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On Topological Structure of Solution Sets for Delay and Functional-Differential Equations

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Abstract. In this paper we characterize the topological structure of global solution sets for classical delay and functional-differential equations in terms of R_δ sets.

Keywords: *Aronszajn's property, global solutions, delay and functional-differential equations*

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1. Introduction

Let us consider the delay differential equation

$$x' = f(t,x) + g\big(t, x(t-\tau(t))\big) \qquad (t \geq t_0 \geq 0) \tag{1}$$

where $f, g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions. The set

$$E_{t_0} = \{t_0\} \cup \{s :\ s = t - \tau(t) \leq t_0 \text{ for } t \geq t_0\}$$

is said to be the initial interval for equation (1) at t_0. In what follows we suppose that E_{t_0} is bounded for every $t_0 \in \mathbb{R}_+$. Recall that for any initial continuous function $x_0 : E_{t_0} \to \mathbb{R}^n$ a function $x = x(t)$ is a solution of (1) on $[t_0, t_0+a)$ for some $0 < a \leq +\infty$ if x is continuous on $E_{t_0} \cup [t_0, t_0 + a)$, satisfies (1) on $(t_0, t_0 + a)$ and

$$x(t) = x_0(t) \qquad \text{for } t \in E_{t_0}. \tag{2}$$

It is well known that under the above assumptions problem (1) has a local solution (see [4, 5]).

Recently Constantin [2] formulated conditions which guarantee the existence of global solutions of equation (1). In this paper we establish that under suitable assumptions the set of all global solutions of equation (1) is a compact R_δ, i.e. it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. In particular, it is non-empty, compact and connected in a suitable space of continuous functions.

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Further, we shall consider more general functional-differential equations of the form

$$x' = f(t,x) + g(t,x_t) \qquad (x_b = \gamma, t \geq b) \tag{3}$$

where $b \in \mathbb{R}$, $\gamma : [-h,0] \to \mathbb{R}^n$ is a continuous function and $x_t(s) = x(t+s)$ for $s \in [-h,0]$. We prove an Aronszajn type theorem for problem (3), too.

Recall that toplogical properties of solution sets for simpler equations of the form

$$x' = g(t,x_t) \qquad (x_t = \gamma, t \geq b) \tag{4}$$

were investigated e.g. by Šeda and Kubáček [12] (Kneser type theorem) and by Kubáček [11] (Aronszajn type theorem). A multi-valued version of Kneser's theorem for an equation of type (4) was proved by Krbec and Kurzweil in [9] (see also references given in [2: p. 85]).

The proofs of our results are based on the following Vidossich type theorem.

Theorem 1 (Kubáček [10]). *Let K be a convex subset of a normed space $(Z, |\cdot|)$, $(Y, \|\cdot\|)$ be a Banach space, and let X be the space of all continuous locally bounded maps $f : K \to Y$ (i.e. bounded on each bounded subset of K) equipped with the topology of locally uniform convergence. Denote*

$$M = \Big\{ x \in X : \|x(t) - h(t)\| \leq p(t) \quad (t \in K) \Big\}$$

where $h \in X$ and $p : K \to \mathbb{R}$ is a non-negative locally bounded continuous function, and let a compact map $T : M \to M$ satisfy the following conditions:

(i) *There exist $t_0 \in K$ and $y_0 \in Y$ such that $\|y_0 - h(t_0)\| \leq p(t_0)$ and $T(x)(t_0) = y_0$ for every $x \in M$.*

(ii) *$T(M)$ is a set of locally equiuniformly continuous maps.*

(iii) *For every $\varepsilon > 0$ and for all $x, y \in M$, $x|_{K_\varepsilon} = y|_{K_\varepsilon}$ implies $T(x)|_{K_\varepsilon} = T(y)|_{K_\varepsilon}$ where $K_\varepsilon = \{t \in K : |t - t_0| \leq \varepsilon\}$.*

Then the set of all fixed points of T is a compact R_δ.

Remark. Note that a slight technical change in the proof of Theorem 1 shows that instead of the set M one can consider the set

$$M' = \Big\{ x \in X : \|x(t) - h(t)\| \leq p(t) \ (t \in K), x(t_0) = y_0, \|y_0 - h(t_0)\| \leq p(t_0) \Big\}.$$

Obviously, in this case condition (i) has the form $T(x)(t_0) = y_0$ for every $x \in M$.

2. Results and proofs

Denote by $\mathcal{R}_0$ the class of continuous scalar functions w such that $w(r) > 0$ for $r \geq \delta > 0$ and $\int_\delta^{+\infty} \frac{ds}{w(s)} = +\infty$.

In what follows we shall need the following result due to Constantin [2: p. 243/Theorem 2.3] concerning continuability of solutions of perturbed ordinary differential equations

$$r' = \varphi(t)w(r) + \psi(t)z(r). \tag{5}$$

Theorem 2. *Assume $\varphi, \psi, z, w : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions such that $z(r) > 0$ and $w(r) > 0$ for all $r \geq \delta \geq 0$. Let $w \in \mathcal{R}_0$ and suppose there exist constants $K, L, M > 0$ such that*

$$z(r) \leq Kw(r) \int_\delta^r \frac{ds}{w(s)} + Mw(r) \qquad (r \geq L \geq 0).$$

Then the solutions of equation (5) are defined in the future.

Now, we prove the following Aronszajn type result for problem (1) - (2).

Theorem 3. *Let $f, g : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions. Suppose there exist continuous functions $\varphi, \psi, z, w : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions of Theorem 2 with w and z non-decreasing on $\mathbb{R}_+$ such that*

$$\left.\begin{aligned} \|f(t,x)\| &\leq \varphi(t)z(\|x\|) \\ \|g(t,x)\| &\leq \psi(t)w(\|x\|) \end{aligned}\right. \qquad \big((t,x) \in \mathbb{R}_+ \times \mathbb{R}^n\big). \tag{6}$$

Moreover, suppose $\sup_{t \in E_{t_0}} \|x_0(t)\| = \|x_0(t_0)\|$. Then the set of all solutions of problem (1) $-$ (2) is a compact R_δ.

Proof. It can be easily verified that problem (1) - (2) is equivalent to the problem

$$\left.\begin{aligned} x(t) &= x_0(t_0) + \int_{t_0}^t \big[f(s,x(s)) + g\big(s, x(s-\tau(s))\big)\big]\, ds \quad (t > t_0) \\ x(s) &= x_0(s) \qquad\qquad\qquad\qquad\qquad\qquad\qquad (s = t - \tau(t) \in E_{t_0}) \end{aligned}\right\}.$$

Let $X = C([t_0, +\infty), \mathbb{R}^n)$ be the space of all continuous functions $[t_0, +\infty) \to \mathbb{R}^n$ with the topology of locally uniform convergence and analogously let $\widetilde{X} = C\big(E_{t_0} \cup [t_0, +\infty), \mathbb{R}^n\big)$. Moreover, let $r : [t_0, +\infty) \to \mathbb{R}_+$ be a solution of equation (5) with initial condition $r(t_0) = \|x_0(t_0)\|$. Remark that in view of Theorem 2 a solution r is defined on the whole interval $[t_0, +\infty)$. Let

$$\widetilde{M} = \Big\{x \in \widetilde{X} : x|_{E_{t_0}} = x_0 \text{ and } \|x(t)\| \leq r(t) \text{ for every } t \geq t_0\Big\}.$$

Define

$$\widetilde{T}(x)(t) = \begin{cases} x_0(t_0) + \int_{t_0}^t \big[f(s,x(s)) + g\big(s, x(s-\tau(s))\big)\big] ds & \text{if } t > t_0 \\ x_0(t - \tau(t)) & \text{if } t - \tau(t) \in E_{t_0} \end{cases}$$

for $x \in \widetilde{M}$ and $t > t_0$. It can be easily verified that x is a solution of system (1) - (2) if and only if x is a fixed point of $\widetilde{T}$. Moreover, it is clear that any such solution x satisfies the inequality $\|x(t)\| \le r(t)$ for every $t \ge t_0$. Now, set

$$V = \big\{x \in X : x(t_0) = x_0(t_0)\big\}$$
$$\widetilde{V} = \big\{x \in \widetilde{X} : x|_{E_{t_0}} = x_0\big\}$$

and define the mapping $P : V \to \widetilde{V}$ by the formula

$$P(x)(t) = \begin{cases} x(t) & \text{if } t > t_0 \\ x_0(t-\tau(t)) & \text{if } t-\tau(t) \in E_{t_0}. \end{cases}$$

It is not difficult to verify that P is a homeomorphism (note that $P^{-1}(x) = x|_{(t_0,+\infty)}$ for $x \in \widetilde{V}$). Consider the operator $T = P^{-1} \circ \widetilde{T} \circ P|_M$, where

$$M = \Big\{x \in X : \|x(t)\| \le r(t) \ (t \ge t_0) \text{ and } x(t_0) = x_0(t_0)\Big\}.$$

Now, we show that T maps M into itself. Indeed, have

$$\begin{aligned} &T(x)(t) \\ &= (P^{-1} \circ \widetilde{T} \circ P)(x)(t) \\ &= P^{-1}\left(\left\{\begin{matrix} x_0(t_0) + \int_{t_0}^t \big[f\big(s, P(x)(s)\big) + g\big(s, P(x)(s-\tau(s))\big)\big]ds & \text{if } t > t_0 \\ x_0(t-\tau(t)) & \text{if } t-\tau(t) \in E_{t_0} \end{matrix}\right\}\right) \\ &= x_0(t_0) + \int_{t_0}^t \big[f(s, x(s)) + g\big(s, P(x)(s-\tau(s))\big)\big]ds \end{aligned}$$

and, by (6),

$$\begin{aligned} \|T(x)(t)\| &\le \|x_0(t_0)\| + \int_{t_0}^t \big[\|f(s,x(s))\| + \big\|g\big(s, P(x)(s-\tau(s))\big)\big\|\big]ds \\ &\le \|x_0(t_0)\| + \int_{t_0}^t \big[\varphi(s)z(\|x(s)\|) + \psi(s)w\big(\big\|P(x)(s-\tau(s))\big\|\big)\big]ds \end{aligned}$$

for $x \in M$ and $t \ge t_0$. Because

$$\begin{aligned} \big\|P(x)(s-\tau(s))\big\| &= \begin{cases} \|x(s-\tau(s))\| & \text{if } s-\tau(s) > t_0 \\ \|x_0(s-\tau(s))\| & \text{if } s-\tau(s) \le t_0 \end{cases} \\ &\le \begin{cases} r(s-\tau(s)) & \text{if } s-\tau(s) > t_0 \\ \|x_0(t_0)\| & \text{if } s-\tau(s) \le t_0 \end{cases} \\ &= \begin{cases} r(s-\tau(s)) & \text{if } s-\tau(s) > t_0 \\ r(t_0) & \text{if } s-\tau(s) \le t_0 \end{cases} \\ &\le r(s) \end{aligned}$$

for $s \geq t_0$, so in view of the assumptions that the functions z and w are non-decreasing we obtain

$$\|T(x)(t)\| \leq \|x_0(t_0)\| + \int_{t_0}^{t} \big[\varphi(s)z(r(s)) + \psi(s)w(r(s))\big]\,ds$$

for $x \in M$ and $t \geq t_0$. Thus T maps M into itself.

Now, let $x \in M$ and $t_1, t_2 \in [t_0, t_0 + a]$ for some $a > 0$, with $t_2 < t_1$. In view of the inequalities

$$\begin{aligned}
&\big\|T(x)(t_1) - T(x)(t_2)\big\| \\
&\quad = \bigg\| \int_{t_2}^{t_1} \big[f(s, x(s)) + g\big(s, P(x)(s - \tau(s))\big)\big]\,ds \bigg\| \\
&\quad \leq \int_{t_2}^{t_1} \big[\varphi(s)z(\|x(s)\|) + \psi(s)w(\|P(x)(s - \tau(s))\|)\big]\,ds \\
&\quad \leq \int_{t_2}^{t_1} \big[\varphi(s)z(r(s)) + \psi(s)w(r(s))\big]\,ds
\end{aligned}$$

it is clear that the family $T(M)$ is locally equiuniformly continuous.

To show the continuity of the mapping T assume $x, x_n \in M$ for $n \in \mathbb{N}$ and $x_n \to x$ (in the sense of the topology of M) and fix $t > t_0$. We have

$$P(x_n)(s - \tau(s)) = \begin{cases} x_n(s - \tau(s)) & \text{if } s - \tau(s) > t_0 \\ x_0(s - \tau(s)) & \text{if } s - \tau(s) \leq t_0. \end{cases}$$

Because $x_n \to x$ uniformly on $[t_0, t]$, so

$$P(x_n)(s - \tau(s)) \to P(x)(s - \tau(s))$$

uniformly on $[t_0, t]$. In view of the Krasnoselskii-Krein lemma [8]

$$g(s, P(x_n)(s - \tau(s))) \to g(s, P(x)(s - \tau(s)))$$

uniformly on $[t_0, t]$. Moreover, $f(s, x_n(s)) \to f(s, x(s))$ uniformly on this interval, so $T(x_n)(t) \to T(x)(t)$. Hence $T(x_n) \to T(x)$ uniformly on every interval $[t_0, t_0 + a]$ $(a > 0)$ which proves the continuity of T.

Further, in view of Ascoli's theorem [6: pp. 80 - 81] we infer that $T(M)$ is relatively compact, so T is a compact mapping. It is clear that T satisfies all conditions of Theorem 1 and therefore the set S of all its fixed points is a compact R_δ. As the homeomorphic image of a compact R_δ set is again a compact R_δ set, so $P^{-1}(S)$ is a compact R_δ set, which completes the proof of Theorem 2 ∎

Now let pass on to problem (3). Let $h > 0$ and $b \in \mathbb{R}$. Denote by $H = C([-h, 0], \mathbb{R}^n)$ the space of all continuous functions $[-h, 0] \to \mathbb{R}^n$ with the topology of uniform convergence and by $X = C([b, +\infty), \mathbb{R}^n)$ and $\widetilde{X} = C([b - h, +\infty), \mathbb{R}^n)$ the spaces of all continuous functions $[b, +\infty) \to \mathbb{R}^n$ and $[b - h, +\infty) \to \mathbb{R}^n$, respectively, with toplogies of locally uniform convergence. For $x \in \widetilde{X}$ and $t \in [b, +\infty)$ denote by $x_t \in H$ the function defined as

$$x_t(s) = x(t + s) \qquad (s \in [-h, 0]).$$

Then obviously $\|x_t\| = \sup_{s \in [-h, 0]} \|x_t(s)\|$.

Our next result is given by the following

Theorem 4. *Let $f : [b, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [b, +\infty) \times H \to \mathbb{R}^n$ be continuous functions. Suppose there exist continuous functions $\varphi, \psi, z, w : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions of Theorem* 2 *with z and w non-decreasing on $\mathbb{R}_+$ and inequalities* (6)*. Moreover, suppose $\gamma \in H$ is such that $\|\gamma\| = \|\gamma(0)\|$. Then the set of all solutions of problem* (3) *is a compact R_δ.*

To prove Theorem 4 it is enough to repeat similar arguments as in the proof of Theorem 3 and therefore we omit its proof.

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