z_0 = \tilde{u}_{n-1}, \quad z_k = G_n z_{k-1} \ \ (k \geq 1), \tag{5.1}$$

namely, it is the first element such that

$$G_{n+1} z_k \ll z_k, \qquad H_n z_k \ll z_k. \tag{5.2}$$

The function $\tilde{u}_n$ is the first element satisfying

$$H_{n+1} y_k \gg y_k, \qquad G_{n+1} y_k \gg y_k \tag{5.3}$$

in the sequence

$$y_0 = \tilde{v}_n, \quad y_k = H_n y_{k-1} \ \ (k \geq 1). \tag{5.4}$$

The existence of the functions $\tilde{v}_n, \tilde{u}_n$ for each n follows from Lemma 3. If $\alpha_1 \leq \beta_2 \leq \alpha_2 \leq \beta_3 \leq \ldots$, then the first relation implies the second one in (5.2) and (5.3).

Theorem 4. *The sequences* $\{\tilde{u}_n\}$ *and* $\{\tilde{v}_n\}$ *converge uniformly to solutions* $u_\star$ *and* $v_\star$ *of problem* (2.3), *respectively, and the relations*

$$\tilde{u}_1 \ll \tilde{u}_2 \ll \ldots \ll \tilde{u}_n \ll \ldots \ll u_\star \ll v_\star \ll \ldots \ll \tilde{v}_n \ll \ldots \ll \tilde{v}_2 \ll \tilde{v}_1 \tag{5.5}$$

hold. The set of all solutions of problem (2.3) *contained in the cone interval* $\langle u_\star, v_\star \rangle$ *is robust stable* (*if* $u_\star = v_\star$, *then* $u_\star$ *is a robust stable solution*).

Now, consider sequences $\{\bar{u}_n\}, \{\bar{v}_n\}$ that converge to a robust stable continuous branch of solutions of problem (2.3). Together with the sequences $\{\bar{u}_n\}, \{\bar{v}_n\}$, we construct an auxiliary sequence of functions $w_n, \bar{u}_n \ll w_n \ll \bar{v}_n$, and sequences of indices

$$1 = i_0 < i_1 < \ldots < i_n < \ldots$$
$$1 = j_0 < j_1 < \ldots < j_n < \ldots$$

such that

$$\bar{u}_n \ll H_{j_n} \bar{u}_n, \quad \bar{v}_n \gg G_{i_n} \bar{v}_n, \quad w_n \ll G_{i_n} w_n \qquad (n \geq 0). \tag{5.6}$$

Set $w_0 = \tilde{x}_0$, $\bar{u}_0 = x_-$ and $\bar{v}_0 = x_+$. Then relations (5.6) are valid for $n = 0$ (the functions x_-, x_+ can be unknown, we do not need to construct them). Further elements of the sequences $\{w_n\}, \{\bar{u}_n\}, \{\bar{v}_n\}$ are defined by induction.

[6] The solutions constructed in this section can differ from the solutions u_* and v_* defined in Lemmas 1 and 2.

Suppose that for some n relations (5.6) hold and the function w_n is already constructed. Lemma 3 implies that the increasing sequence

$$z_0 = w_n, \quad z_m = G_{i_n} z_{m-1} \quad (m \geq 1) \tag{5.7}$$

contains a subsequence $\{z_{m_k}\}$ such that

$$H_{j_n+k} z_{m_k} \ll z_{m_k}, \qquad G_{i_n+1} z_{m_k} \ll z_{m_k}. \tag{5.8}$$

For every $k \geq 1$ consider the decreasing sequence

$$y_0^{(k)} = z_{m_k}, \quad y_\ell^{(k)} = H_{j_n+k} y_{\ell-1}^{(k)} \quad (\ell \geq 1) \tag{5.9}$$

and denote by $\tilde{d}_k$ the first of its elements satisfying $y_\ell^{(k)} \ll H_{j_n+k+1} y_\ell^{(k)}$. The functions $\tilde{d}_1, \tilde{d}_2, \ldots$ can be constructed in the same way as the functions $d_1, d_2, \ldots$ in Theorem 3.

Lemma 4. *The sequence* $\{\tilde{d}_k\}$ *increases, converges to a solution of problem* (2.3), *and*

$$\tilde{d}_{k+1} \ll G_{i_n+1} \tilde{d}_k \tag{5.10}$$

for all sufficiently large k.

Suppose the functions $\tilde{d}_1, \ldots, \tilde{d}_{k_0}, \tilde{d}_{k_0+1}$ are constructed, where k_0 is the smallest of the indices k, for which relation (5.10) holds. Define

$$\bar{u}_{n+1} = \tilde{d}_{k_0}, \qquad j_{n+1} = j_n + k_0 + 1.$$

By construction, $\tilde{d}_{k_0+1} \gg H_{j_{n+1}} \tilde{d}_{k_0+1}$. Therefore the sequence

$$y_0 = \tilde{d}_{k_0+1}, \quad y_m = H_{j_{n+1}} y_{m-1} \quad (m \geq 1) \tag{5.11}$$

decreases and contains a subsequence $\{y_{m_k}\}$ such that

$$H_{j_{n+1}+1} y_{m_k} \gg y_{m_k}, \quad G_{i_n+k} y_{m_k} \gg y_{m_k} \qquad (k \geq 1).$$

Every function y_{m_k} determines the increasing sequence

$$z_0^{(k)} = y_{m_k}, \quad z_\ell^{(k)} = G_{i_n+k} z_{\ell-1}^{(k)} \quad (\ell \geq 1); \tag{5.12}$$

denote by $\bar{d}_k$ the first of its elements satisfying $z_\ell^{(k)} \gg G_{i_n+k+1} z_\ell^{(k)}$.

Lemma 5. *The sequence* $\{\bar{d}_k\}$ *decreases, converges to a solution of problem* (2.3), *and*

$$\bar{d}_{k+1} \gg H_{j_{n+1}+1} \bar{d}_k. \tag{5.13}$$

for every sufficiently large k.

Set

$$\bar{v}_{n+1} = \bar{d}_{k_1}, \quad w_{n+1} = \bar{d}_{k_1+1}, \quad i_{n+1} = i_n + k_1 + 1$$

where k_1 is the smallest k, for which (5.13) holds. This completes the definition of the sequences $\{\bar{u}_n\}$ and $\{\bar{v}_n\}$.

Theorem 5. *The sequences* $\{\bar{u}_n\}$ *and* $\{\bar{v}_n\}$ *converge uniformly to solutions* $u^\star$ *and* $v^\star$ *of problem* (2.3), *respectively, and the relations*

$$\bar{u}_1 \ll \bar{u}_2 \ll \cdots \ll \bar{u}_n \ll \cdots \ll u^\star \ll v^\star \ll \cdots \ll \bar{v}_n \ll \cdots \ll \bar{v}_2 \ll \bar{v}_1 \tag{5.14}$$

are valid. If $u^\star = v^\star$, *then* $u^\star$ *is a robust stable solution. If* $u^\star \neq v^\star$, *then the set of all solutions of problem* (2.3) *contained in the cone interval* $\langle u^\star, v^\star \rangle$ *is a robust stable continuous branch connecting the points* $u^\star$ *and* $v^\star$.

6. Remarks

6.1 Smoothness of solutions. Throughout the paper we consider continuous solutions of problem (2.3). If the boundary $\partial\Omega$ of the domain Ω and the coefficients of operator (2.1) are sufficiently smooth (see, e.g., [12] for exact requirements on smoothness), then for every $\varphi \in C(\Omega)$ and $\alpha > \alpha_0$ the solution $y = A[\alpha]\varphi$ of linear problem (4.1) possesses Hölder continuous first derivatives in Ω, i.e. $y \in C^{(1,\varepsilon)}(\Omega)$ for some $\varepsilon \in (0,1)$. Moreover, if φ is Hölder continuous, then $y = A[\alpha]\varphi$ is a classical regular solution of (4.1). This implies that any continuous solution x_* of nonlinear problem (2.3) with continuous non-linearity f lies in $C^{(1,\varepsilon)}(\Omega)$; if the function f satisfies the Hölder condition

$$|f(t,x) - f(\tau,y)| \le c_1|t-\tau|^\varepsilon + c_2|x-y|^\varepsilon \qquad \big(t,\tau \in \Omega;\ x,y \in \mathbb{R};\ \varepsilon \in (0,1)\big),$$

then every solution x_* is regular, i.e. $x_* \in C(\Omega) \cap C^2(\Omega \backslash \partial\Omega)$.

Observe that solutions of linear problem (4.1) are defined by

$$y(t) = (A[\alpha]\varphi)(t) = \int_\Omega \mathcal{G}(t,\tau;\alpha)\varphi(\tau)\,d\tau$$

where $\mathcal{G}(\cdot,\cdot;\alpha)$ is the corresponding Green function. Therefore iterations (3.6), (3.7) have the form

$$\left.\begin{aligned} y_k^{(n)}(t) &= \int_\Omega \mathcal{G}(t,\tau;\alpha_n)\big(h_n(\tau, y_{k-1}^{(n)}(\tau)) + \alpha_n y_{k-1}^{(n)}(\tau)\big)d\tau \\ z_k^{(n)}(t) &= \int_\Omega \mathcal{G}(t,\tau;\beta_n)\big(g_n(\tau, z_{k-1}^{(n)}(\tau)) + \beta_n z_{k-1}^{(n)}(\tau)\big)d\tau \end{aligned}\right. \qquad (k \ge 1).$$

In the simplest cases, the Green function of problem (4.1) is known explicitly. For example, this is the case for the scalar problem

$$\left.\begin{aligned} -y'' + py' + (q+\alpha)y &= \varphi \\ y(0) = y(1) &= 0 \end{aligned}\right\}$$

with constant coefficients.

6.2 On the existence of upper and lower solutions. Given problem (2.3), an important question is how to construct problems (2.6) that have solutions x_-, x_+ satisfying (2.7). Consider two examples where we use the methods of one-sided estimates [10] and potential bounds [6, 7]. We also refer to the variety of results on upper and lower solutions (see, e.g., [2] and the references therein).

Below we denote by λ_0 the smallest real eigenvalue of the operator L. This eigenvalue is simple and $\mathrm{Re}\lambda \ge \lambda_0$ for every other eigenvalue λ of L (see, e.g., [4]). Suppose that for some $\kappa < \lambda_0$ the one-sided estimate

$$\operatorname{sign} x f(t,x) \le \kappa|x| + q \qquad (t \in \Omega, x \in \mathbb{R}) \tag{6.1}$$

holds, where $q \geq 0$. Then bounds (2.5) can be defined by the formulas

$$h(t,x) = \min\left\{\kappa x - q - 1, f(t,x) - 1\right\}$$
$$g(t,x) = \max\left\{\kappa x + q + 1, f(t,x) + 1\right\}.$$

Relation (6.1) implies that $h(t,x) = \kappa x - q - 1$ for $x \leq 0$ and $g(t,x) = \kappa x + q + 1$ for $x \geq 0$. Denote by x_-, x_+ the solutions of the linear problems

$$Lx = \kappa x - q - 1$$
$$Lx = \kappa x + q + 1,$$

i.e. $x_+ = A[-\kappa](q+1)$, $x_- = -x_+$. Since $-\kappa > -\lambda_0$, the operator $A[-\kappa]$ is positive, [7)] hence

$$x_- \ll 0 \ll x_+ \qquad (6.2)$$

and therefore the functions x_-, x_+ are solutions of problems (2.6). These solutions can be used as initial values for the iteration procedures considered in Sections 4 and 5.

The next example is based on the method to study problems with self-adjoint positive operators by analyzing potential bounds of nonlinear terms. To be simple, consider the scalar problem

$$\left.\begin{aligned} x'' + f(t,x) &= 0 \\ x(0) = x(\pi) &= 0. \end{aligned}\right\} \qquad (6.3)$$

Here $L = -\frac{d}{dt^2}$ and the smallest eigenvalue of L is $\lambda_0 = 1$. Suppose that the primitive of the function f (with respect to the variable x) satisfies the quadratic estimate

$$\int_0^x f(t,\xi)\,d\xi \leq \kappa x^2 + q \qquad (t \in [0,\pi],\ x \in \mathbb{R}) \qquad (6.4)$$

with some $\kappa < \frac{\lambda_0}{2} = \frac{1}{2}$; by the Golomb theorem [3], estimate (6.4) implies the existence of a classical solution of problem (6.3). Define bounds (2.5) of the function f by

$$\begin{aligned} h &= f - c \\ g &= f + c \end{aligned} \qquad (6.5)$$

where $c > \|f(\cdot,0)\|_C$ and therefore

$$h(\cdot,0) \ll 0 \ll g(\cdot,0). \qquad (6.6)$$

From estimate (6.4), the similar quadratic estimates

$$\left.\begin{aligned} \int_0^x g(t,\xi)\,d\xi &\leq \kappa_1 x^2 + q_1 \\ \int_0^x h(t,\xi)\,d\xi &\leq \kappa_1 x^2 + q_1 \end{aligned}\right. \qquad (t \in [0,\pi],\ x \in \mathbb{R}) \qquad (6.7)$$

[7)] It is easily seen that property (i) is valid for $\alpha_0 = -\lambda_0$.

with $\kappa_1 \in (\kappa, \frac{1}{2})$ and $q_1 \geq q + c^2(\kappa_1 - \kappa)^{-1}/4$ follow for the primitives of functions (6.5). Relations (6.6) and (6.7) imply [6] that the auxiliary problems

$$x'' + h(t, x) = 0, \qquad x'' + g(t, x) = 0, \qquad\qquad x(0) = x(\pi) = 0 \tag{6.8}$$

have solutions x_-, x_+ satisfying (6.2). The functions x_-, x_+ are not known, but their norms can be estimated explicitly for given κ_1, q_1.

In both examples, one can choose the function g_1 so that $g_1(\cdot, 0) \gg 0$. Then the operator G_1 takes the identical zero to the non-negative function $A[\beta_1]g_1(\cdot, 0)$, therefore Theorems 3 - 5 are valid for iteration procedures with initial value $\tilde{x}_0 \equiv 0$.

6.3 Example with continuum of solutions. The simplest example of problem (2.3) with robust stable continuum of solutions is

$$Lx = (\lambda_0 - 1)x + \psi(x) \tag{6.9}$$

with

$$\psi(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1. \end{cases}$$

The solutions of problem (6.9) are the functions $x = \xi u_0 \quad (0 \leq \xi \leq 1)$ where u_0 is the non-negative normed eigenfunction of the operator L corresponding to its simple eigenvalue λ_0. The right-hand side of (6.9) satisfies estimate (6.1) with $\kappa = \lambda_0 - 1$, $q = 1$.

Consider the problems

$$Lx = (\lambda_0 - 1)x + (1 - \varepsilon)\psi(x)$$
$$Lx = (\lambda_0 - 1)x + \psi(x) + \varepsilon u_0$$

with $\varepsilon \in (0, 1)$. It is easily seen that the first of them has a unique zero solution. At the same time, the norm of any solution of the second problem is greater than 1. Therefore the identical zero and the function u_0 are included in every robust stable closed subset of the set of solutions of problem (6.9). Thus, all the iteration procedures of Sections 4 and 5 converge to the solutions $u_* = 0, v_* = u_0$ of (6.9) (for any sequences (3.1), (3.2)) and the solution set $\xi u_0 \quad (0 \leq \xi \leq 1)$ of this problem is a robust stable continuous branch. Note that none of the solutions is robust stable itself.

6.4 Other applications. The methods of this paper are applicable to any problem (2.3) with linear operator L satisfying condition (i) and non-linearity satisfying the assumptions of Section 2. The iteration procedures should be constructed in the same way as in Sections 4 and 5, the theorems hold without any change in formulation. For example, this is true if the elliptic operator L is defined by differential expression (2.1) and the Newton boundary condition

$$\frac{\partial x}{\partial \nu} + \theta(t)x = 0 \qquad (t \in \partial\Omega)$$

where ν is the exterior normal at the boundary of the domain Ω and θ is non-negative. Condition (i) is satisfied for periodic problems for equations $-x''+p(t)x'+q(t)x = f(t,x)$ and $x' = f(t,x)$, etc.

There is no need to use theorems of this paper if the function f is Lipschitz continuous or even if it satisfies the lower Lipschitz condition

$$(x-y)(f(t,x)-f(t,y)) \geq c\,(x-y)^2 \qquad (t \in \Omega;\ |x|,|y| \leq \rho). \tag{6.10}$$

Estimate (6.10) implies that the function $f(t,x)+\alpha x$ with any $\alpha > \max\{-c, \alpha_0\}$ increases in x on the segment $[-\rho, \rho]$. Therefore the operator $F[\alpha]$ is monotone on the cone interval $\langle x_-, x_+\rangle$, which it maps into itself. Robust stable fixed points of such operators (or robust stable continual sets of fixed points) can be constructed by the shuttle iteration method.

7. Proof of Theorem 1

7.1 Proof of Lemmas 1 and 2. Consider the sequence $\{u_n\}$. The equality $Lu_n = \hat{h}_n u_n$ is equivalent to

$$u_n = A[\alpha](\hat{h}_n + \alpha I)u_n \tag{7.1}$$

for any $\alpha > \alpha_0$. Therefore

$$u_{n-1} = A[\alpha_n](\hat{h}_{n-1} + \alpha_n I)u_{n-1} \ll A[\alpha_n](\hat{h}_n + \alpha_n I)u_{n-1} = H_n u_{n-1}.$$

Due to the monotonicity of the operator H_n, the relations $u_{n-1} \ll H_n u_{n-1}$, $u_{n-1} \ll x_+$ imply that all the elements of sequence (3.11) satisfy $y_k^{(n)} \gg u_{n-1}$. Therefore [8] $\lim_k y_k^{(n)} = u_n \gg u_{n-1}$, i.e. the sequence $\{u_n\}$ increases.

Fix any $\alpha > \alpha_0$. Since $A[\alpha]$ is a compact operator and the functions h_n are uniformly bounded, it follows from the relations $u_n \in \langle x_-, x_+\rangle$ and (7.1) that the sequence $\{u_n\}$ is compact, hence it converges uniformly to the function $u_* \in \langle x_-, x_+\rangle$. Passing to the limit in (7.1), we obtain $u_* = A[\alpha](\hat{f} + \alpha I)u_*$, therefore u_* is a solution of (2.3). This proves Lemma 1 ∎

The proof of Lemma 2 is by the same argument and we omit it.

7.2 Proof of statement 1 of Theorem 1. By Lemmas 1 and 2,

$$u_n \to u_*, \qquad v_n \to v_*, \qquad u_n \ll u_* \ll v_* \ll v_n. \tag{7.2}$$

Let us show that for any given n the set $\langle u_n, v_n\rangle$ contains a solution of problem (2.8) whenever the function f_1 satisfies (2.9) for a sufficiently small $\delta = \delta_n$. Due to this fact, statement 1 follows from (7.2) if $u_* = v_*$.

Take any n and any $\alpha > \alpha_n, \beta_n$. Due to (3.1) and (3.2),

$$h_n(t,x) < f(t,x) < g_n(t,x) \qquad (t \in \Omega,\ |x| \leq \rho). \tag{7.3}$$

[8] Everywhere we consider the convergence in the space $C(\Omega)$; it is not mentioned sometimes.

Therefore there is a $\delta = \delta_n > 0$ such that (2.9) implies

$$h_n(t,x) \le f_1(t,x) \le g_n(t,x) \qquad (t \in \Omega,\ |x| \le \rho)$$

and hence $\hat{h}_n u \ll \hat{f}_1 u \ll \hat{g}_n u$ for every $u \in \langle u_n, v_n\rangle$. From the estimates $\alpha > \alpha_n, \beta_n$ and (3.4), it follows that the functions $h_n(t,x) + \alpha x$ and $g_n(t,x) + \alpha x$ increase in x, hence

$$\hat{h}_n u_n + \alpha u_n \ll \hat{h}_n u + \alpha u$$
$$\hat{g}_n u + \alpha u \ll \hat{g}_n v_n + \alpha v_n$$

for all $u \in \langle u_n, v_n\rangle$ and therefore

$$\hat{h}_n u_n + \alpha u_n \ll \hat{f}_1 u + \alpha u \ll \hat{g}_n v_n + \alpha v_n.$$

Applying the positive operator $A[\alpha]$ to these relations and using equality (7.1) and the similar equality $v_n = A[\alpha](\hat{g}_n + \alpha I)v_n$, we obtain

$$u_n \ll A[\alpha](\hat{f}_1 u + \alpha u) \ll v_n \qquad (u \in \langle u_n, v_n\rangle),$$

i.e. the completely continuous operator $A[\alpha](\hat{f}_1 + \alpha I)$ maps the cone interval $\langle u_n, v_n\rangle$ into itself. Therefore the equation $x = A[\alpha](\hat{f}_1 + \alpha I)x$ equivalent to problem (2.8) has a solution in $\langle u_n, v_n\rangle$ and statement 1 is proved ∎

7.3 Proof of statement 2. Suppose u_* is an isolated solution of problem (2.3). The relation $u_* = v_*$ follows from statement 3.

Consider any $\varepsilon > 0$ such that u_* is a unique solution of (2.3) in the open ball $B(\varepsilon) = \{u \in C(\Omega) : \|u - u_*\|_C < \varepsilon\}$. We need to show that the rotation [9] $\gamma(I - A[\alpha](\hat{f} + \alpha I), \partial\Gamma)$ of the vector field (3.15) on the boundary $\partial\Gamma$ of some open domain Γ equals 1, where Γ contains the point u_* and the closure $\bar{\Gamma}$ of Γ is contained in $B(\varepsilon)$. Fix any n such that $\langle u_n, v_n\rangle \subset B(\varepsilon)$ and any α satisfying $\alpha > \alpha_n, \beta_n$. Set

$$\begin{aligned} b_1 &= \min_{t\in\Omega, |x|\le\rho} (f(t,x) - h_n(t,x)) \\ b_2 &= \min_{t\in\Omega, |x|\le\rho} (g_n(t,x) - f(t,x)); \end{aligned} \tag{7.4}$$

these numbers are positive due to (7.3). Now, take any $\delta > 0$ such that $\langle u_n - \delta, v_n + \delta\rangle \subset B(\varepsilon)$ and the implications

$$|x - y| \le \delta \quad \Longrightarrow \quad |h_n(t,x) - h_n(t,y) + \alpha(x-y)| < b_1 \tag{7.5}$$
$$|x - y| \le \delta \quad \Longrightarrow \quad |g_n(t,x) - g_n(t,y) + \alpha(x-y)| < b_2 \tag{7.6}$$

are valid for all $t \in \Omega$ and $|x|, |y| \le \rho$. We define $\bar{\Gamma} = \langle u_n - \delta, v_n + \delta\rangle$ where δ is the constant function; Γ is the non-empty interior of this cone interval.

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[9] For the definition, main properties, and methods of application see, e.g., [10] or any book on the topological degree theory.

Let $u \in \bar{\Gamma}$. Since $\alpha > \alpha_n$, the function $h_n(t,x) + \alpha x$ increases in x, therefore

$$\hat{f}u + \alpha u = \hat{h}_n u + \alpha u + (\hat{f} - \hat{h}_n)u \gg (\hat{h}_n + \alpha I)(u_n - \delta) + (\hat{f} - \hat{h}_n)u.$$

Here

$$(\hat{f} - \hat{h}_n)u \gg b_1 \gg (\hat{h}_n + \alpha I)u_n - (\hat{h}_n + \alpha I)(u_n - \delta)$$

due to (7.4) and (7.5), hence $\hat{f}u + \alpha u \gg \hat{h}_n u_n + \alpha u_n$ and consequently,

$$A[\alpha](\hat{f} + \alpha I)u \gg A[\alpha](\hat{h}_n + \alpha I)u_n = u_n \qquad (u \in \bar{\Gamma}).$$

Similarly, the relation $A[\alpha](\hat{f} + \alpha I)u \ll v_n$ for each $u \in \bar{\Gamma}$ follows from (7.4) and (7.6). Thus, the operator $A[\alpha](\hat{f} + \alpha I)$ maps the cone interval $\bar{\Gamma}$ to the smaller cone interval $\langle u_n, v_n \rangle \subset \Gamma$. Therefore vector field (3.15) is linearly homotopic to the vector field $u - u_*$ on the boundary $\partial\Gamma$ of the domain $\bar{\Gamma}$, whence [10)]

$$\gamma(I - A[\alpha](\hat{f} + \alpha I), \partial\Gamma) = \gamma(I - u_*, \partial\Gamma) = 1.$$

This formula is now proved for one value of α. To complete the proof of statement 2, it remains to note that vector fields (3.15) have the same zeros for all $\alpha > \alpha_0$ (their zeros are the solutions of problem (2.3)) and depend continuously on α uniformly with respect to x, therefore the rotation $\gamma(I - A[\alpha](\hat{f} + \alpha I), \partial\Gamma)$ is the same for all $\alpha > \alpha_0$ ∎

7.4 Proof of statement 3. Let $u_* \neq v_*$. Denote by Π_* the set of all solutions of problem (2.3) that lie in $\langle u_*, v_* \rangle$. As it is proved in Subsection 7.2., there is a sequence $\delta_n \to 0$ such that the cone interval $\langle u_n, v_n \rangle$ contains at least one solution of problem (2.8) whenever the function f_1 satisfies (2.9) for $\delta = \delta_n > 0$.

Consider any sequence of problems $Lx = f_n(t,x)$ such that $|f_n(t,x) - f(t,x)| < \delta_n$ $(t \in \Omega, x \in \mathbb{R})$ and a sequence of their solutions $x_n \in \langle u_n, v_n \rangle$. Due to the compactness and continuity of the operator $A[\alpha]$, the equalities $x_n = A[\alpha](\hat{f}_n + \alpha I)x_n$ imply that the sequence $\{x_n\}$ is compact and any its partial limit satisfies $x_* = A[\alpha](\hat{f} + \alpha I)x_*$, $x_* \in \langle u_*, v_* \rangle$, i.e. $x_* \in \Pi_*$. This proves the robust stability of the set Π_*.

It remains to show that Π_* is a continuous branch connecting the points u_* and v_* which is the main part of the proof. Consider any open bounded domain $U \subset C(\Omega)$ such that either $u_* \in U, v_* \notin \bar{U}$ or $v_* \in U, u_* \notin \bar{U}$. We need to show that

$$\Pi_* \cap \partial U \neq \emptyset. \tag{7.7}$$

Consider the following two auxiliary lemmas.

[10)] The rotation $\gamma(I - u_*, \partial U)$ is 1 if $u_* \in U$ and is 0 if $u_* \notin \bar{U}$ for every point u_* and every open bounded domain U; here $\bar{U}$ is the closure of U.

Lemma 6. *For any sufficiently large n and any m the implication*

$$u_n \ll z \ll v_m, \quad H_n z \gg z \quad \Longrightarrow \quad z \notin \partial U \tag{7.8}$$

is valid.

Proof. Due to the monotonicity of the operator H_n, the relations $H_n z \gg z \quad (z \in \langle x_-, x_+\rangle)$ imply that all the elements of sequence (3.11) satisfy $y_k^{(n)} \gg z$, hence the limit u_n of (3.11) satisfies $u_n \gg z$. Therefore the implication

$$u_n \ll z \ll x_+, \quad H_n z \gg z \quad \Longrightarrow \quad z = u_n$$

holds. From $u_n \to u_*$, $u_* \notin \partial U$ it follows that $u_n \notin \partial U$ for every sufficiently large n. This proves (7.8) ∎

Lemma 7. *For every sufficiently large m there is a $n = n(m) \geq m$ such that the implication*

$$u_n \ll z \ll v_m, \quad G_m z \ll z \quad \Longrightarrow \quad z \notin \partial U \tag{7.9}$$

is valid.

Proof. Since $v_m \to v_*$, $v_* \notin \partial U$, the relation $v_m \notin \partial U$ is valid for every sufficiently large m. Let us fix such a m. We show that any sequence $\{z_n\}$ such that

$$u_n \ll z_n \ll v_m, \quad G_m z_n \ll z_n$$

converges to v_m, therefore (7.9) follows from $v_m \notin \partial U$.

The estimate $g_m(t,x) > h_n(t,x)$ implies $(\hat{g}_m + \beta_m I)u_n \gg (\hat{h}_n + \beta_m I)u_n$. Applying to both sides the positive operator $A[\beta_m]$, we obtain $G_m u_n \gg u_n$. From the relations $G_m u_n \gg u_n$ and $G_m v_m = v_m$ it follows that the monotone operator G_m maps the cone interval $\langle u_n, v_m\rangle$ into itself. Therefore $G_m z_n \in \langle u_n, v_m\rangle$ for each n.

Consider any partial limit y_* of the compact sequence $\{y_n\} = \{G_m z_n\} \quad (n \geq 1)$. The relations $y_n \in \langle u_n, v_m\rangle$, $u_n \to u_*$ imply $y_* \in \langle u_*, v_m\rangle$. Also, $y_n = G_m z_n \ll z_n$ implies $G_m y_n \ll y_n$ for each n, hence $G_m y_* \ll y_*$.

From the relations $y_* \in \langle u_*, v_m\rangle$, $G_m y_* \ll y_*$ it follows that all elements of sequence (3.13) with $n = m$ satisfy $z_k^{(m)} \ll y_*$, therefore $v_m = \lim_k z_k^{(m)} \ll y_* \ll v_m$, i.e. $y_* = v_m$. Since this is true for any partial limit y_* of the compact sequence y_n, we conclude that $y_n \to v_m$ and the inequalities $y_n \ll z_n \ll v_m$ imply $z_n \to v_m$ ∎

For any sufficiently large m consider the cone interval $\langle u_n, v_m\rangle$ such that both implications (7.8) and (7.9) are valid; here $n = n(m) \geq m$. This cone interval is a bounded convex closed subset of $C(\Omega)$, therefore there exists a continuous projector P onto this subset:

$$Pz = z, \quad z \in \langle u_n, v_m\rangle, \qquad Pz \in \langle u_n, v_m\rangle, \quad z \in C(\Omega).$$

Consider the completely continuous vector field

$$\Phi(z,\lambda) = z - \lambda G_m P z - (1-\lambda)v_m \qquad (z \in \bar{U})$$

depending on the parameter $\lambda \in [0,1]$. Recall that the operator G_m maps the cone interval $\langle u_n, v_m\rangle$ into itself. Therefore the inclusions $Pz, v_m \in \langle u_n, v_m\rangle$ imply $\lambda G_m Pz + (1-\lambda)v_m \in \langle u_n, v_m\rangle$ for any z and λ, hence every zero z^* of the vector field $\Phi(\cdot,\lambda)$ satisfies

$$z^* = \lambda G_m z^* + (1-\lambda)v_m, \qquad u_n \ll z^* = Pz^* \ll v_m.$$

This means that either $\lambda = 0, z^* = v_m$ or $\lambda > 0, z^* \gg \lambda G_m z^* + (1-\lambda)z^*$. In both cases, $z^* \gg G_m z^*$ and by Lemma 7 $z^* \notin \partial U$, i.e. the vector field $\Phi(\cdot,\lambda)$ is non-degenerate on ∂U for each λ. Therefore the rotation $\gamma(\Phi(\cdot,\lambda), \partial U)$ of this field on the boundary ∂U of the domain U is defined and this rotation is the same for all $\lambda \in [0,1]$. In particular,

$$\gamma(I - G_m P, \partial U) = \gamma(I - v_m, \partial U) \tag{7.10}$$

for $\lambda = 1$ and $\lambda = 0$.

Now consider the vector fields $\Psi(z,\alpha) = z - A[\alpha](\hat{g}_m + \alpha I)Pz$ with the parameter $\alpha \geq \beta_m$. For $\alpha = \beta_m$,

$$\Psi(z,\beta_m) = \Phi(z,1) = z - G_m Pz. \tag{7.11}$$

Since $A[\alpha]$, $\hat{g}_m + \alpha I$ are monotone operators for $\alpha \geq \beta_m$ and $g_m(t,x) > h_n(t,x)$, the inclusion $z \in \langle u_n, v_m\rangle$ implies

$$u_n = A[\alpha](\hat{h}_n + \alpha I)u_n \ll A[\alpha](\hat{g}_m + \alpha I)z \ll A[\alpha](\hat{g}_m + \alpha I)v_m = v_m, \tag{7.12}$$

hence the operator $A[\alpha](\hat{g}_m + \alpha I)$ maps the cone interval $\langle u_n, v_m\rangle$ into itself. Therefore $\Psi(z,\alpha) = 0$ if and only if

$$z = A[\alpha](\hat{g}_m + \alpha I)z, \qquad z = Pz \in \langle u_n, v_m\rangle,$$

i.e. the zeros of the vector field $\Psi(\cdot,\alpha)$ for each α coincide with the solutions of the problem $Lx = g_m(t,x)$ lying in the cone interval $\langle u_n, v_m\rangle$. We see that all the vector fields $\Psi(\cdot,\alpha)$, $\alpha \geq \beta_m$, have the same zeros, hence they are non-degenerate on ∂U together with field (7.11). Since the vector fields $\Psi(\cdot,\alpha)$ depend continuously on α uniformly with respect to z, they have the same rotation on ∂U and (7.10) implies

$$\gamma(I - A[\alpha](\hat{g}_m + \alpha I)P, \partial U) = \gamma(I - v_m, \partial U) \qquad (\alpha \geq \beta_m). \tag{7.13}$$

Similarly one can prove the formula

$$\gamma(I - A[\alpha](\hat{h}_n + \alpha I)P, \partial U) = \gamma(I - u_n, \partial U) \qquad (\alpha \geq \alpha_n). \tag{7.14}$$

Lemma 6 implies that the vector fields $z - \lambda H_n Pz - (1-\lambda)u_n$ are non-degenerate on ∂U for all $\lambda \in [0,1]$ and therefore $\gamma(I - H_n P, \partial U) = \gamma(I - u_n, \partial U)$, i.e. (7.14) is true for $\alpha = \alpha_n$. For $\alpha \geq \alpha_n$ formula (7.14) follows from the fact that the vector fields $z - A[\alpha](\hat{h}_n + \alpha I)Pz$ depend continuously on α and have the same zeros (these zeros are the solutions of the problem $Lx = h_n(t,x)$ lying in the cone interval $\langle u_n, v_m\rangle$).

By assumption, the domain U contains one of the points u_*, v_*, the other point lies outside the set $\bar{U}$. Suppose m and $n = n(m)$ are sufficiently large. Then U contains exactly one of the points u_n, v_m. Therefore one of the rotations $\gamma(I - u_n, \partial U), \gamma(I -$

$v_m, \partial U)$ equals 0, the other rotation equals 1. Fix any $\alpha \geq \alpha_n, \beta_m$. From (7.13), (7.14), the relation

$$\gamma(I - A[\alpha](\hat{g}_m + \alpha I)P, \partial U) \neq \gamma(I - A[\alpha](\hat{h}_n + \alpha I)P, \partial U)$$

follows, consequently any completely continuous deformation connecting the vector fields $z - A[\alpha](\hat{g}_m + \alpha I)Pz$ and $z - A[\alpha](\hat{h}_n + \alpha I)Pz$ is degenerate on ∂U. In particular, the linear deformation

$$\Theta(z, \lambda) = z - A[\alpha](\lambda \hat{g}_m + (1-\lambda)\hat{h}_n + \alpha I)Pz, \quad \lambda \in [0,1]$$

has at least one zero $z = z_m \in \partial U$ for some $\lambda = \lambda_m$. From (7.12) and the similar relations

$$u_n \ll A[\alpha](\hat{h}_n + \alpha I)z \ll v_m \qquad (z \in \langle u_n, v_m \rangle)$$

it follows that the operator $A[\alpha](\lambda \hat{g}_m + (1-\lambda)\hat{h}_n + \alpha I)$ maps the cone interval $\langle u_n, v_m \rangle$ into itself, hence the equality $\Theta(z_m, \lambda_m) = 0$ is equivalent to the relations

$$z_m = A[\alpha](\lambda_m \hat{g}_m + (1-\lambda_m)\hat{h}_{n(m)} + \alpha I)z_m, \quad u_{n(m)} \ll z_m = Pz_m \ll v_m. \tag{7.15}$$

Finally, consider any partial limit z_* of the compact sequence $\{z_m\}$. Relations (3.3) and (7.15) imply $z_* = A[\alpha](\hat{f} + \alpha I)z_*$ and $u_* \ll z_* \ll v_*$, i.e. $z_* \in \Pi_*$. Also, $z_m \in \partial U$ implies $z_* \in \partial U$, which proves that (7.7) is valid and that Π_* is a continuous branch connecting the points u_* and v_*. Theorem 1 is completely proved ∎

8. Proof of Theorems 2 - 4

8.1 Proof of Lemma 3. First note that if the relation $A[\alpha](\hat{\phi} + \alpha I)u \gg u$ holds for some u and $\alpha > \alpha_0$, then it holds also for any $\beta \in (\alpha_0, \alpha)$ in place of α. Indeed, applying the positive operator $(\alpha - \beta)A[\beta]$ to this relation and using the resolvent identity

$$A[\beta] - A[\alpha] = (\alpha - \beta)A[\beta]A[\alpha], \tag{8.1}$$

we obtain

$$A[\beta](\hat{\phi} + \alpha I)u - A[\alpha](\hat{\phi} + \alpha I)u \gg (\alpha - \beta)A[\beta]u$$

and hence

$$A[\beta](\hat{\phi} + \beta I)u \gg A[\alpha](\hat{\phi} + \alpha I)u \gg u.$$

Similarly, if $A[\alpha](\hat{\phi} + \alpha I)u \ll u$, then $A[\beta](\hat{\phi} + \beta I)u \ll u$ for all $\beta \in (\alpha_0, \alpha)$. Therefore it suffices to prove relations (4.4) for sufficiently large α only. Both the relations can be proved in the same way, we prove the first one.

Denote by y_* the limit of the decreasing sequence (3.6). By definition, y_* is a fixed point of the operator $H_n = A[\alpha_n](\hat{h}_n + \alpha_n I)$, hence it is a solution of the problem $Ly = h_n(t, y)$. Fix any $\alpha > \alpha_0, c$. Due to (4.3), the operator $\hat{\phi} + \alpha I$ is monotone on $\langle x_-, x_+ \rangle$, therefore $y_k^{(n)} \gg y_*$ implies

$$A[\alpha](\hat{\phi} + \alpha I)y_k^{(n)} \gg A[\alpha](\hat{\phi} + \alpha I)y_* \qquad (k \geq 0). \tag{8.2}$$

We prove that for all sufficiently large k

$$y_k^{(n)} \ll A[\alpha](\hat{\phi} + \alpha I)y_*, \tag{8.3}$$

then the required relation (4.4) follows from (8.2).

Estimates (4.2) imply $(\hat{\phi}y_* - \hat{h}_n y_*)(t) \geq \delta > 0$ for all $t \in \Omega$. Since $y_* = \lim_k y_k^{(n)}$ and the operators $A[\alpha_n], \hat{h}_n$ are continuous in $C(\Omega)$, for each sufficiently large k the relation

$$\big(I + (\alpha - \alpha_n)A[\alpha_n]\big)\big((\hat{h}_n + \alpha_n I)y_{k-1}^{(n)} - (\hat{h}_n + \alpha_n I)y_*\big) \ll (\hat{\phi} - \hat{h}_n)y_* \tag{8.4}$$

is valid. Let us apply the positive operator $A[\alpha]$ to both sides. Using (8.1), we get

$$A[\alpha_n](\hat{h}_n + \alpha_n I)y_{k-1}^{(n)} - A[\alpha_n](\hat{h}_n + \alpha_n I)y_* \ll A[\alpha](\hat{\phi} - \hat{h}_n)y_*,$$

i.e.

$$y_k^{(n)} \ll A[\alpha](\hat{\phi} - \hat{h}_n)y_* + A[\alpha_n](\hat{h}_n + \alpha_n I)y_*.$$

Due to

$$y_* = A[\alpha_n](\hat{h}_n + \alpha_n I)y_* = A[\alpha](\hat{h}_n + \alpha I)y_*$$

this is equivalent to (8.3) ∎

8.2 Proof of Theorem 2. By definition, sequence (3.11) decreases, therefore $y_{s(n)}^{(n)} \gg \lim_k y_k^{(n)} = u_n$. From the relations

$$y_{s(n-1)}^{(n-1)} \ll H_n y_{s(n-1)}^{(n-1)}, \qquad H_n x_+ \ll x_+, \qquad y_{s(n-1)}^{(n-1)} \ll x_+,$$

it follows that the monotone operator H_n maps the cone interval $\langle y_{s(n-1)}^{(n-1)}, x_+\rangle$ into itself, hence all the elements of sequence (3.11) and its limit u_n lie in this cone interval. Therefore $y_{s(n-1)}^{(n-1)} \ll u_n$ and the proof is complete ∎

8.3 Proof of Theorem 3. By definition of the sequence $\{d_n\}$, the relations

$$d_{n-1} \ll z_{m_{n-1}} \ll z_{m_n} \ll x_+, \qquad d_{n-1} \ll H_n d_{n-1}$$

are valid for each n. Due to the monotonicity of the operator H_n, these relations imply that all elements of sequences (3.11) and (4.8) satisfy $y_k^{(n)} \gg d_{n-1}$, i.e.

$$d_{n-1} \ll (H_n)^k x_+, \quad d_{n-1} \ll (H_n)^k z_{m_n} \qquad (k \geq 0).$$

Passing to the limit in the first relation and setting $k = s(n)$ in the second one, we obtain $d_{n-1} \ll u_n$ and $d_{n-1} \ll d_n$. Since $u_n \ll u_*$, it follows that

$$d_1 \ll d_2 \ll \ldots \ll d_n \ll \ldots \ll u_* \tag{8.5}$$

where u_* is solution (3.12) of problem (2.3). It remains to prove $d_n \to u_*$ if (4.9) holds.

Fix any index ℓ. The relations $z_m \to x_+$ and $g_1(t,x) > h_\ell(t,x)$ imply for every sufficiently large m

$$\big(I + (\alpha_\ell - \beta_1)A[\beta_1]\big)\big((\hat{g}_1 + \beta_1 I)x_+ - (\hat{g}_1 + \beta_1 I)z_{m-1}\big) \ll (\hat{g}_1 - \hat{h}_\ell)x_+.$$

Applying to both sides the operator $A[\alpha_\ell]$ and using (8.1), we get

$$G_1 x_+ - G_1 z_{m-1} \ll A[\alpha_\ell](\hat{g}_1 - \hat{h}_\ell)x_+. \tag{8.6}$$

Here $G_1 x_+ = x_+$ due to (4.9), hence $x_+ = A[\alpha](\hat{g}_1 + \alpha I)x_+$ for every $\alpha > \alpha_0$ and we can rewrite (8.6) as

$$x_+ - z_m \ll x_+ - \alpha_\ell A[\alpha_\ell]x_+ - A[\alpha_\ell]\hat{h}_\ell x_+,$$

i.e. $z_m \gg A[\alpha_\ell](\hat{h}_\ell + \alpha_\ell I)x_+ = H_\ell x_+$. Since the sequence $x_+, H_\ell x_+, (H_\ell)^2 x_+, \ldots$ decreases and converges to u_ℓ, this inequality implies $z_m \gg u_\ell$.

Take any $n = n(\ell)$ such that $n > \ell$ and $z_{m_n} \gg u_\ell$. From (7.1) and $h_n(t,x) > h_\ell(t,x)$ the relations

$$u_\ell = A[\alpha_n](\hat{h}_\ell + \alpha_n I)u_\ell \ll A[\alpha_n](\hat{h}_n + \alpha_n I)u_\ell = H_n u_\ell$$

follow. The inequalities $u_\ell \ll H_n u_\ell$, $u_\ell \ll z_{m_n}$ imply that all elements of sequence (4.8) satisfy $u_\ell \ll y_k^{(n)}$, in particular $u_\ell \ll d_n$. Finally, the relations $u_\ell \to u_*, u_\ell \ll d_{n(\ell)}$ and (8.5) imply $d_n \to u_*$. This completes the proof ∎

8.4 Proof of Theorem 4. Take any $n \geq 2$ and consider the function $\tilde{v}_{n-1}$. By definition, $\tilde{v}_{n-1} \gg H_{n-1}\tilde{v}_{n-1}$, therefore the sequence

$$y_0 = \tilde{v}_{n-1}, \quad y_k = H_{n-1}y_{k-1} \quad (k \geq 1)$$

decreases and its element $\tilde{u}_{n-1}$ satisfies $\tilde{u}_{n-1} \ll \tilde{v}_{n-1}$. Also, by definition,

$$\tilde{u}_{n-1} \ll G_n \tilde{u}_{n-1}, \qquad G_n \tilde{v}_{n-1} \ll \tilde{v}_{n-1}.$$

These relations imply that sequence (5.1) increases and converges uniformly to a solution $\zeta_n \in \langle \tilde{u}_{n-1}, \tilde{v}_{n-1} \rangle$ of the equation $x = G_n x$. Since $\tilde{v}_n$ is an element of this sequence, it follows that $\tilde{u}_{n-1} \ll \tilde{v}_n \ll \zeta_n \ll \tilde{v}_{n-1}$.

Similarly, the relations

$$\tilde{u}_{n-1} \ll H_n \tilde{u}_{n-1}, \qquad H_n \tilde{v}_n \ll \tilde{v}_n$$

imply $\tilde{u}_{n-1} \ll \xi_n \ll \tilde{u}_n \ll \tilde{v}_n$, where ξ_n is the limit of the decreasing sequence (5.4), $\xi_n = H_n \xi_n$. Thus, we have

$$\tilde{u}_1 \ll \xi_2 \ll \ldots \ll \tilde{u}_{n-1} \ll \xi_n \ll \tilde{u}_n \ll \ldots \ll \tilde{v}_n \ll \zeta_n \ll \tilde{v}_{n-1} \ll \ldots \ll \zeta_2 \ll \tilde{v}_1. \tag{8.7}$$

In the same way as in the proof of Theorem 1, the relations

$$\xi_1 \ll \xi_2 \ll \ldots \ll \xi_n \ll \ldots \ll \zeta_n \ll \ldots \ll \zeta_2 \ll \zeta_1, \qquad \xi_n = H_n \xi_n, \ \zeta_n = G_n \zeta_n$$

imply that the sequences $\{\xi_n\}$ and $\{\zeta_n\}$ converge uniformly to solutions $u_\star$ and $v_\star$ of problem (2.3), respectively, the inequality $u_\star \ll v_\star$ holds, and the set of all solutions of problem (2.3) contained in the cone interval $\langle u_\star, v_\star \rangle$ is robust stable. Now all conclusions of Theorem 4 follow from (8.7) ∎

9. Proof of Theorem 5

9.1 Proof of Lemmas 4 and 5. Because both lemmas can be proved in the same way, we prove Lemma 4. The constructions are basically the same as in the corresponding parts of the proofs of Theorems 1 and 4.

Since sequence (5.7) increases and each sequence (5.9) decreases, the relations $\tilde{d}_{k-1} \ll z_{m_{k-1}} \ll z_{m_k}$ are valid for every k. The functions $\tilde{d}_{k-1}$ and z_{m_k} satisfy

$$\tilde{d}_{k-1} \ll H_{j_n+k}\tilde{d}_{k-1}, \qquad H_{j_n+k}z_{m_k} \ll z_{m_k}$$

by definition. These relations imply the inequalities $\tilde{d}_{k-1} \ll \eta_k \ll y_\ell^{(k)}$ for each element of sequence (5.9), where η_k is the limit of this sequence, $\eta_k = H_{j_n+k}\eta_k$. In particular, $\tilde{d}_{k-1} \ll \eta_k \ll \tilde{d}_k$.

From the relations

$$\tilde{d}_1 \ll \eta_2 \ll \tilde{d}_2 \ll \eta_3 \ll \ldots \ll \tilde{d}_{k-1} \ll \eta_k \ll \tilde{d}_k \ll \ldots \ll x_+, \qquad \eta_k = H_{j_n+k}\eta_k,$$

it follows that the sequences $\{\tilde{d}_k\}$ and $\{\eta_k\}$ converge uniformly from below to the same limit $\tilde{d}_*$ and this limit is a solution of problem (2.3).

It remains to prove (5.10). Set $\alpha = \beta_{i_n+1}$. The estimate $g_{i_n+1}(t,x) - f(t,x) \geq \delta > 0$ implies the inequality $(\hat{g}_{i_n+1} + \alpha I)\tilde{d}_k \gg (\hat{f} + \alpha I)\tilde{d}_*$ for each sufficiently large k, therefore

$$G_{i_n+1}\tilde{d}_k = A[\alpha](\hat{g}_{i_n+1} + \alpha I)\tilde{d}_k \gg A[\alpha](\hat{f} + \alpha I)\tilde{d}_* = \tilde{d}_*.$$

Now (5.10) follows from $\tilde{d}_* \gg \tilde{d}_{k+1}$ ∎

9.2 Proof of Theorem 5. The proof of this theorem is close to that of Theorem 1. Thus we give a sketch of it only.

Denote by ζ_n and ξ_{n+1} the limits of the increasing sequence (5.7) and the decreasing sequence (5.11), respectively. From the definition of $\bar{u}_n$ and $\bar{v}_n$ it follows without difficulty that

$$\bar{u}_1 \ll \xi_1 \ll \bar{u}_2 \ll \xi_2 \ll \ldots \ll \zeta_2 \ll \bar{v}_2 \ll \zeta_1 \ll \bar{v}_1. \tag{9.1}$$

These relations and the equalities $\xi_n = H_{j_n}\xi_n, \zeta_n = G_{i_n}\zeta_n$ imply that the functions

$$u^\star = \sup \bar{u}_n = \sup \xi_n$$
$$v^\star = \inf \bar{v}_n = \inf \zeta_n$$

are solutions of problem (2.3), the inequalities (5.14) are valid, and the sequences $\{\bar{u}_n\}, \{\xi_n\}$ converge uniformly to $u^\star$, the sequences $\{\bar{v}_n\}, \{\zeta_n\}$ converge uniformly to $v^\star$.

To prove robust stability of the set $\Pi_\star$ of all solutions of problem (2.3) lying in the cone interval $\langle u^\star, v^\star \rangle$, one can use the argument of Subsections 7.2 and 7.4, it suffices just to replace the functions u_n and v_n by ξ_n and ζ_n, and the operators $\hat{h}_n$ and $\hat{g}_n$ by $\hat{h}_{j_n}$ and $\hat{g}_{i_n}$, respectively.

We need analogs of Lemmas 6 and 7 to prove the last conclusion of the theorem (that $\Pi_\star$ is a continuous branch if $u^\star \neq v^\star$). Suppose $u^\star \neq v^\star$ and consider any open bounded domain U such that either $u^\star \in U, v^\star \notin \bar{U}$ or $v^\star \in U, u^\star \notin \bar{U}$. Consider the sequence

$$\bar{z}_0 = \xi_{n+1}, \quad \bar{z}_\ell = G_{i_n+1}\bar{z}_{\ell-1} \ \ (\ell \geq 1). \tag{9.2}$$

The equality $\xi_{n+1} = H_{j_{n+1}}\xi_{n+1}$ implies that $\xi_{n+1} \ll G_{i_n+1}\xi_{n+1}$, therefore this sequence increases, denote its limit by τ_{n+1}. By definition, $\tau_{n+1} = G_{i_n+1}\tau_{n+1}$.

Lemma 8. *For every sufficiently large n the implications*

$$\begin{aligned} H_{j_{n+1}}x \gg x \quad &\Longrightarrow \quad x \notin \partial U \\ G_{i_n+1}x \ll x \quad &\Longrightarrow \quad x \notin \partial U \end{aligned} \tag{9.3}$$

hold for every $x \in \langle \xi_{n+1}, \tau_{n+1} \rangle$. The sequence $\{\tau_n\}$ converges to $v^\star$.

Proof. If $\xi_{n+1} \ll x$, $G_{i_n+1}x \ll x$, then all elements of sequence (9.2) satisfy $\bar{z}_\ell \ll x$, hence $\tau_{n+1} \ll x$. Therefore the implication

$$\xi_{n+1} \ll x \ll \tau_{n+1}, \quad G_{i_n+1}x \ll x \quad \Longrightarrow \quad x = \tau_{n+1} \tag{9.4}$$

holds. Consider the decreasing sequence (5.9) for $k = k_0 + 1$:

$$y_0^{(k_0+1)} = z_{m_{k_0+1}}, \quad y_\ell^{(k_0+1)} = H_{j_{n+1}} y_{\ell-1}^{(k_0+1)} \ \ (\ell \geq 1).$$

By definition, this sequence contains sequence (5.11), hence they have the same limit ξ_{n+1} and $\xi_{n+1} \ll z_{m_{k_0+1}}$. Also, $G_{i_n+1}z_{m_{k_0+1}} \ll z_{m_{k_0+1}}$ due to (5.8), therefore the relation $\tau_{n+1} \ll z_{m_{k_0+1}}$ holds. If there is a x such that $x \ll \tau_{n+1} \ll z_{m_{k_0+1}}$ and $H_{j_{n+1}}x \gg x$, then $y_\ell^{(k_0+1)} \gg x$ for each ℓ and $\xi_{n+1} = \lim_\ell y_\ell^{(k_0+1)} \gg x$, hence

$$\xi_{n+1} \ll x \ll \tau_{n+1}, \quad H_{j_{n+1}}x \gg x \quad \Longrightarrow \quad x = \xi_{n+1}. \tag{9.5}$$

Now note that relations (5.10) and (9.1) imply

$$\tilde{d}_{k_0+1} \ll G_{i_n+1}\tilde{d}_{k_0} = G_{i_n+1}\bar{u}_{n+1} \ll G_{i_n+1}\xi_{n+1}.$$

Since sequence (5.11) decreases and converges to ξ_{n+1}, all its elements y_m satisfy $\xi_{n+1} \ll y_m \ll y_0 = \tilde{d}_{k_0+1} \ll G_{i_n+1}\xi_{n+1}$. From $\xi_{n+1} \ll y_{m_1} \ll G_{i_n+1}\xi_{n+1}$ it follows that sequence (5.12) with $k = 1$ converges to the same limit τ_{n+1} as sequence (9.2). Both sequences increase and the first of them contains $\bar{d}_1$, hence $\bar{d}_1 \ll \tau_{n+1}$.

By Lemma 5, the sequence $\{\bar{d}_k\}$ decreases, therefore $\bar{v}_{n+1} = \bar{d}_{k_1} \ll \bar{d}_1 \ll \tau_{n+1}$. On the other hand, $\tau_{n+1} \ll z_{m_{k_0+1}}$ and $z_{m_{k_0+1}}$ is an element of sequence (5.7), which increases and converges to ζ_n, hence $\tau_{n+1} \ll \zeta_n$.

The relations $\bar{v}_{n+1} \ll \tau_{n+1} \ll \zeta_n$ and $\bar{v}_n, \zeta_n \to v^\star$ imply $\tau_n \to v^\star$. Since also $\xi_n \to u^\star$, for every sufficiently large n the relations $u^\star, v^\star \notin \partial U$ imply $\xi_{n+1}, \tau_{n+1} \notin \partial U$. Therefore (9.3) follows from (9.4), (9.5) and Lemma 8 is proved ∎

The rest of the proof of Theorem 5 follows exactly the proof of Theorem 1. If n is large enough, Lemma 8 implies the formulas

$$\gamma(I - A[\alpha](\hat{g}_{i_n+1} + \alpha I)P_1, \partial U) = \gamma(I - \tau_{n+1}, \partial U)$$
$$\gamma(I - A[\alpha](\hat{h}_{j_{n+1}} + \alpha I)P_1, \partial U) = \gamma(I - \xi_{n+1}, \partial U)$$

similar to (7.13) and (7.14), where P_1 is a projector onto the cone interval $\langle \xi_{n+1}, \tau_{n+1} \rangle$. Therefore the linear deformation

$$\Theta_1(z, \lambda) = z - A[\alpha](\lambda \hat{g}_{i_n+1} + (1 - \lambda)\hat{h}_{j_{n+1}} + \alpha I)P_1 z$$

has a zero $z_n \in \partial U \cap \langle \xi_{n+1}, \tau_{n+1} \rangle$ for some $\lambda_n \in [0, 1]$ and any partial limit z_* of the compact sequence $\{z_n\}$ satisfies the relations $z_* \in \partial U \cap \langle u^\star, v^\star \rangle$, $z_* = A[\alpha](\hat{f} + \alpha I)z_*$. This completes the proof ∎

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